On the Profinite Topology on Groups And Tameness of Pseudovarieties of Groups

Author:
Khadijeh Alibabaei

Supervisor:
Dr. Jorge Almeida

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FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

Department of Mathematics
University of Porto
Portugal

To My Mother

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Abstract

This thesis concerns three different problems in group theory and semigroup theory.

In the first paper, we show that the wreath product of a finitely generated abelian group with a polycyclic group is a LERF group. This theorem yields as a corollary that the finitely generated free metabelian group is LERF, a result due to Coulbois. We also show that the finitely generated free solvable group of degree three, which is not LERF, does not contain a strictly ascending HNN-extension of a finitely generated group. This settles, in the negative, a question of J. O. Button.

The second part of this thesis is based on the study of pseudovarieties of abelian groups. It has been shown that the proper, non-locally finite pseudovarieties of abelian groups are not tame with respect to the canonical signature. In this paper, we show that every decidable, proper, non-locally finite pseudovariety of abelian groups is completely tame with respect to a further enlarged implicit signature. This theorem yields as a corollary that a pseudovariety of abelian groups is decidable if and only if it is completely tame.

Finally, the last part is based on the study of pseudovarieties of nilpotent groups. It has been shown that for every prime number p, the pseudovariety G_p of all finite p-groups is tame with respect to an implicit signature containing the canonical implicit signature. In this paper we generalize this result and we show that the pseudovariety of all finite nilpotent groups is tame but it is not completely tame.

Contents

Introduction

Several algebraic and combinatorial tools have played an important role in the development of Computer Science and its applications. The theory of formal languages, motivated both by linguistic studies and by the development of computer languages in the 1950's, led to fruitful connections with Mathematics, in which algebraic tools such as semigroups, formal series, wreath products, and the combinatorics of words found an ideal ground for applications.

Eilenberg's treatise [30] reflects already in the mid 1970's a significant development in the area, which both systematized earlier results and fostered further research. On the Computer Science side, finite automata proved to be a simple, yet very powerful model for efficient automatic processes. Their computing power is described by regular languages and thus it became important to determine whether given problems could be handled by restricted types of finite automata, which became a classification problem for regular languages. Eilenberg proposed as a framework for this classification the so-called varieties of languages and showed how they are in natural one-to-one correspondence with pseudovarieties of semigroups. The original Computer Science problem concerning formal languages thus became translated as a question about semigroups: to decide whether a given finite semigroup belongs to a given pseudovariety. Starting in the mid 1980's, Almeida showed how one could use profinite semigroups to handle some such decidability questions [3].

In [37], Hall defined the profinite topology on a group G. In the profinite topology, the set of all normal subgroups with finite index is taken as a base of neighbourhoods of the identity. For every subgroup H of G, the profinite closure of H in G is the intersection of all the subgroups of finite index of G containing H. If every subgroup G is closed in the profinite topology of G, then G is said to be an *extended residually finite* (ERF) group. It is clear that the class of ERF groups is closed under taking subgroups and quotient groups. Mal'cev showed that every polycyclic-by-finite group is ERF (see [39] or [45, p. 148]). Two weaker properties for the profinite topology are as follows: a group is said to be *locally extended residually finite* (LERF)[1] if every finitely generated (f.g.) subgroup of G is closed in the profinite topology of G and a group is said to be *residually finite* (RF) if the trivial subgroup of G is closed in the profinite topology of G. In [39], Mal'cev showed that a finitely presented LERF group has decidable membership problem with respect to its f.g. subgroups. The classes of RF groups and LERF groups are closed under taking subgroups but they are not closed under taking quotients. In [37], Hall showed that a f.g. free group is LERF. The f.g. free solvable groups of solvable degree greater than three are RF groups but not LERF (see [33] and [1]).

In general the properties LERF and RF fail to be preserved by many natural operations. For example, in [33], Gruenberg showed that the wreath product $A \wr B$ of two groups A and B is RF if and only if A and B are RF groups and either A is abelian or B is finite. It is natural to ask under what conditions the wreath product of a group A with a group B is LERF. In general, it is an open question. Since every LERF group is RF, by the Gruenberg theorem, A must be an abelian group. Moreover as the class of LERF groups is closed under taking subgroups, A and B must be LERF. In [27, Proposition 3.19], it is shown that for every f.g. abelian group A, the wreath product $A \wr \mathbb{Z}$ is LERF. In the first chapter, we generalize this result to $A \wr Q$ where Q is a polycyclic group. The fact that every f.g. abelian-by-polycyclic group is RF [35], plays an important role in the proof of our theorem. This theorem yields as a corollary that the f.g. free metabelian group is LERF which was proved before by Coulbois in his Ph.D. thesis.

For determining that a group G is not ERF (respectively LERF), a useful method consists in applying [19, Theorem 1], which says that if a group G contains a strictly ascending HNN-extension $H = \langle B, t \mid tBt^{-1} \subset B \rangle$ with (respectively f.g.) base B and stable letter t then G is not ERF (respectively LERF). This is because in every finite quotient of G the images of the subgroups B and tBt^{-1} are conjugate subgroups and so they are of the same order, whence they must be equal. In [13], Alperin showed that a f.g. solvable group which is not polycyclic contains a subgroup that is a strictly ascending HNN-extension. Hence, by the Mal'cev theorem, a f.g. solvable group is not ERF if and only if it contains a strictly ascending HNN-extension. In [36], it is asked if a f.g. solvable group which is not LERF must contain a f.g. strictly ascending HNN-extension $H = \langle t, B \mid tBt^{-1} \subset B \rangle$ where B is a cyclic subgroup of G. In [20], it is shown that this is not the case; furthermore, there exists a finitely presented solvable group that is not LERF but does not contain a f.g. strictly ascending HNN-extension subgroup $H = \langle t, B \rangle$ where B is a polycyclic subgroup of G. Also, if we take G as a finite non-abelian solvable group then $G \wr \mathbb{Z}$ is a f.g. solvable group and

[1]Some authors call LERF groups "subgroup separable groups" [44, Section 3.7].

by the aforementioned result of [33], this group is not RF and hence, $G \wr \mathbb{Z}$ is not LERF. As mentioned in [20], this group does not contain a strictly ascending HNN-extension of a f.g. group. But the following question is still partially open:

Question. *If G is a f.g. (or finitely presented) solvable group which is RF but not LERF, then must it contain a strictly ascending HNN-extension subgroup $H = \langle t, B \rangle$ where B is a f.g. subgroup of G?*

In the first chapter, we also show that the free solvable group of solvability degree three is a counterexample. It does not completely settle the above question since it is not finitely presented (see [47] or [27]).

Similar to RF and LERF properties, the decidability of pseudovarieties is also not preserved by many operations on pseudovarieties such as semidirect product, join and Mal'cev product [42, 2]. Almeida and Steinberg introduced a refined version of decidability called *tameness* [12]. The tameness property requires the *reducibility* property which is a generalization of the notion of inevitability that Ash introduced to prove the type II conjecture of Rhodes [15].

There are various results using tameness of pseudovarieties to establish the decidability of pseudovarieties obtained by application of the operations of semidirect product, Mal'cev product and join [6, 5, 7, 24, 23, 22, 21].

Also there are connections between tameness and geometry and model theory [32, 31, 34, 9, 10, 16, 49]. So, it is worth finding more examples of tame pseudovarieties.

It has been established that the locally finite pseudovarieties of finite abelian groups and the pseudovariety Ab of all finite abelian groups are tame [11], but in [29] it has been shown that the proper non-locally finite pseudovarieties of abelian groups are not tame with respect to the canonical signature. In the second chapter, we show that such pseudovarieties are tame with respect to a further enlarged implicit signature: to every pseudovariety V of finite abelian groups, we may associate the supernatural number $\pi_{\mathsf{V}} = \operatorname{lcm}(\{n \mid \mathbb{Z}/n\mathbb{Z} \in \mathsf{V}\})$. Conversely, to a supernatural number π, we may associate the pseudovariety Ab_π of all finite abelian groups whose exponents divide π. By [48, Corollary 3.3], a supernatural number π is recursive if and only if Ab_π has decidable membership problem. For a decidable pseudovariety Ab_π, we consider an implicit signature σ_π which depends on the supernatural number π and we show that the decidable pseudovariety Ab_π is tame with respect to σ_π. Since the free σ_π-subalgebra generated by a finite alphabet is not the free abelian group, we prove that the word problem is decidable in this free σ_π-subalgebra, meaning that there is an algorithm to decide whether two elements of this σ_π-algebra represent the same element.

In [12], the notions of *σ-full* and *weakly σ-reducible* pseudovariety were introduced. By [12, Proposition 4.5], every σ-full weakly σ-reducible pseudovariety is σ-reducible. We use this result and we show that every decidable pseudovariety of abelian groups is reducible.

In the third paper, we let σ be the implicit signature obtained by adding to the canonical signature κ, all pseudowords of the form

$$w^{m^\omega} \left(= \lim_{k\to\infty} w^{m^{k!}} \right),$$

where w comes from the signature constructed by Almeida to show that the pseudovariety $\mathbf{G}_p$ of all finite p-groups is tame (p is a prime number) [4], and m is a natural number depending on w. We show that the pseudovariety G_{nil} of all finite nilpotent groups is σ-reducible if and only if for every prime number p, the pseudovariety G_p is σ-reducible. We also show the word problem is decidable in the σ-subalgebra generated by a finite alphabet. This yields as a corollary that the pseudovariety G_{nil} is tame with respect to the systems of equations associated to finite directed graphs but it is not completely tame.

Conclusions

In the first paper, we showed that the wreath product of a finitely generated abelian group with a polycyclic group is a LERF group. This theorem yields as a corollary that the finitely generated free metabelian group is LERF, a result due to Coulbois. We also showed that the finitely generated free solvable group of degree three, which is not LERF, does not contain a strictly ascending HNN-extension of a finitely generated group. This partially settles, in the negative, the following question of J. O. Button:

Question. *If G is a f.g. (or finitely presented) solvable group which is RF but not LERF, then must it contain a strictly ascending HNN-extension subgroup $H = \langle t, B\rangle$ where B is a f.g. subgroup of G?*

Our result does not completely settle the above question since the finitely generated free solvable group of degree three is not finitely presented.

In the second paper, we showed that a pseudovariety of abelian groups is decidable if and only if it is completely tame. It is an open problem whether every decidable pseudovariety is completely tame with respect to some signature. This problem is open even for the pseudovariety of all finite groups.

Finally, in the third paper, we showed the following statement. Let σ be the implicit signature obtained by adding to the canonical signature κ all implicit operations of the form

$$(a_j \circ (w_1, \dots, w_n)^{\omega})^{m^{\omega}} \qquad (m \in \mathbb{N})$$

with $j = 1, \dots, n$, w_i are κ-terms such that $\det M(w_1, \dots, w_n) \neq 0$ and every prime number p dividing $\det M(w_1, \dots, w_n)$ also divides m. Then G_{nil} is σ-graph tame but it is not completely tame. It remains as open problems to find a signature with respect to which G_p or G_{nil} is completely tame. From the proof of Corollary 3.12 in the third paper, we conjecture that if we consider the implicit signature σ_p obtained by adding to the canonical signature κ all implicit operations of the form

$$a_j \circ (w_1, \dots, w_n)^{\omega-1}$$

with $j = 1, \ldots, n$, w_i are κ-terms such that $\det M(w_1, \ldots, w_n) \neq 0 \pmod{p}$, then G_p is completely σ_p-tame.

The following two problems are considered to be very hard: whether the pseudovariety of all finite solvable groups and the pseudovariety of all finite odd order groups are graph tame with respect to some signature. Any results on related problems should be considered progress on so far untractable problems. For prime numbers p and q, our results may be used as first steps to tackle special cases of the preceding problems, such as: whether the semidirect products $\mathsf{G}_p * \mathsf{G}_q$ or $\mathsf{Ab}_p * \mathsf{Ab}_q$ are tame.

[24][24][23][22][21][16][49]

[14][25] [37][39][45, p. 148][44, Section 3.7][39][37] [33] and [1]). [33][27, Proposition 3.19][35][19, Theorem 1] [13][36][20][33][20]

[47]

or [27] [33, Theorem 3.2]

[41, Statement 22.14] [33, Lemma 3.2] [38] [33, Theorem 6.3] [18, Lemma 5.1] [44, Lemma 3.1.5] [19] [35, Theorem 3] [27] [46] [46, Lemma 4.2] [45, statement 1.6.11] [46, Corollary 4.3] [45, Statement 1.3.14] [45, statement 1.6.11] [39] [25] [17]. [45, Statement 6.1.11] [1] [20, Proposition 2.2] [30] [42, 2] [12] [15] [6, 5, 7] [32, 31, 34, 9, 10] [11] [29] [12] [12, Proposition 4.5] [4] [3, Theorem 5.6.1] [8, Proposition 3.1] [12, 15] [26] [4] [11]. [50, 12] [12, Proposition 4.5] [48, Corollary 3.3] [48, Proposition 3.4] [12] [11, Lemma 2.1]. [28, Proposition 2.2] [4, Lemma 6.6] [4, Proposition 2.2] [8, Proposition 3.1] [12, 15] [50, 12] [12, Proposition 4.5] [40] [40, Corollary 3.3] [40, proposition 4.3] [43, Lemma 5.2] [50] [26] [49]

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Paper 1

Paper 2

Paper 3