B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

SÉRIE:

RECHERCHES SUR LES DÉFORMATIONS

Volume LXVIII, no. 2

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Volume LXVIII, no. 2

ŁÓDŹ 2018

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Ministerstwo Nauki
i Szkolnictwa Wyższego

”Bulletin de la Société des sciences et des lettres de Łódź, Série: Recherches sur les déformations” – zadanie finansowane w ramach umowy 597/P-DUN/2017 ze środków Ministra Nauki i Szkolnictwa Wyższego przeznaczonych na działalność upowszechniającą naukę

”Bulletin de la Société des sciences et des lettres de Łódź, Série: Recherches sur les déformations” – task financed under the agreement 597/P-DUN/2017 by the Minister of Science and Higher Education from the funds for science dissemination

PL ISSN 0459-6854; e-ISSN 2450-9329
DOI: 10.26485/0459-6854; DOI: 10.26485/0459-6854/2018/68.2

Wydanie 1
Nakład 80 egz.
Przygotowanie czasopisma do digitalizacji: Justyna Walewska

The journal is indexed in: *Index Copernicus*, *PBN/POL-Index*, *EBSCOhost*, *Zentralblatt MATH*, *MathSciNet (Mathematical Reviews)* and *Referativny Zhurnal*

Druk czasopisma sfinansowany przez Uniwersytet Łódzki

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1. La présente Série du Bulletin de la Société des Sciences et des Lettres de Łódź comprend des communications du domaine des mathématiques, de la physique ainsi que de leurs applications liées aux déformations au sense large.
2. Toute communications est présentée à la séance d'une Commission de la Société par un des members (avec deux opinions de spécialistes designés par la Rédaction). Elle doit lui être adressée directement par l'auteur.
3. L'article doit être écrit en anglais, français, allemand ou russe et débuté par un résumé en anglais ou en langue de la communication présentée. Dans tous les travaux écrits par des auteurs étrangers le titre et le résumé en polonais seront préparés par la rédaction. Il faut fournir le texte original qui ne peut contenir plus de 15 pages (plus 2 copies).
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TITLE – INSTRUCTION FOR AUTHORS SUBMITTING THE PAPERS FOR BULLETIN

Summary

Abstract should be written in clear and concise way, and should present all the main points of the paper. In particular, new results obtained, new approaches or methods applied, scientific significance of the paper and conclusions should be emphasized.

Keywords and phrases:

1. General information

The paper for BULLETIN DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ should be written in LaTeX, preferably in LaTeX 2e, using the style (the file **bull.cls**).

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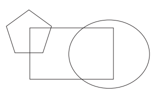
To prepare the LaTeX 2e source file of your paper, copy the template file **instruction.tex** with **Fig1.eps**, give the title of the paper, the authors with their affiliations/addresses, and go on with the body of the paper using all other means and commands of the standard class/style 'bull.cls'.

2.1. Example of a figure

Figures (including graphs and images) should be carefully prepared and submitted in electronic form (as separate files) in Encapsulated PostScript (EPS) format.



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The printed version of an article is primary (comparing with the electronic version). Each contribution submitted is sent for evaluation to two independent referees before publishing.

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References

[1]

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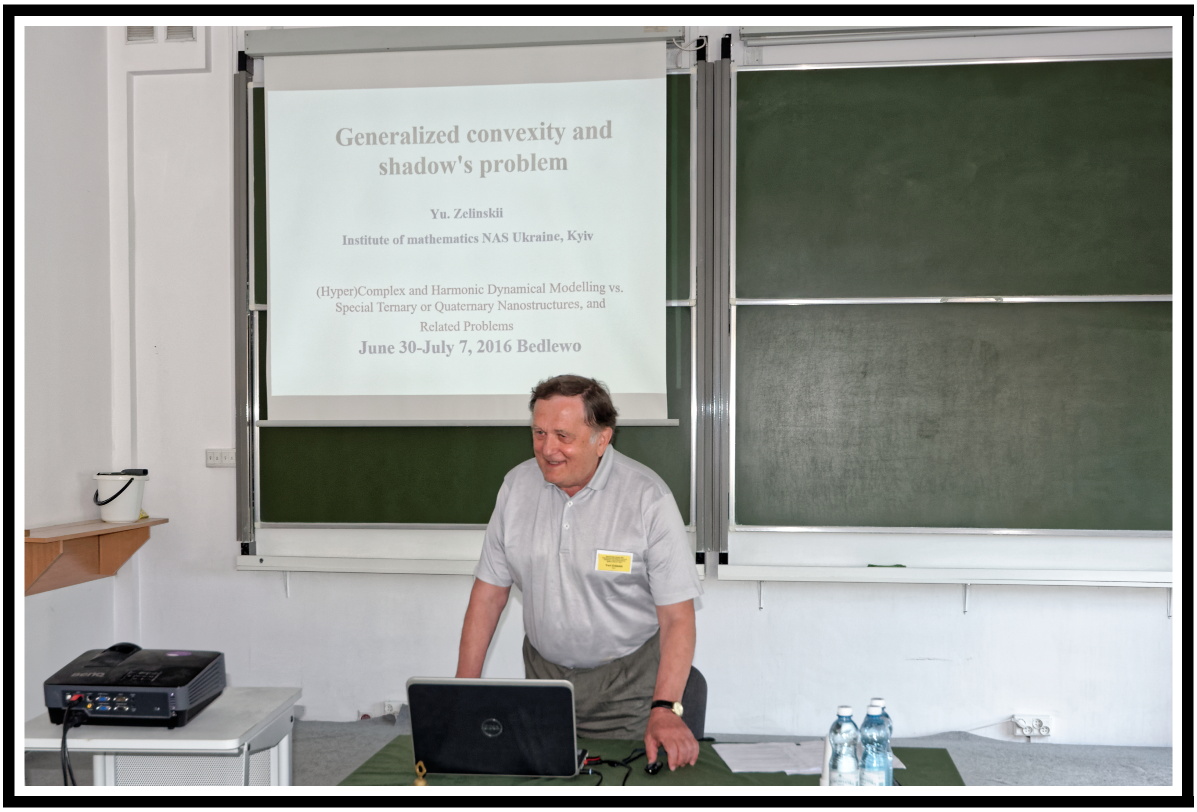
Part I



Yurii Borysovych lecturing durning the Hypercomplex 2016 Conference at Będlewo

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OUR FRIEND PROFESSOR YURII B. ZELINSKII

Professor Yurii Borysovych Zelinskii is a renowned Ukrainian mathematician who developed topological and geometrical methods for solving analytical problems of complex analysis and the theory of mappings.

Yu. B. Zelinskii was born on 22 February 1947 in Borschiv, Ternopil region. His mathematical abilities were displayed when Zelinskii was a schoolboy. He several times won the city and regional mathematical competition, three times participated in the All-Ukranian mathematical competitions, and was a winner of the All-Union competition in Physics by the postal correspondence. In 1965 he graduated from school with a gold medal and joined the Faculty of Mechanics and Mathematics of Taras Shevchenko Kiev State University. In the university Zelinskii was an active participant of the seminar on topological methods of analysis organized by Professor Yu. Yu. Trokhimchuk.

After graduating with honours in 1970 Zelinskii continued his postgraduate studies at the Institute of Mathematics of the Academy of Sciences of Ukraine under the supervision of Professor Trokhimchuk. In 1973 Zelinskii defended his PhD thesis Continuous mappings of manifolds and principles of boundary correspondence (the official classification of the USSR Highest Attestation Committee is Geometry and Topology). The same year he began to work at the Institute of Mathematics of the National Academy of Sciences of Ukraine and in 1989 he defended his Doctorate Thesis "Multivalued mappings in complex analysis". In 2004-2017 Yurii Borysovych was the head of Department of Complex Analysis and Potential Theory of the Institute of Mathematics of the National Academy of Sciences of Ukraine. Zelinskii originated the theory of strongly linear convex sets, which is a complex analogy of real convex analysis. Developing this theory he obtained generalizations of the classical Helly, Caratheodory, Krein-Milman, and some other theorems of classical complex analysis for complex spaces. He also found an approach to investigate generalized convex sets on Grassmanian manifolds.

Yurii Borysovych applied the notion of the local degree of mappings to multivalued mappings of topological manifolds. Using this idea he solved problems posed by Steinhaus and Kosinski in 1950-s on the estimations of dimensions of subsets with fixed multiplicity for mappings of domains on manifolds by known boundary properties of these mappings. Zelinskii also established sufficient conditions for the existence of solutions of multivalued inclusions in Euclidean domains. Special cases of these results are fixed point theorems for multivalued mappings that based on the generalization of the so called acute angle condition. He weakened some conditions of the classical Mobius theorem and obtained new criterions of affinity for mappings of real multidimensional spaces that is strongly invariant on the sets of vertices of rectangular parallelepipeds and the set of vertices of suspensions.

Zelinskii investigated the Ulam problem in the complex case and proved the complex convexity of a compact set in a multidimensional complex space if each intersection of this set with complex hyperplane of a fixed dimension is acyclic in the sense of the triviality of the Cech cogomology groups. The Ulam problem is closely connected with the Mizel-Zamfirescu problem on geometric characterization of the circle. Yurii Borysovych and his postgraduate students obtained some results related to the Mizel-Zamfirescu problem. In particular, it was proved that each convex curve of fixed width satisfying the infinitesimal rectangle condition is a circle.

Applying geometric methods and the theory of multidimensional methods to analytic problems of complex analysis Zelinskii obtained complete topological classification of linearly convex and strongly linear convex domains with smooth boundaries and estimated their cohomology groups.

Scientific contribution of Professor Zelinskii was awarded in 2015 by the Ostrogradskii prize of the National Academy of Sciences of Ukraine. He was a member of the Editorial Boards of the following journals: "Bulletin de la Societe des sciences et des letters de Lodz" (Poland), "International Journal on Engineering Sciences" (India), "Analysis and Applications" (Petrozavodsk, Russia), "Bukovyna Mathematical Journal", "Bulletin of the Taras Shevchenko Scientific Society. Mathematics" (Ukraine). Many years Zelinskii was a member of the dissertation councils at the Institute of Mathematics of the National Academy of Sciences of Ukraine and Fedkovych Chernivtsy National University.

Professor Zelinskii died unexpectedly on 22 July 2017 at the Ukrainain-Polish border. We shall remember him as a gentle cultured men of high quality with fine sense of humour.

Julian Ławrynowicz, Yurii. Yu. Trokhimchuk

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DOI: 10.26485/0459-6854/2018/68.2/1 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 13–24

Dedicated to the memory of
Professor Yurii B. Zelinskii

Andrei V. Pokrovskii and Olga D. Trofymenko

MEAN VALUE THEOREMS FOR SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Summary

We prove a mean value theorem that characterizes continuous weak solutions of homogeneous linear partial differential equations with constant coefficients in Euclidean domains. In this theorem the mean value of a smooth function with respect to a complex Borel measure on an ellipsoid of special form is equal to some linear combination of its partial derivatives at the center of this ellipsoid. The main result of the paper generalizes a well-known Zalcman's theorem.

Keywords and phrases: mean value, linear partial differential operator, weak solution, Fourier-Laplace transform, distribution

1. Introduction

Let $P(D)$ be a linear partial differential operator with constant coefficients in the Euclidean space $\mathbb{R}^n$, $n \geq 1$, and let μ be a complex Borel measure supported in the closed unit ball B of $\mathbb{R}^n$. Zalcman [1] proved the equivalence of the following assertions: (a) for any domain $G \subset \mathbb{R}^n$ and for any complex-valued function $u \in C(G)$,

$$\int u(\mathbf{x} + r\mathbf{t})\, d\mu(\mathbf{t}) = 0$$

for all $\mathbf{x} \in G$ and $r \in (0,\ \mathrm{dist}(\mathbf{x}, \partial G))$ if and only if u is a weak solution of the equation $P(D)f = 0$ in G; (b) the operator $P(D)$ is homogeneous and the functional $F_\mu(\varphi) := \int \varphi(\mathbf{t})\, d\mu(\mathbf{t})$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, in the space $\mathcal{E}'(\mathbb{R}^n)$ is represented in the form $F_\mu = P(D)T$ for some distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ supported in B with $\hat{T}(\mathbf{0}) \neq 0$, where $\hat{T}$ is the Fourier–Laplace transform of T. This result was the first general mean value theorem for solutions of linear partial differential equations, which contains the classical Gauss characterization of harmonic functions by spherical means, the Morera–Carleman characterization of analytic functions of a complex variable by zero integrals $\int f(z)\, dz$ over circles, and some other concrete mean value theorems as special cases. The first author [2] generalized Zalcman's result for the case of quasihomogeneous operators and applied this generalization to the study of removable singularities of solutions of the equation $P(D)f = 0$ with quasihomogeneous semielliptic operator $P(D)$ [3].

On the other hand, the second author studied classes of smooth functions defined in a disk $B(0, R) := \{z \in \mathbb{C} : |z| < R\}$ that satisfy the condition

$$\sum_{p=s}^{m-1} \frac{r^{2p+2}}{(2p+2)(p-s)!p!} \partial^{p-s} \bar{\partial}^p f(z) = \frac{1}{2\pi} \iint\limits_{|\zeta - z| \le r} f(\zeta)(\zeta - z)^s d\xi d\eta, \tag{1}$$

where $R > 0$, $s \in \mathbb{N}_0$, $m \in \mathbb{N}$, $s < m$, $z = x + iy$, $\zeta = \xi + i\eta$ $(x, y, \xi, \eta \in \mathbb{R})$, i is the imaginary unit,

$$\partial f = \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

She proved [4] as a special case of more general result that each function $f \in C^{2(m-1)-s}(B(0, R))$ satisfying this condition for all $r \in (0,\ R)$ and $z \in B(0, R - r)$ is a solution of the equation $\partial^{m-s}\bar{\partial}^m f = 0$.

In the present paper we prove a mean value theorem of Zalcman type that contains all the mentioned results as special cases.

2. Formulation of the main result

Let $n \in \mathbb{N} := \{1, 2, \ldots\}$ and let $\mathbf{M} = (M_1, \ldots, M_n)$ be a vector with positive integer components, $|\mathbf{M}| = M_1 + \ldots + M_n$. To each polynomial $P = P(\mathbf{z})$, $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, with complex-valued coefficients and to each $r > 0$ we assign the differential operator $P(r^{\mathbf{M}}D)$, in which z_k, $k = 1, \ldots, n$, is replaced by $-ir^{M_k}\partial/\partial x_k$. If $\mathbf{M} = (1, \ldots, 1)$, then $P(r^{\mathbf{M}}D) =: P(rD)$. A polynomial $P(\mathbf{z})$ (an operator $P(D) := P(1^{\mathbf{M}}D)$) is said to be $\mathbf{M}$-homogeneous if there is an $l \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ such that $P(\mathbf{z}) \equiv \sum_{\mathbf{k}} a_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}$, where $\mathbf{z}^{\mathbf{k}} := z_1^{k_1} \ldots z_n^{k_n}$ and the sum is taken over the set of all multiindices $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ with $|\mathbf{kM}| := k_1M_1 + \ldots + k_nM_n = l$. For any polynomial $P(\mathbf{z}) \equiv \sum_{\mathbf{k}} a_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}$ we denote by $\deg_{\mathbf{M}} P$ the number $\sup |\mathbf{kM}|$, where the supremum is taken over all multiindices $\mathbf{k} \in \mathbb{N}_0$ with $a_{\mathbf{k}} \neq 0$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$

and $r > 0$ we use the following notation: $r^{\mathbf{M}}\mathbf{x} := (r^{M_1}x_1, \dots . r^{M_n}x_n)$, $B_{\mathbf{M}}(\mathbf{x}, r) := \{\mathbf{x} + r^{\mathbf{M}}\mathbf{t} : \mathbf{t} \in \mathbb{R}^n, |\mathbf{t}| \le 1\}$. If $\mathbf{M} = (1, \dots, 1)$, then $\deg_{\mathbf{M}} P =: \deg P$, $B_{\mathbf{M}}(\mathbf{x}, r) =: B(\mathbf{x}, r)$. Recall that the Fourier–Laplace transform of a distribution $f \in \mathcal{E}'(\mathbb{R}^n)$ is defined by the formula $\hat{f}(\mathbf{z}) := f(e^{-i(\mathbf{x}\cdot\mathbf{z})})$, where $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{z} = x_1 z_1 + \dots + x_n z_n$, and the distribution f acts on the function $e^{-i(x\cdot z)}$ in x. As usual, δ is the Dirac measure, i.e., the unit measure concentrated at the origin.

Let μ be a complex Borel measure supported in $B := B(\mathbf{0}, 1)$ and let $P = P(\mathbf{z})$ and $Q = Q(\mathbf{z})$ be polynomials with complex-valued coefficients ($\mathbf{z} \in \mathbb{C}^n$). Denote by F_μ the functional corresponding to the measure μ in the space $\mathcal{E}'(\mathbb{R}^n)$, i.e., $F_\mu(\varphi) := \int \varphi(\mathbf{t})\, d\mu(\mathbf{t})$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\hat{\mu}(\mathbf{z}) := \hat{F_\mu}(\mathbf{z})$.

Definition 1. *We say that a triple* $(\mathbf{M}, \mu, Q)$ *characterizes continuous weak solutions of the equation* $P(D)f = 0$ *if for any domain* $G \subset \mathbb{R}^n$ *and for any function* $u \in C(G)$ *the following conditions are equivalent:*
(a) u *is a weak solution of the equation* $P(D)f = 0$ *in* G*;*
(b) for all $\varphi \in C_0^\infty(G)$ *and* $r > 0$ *such that* $\operatorname{supp}\varphi + B_{\mathbf{M}}(\mathbf{0}, r) \subset G$ *we have*

$$\int_G u(\mathbf{x})\Big(\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) - Q(-r^{\mathbf{M}}D)\varphi(\mathbf{x})\Big)\, d\mathbf{x} = 0.$$

Here, as usual,

$$\operatorname{supp}\varphi + B_{\mathbf{M}}(\mathbf{0}, r) := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \operatorname{supp}\varphi, \ \ \mathbf{y} \in B_{\mathbf{M}}(\mathbf{0}, r)\}.$$

The main result of this paper is the following theorem.

Theorem 1. *A triple* $(\mathbf{M}, \mu, Q)$ *characterizes continuous weak solutions of the equation* $P(D)f = 0$ *if and only if the polynomial* P *is* $\mathbf{M}$*-homogeneous and there is a distribution* $T \in \mathcal{E}'(\mathbb{R}^n)$ *supported in* B *such that* $\hat{T}(\mathbf{0}) \ne 0$ *and* $F_\mu = P(-D)T + Q(-D)F_\delta$.

3. Auxiliary results

The proof of Theorem 1 is essentially based on Zalcman's arguments [1] and uses the following lemmas.

Lemma 1 [5, Theorem 7.3.2]. *Suppose that* $f \in \mathcal{E}'(\mathbb{R}^n)$ *and* $P(D)$ *is a linear differential operator with constant coefficients. The equation* $P(D)u = f$ *has a distributional solution* $u \in \mathcal{E}'(\mathbb{R}^n)$ *if and only if* $\hat{f}(\mathbf{z})/P(\mathbf{z})$ *is an entire function. In this case the solution is determined uniquely, and the closure of the convex hull of the support of the distribution* u *coincides with that of the distribution* f*.*

Suppose that polynomials P_k, $k \in \mathbb{N}_0$, are given. If a function u satisfies the equalities $P_k(D)u = 0$ in $\mathbb{R}^n$ for each $k \in \mathbb{N}_0$ and

$$u(\mathbf{x}) \equiv g(\mathbf{x})e^{-i(\mathbf{z}\cdot\mathbf{x})} \tag{2}$$

for some polynomial $g(\mathbf{x})$ and $\mathbf{z} \in \mathbb{C}^n$, then we say that u is an exponential solution of the system $P_k(D)f = 0$, $k \in \mathbb{N}_0$.

Lemma 2 [5, Lemma 7.3.7]. *Suppose that $P(D)$ is a linear differential operator with constant coefficients. If $\nu \in \mathcal{E}'(\mathbb{R}^n)$ is a distribution such that $\nu(u) = 0$ for each exponential solution u of the equation $P(D)f = 0$, then $\hat{\nu}(\mathbf{z})/P(-\mathbf{z})$ is an entire function.*

Lemma 3 [6, Theorem 7.6.14]. *Suppose that G is a convex domain in $\mathbb{R}^n$, $q \in \mathbb{N}_0$, and $P_k(D)$, $k = 0, \ldots, q$, is a finite set of linear differential operators with constant coefficients. Then each continuous weak solution of the system $P_k(D)f = 0$, $k = 0, \ldots, q$, in G can be represented in the form of the limit of some sequence of finite linear combinations of exponential solutions of this system, uniformly converging on compact subsets of G.*

Lemma 4 [1, Theorem 3], [2, Lemma 1]. *Suppose that polynomials $P(\mathbf{z})$ and $P_j(\mathbf{z})$, $j \in \mathbb{N}_0$, are such that, for each $j \in \mathbb{N}_0$, either $P_j(\mathbf{z})$ is an $\mathbf{M}$-homogeneous polynomial with $\deg_{\mathbf{M}} P_j = j$ or $P_j(\mathbf{z}) \equiv 0$ ($\mathbf{z} \in \mathbb{C}^n$). Moreover, let $P_j(\mathbf{z}) \not\equiv 0$ for at least one $j \in \mathbb{N}_0$. The system of differential equations $P_j(D)f = 0$, $j \in \mathbb{N}_0$, is equivalent to the equation $P(D)f = 0$ is and only if each of the polynomials $P_j(\mathbf{z})$, $j \in \mathbb{N}_0$, is divisible by the polynomial $P(\mathbf{z})$ and for some number $k \in \mathbb{N}_0$ the polynomial $P_k(\mathbf{z})$ coincides with the polynomial $P(\mathbf{z})$ up to a nonzero constant factor.*

4. Proof of Theorem 1

Suppose that $\mathbf{M} = (M_1, \ldots, M_n)$ ($n \geq 1$) is a vector with positive integer components, μ is a complex Borel measure supported in B, $Q(\mathbf{z})$ ($\mathbf{z} \in \mathbb{C}^n$) is a polynomial, and u is a function of the form (2) in $\mathbb{R}^n$ satisfying the condition

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) = Q(r^{\mathbf{M}}D)u(\mathbf{x}) \tag{3}$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. Let us choose a point $\mathbf{x} \in \mathbb{R}^n$ and expand the function u in the Taylor series around $\mathbf{x}$. Collecting $\mathbf{M}$-homogeneous polynomials in this series, we obtain

$$u(\mathbf{x} + \mathbf{y}) = \sum_{j=0}^{\infty} U_j(\mathbf{y}), \tag{4}$$

where

$$U_j(\mathbf{y}) := \sum_{|\mathbf{kM}|=j} (\mathbf{k}!)^{-1}\partial^{\mathbf{k}}u(\mathbf{x})\mathbf{y}^{\mathbf{k}},$$

$$\mathbf{k}! := k_1! \dots k_n!, \quad \mathbf{y}^{\mathbf{k}} := y_1^{k_1} \dots y_n^{k_n}, \quad \partial^{\mathbf{k}} := \frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}.$$

Similarly, we represent the polynomial Q in the form of the finite sum of $\mathbf{M}$-homogeneous polynomials:

$$Q(\mathbf{y}) = \sum_{j=0}^{d} Q_j(\mathbf{y}), \tag{5}$$

where $d = \deg_{\mathbf{M}} Q$, $Q_j(\mathbf{z})$ is either an $\mathbf{M}$-homogeneous polynomial with $\deg_{\mathbf{M}} Q_j = j$ or $Q_j(\mathbf{z}) \equiv 0$, $Q_j(\mathbf{z}) \equiv 0$ for all $j > d$ ($\mathbf{z} \in \mathbb{C}^n$). The series in (4) converges to $u(\mathbf{x}+\mathbf{y})$ uniformly on compact sets in $\mathbb{R}^n$. Let us choose an arbitrary $r > 0$ and set $\mathbf{y} = r^{\mathbf{M}}\mathbf{t}$, where $\mathbf{t} \in B$. Since the series in (4) converges uniformly, we can integrate both sides of the resultant relation with respect to the measure μ term by term. This yields

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) = \sum_{j=0}^{\infty} U_j(r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) = \sum_{j=0}^{\infty} r^j (R_j(D)u)(\mathbf{x}), \tag{6}$$

where

$$R_j(\mathbf{z}) = \sum_{|\mathbf{kM}|=j} (\mathbf{k}!)^{-1} (i\mathbf{z})^{\mathbf{k}} \int \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}), \quad j \in \mathbb{N}_0, \quad \mathbf{z} \in \mathbb{C}^n. \tag{7}$$

Let $P_j(\mathbf{z}) := R_j(\mathbf{z}) - Q_j(\mathbf{z})$, $j \in \mathbb{N}_0$. Then it follows from (4)-(6) that

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) - Q(r^{\mathbf{M}}D)u(\mathbf{x}) = \sum_{j=0}^{\infty} r^j (P_j(D)u)(\mathbf{x}). \tag{8}$$

Since the condition (3) holds for any $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, we have $P_j(D)u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$.

Let G be a domain in $\mathbb{R}^n$ and let $\varphi \in C_0^{\infty}(G)$. Take $\mathbf{x} \in G$ and $r > 0$ such that $B_{\mathbf{M}}(\mathbf{x}, r) \subset G$. By the Taylor formula with reminder in integral form, for each $l \in \mathbb{N}$ and for all $\mathbf{y} \in B_{\mathbf{M}}(\mathbf{0}, r)$, we have

$$\varphi(\mathbf{x} + \mathbf{y}) = \sum_{|\mathbf{k}|<l} (\mathbf{k}!)^{-1} \partial^{\mathbf{k}} \varphi(\mathbf{x}) \mathbf{y}^{\mathbf{k}} + l \int_0^1 (1-s)^{l-1} \Big(\sum_{|\mathbf{k}|=l} (\mathbf{k}!)^{-1} \partial^{\mathbf{k}} \varphi(\mathbf{x} + s\mathbf{y}) \mathbf{y}^{\mathbf{k}} \Big) ds.$$

By setting $\mathbf{y} = -r^{\mathbf{M}}\mathbf{t}$, $\mathbf{t} \in B$, and rearranging the terms, we obtain

$$\varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t}) = \sum_{j=0}^{p} r^j \Big(\sum_{|\mathbf{kM}|=j} (-1)^{|\mathbf{k}|} (\mathbf{k}!)^{-1} \partial^{\mathbf{k}} \varphi(\mathbf{x}) \mathbf{t}^{\mathbf{k}} \Big) + V_p(r, \mathbf{x}, \mathbf{t}), \tag{9}$$

where $p = p(l)$ is the largest of numbers such that $|\mathbf{kM}| \le p$ implies $|\mathbf{k}| < l$ for any multiindex $\mathbf{k}$; $V_p(r, \mathbf{x}, \mathbf{t}) = o(r^p)$ as $r \to 0$ uniformly in $\mathbf{x} \in \operatorname{supp} \varphi$ and $\mathbf{t} \in B$. It is clear that (9) holds for each $p \in \mathbb{N}_0$. Integrating both sides of (9) with respect to the measure μ, we obtain

$$\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\,d\mu(\mathbf{t}) = \sum_{j=0}^{p} r^j \Big(\sum_{|\mathbf{kM}|=j} (-1)^{|\mathbf{k}|}(\mathbf{k}!)^{-1}\partial^{\mathbf{k}}\varphi(\mathbf{x}) \int \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t})\Big) + W_p(r,\mathbf{x}),$$

or

$$\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\,d\mu(\mathbf{t}) = \sum_{j=0}^{p} r^j (R_j(-D)\varphi)(\mathbf{x}) + W_p(r,\mathbf{x}), \tag{10}$$

where

$$(R_j(-D)\varphi)(\mathbf{x}) = \sum_{|\mathbf{kM}|=j} (-1)^{|\mathbf{k}|}(\mathbf{k}!)^{-1}(-1)^{|\mathbf{k}|}\partial^{\mathbf{k}}\varphi(\mathbf{x}) \int \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}),\ j \in \mathbb{N}_0.$$

$W_p(r,\mathbf{x}) = o(r^p)$ as $r \to 0$ uniformly in $\mathbf{x} \in \operatorname{supp}\varphi$.

Now we assume that a function $u \in C(G)$ satisfies the condition (b) of Definition 1. Then we have from (10) (for sufficiently small $r > 0$)

$$0 = \int_G u(\mathbf{x})\Big(\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\,d\mu(\mathbf{t}) - Q(-r^{\mathbf{M}}D)\varphi(\mathbf{x})\Big)\,d\mathbf{x} = \sum_{j=0}^{p} r^j \int_G u(\mathbf{x})((R_j - Q_j)(-D)\varphi)(\mathbf{x})\,d\mathbf{x} + o(r^p),$$

or

$$0 = \sum_{j=0}^{p} r^j \int_G u(\mathbf{x})(P_j(-D)\varphi)(\mathbf{x})\,d\mathbf{x} + o(r^p) \ \text{ as } \ r \to 0, \tag{11}$$

where $P_j(-D) = R_j(-D) - Q_j(-D)$, $j \in \mathbb{N}_0$. Suppose that at least one of the polynomials $\{P_j(\mathbf{z})\}_{j\in\mathbb{N}_0}$ does not vanish identically and p is the least number such that $P_p(\mathbf{z}) \not\equiv 0$. Dividing both side of (11) by r^p and letting $r \to 0$, we obtain $\int_G u(\mathbf{x})(P_p(-D)\varphi)(\mathbf{x})\,d\mathbf{x} = 0$. Then, proceeding by induction, we have

$$\int_G u(\mathbf{x})(P_j(-D)\varphi)(\mathbf{x})\,d\mathbf{x} = 0 \quad \forall j \in \mathbb{N}_0.$$

If all the polynomials $\{P_j(\mathbf{z})\}_{j\in\mathbb{N}_0}$ are identically zero, then the last assertion is obvious.

Since the function φ was an arbitrary function from $C_0^\infty(G)$ in our arguments, we have that u is a weak solution of the system

$$P_j(D)f = 0, \quad j \in \mathbb{N}_0. \tag{12}$$

Conversely, if u is a weak solution of the system (12) in G, then u satisfies the condition (b) of Definition 1. For exponential solutions this was justified by formula (8). The general case follows from Lemma 3 and the Hilbert Basis Theorem [7], which implies that there is a $j_0 \in \mathbb{N}_0$ such that the system (12) is equivalent to the finite system of differential equations $P_j(D)f = 0$, $j = 0, 1, \dots, j_0$.

To complete the proof of Theorem 1 we should investigate conditions of equivalence of the system (12) and the equation $P(D)f = 0$. The Fourier–Laplace transform $\hat{\mu}(\mathbf{z})$ of the measure μ is an entire function and its Taylor series around the point $\mathbf{z} = 0$ converges absolutely and uniformly on each compact set in $\mathbb{C}^n$. Therefore, by arranging of $\mathbf{M}$-homogeneous polynomials $R_j(-\mathbf{z})$ in this series, we obtain a series that uniformly converges to $\hat{\mu}(\mathbf{z})$ on compact sets in $\mathbb{C}^n$ as follows:

$$\hat{\mu}(\mathbf{z}) = \int e^{-i(\mathbf{z}\cdot\mathbf{t})} d\mu(\mathbf{t}) = \sum_{j=0}^{\infty} R_j(-\mathbf{z}),$$

where the sequence of polynomials $\{R_j(\mathbf{z})\}_{j\in\mathbb{N}_0}$ is defined by (7). Suppose that the triple $(\mathbf{M}, \mu, Q)$ characterizes continuous weak solutions of the equation $P(D)f = 0$. Then this equation is equivalent to the system (12). If $P(\mathbf{z}) \equiv 0$, then $R_j(\mathbf{z}) \equiv Q_j(\mathbf{z})$ for all $j \in \mathbb{N}_0$ and consequently

$$\hat{\mu}(\mathbf{z}) \equiv \sum_{j=0}^{\infty} Q_j(-\mathbf{z}) = Q(-\mathbf{z}).$$

Hence $F_\mu = Q(-D)F_\delta$, which is possible if only if $\deg Q = 0$. Now consider the case $P(\mathbf{z}) \not\equiv 0$. Then there is a number $p \in \mathbb{N}_0$ such that $P_p(\mathbf{z}) \not\equiv 0$. Since the divisors of an $\mathbf{M}$-homogeneous polynomial are also $\mathbf{M}$-homogeneous polynomials, then we have from Lemma 4 that the polynomial $P(\mathbf{z})$ is $\mathbf{M}$-homogeneous. It follows from the fact that the triple $(\mathbf{M}, \mu, Q)$ characterizes continuous weak solutions of the equation $P(D)f = 0$ and from Lemma 2 that $S(\mathbf{z}) := (\hat{\mu}(\mathbf{z}) - Q(-\mathbf{z}))/P(-\mathbf{z})$ is an entire function whence Lemma 1 implies that there is a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ supported in B such that

$$F_\mu = P(-D)T + Q(-D)F_\delta. \tag{13}$$

By applying the Fourier–Laplace transform to both sides of (13), we have $\hat{\mu}(\mathbf{z}) \equiv P(-\mathbf{z})\hat{T}(\mathbf{z}) + Q(-\mathbf{z})$. This means that $S(\mathbf{z}) \equiv T(\mathbf{z})$ and we derive the condition $\hat{T}(\mathbf{0}) \neq 0$ from the fact that an entire function can be uniquely represented by a series of $\mathbf{M}$-homogeneous polynomials uniformly convergent on compact subsets of $\mathbb{C}^n$.

Thus we justify the 'only if' part in Theorem 1. To prove the 'if' part of this theorem suppose that $P(\mathbf{z})$ is an $\mathbf{M}$-homogeneous polynomial, $m = \deg_{\mathbf{M}} P$, and T is a distribution supported in B satisfying (13). In this case $\hat{T}(\mathbf{0}) \neq 0$ need not hold. Let u be an exponential solution of the equation $P(D)f = 0$ in $\mathbb{R}^n$. If $\hat{T}(\mathbf{z}) = \sum_{j=0}^{\infty} T_j(-\mathbf{z})$ is the Taylor series of the entire function $\hat{T}$ around the point $\mathbf{z} = 0$ arranged in $\mathbf{M}$-homogeneous polynomials ($\deg_{\mathbf{M}} T_j = j$ or $T_j(\mathbf{z}) \equiv 0$), then, by comparing the equalities

$$\hat{\mu}(\mathbf{z}) - Q(-\mathbf{z}) = P(-\mathbf{z})T(\mathbf{z}), \qquad \hat{\mu}(\mathbf{z}) - Q(-\mathbf{z}) = \sum_{j=0}^{\infty} P_j(-\mathbf{z}),$$

and (8), we see that $P_j(\mathbf{z}) \equiv 0$ for all $j < m$, $P_{j+m}(\mathbf{z}) \equiv P(\mathbf{z})T_j(\mathbf{z})$ for all $j \in \mathbb{N}_0$, and

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) - Q(r^{\mathbf{M}}D)u(\mathbf{x}) = \sum_{j=0}^{\infty} r^{j+m}(P(D)T_j(D)u(\mathbf{x}) = 0$$

for all $\mathbf{x} \in \mathbb{R}^n$. The case of arbitrary continuous weak solutions of the equation $P(D)f = 0$ is reduced to the case of exponential solutions by applying Lemma 3, the Hilbert Basis Theorem, and integration by parts. The proof of Theorem 1 is completed.

5. Discussion of Theorem 1

Let $Q(\mathbf{z}) \equiv 0$ in Theorem 1. Then the condition (b) of Definition 1 is rewritten in the form

$$\int_G u(\mathbf{x})\Big(\int \varphi(\mathbf{x} - r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t})\Big)\, d\mathbf{x} = 0$$

for all $\varphi \in C_0^\infty(G)$ and $r > 0$ such that $\operatorname{supp}\varphi + B_{\mathbf{M}}(\mathbf{0}, r) \subset G$ whence

$$\int_G \Big(\int u((\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\varphi(\mathbf{x})\, d\mu(\mathbf{t})\Big)\, d\mathbf{x} = 0$$

for all such φ and r. It follows from the Fubini theorem that

$$\int u(\mathbf{x} + r^{\mathbf{M}}\mathbf{t})\, d\mu(\mathbf{t}) = 0. \tag{14}$$

This means that the condition (b) of Theorem 1 is satisfied if and only if (14) holds for all $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$ such that $B_{\mathbf{M}}(\mathbf{x}, r) \subset G$. Hence, for $Q(\mathbf{z}) \equiv 0$, Theorem 1 coincides with Theorem 2 from [2], which generalizes the mentioned Zalcman's result [1, Theorem 4] corresponding to the case $Q(\mathbf{z}) \equiv 0$ and $\mathbf{M} = (1, \dots, 1)$ in Theorem 1.

Now consider the case $n = 2$, $\mathbf{M} = (1, 1)$, and rewrite (1) in the form

$$Q(rD) = \int_B f(z + rt)\, d\mu(t), \tag{15}$$

where G is a domain in $\mathbb{C}$, $f \in C^{2m-2-s}(G)$, $z \in G$, $r > 0$, $B(z, r) \subset G$,

$$Q(z_1, z_2) = \sum_{p=s}^{m-1} (2^{2p-s}\pi(p+1)(p-s)!p!)^{-1}(iz_1 + z_2)^{p-s}(iz_1 - z_2)^p,$$

$d\mu(t) = t^s dt_1 dt_2$, $t = t_1 + it_2$, $t_1, t_2 \in \mathbb{R}$. Introduce the variables $w_1 = iz_1 + z_2$ and $w_2 = iz_1 - z_2$. Then $z_1 = -i(w_1 + w_2)/2$, $z_2 = (w_1 - w_2)/2$, and the Fourier–Laplace transform of μ can be expressed as follows:

$$\hat{\mu}(z_1,z_2) = \int_B e^{-i(z_1t_1+z_2t_2)} t^s dt_1 dt_2$$
$$= \int_B e^{-(w_1+w_2)t_1/2 - i(w_1-w_2)t_2/2} t^s dt_1 dt_2$$
$$= \int_B e^{-w_1(t_1-it_2)/2 - w_2(t_1+it_2)/2} t^s dt_1 dt_2$$
$$= \sum_{k,l=0}^{\infty} (-2)^{k+l}(k!l!)^{-1} w_1^k w_1^l \int_B (t_1-it_2)^k (t_1+it_2)^{l+s} dt_1 dt_2$$
$$= \sum_{p=s}^{\infty} (-2)^{2p-s} s((p-s)!p!)^{-1} (iz_1+z_2)^{p-s}(iz_1-z_2)^p \int_B |t|^{2p} dt_1 dt_2$$
$$= \sum_{p=s}^{\infty} (-2)^{2p-s} ((p-s)!p!)^{-1} (iz_1+z_2)^{p-s}(iz_1-z_2)^p 2\pi(2p+2)^{-1}.$$

This chain of equalities shows that there is a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ supported in B such that $\hat{T}(0,0) \neq 0$ and

$$\hat{\mu}(z_1,z_2) - Q(-z_1,-z_2) \equiv (-2)^{-(2m-s)}(iz_1+z_2)^{m-s}(iz_1-z_2)^m \hat{T}(z_1,z_2).$$

Theorem 1 implies that the triple $(\mathbf{M},\mu,Q)$ characterizes continuous weak solutions of the equation $\partial^{m-s}\bar{\partial}^m f = 0$. Since the differential operator $\partial^{m-s}\bar{\partial}^m$ is elliptic and consequently its distributional and classical solutions coincide, then we show that a function $f \in C^{2(m-1)-s}(G)$ satisfies the condition (15) for all $z \in G$ and $r \in (0, \operatorname{dist}(z,\partial G))$ if and only if f is a solution of the equation $\partial^{m-s}\bar{\partial}^m f = 0$ in G.

Note that the conditions in the 'only if' part of the last assertion can be essentially weakened. Namely, let $m \in \mathbb{N}$, $s \in \mathbb{N}_0$, $s < m$, and let

$$J_{s+1}(z) := \left(\frac{z}{2}\right)^{s+1} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(s+p+2)} \left(\frac{z}{2}\right)^{2p} \qquad (z \in \mathbb{C})$$

be the Bessel function. For $r > 0$ denote by Z_r the set of all zeros of the entire function

$$g_{s,m,r}(z) := \frac{J_{s+1}(zr)}{(zr)^{s+1}} - \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}}$$

belonging to $\mathbb{C} \setminus \{0\}$. Let r_1, r_2, R be positive numbers. The following result was proved in [4]: (a) if $R > r_1 + r_2$, $Z_{r_1} \cap Z_{r_2} = \varnothing$, $f \in C^{2m-2-s}(B(0,R))$, and the condition (1) holds for all $r \in \{r_1,r_2\}$ and $z \in B(0,R-r)$, then f belongs to the class $C^\infty(B(0,R))$ and satisfies the differential equation $\partial^{m-s}\bar{\partial}^m f = 0$; (b) if $\max\{r_1,r_2\} < R < r_1+r_2$ or $Z_{r_1} \cap Z_{r_2} \neq \varnothing$, then there exists a function $f \in C^\infty(B(0,R))$ satisfying the condition (1) for all $r \in \{r_1,r_2\}$ and $z \in B(0,R-r)$ that is not a solution of this equation in $B(0,R)$.

In the case $m = 1$ and $s = 0$ assertions (a) and (b) coincide with assertions (1) and (4) of Theorem 5.4 from [8, p. 399] for $n = 2$, respectively, where the local version of the classical Delsarte's two-radii theorem [9] characterizing harmonic functions in $\mathbb{R}^n$ is presented.

Acknowlegements

The first author was supported by the Polish-Ukrainain grant No. 39/2014 *Topological-analytical methods in complex and hypercomplex analysis* of the Polish Academy of Sciences and the National Academy of Sciences of Ukraine. The second author was supported by the Fundamental Research Programme funded by the Ministry of Education and Science of Ukraine (project 0115U000136).

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Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 27, 2017.

TWIERDZENIE O WARTOŚCI ŚREDNIEJ DLA ROZWIĄZAŃ LINIOWYCH RÓWNAŃ RÓŻNICZKOWYCH O POCHODNYCH CZĄSTKOWYCH O STAŁYCH WSPÓŁCZYNNIKACH

S t r e s z c z e n i e

Wykazujemy twierdzenie o wartości średniej, które charakteryzuje ciągłe słabe rozwiązania jednorodnych liniowych równań różniczkowych cząstkowych o stałych współczynnikach w obszarach euklidesowych. W twierdzeniu tym wartość średnia funkcji gładkiej względem zespolonej miary borelowskiej na pewnej elipsoidzie specjalnej postaci jest równa pewnej kombinacji liniowej jej pochodnych cząstkowych w środku tej elipsoidy. Główny wynik pracy uogólnia znane twierdzenie Zalcmana

Słowa kluczowe: wartość średnia liniowego operatora różniczkowego cząstkowego, słabe rozwiązanie, transformata Fouriera-Laplace'a, dystrybucja

DOI: 10.26485/0459-6854/2018/68.2/2 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 25–36

Dedicated to the memory of
Professor Yurii B. Zelinskii

Sergiy Anatoliyovych Plaksa and Vitalii Stanislavovych Shpakivskyi

INTEGRAL THEOREMS FOR MONOGENIC FUNCTIONS IN AN INFINITE-DIMENSIONAL SPACE WITH A COMMUTATIVE MULTIPLICATION

Summary

We consider monogenic functions taking values in a topological vector space being an expansion of a certain infinite-dimensional commutative Banach algebra associated with the three-dimensional Laplace equation. We establish also integral theorems for monogenic functions taking values in the mentioned algebra and the mentioned topological vector space.

Keywords and phrases: Laplace equation, spatial potentials, harmonic algebra, topological vector space, differentiable in the sense of Gâteaux function, monogenic function, Cauchy-Riemann conditions

1. Introduction

A commutative algebra $\mathbb{A}$ with unit is called harmonic (see [1, 2, 3, 4]) if in $\mathbb{A}$ there exists a triad of linearly independent vectors e_1, e_2, e_3 satisfying the relations

$$e_1^2 + e_2^2 + e_3^2 = 0, \qquad e_k^2 \neq 0,\ k = 1, 2, 3\,.$$

Such a triad e_1, e_2, e_3 is also called harmonic.

In the papers [1, 2, 3, 4, 5, 6, 7, 8] harmonic algebras are used for constructions of spatial *harmonic* functions, i.e. doubly continuously differentiable functions $u(x, y, z)$

satisfying the three-dimensional Laplace equation

$$\Delta_3 u(x,y,z) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x,y,z) = 0\,. \tag{1}$$

I. Mel'nichenko [3, 4] found all three-dimensional harmonic algebras and developed a method for finding all harmonic bases in these algebras. But it is impossible to obtain all solutions of equation (1) in the form of components of differentiable in the sense of Gâteaux functions taking values in finite-dimensional commutative algebras (see, e.g., [4, p. 43]).

In the papers [4, 6] spherical functions are obtained as the first components of decompositions of corresponding analytic functions with respect to the basis of an infinite-dimensional commutative Banach algebra $\mathbb{F}$. To obtain all solutions of equation (1) in the form of components of differentiable in the sense of Gâteaux functions, in the papers [7] we included corresponding algebras in topological vector spaces.

In the paper [8] we constructed spatial harmonic functions in the form of principal extensions of analytic functions of a complex variable into a complexification $\mathbb{F}_{\mathbb{C}}$ of the algebra $\mathbb{F}$. We considered special extensions of differentiable in the sense of Gâteaux functions with values in a topological vector space $\widetilde{\mathbb{F}}_{\mathbb{C}}$ being an expansion of the algebra $\mathbb{F}_{\mathbb{C}}$. Moreover, we considered also relations between the mentioned extensions and spatial potentials, in particular, axial-symmetric potentials.

For monogenic functions given in an infinite-dimensional algebra or a topological vector space associated with axial-symmetric potentials, analogues of classical integral theorems of complex analysis was proved in the paper [9].

In the present paper, using ideas of the paper [9], we prove integral theorems for monogenic functions taking values in an infinite-dimensional algebra $\mathbb{F}_{\mathbb{C}}$ and a topological vector space $\widetilde{\mathbb{F}}_{\mathbb{C}}$.

2. An infinite-dimensional algebra $\mathbb{F}_{\mathbb{C}}$

Consider an infinite-dimensional commutative associative Banach algebra over the field of real numbers $\mathbb{R}$, namely:

$$\mathbb{F} := \left\{a = \sum_{k=1}^{\infty} a_k e_k : a_k \in \mathbb{R}, \sum_{k=1}^{\infty} |a_k| < \infty\right\}$$

with the norm $\|a\|_{\mathbb{F}} := \sum\limits_{k=1}^{\infty} |a_k|$ and the basis $\{e_k\}_{k=1}^{\infty}$, where the multiplication table for the basis elements is of the following form:

$$e_n e_1 = e_n, \qquad e_{2n+1}e_{2n} = \frac{1}{2}\, e_{4n} \quad \forall\, n \geq 1\,,$$

$$e_{2n+1}e_{2m} = \frac{1}{2}\left(e_{2n+2m} - (-1)^m e_{2n-2m}\right) \quad \forall\, n > m \geq 1\,,$$

$$e_{2n+1}e_{2m}=\frac{1}{2}\Big(e_{2n+2m}+(-1)^n e_{2m-2n}\Big)\quad \forall\, m>n\geq 1\,,$$

$$e_{2n+1}e_{2m+1}=\frac{1}{2}\Big(e_{2n+2m+1}+(-1)^m e_{2n-2m+1}\Big)\quad \forall\, n\geq m\geq 1\,,$$

$$e_{2n}e_{2m}=\frac{1}{2}\Big(-e_{2n+2m+1}+(-1)^m e_{2n-2m+1}\Big)\quad \forall\, n\geq m\geq 1\,.$$

This algebra was proposed in the paper [4] (see also [7]). Inasmuch as $e_1^2+e_2^2+e_3^2=0$, the algebra $\mathbb{F}$ is harmonic and the vectors e_1, e_2, e_3 form a harmonic triad.

Now, consider a complexification $\mathbb{F}_{\mathbb{C}}:=\mathbb{F}\oplus i\mathbb{F}\equiv\{a+ib\,:\,a,b\in\mathbb{F}\}$ of the algebra $\mathbb{F}$ such that the norm in $\mathbb{F}_{\mathbb{C}}$ is given as $\|c\|:=\sum\limits_{k=1}^{\infty}|c_k|$, where $c=\sum\limits_{k=1}^{\infty}c_k e_k$, $c_k\in\mathbb{C}$, and $\mathbb{C}$ is the field of complex numbers.

Note that the algebra $\mathbb{F}_{\mathbb{C}}$ is isomorphic to the algebra $\mathbf{F}_{\mathbb{C}}$ of absolutely convergent trigonometric Fourier series

$$c(\theta)=c_0+\sum_{k=1}^{\infty}\big(a_k\, i^k\cos k\theta+b_k\, i^k\sin k\theta\big)$$

with $c_0, a_k, b_k\in\mathbb{C}$ and the norm $\|c\|_{\mathbf{F}_{\mathbb{C}}}:=|c_0|+\sum\limits_{k=1}^{\infty}\big(|a_k|+|b_k|\big)$. In this case, we have the isomorphism $e_{2k-1}\leftrightarrow i^{k-1}\cos{(k-1)\theta}$, $e_{2k}\leftrightarrow i^k\sin k\theta$ between elements of bases.

3. Monogenic and analytic functions taking values in the algebra $\mathbb{F}_{\mathbb{C}}$

Below, we shall consider functions given in subsets of the linear manifold $E_4:=\{\xi=xe_1+s\,ie_1+ye_2+ze_3\;:\;x,s,y,z\in\mathbb{R}\}$ containing the complex plane $\mathbb{C}$. With a set $Q\subset\mathbb{R}^4$ we associate the set $Q_\xi:=\{\xi=xe_1+s\,ie_1+ye_2+ze_3:(x,s,y,z)\in Q\}$ in E_4. In what follows, $\xi=xe_1+s\,ie_1+ye_2+ze_3$.

A function $\Psi:Q_\xi\to\mathbb{F}_{\mathbb{C}}$ is called *analytic* in a domain Q_ξ if in a certain neighborhood of each point $\xi_0\in Q_\xi$ it can be represented in the form of the sum of convergent power series

$$\Psi(\xi)=\sum_{k=1}^{\infty}c_k(\xi-\xi_0)^k,\quad c_k\in\mathbb{F}_{\mathbb{C}}.\tag{2}$$

A continuous function $\Phi:Q_\xi\to\mathbb{F}_{\mathbb{C}}$ is called *monogenic* in a domain $Q_\xi\subset E_4$ if Φ is differentiable in the sense of Gâteaux in every point of Q_ξ, i. e., if for every $\xi\in Q_\xi$ there exists an element $\Phi'(\xi)\in\mathbb{F}_{\mathbb{C}}$ such that

$$\lim_{\varepsilon\to 0+0}\left(\Phi(\xi+\varepsilon h)-\Phi(\xi)\right)\varepsilon^{-1}=h\Phi'(\xi)\qquad\forall\, h\in E_4\,.\tag{3}$$

It is obvious that an analytic function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ is monogenic in the domain Q_ξ and its derivative $\Phi'(\xi)$ is also monogenic in Q_ξ.

Below, we establish sufficient conditions for a monogenic function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ to be analytic in a domain $Q_\xi \subset E_4$.

Let us emphasize that in the case where a monogenic function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ has the continuous Gâteaux derivatives Φ', Φ'', it satisfies the identity $\Delta_3\Phi(\xi) \equiv 0$ because

$$\Delta_3\Phi(\xi) \equiv \Phi''(\xi)\ (e_1^2 + e_2^2 + e_3^2) \equiv 0\,.$$

Thus, for every component $U_k : Q \to \mathbb{C}$ of the decomposition

$$\Phi(\xi) = \sum_{k=1}^{\infty} U_k(x, s, y, z)\ e_k \tag{4}$$

of such a function Φ, the functions $\mathrm{Re}\, U_k(x, s, y, z)$, $\mathrm{Im}\, U_k(x, s, y, z)$ are spatial harmonic functions for every fixed s.

We say that the functions $U_k : Q \to \mathbb{C}$ of the decomposition (4) are $\mathbb{R}$-*differentiable* in Q if for all points $(x, s, y, z) \in Q$ the following relations are true:

$$U_k(x + \Delta x, s + \Delta s, y + \Delta y, z + \Delta z) - U_k(x, s, y, z) =$$
$$= \frac{\partial U_k}{\partial x}\Delta x + \frac{\partial U_k}{\partial s}\Delta s + \frac{\partial U_k}{\partial y}\Delta y + \frac{\partial U_k}{\partial z}\Delta z + o(\|\Delta\xi\|)\,,$$
$$\Delta\xi := e_1\Delta x + ie_1\Delta s + e_2\Delta y + e_3\Delta z \to 0.$$

The following theorem can be proved similarly to Theorem 4.1 [6].

Theorem 1. *Let a function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ be continuous in a domain $Q_\xi \subset E_4$ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition (4) be $\mathbb{R}$-differentiable in Q. In order the function Φ be monogenic in the domain Q_ξ, it is necessary and sufficient that the conditions*

$$\frac{\partial\Phi}{\partial s} = \frac{\partial\Phi}{\partial x}\, i\,, \qquad \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial x}\, e_2\,, \qquad \frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial x}\, e_3 \tag{5}$$

be satisfied in Q_ξ and the following relations be fulfilled in Q:

$$\sum_{k=1}^{\infty}\left|\frac{\partial U_k(x, s, y, r)}{\partial x}\right| < \infty, \tag{6}$$

$$\lim_{\varepsilon\to 0+0} \sum_{k=1}^{\infty} \Bigg| U_k(x + \varepsilon h_1, s + \varepsilon h_2, y + \varepsilon h_3, r + \varepsilon h_4) - U_k(x, s, y, r) -$$
$$-\frac{\partial U_k(x, s, y, r)}{\partial x}\ \varepsilon h_1 - \frac{\partial U_k(x, s, y, r)}{\partial s}\ \varepsilon h_2 - \frac{\partial U_k(x, s, y, r)}{\partial y}\ \varepsilon h_3 -$$
$$-\frac{\partial U_k(x, s, y, r)}{\partial r}\ \varepsilon h_4 \Bigg|\ \varepsilon^{-1} = 0 \qquad \forall\, h_1\,, h_2\,, h_3\,, h_4 \in \mathbb{R}\,. \tag{7}$$

Note that the first of conditions (5) means that every function U_k from the equality (4) is holomorphic with respect to the variable $x + is$ for each fixed pair (y, z).

4. Integral theorems for monogenic functions taking values in the algebra $\mathbb{F}_{\mathbb{C}}$

In the paper [10] for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. The convexity of the domain in the mentioned results from [10] is withdrawn by E. K. Blum [11].

Below we establish similar results for monogenic functions $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ given only in a domain Q_ξ of the linear manifold E_4 instead of domain of whole algebra. Let us note that *a priori* the differentiability of the function Φ in the sense of Gâteaux is a restriction weaker than the differentiability of this function in the sense of Lorch. Let us also note that in the paper [9] similar results were established for monogenic functions in an other infinite-dimensional algebra associated with axial-symmetric potentials.

In the case where Γ is a Jordan rectifiable curve in $\mathbb{R}^4$ we shall say that Γ_ξ is also a Jordan rectifiable curve. For a continuous function $\Phi : \Gamma_\xi \to \mathbb{F}_{\mathbb{C}}$ of the form (4), where $(x, s, y, r) \in \Gamma$ and $U_k : \Gamma \to \mathbb{C}$, we define an integral along the curve Γ_ξ with $d\xi := e_1\, dx + ie_1\, ds + e_2\, dy + e_3\, dz$ by the equality

$$\int_{\Gamma_\xi} \Phi(\xi) d\xi := \sum_{k=1}^{\infty} e_k \int_{\Gamma} U_k(x, s, y, z) dx + i \sum_{k=1}^{\infty} e_k \int_{\Gamma} U_k(x, s, y, z) ds +$$

$$+ \sum_{k=1}^{\infty} e_2 e_k \int_{\Gamma} U_k(x, s, y, z) dy + \sum_{k=1}^{\infty} e_3 e_k \int_{\Gamma} U_k(x, s, y, z) dz \qquad (8)$$

in the case where the series on the right-hand side of the equality are elements of the algebra $\mathbb{F}_{\mathbb{C}}$.

Theorem 2. *Let $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ be a monogenic function in a domain Q_ξ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *have continuous partial derivatives in Q. Then for every closed Jordan rectifiable curve $\Gamma_\xi \subset Q_\xi$ homotopic to a point in Q_ξ, the following equality holds:*

$$\int_{\Gamma_\xi} \Phi(\xi) d\xi = 0\,. \qquad (9)$$

Proof. Using the Stokes formula and the equalities (5), we obtain the equality

$$\int\limits_{\partial\triangle_\xi} \Phi(\xi)d\xi = 0 \tag{10}$$

for the boundary $\partial\triangle_\xi$ of every triangle $\triangle_\xi$ such that $\overline{\triangle_\xi} \subset Q_\xi$. Now, we can complete the proof similarly to the proof of Theorem 3.2 [11]. The theorem is proved. □

For functions $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ the following Morera theorem can be established in the usual way.

Theorem 3. *If a function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ is continuous in a domain Q_ξ and satisfies the equality* (10) *for every triangle $\triangle_\xi$ such that $\overline{\triangle_\xi} \subset Q_\xi$, then the function Φ is monogenic in the domain Q_ξ.*

Let $\tau := we_1 + \hat{y}e_2 + \hat{z}e_3$ where $w \in \mathbb{C}$ and $\hat{y}\,, \hat{z} \in \mathbb{R}$. Generalizing a resolvent resolution (cf. the equality (5) in [8]), we obtain

$$(\tau-\xi)^{-1} = \frac{1}{\sqrt{(w-\tau_1)(w-\tau_2)}}\Bigg(e_1 + \sum_{k=1}^{\infty} i^k \left(u_2^{-k} + u_1^k\right) e_{2k+1} +$$

$$+ \sum_{k=1}^{\infty} i^{k-1}\left(u_2^{-k} - u_1^k\right) e_{2k}\Bigg), \qquad w \notin s[\tau_1, \tau_2], \tag{11}$$

where

$$\tau_1 := x + is - i\sqrt{(y-\hat{y})^2 + (z-\hat{z})^2}, \qquad \tau_2 := x + is + i\sqrt{(y-\hat{y})^2 + (z-\hat{z})^2},$$

$$u_1 := \frac{(w-x-is) - \sqrt{(w-\tau_1)(w-\tau_2)}}{(y-\hat{y}) + i(z-\hat{z})},$$

$$u_2 := \frac{(w-x-is) + \sqrt{(w-\tau_1)(w-\tau_2)}}{(y-\hat{y}) + i(z-\hat{z})},$$

$s[\tau_1, \tau_2]$ is the segment connecting the points τ_1, τ_2, and $\sqrt{(w-\tau_1)(w-\tau_2)}$ is that continuous branch of the function

$$G(w) = \sqrt{(w-\tau_1)(w-\tau_2)}$$

analytic outside of the cut along the segment $s[\tau_1, \tau_2]$ for which $G(w) > 0$ for any $w > x$. Let us note that one should to set $u_1^k = 0$ and $u_2^{-k} = 0$ by continuity in the equality (11) for that $w \notin s[\tau_1, \tau_2]$ for which $\hat{y} = y$ and $\hat{z} = z$.

Thus, for every ξ the element $\left(\tau - \xi\right)^{-1}$ exists for all

$$\tau \notin S(\xi) := \Big\{\tau = we_1 + \hat{y}e_2 + \hat{z}e_3 :$$

$$\mathrm{Re}\, w = x,\ |\mathrm{Im}\, w - s| \le \sqrt{(y-\hat{y})^2 + (z-\hat{z})^2}\Big\}.$$

Now, the next theorem can be proved similarly to Theorem 5 [12].

Theorem 4. *Suppose that Q is a domain convex in the direction of the axes Oy, Oz. Suppose also that $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ is a monogenic function in the domain Q_ξ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *have continuous partial derivatives in Q. Then for every point $\xi \in Q_\xi$ the following equality is true:*

$$\Phi(\xi) = \frac{1}{2\pi i} \int\limits_{\Gamma_\xi} \Phi(\tau)\big(\tau - \xi\big)^{-1} d\tau\,, \tag{12}$$

where Γ_ξ is an arbitrary closed Jordan rectifiable curve in Q_ξ, which surrounds once the set $S(\xi)$ and is homotopic to the circle $\{\tau = we_1 + \hat{y}e_2 + \hat{z}e_3 : |w - x - is| = R,\ \hat{y} = y, \hat{z} = z\}$ contained completely in Ω_ξ.

Using the formula (12), we obtain the Taylor expansion of monogenic function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ in the usual way (see., for example, [13, p. 107]) in the case where the conditions of Theorem 4 are satisfied. Thus, in this case, $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ is an analytic function. In addition, in this case, an uniqueness theorem for monogenic functions can also be proved in the same way as for holomorphic functions of the complex variable (cf. [13, p. 110]).

Thus, the following theorem is true:

Theorem 5. *Let $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ be a continuous function in a domain Q_ξ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *have continuous partial derivatives in Q. Then the function Φ is monogenic in Q_ξ if and only if one of the following conditions is satisfied:*

(I) *the conditions* (5) *are satisfied in Q_ξ and the relations* (6), (7) *are fulfilled in Q;*

(II) *the function Φ satisfies the equality* (10) *for every triangle $\triangle_\xi$ such that $\overline{\triangle_\xi} \subset Q_\xi$;*

(III) *the function Φ is analytic in the domain Q_ξ.*

5. Monogenic functions with values in a topological vector space $\widetilde{\mathbb{F}}_\mathbb{C}$ containing the algebra $\mathbb{F}_\mathbb{C}$

Let us insert the algebra $\mathbb{F}_\mathbb{C}$ in the topological vector space

$$\widetilde{\mathbb{F}}_\mathbb{C} := \Big\{ g = \sum_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{C} \Big\}$$

with the topology of coordinate-wise convergence. Note that $\widetilde{\mathbb{F}}_\mathbb{C}$ is not an algebra because the product of elements $g_1, g_2 \in \widetilde{\mathbb{F}}_\mathbb{C}$ is defined not always. At the same time, for each $g = \sum\limits_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{F}}_\mathbb{C}$ and $\xi = (x+is)e_1 + ye_2 + ze_3$ with $x, s, y, z \in \mathbb{R}$ it is easy to define the product

$$g\,\xi \equiv \xi\, g := (x+is)\sum_{k=1}^{\infty} c_k e_k + y\Bigg(-\frac{c_2}{2}\,e_1 + \left(c_1 - \frac{c_5}{2}\right) e_2 - \frac{c_4}{2}\,e_3 +$$

$$+\frac{1}{2}\sum_{k=2}^{\infty}(c_{2k-1}-c_{2k+3})\,e_{2k} - \frac{1}{2}\sum_{k=2}^{\infty}(c_{2k-2}+c_{2k+2})\,e_{2k+1}\Bigg)+$$

$$+z\Bigg(-\frac{c_3}{2}\,e_1 - \frac{c_4}{2}\,e_2 + \left(c_1 - \frac{c_5}{2}\right) e_3 + \frac{1}{2}\sum_{k=4}^{\infty}(c_{k-2}-c_{k+2})\,e_k\Bigg).$$

In the paper [8], we proved that monogenic functions given in domains of the linear manifold $\{\zeta = xe_1 + ye_2 + ze_3 \,:\, x,y,z \in \mathbb{R}\}$ and taking values in the space $\widetilde{\mathbb{F}}_{\mathbb{C}}$ can be extended to monogenic functions given in domains of the linear manifold E_4.

We shall consider functions $\Phi : Q_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ for which the functions $U_k : Q \to \mathbb{C}$ in the decomposition (4) are $\mathbb{R}$-differentiable in the domain Q. Such a function Φ is continuous in Q_ξ and, therefore, we call Φ a *monogenic* function in Q_ξ if $\Phi'(\xi) \in \widetilde{\mathbb{F}}_{\mathbb{C}}$ in the equality (3).

The next theorem is similar to Theorem 1, where the necessary and sufficient conditions for a function $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ to be monogenic include additional relations (6), (7) conditioned by the norm of absolute convergence in the algebra $\mathbb{F}_{\mathbb{C}}$.

Theorem 6. *Let a function $\Phi : Q_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ be of the form* (4) *and the functions $U_k : Q \to \mathbb{C}$ be $\mathbb{R}$-differentiable in Q. In order the function Φ be monogenic in the domain Q_ξ, it is necessary and sufficient that the conditions* (5) *be satisfied in Q_ξ.*

For a continuous function $\Phi : \Gamma_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ of the form (4), we define an integral along a Jordan rectifiable curve Γ_ξ by the equality (8) in the case where the series on the right-hand side of this equality are elements of the space $\widetilde{\mathbb{F}}_{\mathbb{C}}$.

In the next theorem, for the sake of simplicity, we suppose that the curve Γ_ξ is the piece-smooth edge of a piece-smooth surface. In this case the following statement is a result of the Stokes formula and the equalities (5).

Theorem 7. *Suppose that $\Phi : Q_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ is a monogenic function in a domain Q_ξ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *have continuous partial derivatives in Q. Suppose also that Σ is a piece-smooth surface in Q with the piece-smooth edge Γ. Then the equality* (9) *holds.*

Let us define the product $gh \equiv hg$ for each $g = \sum\limits_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{F}}_{\mathbb{C}}$ and $h =$

$\sum\limits_{k=1}^{\infty} t_k e_k \in \mathbb{F}_{\mathbb{C}}$ in the case where the sequence $\{c_k\}_{k=1}^{\infty}$ is bounded:

$$gh \equiv hg := \left(c_1 t_1 + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^{[k/2]} c_k t_k \right) e_1 +$$

$$+ \left(c_2 t_1 + \left(c_1 + \frac{c_5}{2} \right) t_2 + \frac{-c_4}{2} \, t_3 + \frac{1}{2} \sum_{k=4}^{\infty} (-1)^{\left[\frac{k-1}{2}\right]} \left(c_{k-2+(-1)^k} + c_{k+2+(-1)^k} \right) \right) e_2 +$$

$$+ \left(c_3 t_1 + \frac{-c_4}{2} \, t_2 + \left(c_1 - \frac{c_5}{2} \right) t_3 + \frac{1}{2} \sum_{k=4}^{\infty} (-1)^{\left[\frac{k-2}{2}\right]} \left(c_{k-2} - c_{k+2} \right) \right) e_3 +$$

$$+ \sum_{m=4}^{\infty} \Upsilon_m \, e_m \, ,$$

where the constants Υ_m are defined by the next relations in four following cases:

1) if m is of the form $m = 4r$ with natural r, then

$$\Upsilon_m = c_m t_1 + \frac{1}{2} \sum_{k=2}^{m-1} \left(c_{m-k+1} + (-1)^{\left[\frac{k-1}{2}\right]} c_{m+k+(-1)^k} \right) t_k +$$

$$+ \left(c_1 - \frac{c_{2m+1}}{2} \right) t_m + \frac{c_{2m}}{2} \, t_{m+1} +$$

$$+ \frac{1}{2} \sum_{k=m+2}^{\infty} (-1)^{\left[\frac{k+1}{2}\right]} \left(c_{k-m+(-1)^k} - c_{k+m+(-1)^k} \right) t_k \, ;$$

2) if m is of the form $m = 4r - 1$ with natural r, then

$$\Upsilon_m = c_m t_1 + \frac{1}{2} \sum_{k=2}^{m-2} \left((-1)^{k-1} c_{m-k-(-1)^k} + (-1)^{\left[\frac{k}{2}\right]} c_{m+k-1} \right) t_k -$$

$$- \frac{c_{2m-2}}{2} \, t_{m-1} + \left(c_1 - \frac{c_{2m-1}}{2} \right) t_m +$$

$$+ \frac{1}{2} \sum_{k=m+1}^{\infty} (-1)^{\left[\frac{k-2}{2}\right]} \left(c_{k-m+1} - c_{k+m-1} \right) t_k \, ;$$

3) if m is of the form $m = 4r - 2$ with natural r, then

$$\Upsilon_m = c_m t_1 + \frac{1}{2} \sum_{k=2}^{m-1} \left(c_{m-k+1} + (-1)^{\left[\frac{k-1}{2}\right]} c_{m+k+(-1)^k} \right) t_k +$$

$$+ \left(c_1 + \frac{c_{2m+1}}{2} \right) t_m - \frac{c_{2m}}{2} \, t_{m+1} +$$

$$+ \frac{1}{2} \sum_{k=m+2}^{\infty} (-1)^{\left[\frac{k-1}{2}\right]} \left(c_{k-m+(-1)^k} + c_{k+m+(-1)^k} \right) t_k \, ;$$

4) if m is of the form $m = 4r - 3$ with natural r, then

$$\Upsilon_m = c_m t_1 + \frac{1}{2}\sum_{k=2}^{m-2}\Big((-1)^{k-1}c_{m-k-(-1)^k} + (-1)^{\left[\frac{k}{2}\right]}c_{m+k-1}\Big)t_k +$$

$$+\frac{c_{2m-2}}{2}\,t_{m-1} + \Big(c_1 + \frac{c_{2m-1}}{2}\Big)t_m + \frac{1}{2}\sum_{k=m+1}^{\infty}(-1)^{\left[\frac{k}{2}\right]}\left(c_{k-m+1} + c_{k+m-1}\right)t_k\,.$$

In the case where Γ is a piece-smooth curve (or Σ is a piece-smooth surface) in $\mathbb{R}^4$ we shall say that Γ_ξ is also a piece-smooth curve (or Σ_ξ is also a piece-smooth surface, respectively). We say that a domain $Q \subset \mathbb{R}^4$ is *convex in the direction of the plane* $\{(\hat{x},\hat{s},\hat{y},\hat{z}) : \hat{x},\hat{s} \in \mathbb{R},\ \hat{y} = y, \hat{z} = z\}$ if Q contains any segment that is parallel to the mentioned plane and connects two points of the domain Q.

The next theorem can be proved similarly to Theorem 5 in [12].

Theorem 8. *Suppose that Q is a domain convex in the direction of the plane $\{(\hat{x},\hat{s},\hat{y},\hat{z}) : \hat{x},\hat{s} \in \mathbb{R},\ \hat{y} = y, \hat{z} = z\}$. Suppose also that $\Phi : Q_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ is a monogenic function in the domain Q_ξ, and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *form an uniformly bounded family and have continuous partial derivatives in Q. Then for every point $\xi \in Q_\xi$ the equality* (12) *holds, where Γ_ξ is a piece-smooth curve that surrounds once the set $S(\xi)$ and, in addition, Γ_ξ and the circle $\{\tau = we_1 + \hat{y}e_2 + \hat{z}e_3 : |w - x - is| = R,\ \hat{y} = y, \hat{z} = z\}$ are edges of a piece-smooth surface Σ_ξ contained completely in Ω_ξ.*

Acknowledgments

This research is partially supported by Grant of Ministry of Education and Science of Ukraine (Project No. 0116U001528).

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Presented by Andrzej Łuczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on April 16, 2018.

TWIERDZENIA CAŁKOWE DLA FUNKCJI MONOGENICZNYCH W PRZESTRZENI NIESKOŃCZENIE-WYMIAROWEJ Z MNOŻENIEM PRZEMIENNYM

S t r e s z c z e n i e

Rozpatrujemy funkcje o wartościach w wektorowej przestrzeni topologicznej będącej rozszerzeniem pewnej nieskończenie-wymiarowej przemiennej algebry Banacha stowarzyszonej z trójwymiarowym równaniem Laplace'a. Uzyskujemy twierdenia całkowe dla funkcji monogenicznych o wartościach we wspomnianej algebrze i we wspomnianej wektorowej przestrzeni topologicznej.

Słowa kluczowe: równanie Laplace'a, potencjaly przestrzenne, algebra harmoniczna, przestrzeń wektorowa topologiczna, różniczkowalność w sensie Gâteaux, funkcja monogeniczna, warunki Cauchy'ego-Riemanna

DOI: 10.26485/0459-6854/2018/68.2/3 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 37–44

Dedicated to the memory of
Professor Yurii B. Zelinskii

Aleksandr Bakhtin, Liudmyla Vyhivska, and Iryna Denega

INEQUALITY FOR THE INNER RADII OF SYMMETRIC NON-OVERLAPPING DOMAINS

Summary

The paper deals with the following problem stated in [1] by V.N. Dubinin and earlier in different form by G.P. Bakhtina [2]. Let $a_0 = 0$, $|a_1| = \ldots = |a_n| = 1$, $a_k \in B_k \subset \overline{\mathbb{C}}$, where $B_0, \ldots, B_n$ are non-overlapping domains, and $B_1, \ldots, B_n$ are symmetric domains about the unit circle. Find the exact upper bound for $r^{\gamma}(B_0, 0) \prod\limits_{k=1}^{n} r(B_k, a_k)$, where $r(B_k, a_k)$ is the inner radius of B_k with respect to a_k. For $\gamma = 1$ and $n \geq 2$ this problem was solved by L.V. Kovalev [3, 4]. In the present paper it is solved for $\gamma_n = 0,25n^2$ and $n \geq 4$ under the additional assumption that the angles between neighboring line segments $[0, a_k]$ do not exceed $2\pi/\sqrt{2\gamma}$.

Keywords and phrases: inner radius of domain, non-overlapping domains, radial system of points, separating transformation, quadratic differential, Green's function

In geometric function theory of a complex variable problems maximizing the product of inner radii of non-overlapping domains are well known [1–10]. One of the such problems is considered in the article.

Let $\mathbb{N}$, $\mathbb{R}$ be a sets of natural and real numbers, respectively, $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ be an expanded complex plain or a sphere of Riemann, $\mathbb{R}^+ = (0, \infty)$. Let $r(B, a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to the point $a \in B$ (see, f.e. [1–5]). The inner radius of the domain B is associated with the generalized

Green function $g_B(z,a)$ of the domain B by the relations

$$g_B(z,a) = -\ln|z-a| + \ln r(B,a) + o(1), \quad z \to a,$$

$$g_B(z,\infty) = \ln|z| + \ln r(B,a) + o(1), \quad z \to \infty.$$

Let U_1 be a unit circle $|w| \le 1$.

The system of non-overlapping domains is called a finite set of arbitrary domains $\{B_k\}_{k=0}^n$, $n \in \mathbb{N}$, $n \ge 2$ such that $B_k \subset \overline{\mathbb{C}}$, $B_k \cap B_m = \emptyset$, $k \ne m$, $k,m = \overline{0,n}$.

Further we consider the following system of points $A_n := \{a_k \in \mathbb{C}, k = \overline{1,n}\}$, $n \in \mathbb{N}$, $n \ge 2$, satisfying the conditions $|a_k| \in \mathbb{R}^+$, $k = \overline{1,n}$ and $0 = \arg a_1 < \arg a_2 < \dots < \arg a_n < 2\pi$. Denote by

$$P_k = P_k(A_n) := \{w : \arg a_k < \arg w < \arg a_{k+1}\}, \quad a_{n+1} := a_1,$$

$$\alpha_k := \frac{1}{\pi}\arg\frac{a_{k+1}}{a_k}, \quad \alpha_{n+1} := \alpha_1, \quad k = \overline{1,n}, \quad \sum_{k=1}^n \alpha_k = 2.$$

Consider the following problem.

Problem. Let $a_0 = 0$, $|a_1| = \ldots = |a_n| = 1$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, where $B_0, \dots, B_n$ are pairwise non-overlapping domains and $B_1, \dots, B_n$ are symmetric domains with respect to the unit circle. Find the exact upper bound of the product

$$I_n(\gamma) = r^\gamma(B_0,0)\prod_{k=1}^n r(B_k,a_k).$$

For $\gamma = 1$ the problem was formulated as an open problem in the paper [1]. L.V. Kovalev solved the problem for $n \ge 2$ and $\gamma = 1$ [3, 4]. The following theorem substantially complements the results of the papers [2, 3, 4].

Theorem 1. *Let $n \in \mathbb{N}$, $n \ge 2$, $\gamma \in (0, \gamma_n]$, $\gamma_2 = 1,49$, $\gamma_3 = 3,01$, $\gamma_n = 0,25\,n^2$, $n \ge 4$. Then for any different points of a unit circle $|w| = 1$ such that $0 < \alpha_k \le 2/\sqrt{2\gamma}$, $k = \overline{1,n}$ and for any different system of non-overlapping domains B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, where the domains B_k, $k = \overline{1,n}$, have symmetry with respect to the unit circle $|w| = 1$, the following inequality holds*

$$r^\gamma(B_0,0)\prod_{k=1}^n r(B_k,a_k) \le \left(\frac{4}{n}\right)^n \frac{\left(\frac{2\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left|1-\frac{2\gamma}{n^2}\right|^{\frac{n}{2}+\frac{\gamma}{n}}} \left|\frac{n-\sqrt{2\gamma}}{n+\sqrt{2\gamma}}\right|^{\sqrt{2\gamma}}. \tag{1}$$

Equality in this inequality is achieved when a_k and B_k, $k = \overline{0,n}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + 2(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\,dw^2. \tag{2}$$

Proving the theorem 1. Consider the system of functions

$$\pi_k(w) = \left(e^{-i \arg a_k} w\right)^{\frac{1}{\alpha_k}}, \quad k = \overline{1,n}.$$

The family of functions $\{\pi_k(w)\}_{k=1}^{n}$ is called admissible for separating transformation of domains B_k, $k = \overline{0,n}$ with respect to angles $\{P_k\}_{k=1}^{n}$.

Let $\Omega_k^{(1)}$, $k = \overline{1,n}$, denote the domain of the plane $\mathbb{C}_\zeta$, obtained as a result of the union of the connected component of the set $\pi_k(B_k \bigcap \overline{P}_k)$, containing the point $\pi_k(a_k)$ with the own symmetric reflection with respect to the real axis. In turn, by $\Omega_k^{(2)}$, $k = \overline{1,n}$, one denotes the domain of the plain $\mathbb{C}_\zeta$, which are obtained as a result of the union of the connected component of the set $\pi_k(B_{k+1} \bigcap \overline{P}_k)$, containing the point $\pi_k(a_{k+1})$ with the own symmetric reflection with respect to the real axis, $B_{n+1} := B_1$, $\pi_n(a_{n+1}) := \pi_n(a_1)$. Moreover, we denote $\Omega_k^{(0)}$ as the domain of the plane $\mathbb{C}_\zeta$, obtained as a result of the union of the connected component of the set $\pi_k(B_0 \bigcap \overline{P}_k)$ containing the point $\zeta = 0$ with the own symmetric reflection with respect to the real axis. Denote by

$$\pi_k(a_k) := \omega_k^{(1)} = 1, \quad \pi_k(a_{k+1}) := \omega_k^{(2)} = -1, \quad k = \overline{1,n}.$$

From the definition of the function π_k, it follows that

$$|\pi_k(w) - 1| \sim \frac{1}{\alpha_k} \cdot |w - a_k|, \quad w \to a_k, \quad w \in \overline{P_k},$$

$$|\pi_k(w) + 1| \sim \frac{1}{\alpha_k} \cdot |w - a_{k+1}|, \quad w \to a_{k+1}, \quad w \in \overline{P_k},$$

$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w \to 0, \quad w \in \overline{P_k}.$$

Further, using the result of the papers [1, 2], we obtain the inequalities

$$r\left(B_k, a_k\right) \leq \left[\alpha_k r\left(\Omega_k^{(1)}, 1\right) \cdot \alpha_{k-1} r\left(\Omega_k^{(2)}, -1\right)\right]^{\frac{1}{2}}, \quad k = \overline{1,n}, \tag{3}$$

$$r\left(B_0, 0\right) \leq \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(0)}, 0\right)\right]^{\frac{1}{2}}. \tag{4}$$

From inequalities (3) and (4), and using the technique developed in [5, p. 269–274], we obtain

$$I_n(\gamma) \leq \prod_{k=1}^{n} \left[r\left(\Omega_k^{(0)}, 0\right)\right]^{\frac{\gamma\alpha_k^2}{2}} \prod_{k=1}^{n} \left[\alpha_{k-1} r\left(\Omega_k^{(2)}, -1\right) \alpha_k r\left(\Omega_k^{(1)}, 1\right)\right]^{\frac{1}{2}} =$$

$$= \left(\prod_{k=1}^{n} \alpha_k\right) \left[\prod_{k=1}^{n} r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)}, 0\right) r\left(\Omega_k^{(1)}, 1\right) r\left(\Omega_k^{(2)}, -1\right)\right]^{\frac{1}{2}}. \tag{5}$$

Further, consider the product of three domains

$$r^{\gamma\alpha_k^2}(G_0, 0) r(G_1, 1) r(G_2, -1).$$

To the domains G_0, G_1, G_2 we again apply separating transformation. Let

$$T_k := \{z : (-1)^{k+1} Im\, z > 0\}, \quad k \in \{1, 2\},$$

$$D_1 = \overline{T_1} \cap U_1, \quad D_2 = \overline{\mathbb{C}} \backslash U_1 \cap \overline{T_1}, \quad D_3 = \overline{T_2} \cap U_1, \quad D_4 = \overline{\mathbb{C}} \backslash U_1 \cap \overline{T_2},$$

$$\beta(z) = \frac{2z}{1+z^2}.$$

From the definition of the function $\beta(z)$, it follows that

$$|\beta(z)| \sim 2|z|, \quad z \to 0, \quad z \in \overline{T_k},$$

$$|\beta(z) - 1| \sim \frac{1}{2}|z-1|^2, \quad z \to 1, \quad z \in \overline{T_k},$$

$$|\beta(z) + 1| \sim \frac{1}{2}|z+1|^2, \quad z \to -1, \quad z \in \overline{T_k}.$$

The result of separating transformation the domain G_0 with respect to the function $\beta(z)$ and the system of domains $\{\overline{D}_k\}_{k=1}^4$ denote by $G_0^{(k)}$, $k = \overline{1,4}$; besides, the result of separating transformation the domain G_j, $j \in \{1, 2\}$, with respect to the function $2z/(1+z^2)$ and the system of domains $\{\overline{D}_k\}_{k=1}^4$ denote by $G_1^{(k)}, G_2^{(k)}$, $k = \overline{1,4}$. Further, we obtain the inequalities

$$r(G_0, 0) \le \left[\frac{1}{2} r\left(G_0^{(1)}, 0\right) \cdot \frac{1}{2} r\left(G_0^{(2)}, 0\right)\right]^{\frac{1}{2}},$$

$$r(G_1, 1) \le \left[2r\left(G_1^{(1)}, 1\right) 2r\left(G_1^{(2)}, 1\right) 2r\left(G_1^{(3)}, 1\right) 2r\left(G_1^{(4)}, 1\right)\right]^{\frac{1}{8}},$$

$$r(G_2, -1) \le \left[2r\left(G_2^{(1)}, -1\right) 2r\left(G_2^{(2)}, -1\right) 2r\left(G_2^{(3)}, -1\right) 2r\left(G_2^{(4)}, -1\right)\right]^{\frac{1}{8}}.$$

Since the domains G_1, G_2, have symmetry with respect to the unit circle, then

$$r^{\alpha_k^2 \gamma}(G_0, 0)\, r(G_1, 1)\, r(G_2, -1) \le$$

$$\le 2^{1-\alpha_k^2\gamma} \left[r^{2\alpha_k^2\gamma}\left(G_0^{(1)}, 0\right) r\left(G_1^{(1)}, 1\right) r\left(G_2^{(1)}, -1\right)\right]^{\frac{1}{2}} \times$$

$$\times \left[r^{2\alpha_k^2\gamma}\left(G_0^{(3)}, 0\right) r\left(G_1^{(3)}, 1\right) r\left(G_2^{(3)}, -1\right)\right]^{\frac{1}{2}}.$$

In case $2\alpha_k^2\gamma \le 4$, using paper [1], we obtain

$$r^{2\alpha_k^2\gamma}\left(G_0^{(s)}, 0\right) r\left(G_1^{(s)}, 1\right) r\left(G_2^{(s)}, -1\right) \le$$

$$\le \frac{2^{2\gamma\alpha_k^2+6}(\alpha_k\sqrt{2\gamma})^{2\gamma\alpha_k^2}}{(2-\alpha_k\sqrt{2\gamma})^{\frac{1}{2}(2-\alpha_k\sqrt{2\gamma})^2}(2+\alpha_k\sqrt{2\gamma})^{\frac{1}{2}(2+\alpha_k\sqrt{2\gamma})^2}}, \quad s \in \{1, 3\}.$$

Equality in this inequality is achieved when $G_0^{(s)}$, $G_1^{(s)}$, $G_2^{(s)}$ are circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{(4-2\alpha_k^2\gamma)z^2 + 2\alpha_k^2\gamma}{z^2(z^2-1)^2}\, dz^2 \tag{6}$$

($0 \in G_0^{(s)}$, $1 \in G_1^{(s)}$, $-1 \in G_2^{(s)}$, $s \in \{1,3\}$). Since $\alpha_k^2\gamma \leq 2$, then according to the papers [1, 6], the inequality holds

$$r^{\gamma}(B_0,0)\prod_{k=1}^{n} r(B_k,a_k) \leq \left(\frac{1}{\sqrt{2\gamma}}\right)^n \prod_{k=1}^{n}\left(\alpha_k\sqrt{2\gamma}\right) 2^{\frac{1-\alpha_k^2\gamma}{2}} \times$$

$$\times \left[\frac{2^{2\gamma\alpha_k^2+6}(\alpha_k\sqrt{2\gamma})^{2\gamma\alpha_k^2}}{(2-\alpha_k\sqrt{2\gamma})^{\frac{1}{2}(2-\alpha_k\sqrt{2\gamma})^2}(2+\alpha_k\sqrt{2\gamma})^{\frac{1}{2}(2+\alpha_k\sqrt{2\gamma})^2}}\right]^{\frac{1}{4}} =$$

$$= \left(\frac{1}{\sqrt{2\gamma}}\right)^n \prod_{k=1}^{n}\left[\frac{2^8\,(\alpha_k\sqrt{2\gamma})^{2\gamma\alpha_k^2+4}}{(2-\alpha_k\sqrt{2\gamma})^{\frac{1}{2}(2-\alpha_k\sqrt{2\gamma})^2}(2+\alpha_k\sqrt{2\gamma})^{\frac{1}{2}(2+\alpha_k\sqrt{2\gamma})^2}}\right]^{\frac{1}{4}}.$$

Consider the function

$$\Psi(x) = 2^8 \cdot x^{x^2+4} \cdot (2-x)^{-\frac{1}{2}(2-x)^2} \cdot (2+x)^{-\frac{1}{2}(2+x)^2},$$

where $x = \alpha_k\sqrt{2\gamma}$, $x \in (0,2]$.

Consider the extremal problem

$$\prod_{k=1}^{n}\Psi(x_k) \longrightarrow \max, \quad \sum_{k=1}^{n} x_k = 2\sqrt{2\gamma},$$

$$x_k = \alpha_k\sqrt{2\gamma}, \quad 0 < x_k \leq 2.$$

Let $F(x) = \ln(\Psi(x))$ and $X^{(0)} = \left\{x_k^{(0)}\right\}_{k=1}^{n}$ is any extremal point above the indicated problem. Repeating the arguments of [6], we obtain the statement: if $0 < x_k^{(0)} < x_j^{(0)} < 2$, then the following equalities hold $F'(x_k^{(0)}) = F'(x_j^{(0)})$, and when some $x_j^{(0)} = 2$, then for any $x_k^{(0)} < 2$, $F'(x_k^{(0)}) \leq F'(2)$, where $k,j = \overline{1,n}$, $k \neq j$,

$$F'(x) = 2x\ln x + (2-x)\ln(2-x) - (2+x)\ln(2+x) + \frac{4}{x}$$

(see. Fig. 1).

We verify that assertion is correct: if the function $Z(x_1,\ldots,x_n) = \sum\limits_{k=1}^{n} F(x_k)$ reaches a maximum at the point $(x_1^{(0)},\ldots,x_n^{(0)})$ with conditions $0 < x_k^{(0)} \leq 2$, $k = \overline{1,n}$, $\sum\limits_{k=1}^{n} x_k^{(0)} = 2\sqrt{2\gamma}$, then

$$x_1^{(0)} = x_2^{(0)} = \ldots = x_n^{(0)}.$$

For the simplicity, let $x_1^{(0)} \leq x_2^{(0)} \leq \ldots \leq x_n^{(0)}$. The function

$$F''(x) = \ln\left(\frac{x^2}{4-x^2}\right) - \frac{4}{x^2}$$

strictly ascending by $(0,2)$ and exist $x_0 \approx 1,768828$ such that

$$sign F''(x) \equiv sign(x - x_0).$$

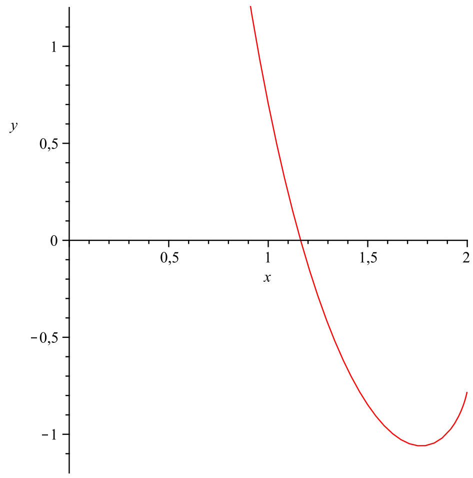

Fig. 1. Graph of the function $y = F'(x)$

Taking into consideration properties of the function $F'(x)$, the condition of theorem and relying on the method developed in [6], we obtain that for $F'(x)$ the inequality $(x_1 - 1,45)\,n + (x_2 - x_1) > 0$ always holds for $n \geq 4$. Hence $nx_1 + (x_2 - x_1) > 1,45\,n$. And, finally, we get

$$(n-1)x_1 + x_2 > 1,45n = 2\sqrt{2\gamma_n}, \quad \gamma_n = 0,25\,n^2, \quad n \geq 4.$$

So, in the case $n \geq 4$ the set of points $\left\{x_k^{(0)}\right\}_{k=1}^{n}$ can not be extreme under the condition $x_n^{(0)} \in (x_0,\, 2]$. Thus, for an extreme set $\left\{x_k^{(0)}\right\}_{k=1}^{n}$ is possible only the case when $x_k^{(0)} \in (0,\, x_0]$, $k = \overline{1, n}$, and $x_1^{(0)} = x_2^{(0)} = \ldots = x_n^{(0)}$. For any $\gamma < \gamma_n$, $n \geq 4$, all previous arguments remain.

Further, let $F'(x) = t$, $y_0 \leq t \leq -0,78$, $y_0 \approx -1,059$. Consider the following values t: $t_1 = -0,78$, $t_2 = -0,80$, $t_3 = -0,85$, $t_4 = -0,90$, $\cdots$, $t_{11} = -1,05$, $t_{12} = -1,059$. One finds the solution of equation $F'(x) = t_k$, $k = \overline{1,12}$. For any $t_k \in [y_0, -0,78)$ the equation has two solutions: $x_1(t) \in (0, x_0]$, $x_2(t) \in (x_0, 2]$, $x_0 \approx 1,768828$. Direct calculations are presented in the table below.

Taking into consideration properties of the function $F'(x)$ and the condition of theorem, we obtain the following inequality

$$\sum_{k=1}^{n} x_k(t) > (n-1)x_1(t_k) + x_2(t_{k+1}) \geq$$

$$\geq \min_{1 \leq k \leq 11} \left((n-1)x_1(t_k) + x_2(t_{k+1})\right) = 2\sqrt{2\gamma_n},$$

where $t_k \leq t \leq t_{k+1}$, $k = \overline{1,11}$. So, we have that for the extremal set $X^{(0)}$ the only case is possible where $\left\{x_k^{(0)}\right\}_{k=1}^{n} \in (0,\, x_0]$, $x_0 \approx 1,7688283$, and therefore $x_1^{(0)} = x_2^{(0)} = \cdots = x_n^{(0)}$.

k	t_k	$x_1(t_k)$	$x_2(t_k)$	$x_1(t_k)+x_2(t_{k+1})$	$2x_1(t_k)+x_2(t_{k+1})$
1	-0,78	1,458417	1,998914		
2	-0,80	1,470034	1,994779	3,453196	4,911613
3	-0,85	1,501193	1,980165	3,450199	4,920233
4	-0,90	1,536275	1,959964	3,461157	4,962350
5	-0,95	1,577242	1,932788	3,469063	5,005338
6	-1,00	1,628755	1,894239	3,471481	5,048723
7	-1,01	1,641325	1,884177	3,512932	5,141687
8	-1,02	1,655169	1,872815	3,514140	5,155465
9	-1,03	1,670801	1,859641	3,514810	5,169979
10	-1,04	1,689217	1,843656	3,514457	5,185258
11	-1,05	1,712998	1,822285	3,511502	5,200719
12	-1,059	1,768589	1,769066	3,482064	5,195062

Finally, we have the relation

$$r^{\gamma}(B_0,0)\prod_{k=1}^{n} r(B_k,a_k) \leq \left(\frac{1}{\sqrt{2\gamma}}\right)^{n}\left[\Psi\left(\frac{2}{n}\sqrt{2\gamma}\right)\right]^{\frac{n}{4}}.$$

Using the specific expression for $\Psi(x)$ and simple transformations, we obtain the inequality (1). In this way, the main inequality of theorem 1 is proved. Realizing in (6) the change of variable by the formula $z = 2w^{\frac{n}{2}}/(1+w^n)$, we obtain the quadratic differential (2). The equality sign is verified directly. The theorem 1 is proved.

Acknowlegements

The authors acknowledge the Polish-Ukrainian grant no. 39/2014 *Topological-analytical methods in complex and hypercomplex analysis* of the Polish Academy of Sciences and the National Academy of Sciences of Ukraine.

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Presented by Andrzej Łuczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on May 14, 2018.

NIERÓWNOŚĆ NA WEWNĘTRZNE PROMIENIE SYMETRYCZNYCH NIE ZACHODZĄCYCH NA SIEBIE OBSZARÓW

S t r e s z c z e n i e

Praca dotyczy zagadnienia postawionego przez V. N. Dubinina [1], a przedtem w innej postaci przez G. P. Bakhtina [2]. Niech $a_0 = 0$, $|a_1| = \ldots = |a_n| = 1$, $a_k \in B_k \subset \overline{\mathbb{C}}$, gdzie $B_0, \ldots, B_n$ są nie zachodzącymi na siebie obszarami, przy czym obszary $B_1, \ldots, B_n$ są symetryczne względem okręgu jednostkowego. Problem polega na znalezieniu dokładnego kresu górnego dla $r^\gamma(B_0, 0) \prod\limits_{k=1}^{n} r(B_k, a_k)$, gdzie $r(B_k, a_k)$ jest promieniem wewnętrznym obszaru B_k względem punktu a_k. Dla $\gamma = 1$ i $n \geq 2$ problem został rozwiązany przez L.V. Kovaleva [3, 4]. W obecnej pracy problem został rozwiązany dla $\gamma_n = 0,25n^2$ i $n \geq 4$ przy dodatkowym założeniu, że kąty między sąsiednimi odcinkami $[0, a_k]$ nie przekraczają $2\pi/\sqrt{2\gamma}$.

Słowa kluczowe: promień wewnętrzny obszaru, obszary nie zachodzące na siebie, promienisty układ punktów, transformacja rozdzielająca, różniczka kwadratowa, funkcja Greena

DOI: 10.26485/0459-6854/2018/68.2/4 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 45–52

Dedicated to the memory of
Professor Yurii B. Zelinskii

Anatoly S. Serdyuk and Tetiana A. Stepanyuk

LEBESGUE-TYPE INEQUALITIES FOR THE FOURIER SUMS ON CLASSES OF GENERALIZED POISSON INTEGRALS

Summary

For functions from the set of generalized Poisson integrals $C^{\alpha,r}_{\beta}L_p$, $1 \leq p < \infty$, we obtain upper estimates for the deviations of Fourier sums in the uniform metric in terms of the best approximations of the generalized derivatives $f^{\alpha,r}_{\beta}$ of functions of this kind by trigonometric polynomials in the metric of the spaces L_p. Obtained estimates are asymptotically best possible.

Keywords and phrases: Lebesgue-type inequalities, Fourier sums, generalized Poisson integrals, best approximations

Let L_p, $1 \leq p < \infty$, be the space of 2π–periodic functions f summable to the power p on $[0, 2\pi)$, in which the norm is given by the formula $\|f\|_p = \Big(\int\limits_0^{2\pi} |f(t)|^p dt \Big)^{\frac{1}{p}}$; L_∞ be the space of measurable and essentially bounded 2π–periodic functions f with the norm $\|f\|_\infty = \operatorname{ess\,sup}\limits_t |f(t)|$; C be the space of continuous 2π–periodic functions f, in which the norm is specified by the equality $\|f\|_C = \max\limits_t |f(t)|$.

Denote by $C^{\alpha,r}_{\beta}L_p,\ \alpha > 0,\ r > 0,\ \beta \in \mathbb{R},\ 1 \leq p \leq \infty$, the set of all 2π–periodic functions, representable for all $x \in \mathbb{R}$ as convolutions of the form (see, e.g., [1, p. 133])

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt,\ a_0 \in \mathbb{R}, \quad \varphi \perp 1, \tag{1}$$

where $\varphi \in L_p$ and $P_{\alpha,r,\beta}(t)$ are fixed generated kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\big(kt - \frac{\beta\pi}{2}\big), \quad \alpha, r > 0, \quad \beta \in \mathbb{R}. \tag{2}$$

The kernels $P_{\alpha,r,\beta}$ of the form (2) are called generalized Poisson kernels. For $r = 1$ and $\beta = 0$ the kernels $P_{\alpha,r,\beta}$ are usual Poisson kernels of harmonic functions.

If the functions f and φ are related by the equality (1), then function f in this equality is called generalized Poisson integral of the function φ. The function φ in equality (1) is called as generalixed derivative of the function f and is denoted by $f_\beta^{\alpha,r}$.

The set of functions f from $C_\beta^{\alpha,t}L_p$, $1 \le p \le \infty$, such that $f_\beta^{\alpha,r} \in B_p^0$, where

$$B_p^0 = \{\varphi :\ ||\varphi||_p \le 1,\ \varphi \perp 1\},\ 1 \le p \le \infty,$$

we will denote by $C_{\beta,p}^{\alpha,r}$.

Let $E_n(f)_{L_p}$ be the best approximation of the function $f \in L_p$ in the metric of space L_p, $1 \le p \le \infty$, by the trigonometric polynomials t_{n-1} of degree $n-1$, i.e.,

$$E_n(f)_{L_p} = \inf_{t_{n-1}} \|f - t_{n-1}\|_{L_p}.$$

Let $\rho_n(f;x)$ be the following quantity

$$\rho_n(f;x) := f(x) - S_{n-1}(f;x), \tag{3}$$

where $S_{n-1}(f;\cdot)$ are the partial Fourier sums of order $n-1$ for a function f.

Least upper bounds of the quantity $\|\rho_n(f;\cdot)\|_C$ over the classes $C_{\beta,p}^{\alpha,r}$, we denote by $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$, i.e.,

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = \sup_{f \in C_{\beta,p}^{\alpha,r}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_C,\ r > 0,\ \alpha > 0,\ 1 \le p \le \infty. \tag{4}$$

Asymptotic behaviour of the quantities $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$ of the form (4) was studied in [2]–[10].

In [11]–[13] it was found the analogs of the Lebesgue inequalities for functions $f \in C_\beta^{\alpha,r}L_p$ in the case $r \in (0,1)$ and $p = \infty$, and also in the case $r \ge 1$ and $1 \le p \le \infty$, where the estimates for the deviations $\|f(\cdot) - S_{n-1}(f;\cdot)\|_C$ are expressed in terms of the best approximations $E_n(f_\beta^{\alpha,r})_{L_p}$. Namely, in [11] it is proved that the following best possible inequalitiy holds

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le \Big(\frac{4}{\pi^2}\ln n^{1-r} + O(1)\Big)e^{-\alpha n^r} E_n(f_\beta^{\alpha,r})_{L_\infty}, \tag{5}$$

where $O(1)$ is a quantity uniformly bounded with respect to n, β and $f \in C_\beta^{\alpha,r}L_\infty$.

The present paper is a continuation of [11], [12], and is devoted to getting asymptotically best possible analogs of Lebesgue-type inequalities on the sets $C^{\alpha,r}_{\beta}L_p$, $r \in (0,1)$ and $p \in [1,\infty)$. This case was not considered yet. Let formulate the results of the paper.

By $F(a,b;c;d)$ we denote Gauss hypergeometric function

$$F(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

$$(x)_k := \frac{x}{2}\Big(\frac{x}{2}+1\Big)\Big(\frac{x}{2}+2\Big)...\Big(\frac{x}{2}+k-1\Big).$$

For arbitrary $\alpha > 0$, $r \in (0,1)$ and $1 \le p \le \infty$ we denote by $n_0 = n_0(\alpha,r,p)$ the smallest integer n such that

$$\frac{1}{\alpha r}\frac{1}{n^r} + \frac{\alpha r \chi(p)}{n^{1-r}} \le \begin{cases} \frac{1}{14}, & p=1, \\ \frac{1}{(3\pi)^3} \cdot \frac{p-1}{p}, & 1<p<\infty, \\ \frac{1}{(3\pi)^3}, & p=\infty, \end{cases} \tag{6}$$

where $\chi(p) = p$ for $1 \le p < \infty$ and $\chi(p) = 1$ for $p = \infty$.

The following statement holds.

Theorem 1. *Let $0 < r < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then in the case $1 < p < \infty$ for any function $f \in C^{\alpha,r}_{\beta}L_p$ and $n \ge n_0(\alpha,r,p)$, fthe following inequality is true:*

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma_{n,p}\Bigg(\Bigg(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Bigg)\frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Bigg)\Bigg) E_n(f^{\alpha,r}_{\beta})_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{7}$$

where $F(a,b;c;d)$ is Gauss hypergeometric function, and in the case $p = 1$ for any function $f \in C^{\alpha,r}_{\beta}L_1$ and $n \ge n_0(\alpha,r,1)$, the following inequality is true:

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le e^{-\alpha n^r} n^{1-r} \Big(\frac{1}{\pi \alpha r} + \gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r} + \frac{1}{n^{1-r}}\Big)\Big) E_n(f^{\alpha,r}_{\beta})_{L_1}. \tag{8}$$

In (7) and (8), the quantity $\gamma_{n,p} = \gamma_{n,p}(\alpha,r,\beta)$ is such that $|\gamma_{n,p}| \le (14\pi)^2$.

Proof of Theorem 1. Let $f \in C^{\alpha,r}_{\beta}L_p$, $1 \le p \le \infty$. Then, at every point $x \in \mathbb{R}$ the following integral representation is true:

$$\rho_n(f;x) = f(x) - S_{n-1}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{\alpha,r}_{\beta}(t) P^{(n)}_{\alpha,r,\beta}(x-t))dt, \tag{9}$$

where

$$P^{(n)}_{\alpha,r,\beta}(t) := \sum_{k=n}^{\infty} e^{-\alpha k^r} \cos\Big(kt - \frac{\beta\pi}{2}\Big), \ 0<r<1, \ \alpha>0, \ \beta \in \mathbb{R}. \tag{10}$$

The function $P^{(n)}_{\alpha,r,\beta}(t)$ is orthogonal to any trigonometric polynomial t_{n-1} of degree not greater than $n-1$. Hence, for any polynomial t_{n-1} from we obtain

$$\rho_n(f;x) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \delta_n(t) P^{(n)}_{\alpha,r,\beta}(x-t)dt, \tag{11}$$

where

$$\delta_n(x) = \delta_n(\alpha, r, \beta, n; x) := f^{\alpha,r}_{\beta}(x) - t_{n-1}(x). \tag{12}$$

Further we choose the polynomial t^*_{n-1} of the best approximation of the function $f^{\alpha,r}_{\beta}$ in the space L_p, i.e., such that

$$\|f^{\alpha,r}_{\beta} - t^*_{n-1}\|_p = E_n(f^{\alpha,r}_{\beta})_{L_p}, \quad 1 \le p \le \infty,$$

to play role of t_{n-1} in (11). Thus, by using the inequality

$$\Big\|\int\limits_{-\pi}^{\pi} K(t-u)\varphi(u)du\Big\|_C \le \|K\|_{p'}\|\varphi\|_p, \tag{13}$$

$$\varphi \in L_p, \quad K \in L_{p'}, \quad 1 \le p \le \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

(see, e.g., [14, p. 43]), we get

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le \frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_{p'} E_n(f^{\alpha,r}_{\beta})_{L_p}. \tag{14}$$

For arbitrary $\upsilon > 0$ and $1 \le s \le \infty$ assume

$$\mathcal{I}_s(\upsilon) := \Big\|\frac{1}{\sqrt{t^2+1}}\Big\|_{L_s[0,\upsilon]}, \tag{15}$$

where

$$\|f\|_{L_s[a,b]} = \begin{cases} \Big(\int\limits_a^b |f(t)|^s dt\Big)^{\frac{1}{s}}, & 1 \le s < \infty, \\ \operatorname*{ess\,sup}\limits_{t\in[a,b]} |f(t)|, & s = \infty. \end{cases}$$

It follows from the paper [9] for arbitrary $r \in (0,1)$, $\alpha > 0$, $\beta \in \mathbb{R}$, $1 \le s \le \infty$, $\frac{1}{s} + \frac{1}{s'} = 1$, $n \in \mathbb{N}$ and $n \ge n_0(\alpha, r, s')$ the following estimate holds

$$\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_s = e^{-\alpha n^r} n^{\frac{1-r}{s'}}\Bigg(\frac{\|\cos t\|_s}{\pi^{1+\frac{1}{s}}(\alpha r)^{\frac{1}{s'}}}\mathcal{I}_s\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) +$$
$$+\delta^{(1)}_{n,s}\Big(\frac{1}{(\alpha r)^{1+\frac{1}{s'}}}\mathcal{I}_s\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{s'}}}\Big)\Bigg), \tag{16}$$

where the quantity $\delta^{(1)}_{n,s} = \delta^{(1)}_{n,s}(\alpha, r, \beta)$, satisfies the inequality $|\delta^{(1)}_{n,s}| \le (14\pi)^2$.

Substituting $s = p' = \infty$, from 14 and (16) we get (8).

Further, according to [9] for $n \geq n_0(\alpha, r, s')$, $1 < s < \infty$, $\frac{1}{s} + \frac{1}{s'} = 1$, the following equality takes place

$$\mathcal{I}_s\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) = F^{\frac{1}{s}}\Big(\frac{1}{2}, \frac{3-s}{2}; \frac{3}{2}; 1\Big) + \frac{\Theta^{(1)}_{\alpha,r,s',n}}{s-1}\Big(\frac{\alpha r}{\pi n^{1-r}}\Big)^{s-1}, \tag{17}$$

where $|\Theta^{(1)}_{\alpha,r,s',n}| < 2$.

Let now consider the case $1 < p < \infty$.

Formulas (16) and (17) for $s = p'$ and $n \geq n_0(\alpha, r, p)$ imply

$$\frac{1}{\pi}\big\|\mathcal{P}_{\alpha,r,n}\big\|_{p'} = e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma^{(1)}_{n,p}\Big(\frac{1}{p'-1}\frac{(\alpha r)^{\frac{p'-1}{p}}}{n^{(1-r)(p'-1)}} + \frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big)\Bigg) =$$

$$= e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma^{(2)}_{n,p}\Bigg(\Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{(1-r)(p'-1)}} + \frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Bigg)\Bigg), \tag{18}$$

where the quantities $\delta^{(i)}_{n,p} = \delta^{(i)}_{n,p}(\alpha, r, \beta)$, satisfiy the inequality $|\delta^{(i)}_{n,p}| \leq (14\pi)^2$, $i = 1, 2$. Estimate (7) follows from (14) and (18). Theorem 1 is proved. $\square$

It should be noticed, that estimates (7) and (8) are asymptotically best possible on the classes $C^{\alpha,r}_{\beta,p}$, $1 \leq p < \infty$.

If $f \in C^{\alpha,r}_{\beta,p}$, then $\|f^{\alpha,r}_\beta\|_p \leq 1$, and $E_n(f^{\alpha,r}_\beta)_{L_p} \leq 1$, $1 \leq p < \infty$. Considering the least upper bounds of both sides of inequality (7) over the classes $C^{\alpha,r}_{\beta,p}$, $1 < p < \infty$, we arrive at the inequality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \leq e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma_{n,p}\Bigg(\Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Bigg)\Bigg) E_n(f^{\alpha,r}_\beta)_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{19}$$

Comparing this relation with the estimate of Theorem 4 from [9] (see also [10]), we conclude that inequality (7) on the classes $C^{\alpha,r}_{\beta,p}$, $1 < p < \infty$, is asymptotically best possible.

In the same way, the asymptotical sharpness of the estimate (8) on the class $C^{\alpha,r}_{\beta,1}$ follows from comparing inequality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \leq e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{\pi\alpha r} + \gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r} + \frac{1}{n^{1-r}}\Big)\Big) E_n(f^{\alpha,r}_\beta)_{L_1} \tag{20}$$

and formula (9) from [8].

Acknowledgements

Second author is supported by the Austrian Science Fund FWF project F5503 (part of the Special Research Program (SFB) Quasi-Monte Carlo Methods: Theory and Applications)

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Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on May 14, 2018.

NIERÓWNOŚĆ TYPU LEBESGUE'A DLA SUM FOURIERA NA KLASACH UOGÓLNIONYCH CAŁEK POISSONA

S t r e s z c z e n i e

Dla funkcji ze zbioru uogólnionych całek Poissona $C^{\alpha,r}_{\beta}L_p$, $1 \leq p < \infty$, otrzymujemy górne oszacownie dla odchyleń sum Fouriera w jednostajnej metryce w terminach najlepszej aproksymacji uogólnionych pochodnych $f^{\alpha,r}_{\beta}$ funkcji tego typu w metryce przestrzeni L_p. Uzyskane oszacowania są asymptotycznie najlepsze z możliwych.

Słowa kluczowe: nierówności typu Lebesgue'a, sumy Fouriera, uogólnione całki Poissona, najlepsze przybliżenia

DOI: 10.26485/0459-6854/2018/68.2/5 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 53–60

Dedicated to the memory of
Professor Yurii B. Zelinskii

Bogdan A. Klishchuk

WEAKLY *m*-CONVEX SETS AND THE SHADOW PROBLEM

Summary

In this paper we study some properties of weakly m-convex sets in n-dimensional Euclidean space. We obtain estimates for different variants of the shadow problem at a fixed point. We discuss unsolved questions related to this problem.

Keywords and phrases: m-convex set, weakly m-convex set, Grassmann manifold, conjugate set, shadow problem, 1-hull of family of sets

1. Introduction

The purpose of this paper is to study different variants of a problem which can be called the shadow problem at a fixed point. We construct an example giving a lower estimate to create a shadow at a point tangent to the sphere S^2 in the space $\mathbb{R}^3$.

Further, under m-dimensional planes we mean m-dimensional affine subspaces of the Euclidean space $\mathbb{R}^n$.

Definition 1.1. We say that the set $E \subset \mathbb{R}^n$ is *m-convex* with respect to the point $x \in \mathbb{R}^n \backslash E$ if there exists an m-dimensional plane L such that $x \in L$ and $L \cap E = \varnothing$.

Definition 1.2. We say that the open set $G \subset \mathbb{R}^n$ is *weakly m-convex* if it is m-convex with respect to each point $x \in \partial G$ belonging to the boundary of the set G. Any set $E \subset \mathbb{R}^n$ is *weakly m-convex* if it can be approximated from outside by the family of open weakly m-convex sets.

It is easy to construct examples of weakly m-convex set which is not m-convex.

Example 1.1. Let $D = \{(x, y) \mid (|x| < 3, 3 < |y| < 9) \vee (3 < |x| < 9, |y| < 3)\}$ be a set consisting of four open squares. This set is weakly 1-convex, but is not an 1-convex set.

Example 1.2. Let $B = \{(x, y) \mid x^2 + y^2 < 1\}$ be an open circle in the plane. We choose three points of a circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ and consider a simplex σ with vertices at these points. It is easy to see that a set $E = B \setminus \sigma$ is also weakly 1-convex set, but it is not an 1-convex.

2. Properties of weakly m-convex sets

In this section we study properties of m-convex sets. The next proposition was proved by Yu. Zelinskii in [5].

Proposition 2.1. *If E_1 and E_2 is weakly k-convex and weakly m-convex set respectively, $k \leqslant m$, then a set $E = E_1 \cap E_2$ is weakly k-convex set.*

Let $G(n, m)$ be Grassmann manifold of m-dimensional planes in $\mathbb{R}^n$[2].

Definition 2.1. A set E^* is called *conjugate* to a set E if E^* is a subset of a set consisting of m-dimensional planes in $G(n, m)$ that don't intersect the set E.

Now we prove the following theorem.

Theorem 2.1. *If K is weakly m-convex compact set and a set K^* is connected then for the section of K by arbitrary $(n - m)$-dimensional plane L the set $L \backslash K \cap L$ is connected.*

Proof. As was proved in the proposition 2 [5] the set K^* is an open set, so any two of its points can be connected by a continuous arc in K^*. Suppose that there exists an $(n - m)$-dimensional plane L for which the set $L \backslash K \cap L$ is not connected. Thus the intersection $K \cap L$ is a carrier of some non-zero $(n - m - 1)$-dimensional chain z [2].

Let a point x belong to a bounded component of the set $L \backslash K \cap L$. Such points exist because of the compactness of K. From the weak m-convexity of K it follows that an m-dimensional plane l_1 which does not intersect K passes through the point x. Now we take other m-dimensional plane l_2 outside of some sufficiently large ball containing the compact K.

If we compactificate the space $\mathbb{R}^n$ to a sphere S^n by an infinitely remote point then we obtain two m-dimensional chains $w_1 = l_1 \cup (\infty)$ and $w_2 = l_2 \cup (\infty)$ from which the first chain is affected by the chain z and the second is not. On the one hand, these chains can not be translated into one another by homotopy which would not intersect a chain z and therefore a set $K \cap L$.

On the other hand, from the fact that the set K^* is connected follows the existence in K^* of pairs of points y_1, y_2 which define the planes l_1 and l_2 respectively and connected by an arc in K^*. The points of this arc define the homotopy of the plane l_1 in l_2 that has no common points with the set $K \cap L$. The resulting contradiction completes the proof of our theorem. □

3. The shadow problem

Now we study different variants of the shadow problem in n-dimensional Euclidean space.

For every set $E \subset \mathbb{R}^n$ we can consider the minimal m-convex set containing E and call it *m-convex hull* of a set E.

Introducing the concept of m-convex hull of a set E, we obtain the next problem: to find the criterion that the point $x \in \mathbb{R}^n \setminus E$ belongs to the m-convex hull of the set E. For a case of 1-convex hull of a set that is a union of some set of balls the problem was formulated by G. Khudaiberganov and named *the shadow problem* [1].

The shadow problem. What is the minimum number of mutually disjoint closed or open balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere with condition that any straight line passing through the center of the sphere intersects at least one of these balls?

In other words, this problem can be formulated as follows. What is the minimum number of mutually disjoint closed or open balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere with condition that the center of the sphere belongs to an 1-convex hull of the family of these balls?

G. Khudaiberganov proved that in the case $n = 2$ two discs are sufficient to create a shadow in the center of a circle. He assumed that for $n > 2$ the minimum number of such balls equals n. Subsequently, professor Yu. Zelinskii [8] proved that in the case $n = 3$ three balls are not enough to create a shadow for the center of the sphere. At the same time the four balls create the shadow. In the general case it is sufficient $n + 1$ balls.

Theorem 3.1. *There exist two closed (open) balls with centers on the unit circle and of radii smaller than one with condition that the center of the circle belongs to an 1-hull of these balls.*

Theorem 3.2. *In order that the center of a sphere S^{n-1} in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n > 2)$ belongs to an 1-convex hull of a family of mutually disjoint open (closed) balls of radii whose values do not exceed (smaller than) of the radius of the sphere and with centers on the sphere it is necessary and sufficient $(n + 1)$ balls.*

Note that professor Yu. Zelinskii generalized the shadow problem for an arbitrary point inside the sphere.

Problem 3.1. What is the minimum number of mutually disjoint closed or open balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere with condition that the interior of the sphere belongs to an 1-convex hull of the family of these balls?

He obtained [9] the solution of this problem in a case $n = 2$.

Theorem 3.3. *In order that an interior of a circle belongs to an 1-convex hull of a family of mutuall disjoint open or closed discs with centers on the circle and of radii smaller than the radius of the circle it is sufficient 3 discs.*

In a case where the point doesn't necessarily belong to some sphere, the following theorem obtained by Yu. Zelinskii is true [7].

Theorem 3.4. *In order that a chosen point in the n-dimensional Euclidean space $\mathbb{R}^n$ for $n \geqslant 2$ belongs to an 1-hull of a family of open (closed) balls that do not contain this point and do not intersect pairwise it is necessary and sufficient n balls.*

Note that where balls are of the same radius we have the next result[10].

Theorem 3.5. *Any set consisting of three balls of the same radius which do not intersect pairwise forms an 1-convex set in the three-dimensional Euclidean space $\mathbb{R}^3$.*

Now we consider a set consisting of three balls in the space $\mathbb{R}^n$. The following statement is true.

Theorem 3.6. *For an arbitrary point of the space $\mathbb{R}^n \setminus \bigcup_{i=1}^{3} B_i$, where B_1, B_2, B_3 are three balls of the same radius that do not intersect pairwise and do not pass through this point, there exists an $(n-2)$-dimensional plane containing this point and does not intersect any of the balls.*

Proof. Let B_1, B_2, B_3 be three balls of the same radius that do not intersect pairwise and do not pass through some point $x \in \mathbb{R}^n$. Let us construct a three-dimensional plane L passing through three centers of the balls and a point x. The intersections of the selected balls with the plane L are three three-dimensional balls B_1', B_2', B_3'. Then according to Theorem 3.5, in the plane L there exists a straight line l which does not intersect any of these balls.

Now we consider the orthogonal complement L_1 of a plane L in the space $\mathbb{R}^n$. This is an $(n-3)$-dimensional plane. Obviously, the Cartesian product $l \times L_1$ is an $(n-2)$-dimensional plane passing through the point x and does not intersect any of the balls B_1, B_2, B_3. The proof is completed. $\square$

Definition 3.1. We say that a family of sets $\Im = \{F_\alpha\}$ creates a shadow tangent to the manifold M at the point $x \in M$ if every straight line tangent to the manifold M

at the point $x \in M \setminus \bigcup_{\alpha} F_\alpha$ has a non-empty intersection at least with one of the sets F_α belonging to the family $\Im$.

Now we formulate the shadow problem for points of the sphere S^{n-1} which don't belong to the union of the balls with respect to the straight lines tangent to the sphere.

Problem 3.2. What is the minimum number of mutually disjoint closed or open balls $\{B_i\}$ in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere which provide a shadow tangent to the sphere S^{n-1} at each point $x \in S^{n-1} \setminus \bigcup_i B_i$?

Lemma 3.1. *We consider an equilateral triangle in the Euclidean plane $\mathbb{R}^2$. If we choose three circles B_i, $i = 1, 2, 3$, with centers at the vertices of this triangle and of a radius equals to half of the height of the triangle then every straight line passing through an arbitrary point $x \in (\bigcup_{i=1}^{3} B_i)^* \setminus \bigcup_{i=1}^{3} B_i$, where $(\bigcup_{i=1}^{3} B_i)^*$ is a convex hull of the set $\bigcup_{i=1}^{3} B_i$, intersects at least one of the selected circles.*

Proof. Without loss of generality we take an unit circle with the center at the origin and consider an equilateral triangle with vertices in points $(0, 1)$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ inscribed in the circle. Now we take discs B_1, B_2, B_3 of a radius $\frac{3}{4}$ in each vertex of the triangle. We note that the circumscribed circle of this triangle lies in a convex hull of these three circles.

It is easy to see that any straight line passing through a point $x \in (\bigcup_{i=1}^{3} B_i)^* \setminus \bigcup_{i=1}^{3} B_i$, where $(\bigcup_{i=1}^{3} B_i)^*$ is a convex hull of the set $\bigcup_{i=1}^{3} B_i$, intersects at least one of the three selected discs. By increasing the radii of the selected discs, we obtain that the lemma is true for three open circles of a fixed radius.

Lemma 3.1 gives an answer on the problem 3.2 in the case $n = 2$. □

This result shows that in a three-dimensional case for an arbitrary point of a sphere it is possible to select three balls touching pairwise and creating a shadow at all points of a curvilinear triangle created on the sphere by these balls. Note that the harmonization of such construction for the whole sphere requires additional considerations. This is shown in the following example.

Example 3.1. There exists a set consisting of 14 open (closed) balls that do not intersect pairwise with centers on a sphere $S^2 \subset \mathbb{R}^3$ that can not provide a shadow tangent to the sphere S^2 at each point $x \in S^2 \setminus \bigcup_{i=1}^{14} B_i$.

Without loss of generality we can assume that the chosen sphere S^2 has center at the origin and its radius equals 1. We take a cube with vertices in points ($x = \pm 1/\sqrt{3}, y = \pm 1/\sqrt{3}, z = \pm 1/\sqrt{3}$) inscribed in this sphere. The length of an edge of the cube is equal to $a = 2/\sqrt{3}$.

Now we choose eight open balls with centers at the vertices of the cube and radius $r = 1/\sqrt{3} \approx 0.577$ which equals to half of the cube's edge. We add to this collection six new open balls with centers at an intersection of the rays going from the origin and passing through the center of the face of the cube with the sphere S^2. Radii of these balls equal $r = \sqrt{2 - 2/\sqrt{3}} - 1/\sqrt{3}$. Each of them touches exactly up to four previously selected balls. This collection of balls of two different radii covers the sphere. As the calculations show, this set of balls is not sufficient to create a shadow tangent to the sphere S^2 at each point $x \in S^2 \setminus \bigcup\limits_{i=1}^{14} B_i$.

Note that the constructed set of balls gives a lower estimate of the required number of balls. The question on an upper estimate remains open.

4. Open problems

Unfortunately, Theorem 3.3 gives the solution of the problem 3.1 only in the case $n = 2$. The question on the solution of this problem in higher dimensions remains open.

Question 4.1. What is the minimum number of mutually disjoint closed or open balls in n-dimensional Euclidean space with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere with condition that the interior of the sphere belongs to an 1-convex hull of the family of these balls?

Finally, Lemma 3.1 gives the answer on the Problem 3.2 for $n = 2$. At the same time, Example 3.1 gives a lower estimate in the case $n = 3$. The questions on an upper estimate for $n = 3$ and solution of the problem in the case $n > 3$ are open.

Question 4.2. What is the minimum number of mutually disjoint closed or open balls B_i in the space $\mathbb{R}^n$ ($n > 3$) with centers on the sphere S^{n-1} and of radii smaller than the radius of the sphere which provide a shadow tangent to the sphere S^{n-1} at each point $x \in S^{n-1} \setminus \bigcup\limits_i B_i$?

Acknowledgements

The author acknowledges the Polish-Ukrainain grant No. 39/2014 *Topological-analytical methods in complex and hypercomplex analysis* of the Polish Academy of Sciences and the National Academy of Sciences of Ukraine.

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Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on April 16, 2018.

ZBIORY SŁABO m-WYPUKŁE I PROBLEM CIENIA

S t r e s z c z e n i e

Badamy własności zbiorów słabo m-wypukłych w n-wymiarowej przestrzeni euklidesowej. Uzyskujemy oszacowania dla różnych wariantów problemu cienia w ustalonym punkcie. Analizujemy również kilka nierozwiązanych zagadnień.

Słowa kluczowe: zbiór m-wypukły, zbiór słabo m-wypukły, rozmaitość Grassmanna, zbiór sprzężony, problem cienia, 1-otoczka rodziny zbiorów

DOI: 10.26485/0459-6854/2018/68.2/6 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 61–70

Dedicated to the memory of
Professor Yurii B. Zelinskii

Sergii Favorov and Natalia Girya

CONVERGENCE OF DIRICHLET SERIES ON A FINITE-DIMENSIONAL SPACE

Summary

We consider conditions for convergence of Dirichlet series on a finite-dimensional space in Stepanov's metric. Also, we obtain some applications for Stepanov's and Besicovitch's almost periodic functions.

Keywords and phrases: Dirichlet series, exponents of a Dirichlet series, Fourier series, Stepanov's metric, Besicovitch's metric, almost periodic function

Consider a Dirichlet series $\sum_k a_k e^{\lambda_k z}$, $a_k \in \mathbb{C}$, $\lambda_k \in \mathbb{R}$. In the paper [4] and [5], V. Stepanov obtained the following result:

Theorem S. Suppose that $\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$. If $\lambda_{k+1} - \lambda_k > \alpha > 0$, $k \in \mathbb{Z}$, α does not depend on n, then the sums $S_N(x) = \sum_{k=-N}^{N} a_k e^{i\lambda_k x}$ form a Cauchy sequence with respect to the integral metric, namely

$$\sup_{y\in\mathbb{R}} \left(\int_y^{y+1} |S_M - S_N|^2 dx \right)^{\frac{1}{2}} \to 0 \quad M, N \to \infty.$$

The quantity

$$D_{S_l^p}[f(x),g(x)] = \sup_{x\in\mathbb{R}}[\frac{1}{l}\int\limits_x^{x+l} |f(y)-g(y)|^p dy]^{\frac{1}{p}}, \quad p\ge 1,$$

is called Stepanov's distance of order p ($p \ge 1$) associated with length l ($l>0$). The corresponding metric is called Stepanov's one.

Here we assume that functions $f(x), g(x)$ are pth power integrable on each segment. Note that Stepanov's distances are equivalent for various $l>0$; the space of functions with finite Stepanov's norm $D_{S_l^p}[f(x),0]$ is complete (see [4]).

In our paper we prove an analogue of Theorem S on the space $\mathbb{R}^d$. In one-dimensional case our result is stronger than Theorem S.

We need some definitions and notations.

Let $B(x_0,r)$ be the open ball with center at the point $x_0\in\mathbb{R}^d$ and radius $r>0$, $\langle t,x\rangle$ be the scalar product on $\mathbb{R}^d$, and ω_d be the volume of a unit ball in $\mathbb{R}^d$.

Suppose that $f\colon \mathbb{R}^d\to\mathbb{C}$, $g\colon \mathbb{R}^d\to\mathbb{C}$ are measurable and L^p-integrable functions on each compact set.

Definition 1.

$$D_{S_l^p}[f(x),g(x)] = \sup_{x\in\mathbb{R}^d}\left[\frac{1}{\omega_d l^d}\int\limits_{B(x,l)} |f(y)-g(y)|^p dy\right]^{\frac{1}{p}}, \quad p\ge 1.$$

The metrics generating by these distances with different $l>0$ are equivalent and complete, therefore we will take $l=1$ and write D_{S^p} instead of $D_{S_1^p}$. Such distance is called Stepanov's metric.

By $SH(\mathbb{R}^d)$ denote the Schwartz space of smooth functions $f(x)$, $x\in\mathbb{R}^d$, with the following property: for any $m=(m_1,m_2,...,m_d)\in(\mathbb{N}\cup\{0\})^d$ and for any $k\in\mathbb{N}$ the equality $\left(\frac{\partial^{m_1+m_2+\ldots+m_d}}{\partial x^{m_1}\partial x^{m_2}\ldots\partial x^{m_d}} f\right)(x) = \overline{o}\left(\frac{1}{|x|^k}\right), x\to\infty$ holds true.

Definition 2. (see [6]) The function $\widehat{f}(t) = \int\limits_{\mathbb{R}^d} f(x)e^{-i\langle t,x\rangle}dx$, $t\in\mathbb{R}^d$, is called the Fourier transform of $f(x)\in L^1(\mathbb{R}^d)$.

It is known (see, for example, [6], [8]), that the Fourier transform is the automorphism on $SH(\mathbb{R}^d)$.

Let $\{(a_n,\lambda_n)\}_{n=1}^\infty$ be a set of pairs where $a_n\in\mathbb{C}$, $\lambda_n\in\mathbb{R}^d$. Let $\Lambda = \bigsqcup\limits_{j=1}^{\infty}\Lambda_j$ be a partition of the set $\Lambda=\{\lambda_n\}_{n=1}^\infty$ with the property $\operatorname{diam}\Lambda_j<1$, $\quad j=1,2,...$. Denote $S_N(x) = \sum\limits_{k=1}^N a_k e^{i\langle\lambda_k,x\rangle}$.

Theorem 1. *Suppose* $a_n > 0, 0 < r < \infty$. *Then*

$$\sum_{j=1}^{\infty}\left(\sum_{\lambda_n\in\Lambda_j} a_n\right)^2 \leq C_1 \sup_N \int\limits_{B(0;r)} |S_N(x)|^2 dx,$$

where $C_1 = C_1(r,d)$.

Proof. Let $\varphi(x) \in SH(\mathbb{R}^d)$ be an even nonnegative function such that $\operatorname{supp}\varphi(x) \subset B(0,\frac{r}{2})$. Put $\psi(x) = \frac{1}{\delta^d}(\varphi * \varphi)(\frac{x}{\delta})$ for $\delta \in (0,1)$. Clearly, $\operatorname{supp}\psi(x) \subset B(0,\delta r)$ and $\widehat{\psi}(t) = |\widehat{\varphi}(\delta t)|^2 \geq 0,\ \widehat{\psi}(0) > 0$ and

$$\widehat{\psi}(t) \geq \varepsilon > 0, \quad t \in B(0,1) \tag{1}$$

for appropriate δ.

Let $M = \sup\limits_{\mathbb{R}^d}\psi(x)$. We have the following sequence of inequalities:

$$\int\limits_{B(0;r)} |S_N(x)|^2 dx \geq$$

$$\geq M^{-1}\int\limits_{\mathbb{R}^d}\psi(x)|S_N(x)|^2 dx = M^{-1}\int\limits_{\mathbb{R}^d}\psi(x)\sum_{n=1}^{N}\sum_{l=1}^{N} a_n a_l e^{i\langle\lambda_n-\lambda_l,\, x\rangle}dx =$$

$$= M^{-1}\sum_{n=1}^{N}\sum_{l=1}^{N} a_n a_l \int\limits_{\mathbb{R}^d}\psi(x)e^{i\langle\lambda_n-\lambda_l,\, x\rangle}dx = M^{-1}\sum_{n=1}^{N}\sum_{l=1}^{N} a_n a_l\widehat{\psi}(\lambda_l-\lambda_n).$$

Since $\widehat{\psi}(t) \geq 0$ we omit all the terms where the elements λ_n, λ_k belong to different sets Λ_j and get the following inequalities:

$$M^{-1}\sum_{n=1}^{N}\sum_{l=1}^{N} a_n a_l\widehat{\psi}(\lambda_l-\lambda_n) \geq M^{-1}\sum_j \sum_{\substack{1\leq n,l\leq N\\ \lambda_n,\lambda_k\in\Lambda_j}} a_n a_l\widehat{\psi}(\lambda_l-\lambda_n) \geq$$

$$\geq M^{-1}\varepsilon\sum_j \sum_{\substack{1\leq n,l\leq N\\ \lambda_n,\lambda_l\in\Lambda_j}} a_n a_l = M^{-1}\varepsilon\sum_j\left(\sum_{\substack{1\leq n\leq N\\ \lambda_n\in\Lambda_j}} a_n\right)^2.$$

Thus,

$$\sum_j\left(\sum_{\lambda_n\in\Lambda_j} a_n\right)^2 \leq C_1\sup_N \int\limits_{B(0,r)} |S_N(x)|^2 dx.$$

This completes the proof of the Theorem. □

Define $T_m = \{(j,l) : m \le \mathrm{dist}(\Lambda_j, \quad \Lambda_l) < m+1\}$. Note that $\mathbb{N}^2 = \bigsqcup_{m=0}^{\infty} T_m$.

Let $\{B(x_j,1)\}$ be a set of balls such that multiplicities of their intersections do not exceed h and $\Lambda_j \subset B(x_j,1)$ for all $j \in \mathbb{N}$. Note that for a fixed k and any j such that $B(x_k,2) \cap B(x_j,2) \ne \emptyset$ we have $|x_j - x_k| < 4$ and $B(x_j,1) \subset B(x_k,5)$. Let M be a number of such balls $B(x_j,1)$. The sum of volumes of these balls is at most $M\omega_d$. Clearly, $M\omega_d \le h5^d\omega_d$, therefore multiplicities of the system of the balls $B(x_j,2)$ bound by $H = h5^d$. Replace each ball $B(x_j,1)$ by some ball $B(x'_j,1)$ with $x'_j \in \Lambda_j \subset B(x_j,1)$. Note that $\Lambda_j \subset B(x'_j,1)$. Since $B(x'_j,1) \subset B(x_j,2)$, we see that multiplicities of intersections of the system $\{B(x'_j,1)\}$ are bounded by H. Hence we may suppose that $x_j \in \Lambda_j$.

Lemma. *For any $l, m \in \mathbb{N}$ the number of elements of the set $\{k \in \mathbb{N} \colon (k,l) \in T_m\}$ does not exceed $C_2 H m^{d-1}$, $C_2 = C_2(d)$.*

Proof. Let $(k,l) \in T_m$. We have $m \le \mathrm{dist}(\Lambda_k, \Lambda_l) \le |x_k - x_l| \le \mathrm{dist}(\Lambda_k, \Lambda_l) + 2 \le m+3$. Therefore, all balls $B(x_k,1)$ with $(k,l) \in T_m$ are contained in the spherical layer $\{x : m-1 \le |x - x_l| \le m+4\}$. The volume of this spherical layer is $\omega_d((m+4)^d - (m-1)^d) \le C_2\omega_d m^{d-1}$, where C_2 depends on d only.

Hence a common value of the set T_m of balls $B(x_k,1)$ with $(l,k) \in T_m$ does not exceed $C_2 H m^{d-1}$. ☐

Theorem 2. *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$, $\Lambda = \bigsqcup_{j=1}^{\infty} \Lambda_j$, $\mathrm{diam}\,\Lambda_j < 1, j = 1,2,\dots$. Suppose that $\Lambda_j \subset B(x_j,1)$, $x_j \in \Lambda_j$ and the multiplicities of intersections of the balls $B(x_j,1)$ do not exceed h, also suppose that $\sum_{j=1}^{\infty}\left(\sum_{\lambda_n \in \Lambda_j} |a_n|\right)^2 = K^2 < \infty$ for some $a_n \in \mathbb{C}$.*

Then the following conditions are fulfilled:

$$a)\ \ D_{S^2}[S_N(x), 0] \le C_3 K,$$

where $S_N(x) = \sum_{k=1}^{N} a_k e^{i\langle\lambda_k, x\rangle}$, C_3 does not depend on N.

$$b)\ \ \lim_{M,N\to\infty} D_{S^2}[S_N(x), S_M(x)] = 0,$$

therefore the series $\sum_k a_k e^{i\langle\lambda_k, x\rangle}$ converges in the metric D_{S^2}.

Proof. Let $\varphi(x) \in \ SH(\mathbb{R}^d)$ be a function such that $\varphi(x) = 1,\ x \in B(0;1)$ and $\mathrm{supp}\varphi(x) \subset B(0,2),\ 0 \le \varphi(x) \le 1$.

Then

$$\int_{B(y;1)} |S_N(x)|^2 dx \le \int_{\mathbb{R}^d} \varphi(x-y) \sum_{1\le k\le N} \sum_{1\le l\le N} a_k \overline{a}_l e^{i\langle\lambda_k - \lambda_l,\, x\rangle} dx =$$

$$= \sum_{1\le k\le N}\sum_{1\le l\le N} a_k\overline{a}_l \int\limits_{\mathbb{R}^d} \varphi(x)e^{i\langle\lambda_k-\lambda_l,\, x+y\rangle}dx \le$$

$$\le \sum_{1\le k\le N}\sum_{1\le l\le N} |a_k||\overline{a}_l| \left|\int\limits_{\mathbb{R}^d} \varphi(x)e^{i\langle\lambda_k-\lambda_l,\, x+y\rangle}dx\right| =$$

$$= \sum_{1\le k\le N}\sum_{1\le l\le N} |a_k||a_l||\widehat{\varphi}(\lambda_l-\lambda_k)|.$$

Since $\widehat{\varphi}\in\ SH(\mathbb{R}^d)$, we get $|\widehat{\varphi}(x)|\le C_4\min\{1, \frac{1}{|x|^{d+1}}\}$. After appropriate rearrangement of the summands

$$\sum_{1\le k\le N}\sum_{1\le l\le N} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)|$$

we get:

$$\sum_{1\le k\le N}\sum_{1\le l\le N} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| =$$

$$=\sum_{j}\sum_{\substack{1\le k,\, l\le N\\ \lambda_k,\, \lambda_l\in\Lambda_j}} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)|+$$

$$+\sum_{m=1}^{\infty}\sum_{(j,\, p)\in T_m}\sum_{\substack{1\le k,\, l\le N\\ \lambda_k\in\Lambda_j,\, \lambda_l\in\Lambda_p}} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| = \Sigma_1+\Sigma_2.$$

We estimate the sums Σ_1 and Σ_2 separately.

We have $|\widehat{\varphi}(\lambda_k-\lambda_l)|\le C_4$ for any j under the condition $\lambda_k,\lambda_l\in\Lambda_j$. Hence the next bound for Σ_1 holds:

$$\sum_{\substack{1<k,l<N\\ \lambda_k,\lambda_l\in\Lambda_j}} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| \le C_4\sum_{\lambda_k\in\Lambda_j}|a_k|\sum_{\lambda_l\in\Lambda_j}|a_l| = C_4\left(\sum_{\lambda_k\in\Lambda_j}|a_k|\right)^2,$$

Therefore,

$$\Sigma_1\le C_4K^2. \tag{2}$$

Further, for each fixed $m\ge 1$:

$$\sum_{(j,p)\in T_m}\sum_{\substack{1\le k,\, l\le N\\ \lambda_k\in\Lambda_j,\, \lambda_l\in\Lambda_p}} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| \le C_4\frac{1}{m^{d+1}}\sum_{(j,p)\in T_m}\sum_{\substack{1\le k\le N\\ \lambda_k\in\Lambda_j}}|a_k|\sum_{\substack{1\le l\le N\\ \lambda_l\in\Lambda_p}}|a_l| \le$$

$$\leq \frac{1}{2}C_4\frac{1}{m^{d+1}}\sum_{(j,p)\in T_m}\left(\left(\sum_{\lambda_k\in\Lambda_j}|a_k|\right)^2+\left(\sum_{\lambda_l\in\Lambda_p}|a_l|\right)^2\right) \tag{3}$$

Using Lemma and replacing the summation over p such that $(j,p)\in T_m$ by the summation over all $s\in\mathbb{N}$, we obtain the following estimate for (3):

$$\frac{C_2C_4}{2}\frac{m^{d-1}}{m^{d+1}}\sum_s\left(\left(\sum_{\lambda_k\in\Lambda_s}|a_k|\right)^2+\left(\sum_{\lambda_l\in\Lambda_s}|a_l|\right)^2\right)=\frac{C_2C_4}{m^2}\sum_s\left(\sum_{\lambda_l\in\Lambda_s}|a_l|\right)^2.$$

Therefore,

$$\Sigma_2\leq C_5K^2. \tag{4}$$

Finally, taking into account (2) and (4), we obtain

$$\int\limits_{B(y;1)}|S_N(x)|^2dx\leq C_6\cdot K^2,$$

where C_6 does not depend on N. Hence, $D_{S^2}[S_N(x)]\leq C_3\cdot K$, where C_3 does not depend on N, so the proposition a) is proved.

Prove the proposition b). Let $K_N^2=\sum\limits_j\left(\sum\limits_{\substack{1\leq k\leq N\\ \lambda_k\in\Lambda_j}}|a_k|\right)^2$. Actually we have just proved the inequality

$$\sup_y\int\limits_{B(y,1)}|S_N(x)|^2dx\leq(C_3K_N)^2. \tag{5}$$

Substituting the sum $S_N(x)-S_M(x)$ for $S_N(x)$ in inequality (5), we get

$$D_{S^2}[S_N(x),S_M(x)]\leq C_3^2(K_N^2-K_M^2),$$

here $K_N^2-K_M^2=\sum\limits_j(\sum\limits_{\substack{M\leq n\leq N\\ \lambda_n\in\Lambda_j}}|a_k|)^2$.

Prove that $(K_N^2-K_M^2)\to 0$ as $N,M\to\infty$. Assume that M is sufficiently large. By the condition $\sum\limits_j\left(\sum\limits_{\lambda_n\in\Lambda_j}|a_n|\right)^2=K^2$, for each $\varepsilon>0$ there exists $q\in\mathbb{N}$ (q does not depend on M and on N) such that $\sum\limits_{j=q+1}^{\infty}\left(\sum\limits_{\substack{M\leq n\leq N\\ \lambda_n\in\Lambda_j}}|a_n|\right)^2\leq\frac{\varepsilon}{2}$.

Next, for each fixed $1 \le j \le q$ there exists M such that the inequality

$$\left(\sum_{\lambda_n \in \Lambda_j} |a_n|\right)^2 \le \frac{\varepsilon}{2q}$$

is satisfied for $n > M$. Then $\sum\limits_{j=1}^{q} \left(\sum\limits_{\substack{M \le n \le N \\ \lambda_n \in \Lambda_j}} |a_n|\right)^2 \le q \cdot \frac{\varepsilon}{2q} = \frac{\varepsilon}{2}$. Hence, for each $\varepsilon > 0$ we obtain $(K_N^2 - K_M^2) \le \varepsilon$. This completes the proof. □

Remark 1. Theorem 2 is true for $\operatorname{diam} \Lambda_j \le r, j = 1, 2, \ldots$, and for the balls of radius $R \ge r$.

Suppose that there exists a set of balls $\{B(x_j, R)\}$ such that multiplicities of intersections of the balls do not exceed h, and the numbers of points $\lambda \in \Lambda$ contained in $B(x_j, R)$ are uniformly bounded.

Put $\Lambda_1 = \Lambda \cap B(x_1, R),\ \Lambda_2 = \Lambda \cap B(x_1, R) \setminus \Lambda_1,\ \Lambda_j = (\Lambda \cap B(x_1, R)) \setminus \bigcup\limits_{k=1}^{j-1} \Lambda_k$. The sets Λ_j satisfy all the conditions of Theorem 2 and for any j the number of elements Λ_j does not exceed some bound $s < \infty$.

Clearly, $\sum\limits_{n=1}^{\infty} |a_n|^2 < \infty$ implies $\sum\limits_{j=1} \left(\sum\limits_{\lambda_n \in \Lambda_j} |a_n|\right)^2 \le \sum\limits_{j=1} s \sum\limits_{\lambda_n \in \Lambda_j} |a_n|^2 < \infty$.

We get the following consequence of Theorem 2:

Theorem 3. *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $\{B(x_j, R)\}$ be a set of balls such that multiplicities of intersections of the balls do not exceed h. Suppose that numbers of elements of the sets $\Lambda \cap B(x_j, R)$ are uniformly bounded for all $j \in \mathbb{N}$. If for some $a_n \in \mathbb{C}$ $\sum\limits_{n=1}^{\infty} |a_n|^2 < \infty$, then the following conditions are fulfilled:*

$$a)\ \sup_N S_N(x) < \infty,$$

here $S_N(x) = \sum\limits_{k=1}^{N} a_k e^{i\langle \lambda_k, x\rangle}$.

$$b)\ \lim_{M,N \to \infty} D_{S^2}[S_N(x), S_M(x)] = 0.$$

Consider some applications of the obtained results.

Definition 3. (see [2] for the case d=1). Function $f(x)\colon \mathbb{R}^d \to \mathbb{C}$ is called Stepanov's almost periodic function of order p (S^p-almost periodic function) if there exists a sequence of finite exponential sums $S_n(x) = \sum\limits_j c_j e^{i\langle \lambda_j, x\rangle},\ c_j \in \mathbb{C},\ \lambda_j \in \mathbb{R}^d$, such that

$$\lim_{n \to \infty} D_{S^p}[f(x), S_n(x)] = 0.$$

To each S^p-almost periodic function $f(x),\ x \in \mathbb{R}^d$, we associate the Fourier series

$$f(x) \sim \sum_{\lambda \in \mathbb{R}^d} a(\lambda, f) e^{i\langle \lambda, x\rangle},$$

where $a(\lambda, f) = \lim\limits_{T\to\infty} \frac{1}{\omega_d T^d} \int\limits_{B(0,\,T)} f(x) e^{-i\langle\lambda,x\rangle} dx.$

Definition 4. (see [2] for the case $d = 1$ and [3] for the case $d > 1$) The spectrum of function $f(x)$ is the set $spf = \{\lambda \in \mathbb{R}^d : a(\lambda, f) \neq 0\}$.

It is well known (for the case $d = 1$ see [2], the proof for the case $d > 1$ can be treated in the same way) that spectrum of S^p-a.p.function is at most countable. The properties of the spectrum of the almost periodic functions in various metrics were considered in [7]. There were considered Stepanov's, Weil's and Besicovitch's almost periodic functions on $\mathbb{R}^d$.

Theorem 4. *For any set of pairs $\{(a_n, \lambda_n)\}_{n=1}^{\infty}$ that satisfy the conditions of Theorem 2 there exists S^2– almost periodic function $f(x)$ with Fourier series $\sum\limits_n a_n e^{i\langle\lambda_n,x\rangle}$.*

Proof. It follows from the completeness of the metric D_{S^2} and Theorem 2 that the sums $\sum\limits_{n\le N} a_n e^{i\langle\lambda_n,x\rangle}$ converge to $f(x)$ with respect to the metric D_{S^2}. □

Also we get

Theorem 5. *For any set of pairs $\{(a_n, \lambda_n)\}_{n=1}^{\infty}$ that satisfy the conditions of Theorem 3 there exists S^2– almost periodic function $f(x)$ with Fourier series $\sum\limits_n a_n e^{i\langle\lambda_n,x\rangle}$.*

Let the functions $f\colon \mathbb{R}^d \to \mathbb{C}$, $g\colon \mathbb{R}^d \to \mathbb{C}$ be measurable and L^p-integrable on each compact in $\mathbb{R}^d$.

Generalizing the definition of Besikovitch's distance (see [1]) for the function on $\mathbb{R}^d$ we have the following definition.

Definition 5. Put

$$D_{B^p}[f(x), g(x)] = \left\{ \overline{\lim_{T\to\infty}} \frac{1}{\omega_d T^n} \int\limits_{B(0,T)} |f(y) - g(y)|^p dy \right\}^{\frac{1}{p}}, \quad p \geq 1,$$

the metric generated by this distance is called Besicovitch's distance of order p.

Definition 6. (see [1] for the case d=1) Function $f(x)\colon \mathbb{R}^d \to \mathbb{C}$ is called Besicovitch's almost periodic function of order p (B^p–almost periodic function) if there exists a sequence of finite exponential sums $S_n(x) = \sum\limits_j c_j e^{i\langle\lambda_j,x\rangle}, c_j \in \mathbb{C}, \lambda_j \in \mathbb{R}^d$, such that

$$\lim_{n\to\infty} D_{B^p}[f(x), S_n(x)] = 0.$$

Each B^p–almost periodic function $f(x),\ x\in\mathbb{R}^d$, has at most countable spectrum

$$\mathrm{sp} f=\{\lambda\colon a(\lambda,f)=\lim_{T\to\infty}\frac{1}{\omega_d T^d}\int\limits_{B(0,\,T)} f(x)e^{-i\langle\lambda,x\rangle}dx\neq 0\}.$$

Moreover, for each B^2 – almost periodic function f we have

$$\sum_{\lambda_n\in \mathrm{sp} f}|a(\lambda_n,f)|^2<\infty.$$

The proof is similarly to the case $d=1$.

Hence we obtain

Theorem 6. *Let $f(x),\ x\in\mathbb{R}^d$, be B^2– almost periodic function with the spectrum $\Lambda=\{\lambda_k\}_{k=1}^\infty$. Suppose that there exists a set of balls $\{B(x_j,R)\}$ such that the multiplicities of intersections do not exceed h, and numbers of elements $\lambda\in\Lambda\cap B(x_j,R)$ is uniformly bounded. Then the function $f(x)$ is S^2 – almost periodic.*

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Presented by Władysław Wilczyński at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 25, 2018.

O ZBIEŻNOŚCI SZEREGU DIRICHLETA W PRZESTRZENI SKOŃCZENIE WYMIAROWEJ

S t r e s z c z e n i e

Rozważamy warunki zbieżności szeregów Dirichleta w przestrzeni skończenie wymiarowej przy metryce Stepanova. Uzyskujemy też pewne zastosowania dla funkcji prawie okresowych Stepanova i Besicovitcha.

Słowa kluczowe: szereg Dirichleta, wykładniki w szeregu Dirichleta, szereg Fouriera, metryka Stepanova, metryka Besicovitcha, funkcje prawie okresowe

DOI: 10.26485/0459-6854/2018/68.2/7 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 71–76

Dedicated to the memory of
Professor Yurii B. Zelinskii

Serhii V. Gryshchuk

ON SOME CASES OF PLANE ORTHOTROPY

Summary

There are considered some cases of plane orthotropy in the absence of body forces. Then every function from a pair-solution of the equilibrium system of equations with respect to displacements satisfies the elliptic fouth-order equation of the type:

$$\left(\alpha_1\frac{\partial^4}{\partial x^4}+\alpha_2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)w(x,y)=0,$$

with certain real $\alpha_k\neq 0$, $k=1,2$.

Keywords and phrases: the generalized Hook's law, a plane orthotropy, the equilibrium system

1. Introduction

As well-known (cf., e.g., [1, 2, 3]), in the case of isotropic plane deformations with the absence of body forces a function (displacement) u or v from a pair-solution $(u(x,y),v(x,y))$ of the equilibrium system of equations in displacements

$$\begin{cases} (\lambda+\mu)\left(\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 v(x,y)}{\partial x\partial y}\right)+\mu\Delta_2 u(x,y)=0, \\ (\lambda+\mu)\left(\frac{\partial^2 u(x,y)}{\partial x\partial y}+\frac{\partial^2 v(x,y)}{\partial y^2}\right)+\mu\Delta_2 v(x,y)=0\,\forall(x,y)\in D, \end{cases}\tag{1}$$

as well as the stress Airy's function, satisfies the biharmonic equation: $(\Delta_2)^2w(x,y)=0$, where $\Delta_2:=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the 2-D Laplasian, D is a domain of the Cartesian plane xOy, λ and μ are the Lamé constants.

Similar results for anisotrophic solid body are not well-known. One of the reason of this fact is a difficulty (due to a variety of cofficients) of the generalized Hooke's law expressing strains via stresses in a linear form.

The aim of this paper is to prove analogous (to the isotropic case) statements for some cases of an elastic anisotrophic homogeneous plane solid body — a plane orthotropic body, or briefly, a plane orthotropy. We will restrict our attention on some simple but interesting cases of orthotropy.

2. Notations and preliminaries

Let $\mathbb{R}^{3\times 3}$ be a set of all real 3×3 matrices, $A \in \mathbb{R}^{3\times 3}$, $\det A$ is a determinant of A. If $\det A \neq 0$ then there exists the inverse matrix $B = A^{-1}$ such that $AB = BA = 1$, where 1 is the unity matrix. By $\mathbb{R}_+^{3\times 3}$ we define all matrices of $\mathbb{R}^{3\times 3}$ which are symmetric and positive defined. A symbol $\overleftarrow{\vartheta}$ defines a vector-column having three real coordinates ϑ_k, $k = 1, 2, 3$.

Let a model of an elastic anisotropic medium occupied a domain D of the Cartesian plane xOy be a homogeneous (cf., e.g., [4, p. 25]) plane orthotropic (cf., e.g., [4, p. 35]) body.

Let $\overleftarrow{\varepsilon}$ has coordinates equal to strains (cf., e.g., [4, p. 18]):

$$\varepsilon_1 := \varepsilon_x,\, \varepsilon_2 := \varepsilon_x,\, \varepsilon_3 = \gamma_{xy}.$$

Let $\overleftarrow{\sigma}$ has coordinates equal to stresses (cf., e.g., [4, p. 16]):

$$\sigma_1 := \sigma_x,\, \sigma_2 := \sigma_y,\, \sigma_3 := \tau_{xy}.$$

The generalized Hooke's law for our model has two equivalent forms (cf., e.g., [4, § 3], [5, § 4.1.3]):

$$\overleftarrow{\varepsilon} = A\overleftarrow{\sigma},\, \overleftarrow{\sigma} = A^{-1}\overleftarrow{\varepsilon}, \tag{2}$$

with $A \in \mathbb{R}_+^{3\times 3}$ of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{pmatrix}, \tag{3}$$

where

$$a_{11} > 0,\, a_{11}a_{22} - (a_{12})^2 > 0,\, a_{66} > 0. \tag{4}$$

Unequalities (4) follows from the Sylvester's criterion of positive definiteness of the matrix (3).

A numbers a_{ij} and A_{ij} , $k \leq m$, $k, m = 1, 2, 6$, are called *elastic constants* ([4, p. 27]). They are constants in D due to the homogeneity of the solid body.

Consider notations for elements of A^{-1}:

$$A^{-1} =: \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{pmatrix}, \tag{5}$$

where A_{km} satisfy (4) with $a_{km} := A_{km}$, $k \leq m$, $k, m = 1, 2, 6$.

A *stress function* ([6, p. 21] with $\overline{U} \equiv 0$) is a function w satisfying relations:

$$\sigma_x(x_0, y_0) = \frac{\partial^2 w}{\partial y^2}(x_0, y_0),\ \sigma_y(x_0, y_0) = \frac{\partial^2 w}{\partial x^2}(x_0, y_0),$$

$$\tau_{xy}(x_0, y_0) = -\frac{\partial^2 w}{\partial x \partial y}(x_0, y_0)\, \forall (x_0, y_0) \in D.$$

In the absence of body forces, the stress function $w(x, y)$ satisfies the elliptic fouth-order equation (“ the stress equation”, cf., e.g., [6, p. 27] with $a_{16} = a_{26} = 0$):

$$\left(a_{22} \frac{\partial^4}{\partial x^2} + (2a_{12} + a_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} + a_{11} \frac{\partial^4}{\partial y^4} \right) w(x, y) = 0. \tag{6}$$

The equilibrium system of equations with respect to the displacement vector $(u(x, y), v(x, y))$ has a form (cf., e.g., [4, p. 75]):

$$\begin{cases} \left(A_{11} \frac{\partial^2}{\partial x^2} + A_{66} \frac{\partial^2}{\partial y^2} \right) u(x, y) + (A_{12} + A_{66}) \frac{\partial^2 v(x,y)}{\partial x \partial y} = 0, \\ \left(A_{66} \frac{\partial^2}{\partial x^2} + A_{22} \frac{\partial^2}{\partial y^2} \right) v(x, y) + (A_{12} + A_{66}) \frac{\partial^2 u(x,y)}{\partial x \partial y} = 0, \end{cases} \tag{7}$$

where all $(x, y) \in D$; A_{km}, $k \leq m$, $k, m = 1, 2, 6$, are defined in (5).

3. Cases of orthotropy and solutions of their equilibrium systems and stress equation

Consider the following equation (particular case of (6)):

$$l_{0,p} w(x, y) \equiv$$

$$\equiv \left((2p - 1) \frac{\partial^4}{\partial x^4} + 2p \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) w(x, y) = 0\ \forall (x, y) \in D, \tag{8}$$

where $p \neq 1$ is a real parameter.

Consider an orthotropy with

$$a_{11} = a_{12} = 1,\ a_{22} = 2p - 1,\ a_{16} = a_{26} = 0,\ a_{66} = 2(p - 1). \tag{9}$$

Then the equation (8) is a stress equation. It is easy to check that the matrix (3) is positive defened only for $p > 1$. So a case $p < 1$ has no elastic meaning and we are to investigate a case $p > 1$. Calculating the inverse matrix A^{-1} we find:

$$A_{11} = \frac{2p - 1}{2(p - 1)},\ A_{12} = -\frac{1}{2(p - 1)},\ A_{22} = A_{66} = -A_{12}. \tag{10}$$

Since $A_{12} + A_{66} = 0$ a system (7) takes a form

$$\begin{cases} \frac{1}{2(p-1)}\, l_{1,p} u(x,y) = 0, \\ \frac{1}{2(p-1)}\, \Delta_2 v(x,y) = 0\,\forall (x,y) \in D, \end{cases} \tag{11}$$

where $l_{1,p} := (2p-1)\,\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Taking into account that the operator (8) can be factorisated in the form:

$$l_{1,p} = l_{1,p} \circ \Delta_2 = \Delta_2 \circ l_{1,p}$$

($l_1 \circ l_2$ is a symbol of composition of operators l_1 and l_2), we see that if a pair (u,v) is a solution of (11) then $w := u$ or $w := v$ is a solution of the equation (8). So we proved the following theorem.

Theorem 1. *Let $p > 1$, an orthotropy is defined by* (2), (9). *Then every displacement-function from a pair of solution of the eqilibrium system* (11) *satisfies the equation* (8).

Now consider another cases of orthotropy for which an equilibrium equation splits onto two equations containing except of operators of the type $l_{1,p}$ an extra term-operator $\frac{\partial^2}{\partial x \partial y}$ acted to another unknown function and has a non-zero coefficient.

Let p be an arbitrary fixed number: $0 < p < 1$.

Take into consideration the plane orthotropy:

$$a_{11} = a_{22} = 1,\ a_{16} = a_{26} = 0,\ a_{66} = 2(p - a_{12}),\ -1 < a_{12} < p. \tag{12}$$

An a_{12} belongs to such measures due to the positiveness of the matrix (3). Therefore, we have:

$$A_{11} = A_{22} = \frac{1}{1-a_{12}^2},\ A_{21} = A_{12} = -\frac{a_{12}}{1-a_{12}^2},\ A_{66} = \frac{1}{2(p-a_{12})}.$$

The equilibrium system (7) gets a form:

$$\begin{cases} \frac{1}{1-a_{12}^2}\frac{\partial^2}{\partial x^2}u(x,y) + \frac{1}{2(p-a_{12})}\frac{\partial^2}{\partial y^2}u(x,y) + \\ + \left(-\frac{a_{12}}{1-a_{12}^2} + \frac{1}{2(p-a_{12})}\right)\frac{\partial^2 v(x,y)}{\partial x \partial y} = 0, \\ \frac{1}{2(p-a_{12})}\frac{\partial^2}{\partial x^2}u(x,y) + \frac{1}{1-a_{12}^2}\frac{\partial^2}{\partial y^2}v(x,y) + \\ + \left(-\frac{a_{12}}{1-a_{12}^2} + \frac{1}{2(p-a_{12})}\right)\frac{\partial^2 u(x,y)}{\partial x \partial y} = 0, \end{cases} \tag{13}$$

where all $(x,y) \in D$.

Consider the following ("stress") equation:

$$l_{2,p} w(x,y) \equiv$$

$$\equiv \left(\frac{\partial^4}{\partial x^4} + 2p\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w(x,y) = 0\,\forall (x,y) \in D. \tag{14}$$

The system (13) is equivalent to the following system:

$$\begin{cases} B_{11}\frac{\partial^2 u(x,y)}{\partial x^2} + B_{12}\frac{\partial^2 u(x,y)}{\partial y^2} + \frac{\partial^2 v(x,y)}{\partial x \partial y} = 0, \\ B_{21}\frac{\partial^2 v(x,y)}{\partial x^2} + B_{22}\frac{\partial^2 v(x,y)}{\partial y^2} + \frac{\partial^2 u(x,y)}{\partial x \partial y} = 0 \, \forall (x,y) \in D. \end{cases} \tag{15}$$

where

$$B_{11} = B_{22} := \frac{2\,(p - a_{12})}{(a_{12} - p)^2 + 1 - p^2},$$

$$B_{12} = B_{21} := \frac{1 - (a_{12})^2}{(a_{12} - p)^2 + 1 - p^2}.$$

Theorem 2. *Let* $0 < p < 1$*, an orthotropy is defined by* (2), (12)*. Then every displacement-function from a pair of solution of the eqilibrium system* (15) *satisfies the equation* (14).

Proof. Acting by the differential operator $\frac{\partial^2}{\partial x \partial y}$ on the second equation of (15) and substituting to the obtained equation an expression of $\frac{\partial^2 v}{\partial x \partial y}$, we arrive at the equation:

$$\frac{\partial^4 u(x,u)}{\partial x^4} + C_2 \frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} + C_3 \frac{\partial^4 u(x,y)}{\partial y^4} \, \forall (x,y) \in D, \tag{16}$$

where

$$C_3 = \frac{B_{22}B_{12}}{B_{11}B_{21}} \equiv 1, \; C_2 := \frac{B_{11}B_{22} + B_{12}B_{21} - 1}{B_{11}B_{21}}.$$

So, to prove Theorem we need to check the equality $C_2 = 2p$. In terms of p and a_{12} the relation $C_2 = 2p$ can be rewritten in the form:

$$\alpha^2 + \beta^2 + 2p\alpha\beta = (\alpha + a_{12}\beta)^2 \,,$$

where $\alpha := 1 - a_{12}^2$, $\beta := 2(a_{12} - p)$. By doing simple algebraic transformation, the last one is equivalent to the relation

$$1 - a_{12}^2 = 2(a_{12} - p)\frac{\alpha}{\beta},$$

which with use of the definitions of α and β is an identity. So, we proved that if (u, v) is a solution of (15) then v satisfies the equation (14).

A similar statement for v can be proved analogously. The theorem is proved. □

Acknowlegements

The research is partically supported by the Polish-Ukrainain grant No. 39/2014 *Topological-analytical methods in complex and hypercomplex analysis* of the Polish Academy of Sciences and the National Academy of Sciences of Ukraine and by the Fundamental Research Programme funded by the Ministry of Education and Science of Ukraine (project No. 0116U001528).

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 27, 2018.

O PEWNYCH PRZYPADKACH PŁASKIEJ ORTHOTROPII

S t r e s z c z e n i e

Rozpatrywane są pewne przypadki płaskiej orthotropii przy założeniu braku oddziaływania sił ciała. Wówczas każda funkcja z pary rozwiązań układu równowagi równań ze względu na przemieszczenia spełnia równanie eliptyczne czwartego rzędu typu:

$$\left(\alpha_1 \frac{\partial^4}{\partial x^4} + \alpha_2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w(x, y) = 0,$$

z pewnymi rzeczywistymi stałymi $\alpha_k \neq 0$, $k = 1, 2$.

Słowa kluczowe: uogólnione prawo Hooke'a, orthotropia płaska, układ równowagi

DOI: 10.26485/0459-6854/2018/68.2/8 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 77–84

Dedicated to the memory of
Professor Yurii B. Zelinskii

Tatiana M. Osipchuk

ON SYSTEM OF BALLS WITH EQUAL RADII GENERATING SHADOW AT A POINT

Summary

Problems related to the determination of the minimal number of balls that generate a shadow at a fixed point in the multi-dimensional Euclidean space $\mathbb{R}^n$ are considered in the present work. Here, the statement "a system of balls generate shadow at a point" means that any line passing through the point intersects at least one ball of the system. The minimal number of pairwise-disjoint balls with equal radii in $\mathbb{R}^n$ which do not contain a fixed point of the space and generate shadow at the point is indicated in the work.

Keywords and phrases: convex set, problem of shadow, system of balls, sphere, multi-dimensional real Euclidean space

1. Introduction

In 1982 G. Khudaiberganov [1] proposed the problem of shadow.

Let us consider n-dimensional real Euclidean space $\mathbb{R}^n$ and an open (closed) ball $B(x_0, r) \subset \mathbb{R}^n$ with radius r and center x as the set of all points of distance less than (less than or equal to) r away from x ([2]). It is also called an *n-dimensional ball.* A set of all points in $\mathbb{R}^n$ of the same distance is a sphere S^{n-1} ([2]).

Let x be a fixed point in the real multi-dimensional Euclidean space $\mathbb{R}^n$. We say that a system of balls $\{B_i : i \in \mathbb{N}\} \subset \mathbb{R}^n$ not containing x *generates a shadow at this point* if any straight line passing through x intersects at least one ball of the system. So, the problem of shadow can be formulated as follows: *To find the minimal* *number of pairwise disjoint, open (closed) balls in* $\mathbb{R}^n$ *centered on a sphere* S^{n-1}*, not containing the sphere center, and generating shadow at the sphere center.*

This problem was solved by G. Khudaiberganov for the case $n = 2$: it was proved that two balls are sufficient for a circumference on the plane. For all that, it was also made the assumption that for the case $n > 2$ the minimal number of such balls is exactly equal to n.

Twenty years passed and Yu. Zelinskii became interested in this problem. In [3], he and his students proved that three balls are not sufficient for the case $n = 3$, but it is possible to generate a shadow at the center of a sphere with four balls. In their work it is also proved that for the general case the minimal number is $n + 1$ balls, so the complete answer to this problem for a collection of closed and open balls was obtained. Thus, G. Khudaiberganov's assumption was wrong. In [3], it is also proposed another method of solving the problem for the case $n = 2$ which gives some numerical estimates.

Since 2015, a group of mathematicians leading by Yu. Zelinskii has been working on a series of problems similar to the problem of shadow and their generalizations in the Institute of Mathematics of the National Academy of Sciences of Ukraine. In [4], [5], [9] one can find the review of problems and their solutions related to the problem of shadow. One of these problems is the following:

Problem 1. ([6]) *Let* x *be a fixed point of the space* $\mathbb{R}^n$*,* $n \geq 2$*. What is the minimal number* $m(n)$ *of pairwise disjoint, open (closed) balls with equal radii in* $\mathbb{R}^n$ *not containing* x *and generating shadow at* x*?*

It is not difficult to show that the minimal number of the balls in the plane is two. In [6], an example of four pairwise disjoint, open (closed) balls with equal radii in space $\mathbb{R}^3$ not containing a fixed point of the space and generating shadow at this point is constructed. In [7] it is proved that non three of such balls in $\mathbb{R}^3$ generate shadow at a fixed point of space. Thus, Problem 1 is solved for space $\mathbb{R}^3$, and the answer is $m(3) = 4$.

Moreover, in [6] it is proved that there does not exist a system of pairwise disjoint, open (closed) balls with equal radii in space $\mathbb{R}^3$ centered on a fixed sphere, not containing the sphere center, and generating shadow at the sphere center.

In the present work, Problem 1 is solved as $n \geq 3$. The following section holds auxiliary results concerning the problem.

The author expresses gratitude to her teacher Professor Yurii Zelinskii for the setting of interesting problems and maintaining author's interest in mathematics through scientific discussions and advices. This work is dedicated to the memory of Professor Yurii Zelinskii.

2. Auxiliary results

The results of this section are formulated as lemmas since they are auxiliary within the scope of this paper.

As it was mentioned in the Introduction, the following lemma is true.

Lemma 1. ([7]) *Let $n = 3$; then $m(n) = 4$ for any fixed point $x \in \mathbb{R}^3$.*

In [8] a shadow problem for a system of balls with centers freely placed in $\mathbb{R}^n$ without restrictions on their radii is considered. So, the following lemma gives lower estimate for the number of non-overlapping balls that do not contain a fixed point in the space and generate shadow at this point.

Lemma 2. ([8]) *The minimal number of open (closed) non-overlapping balls not containing a fixed point in the space $\mathbb{R}^n$, $n \geq 2$, and generating shadow at this point is equal to n.*

The following lemma will be frequently used.

Lemma 3. ([9]) *Let two open (closed) non-overlapping balls $\{B_i = B(r_i) : i = 1, 2\} \subset \mathbb{R}^n$ with centers on a sphere $S^{n-1}(r)$ and with radii r_1, r_2 such that $r > r_1 \geq r_2$ be given. Then every ball homothetic to B_1 relative to the sphere center with coefficient of homothety k_1 does not intersect every ball homothetic to B_2 relative to the sphere center with coefficient of homothety k_2, if $k_1 \leq k_2$.*

In [3], the following example of system of $n+1$ balls in $\mathbb{R}^n$ satisfying the conditions of Khudaiberganov's shadow problem is given.

Example 1. ([3]) Suppose a is the half-length of the edge of an n-dimensional regular simplex (see [2]). Let us consider a system of $n + 1$ open balls $\{B_i : i = 1, \dots n + 1\} \subset \mathbb{R}^n$ with correspondent radii $r_1 = a + \varepsilon$, $r_2 = a - \varepsilon/2$, $r_3 = a - \varepsilon/2^2$, $r_4 = a - \varepsilon/2^3$, …, $r_{n+1} = a - \varepsilon/2^n$, where ε is sufficiently small. Let us place centers of the balls at the vertexes of a simplex such that the balls touch each other. This simplex is slightly different from the regular one and can be inscribed into a sphere. Thus, the system of open balls generate shadow at the sphere center. Let us consider the closures of the balls $\{\overline{B}_i : i = 1, \dots n+1\}$. If we slightly reduce these closed balls, then new system of closed balls generates shadow at the sphere center by continuity.

Using Example 1, in [10] an example of system of $n+1$ balls in $\mathbb{R}^n$ satisfying the conditions of Problem 1 is built as follows.

Let us fix a point $x \in \mathbb{R}^n$ and let us consider open (closed) balls $\{B_i : i = 1, \dots n+1\}$, of Example 1, placed on the sphere with the sphere center at x. Let us apply homothety to each open (closed) ball B_i with respective coefficient of homothety $k_i = r_1/r_i$, $i = 1, \dots n + 1$. Then $k_1 < \dots < k_i < k_{i+1} < \dots < k_{n+1}$ and the obtained system consists of $n+1$ balls with the same radii that are equal to r_1. Since $r_1 > \dots > r_i > r_{i+1} > \dots > r_{n+1}$, new balls are pairwise disjoint by Lemma 3, do not contain x and generate shadow at x by the constructions. So, the following lemma is true.

Lemma 4. ([10]) *Let $n \geq 2$; then $m(n) \leq n+1$ for any fixed point $x \in \mathbb{R}^n$.*

3. Main results

We need the following definitions for this section.

Any m-dimensional affine subspace of the space $\mathbb{R}^n$, $m < n$, is called an *m-dimensional plane.* An $(n-1)$-dimensional plane is called a *hyperplane.*

Theorem 1. *Let $n \geq 3$; then $m(n) > n$ for any fixed point $x \in \mathbb{R}^n$.*

Proof. The case as $n = 3$ holds by Lemma 1. Let us prove this theorem for $n > 3$ using the method of mathematical induction. We consider the space $\mathbb{R}^4$ with points $x = (x_1, x_2, x_3, x_4)$. Let us fix a point $x_0 \in \mathbb{R}^4$. By Lemma 2, the number of non-overlapping, open (closed) balls generating shadow at x is not less than four. Suppose there exists a system of four non-overlapping four-dimensional open (closed) balls $\{B_i(r, a^i) : i = \overline{1,4}\}$ with the same radii equal to r which does not contain point x_0 and generates shadow at x_0 (see Fig. 1).

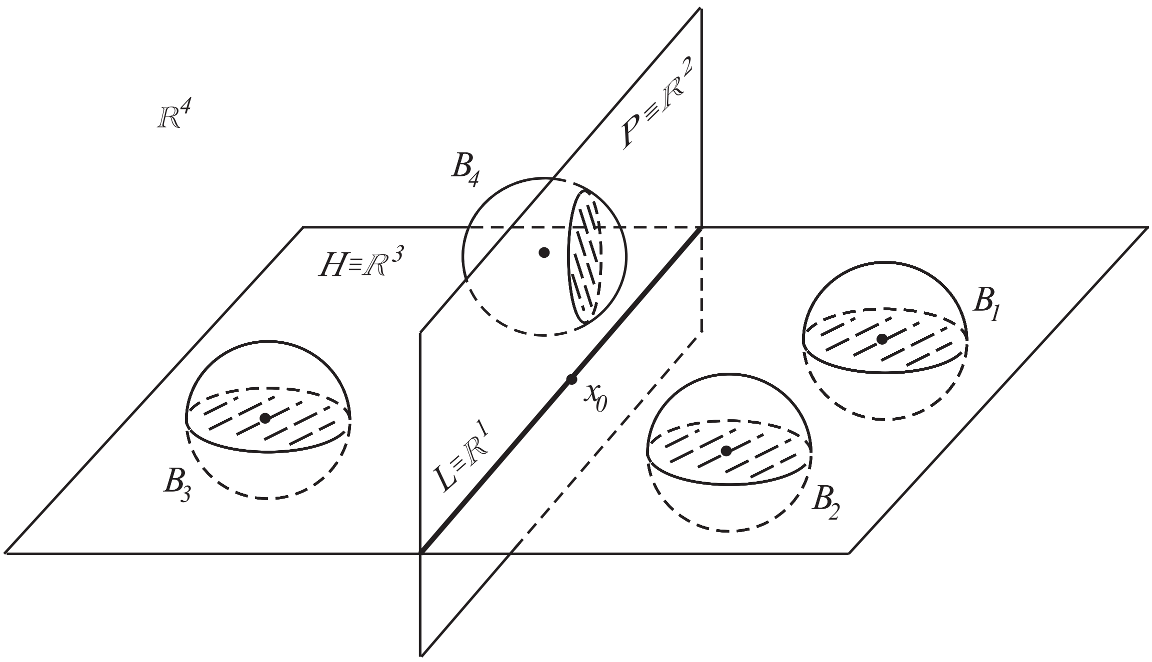

Fig. 1

Then let us draw a three-dimensional plane H (hyperplane) through the point x and the centers of balls B_1, B_2, B_3. Without loss of generality, we can choose a coordinate system such that H is the coordinate hyperplane $x_4 = 0$. Then open (closed) balls B_i, $i = 1, 2, 3$, can be described as follows:

$$B_i := \left\{x \in \mathbb{R}^4 : (x_1 - a_1^i)^2 + (x_2 - a_2^i)^2 + (x_3 - a_3^i)^2 + x_4^2 < r^2\right\}$$

$$\left(B_i := \left\{x \in \mathbb{R}^4 : (x_1 - a_1^i)^2 + (x_2 - a_2^i)^2 + (x_3 - a_3^i)^2 + x_4^2 \le r^2\right\}\right),$$

for $i = \overline{1,3}$. The intersection of hyperplane H and balls B_1, B_2, B_3 gives three-dimensional balls with the same radii that are equal to r. By Lemma 1 none three non-overlapping open (closed) balls with equal radii in space $\mathbb{R}^3$ generate shadow at point x_0. Thus, there exists a straight line L (1-dimensional plain) in hyperplane H passing through x_0 and not intersecting any of the three-dimensional balls. Without loss of generality, suppose L coincides with the coordinate axis x_1, i.e

$$L := \left\{x \in \mathbb{R}^4 : \left\{\begin{array}{c} x_2 = 0, \\ x_3 = 0, \\ x_4 = 0. \end{array}\right.\right\}$$

Then $L \cap B_i = \emptyset$, $i = \overline{1,3}$, i.e

$$B_i : (x_1 - a_1^i)^2 \ge r^2 - a_2^i - a_3^i,\ i = \overline{1,3}$$
$$\left(B_i : (x_1 - a_1^i)^2 > r^2 - a_2^i - a_3^i,\ i = \overline{1,3}\right).$$

We claim that the two-dimensional plane

$$P := \left\{x \in \mathbb{R}^4 : \left\{\begin{array}{c} x_2 = 0, \\ x_3 = 0 \end{array}\right.\right\}$$

does not intersect any of the initial four-dimensional balls B_1, B_2, B_3. Indeed,

$$B_i : (x_1 - a_1^i)^2 + x_4^2 \ge (x_1 - a_1^i)^2 \ge r^2 - a_2^i - a_3^i,\ i = \overline{1,3}.$$
$$\left(B_i : (x_1 - a_1^i)^2 + x_4^2 \ge (x_1 - a_1^i)^2 > r^2 - a_2^i - a_3^i,\ i = \overline{1,3}\right).$$

The intersection $P \cap B_4$ is a disk. By Lemma 2, one disk does not generate shadow at x_0 in the two-dimensional plane P. Thus, in the space $\mathbb{R}^4$ four non-overlapping open (closed) balls with equal radii not containing a point of the space do not generate shadow at the point.

Suppose none $n-1$ non-overlapping, open (closed) balls with equal radii in the space $\mathbb{R}^{n-1}$ not containing a fixed point of the space generate shadow at the point.

Let us consider the space $\mathbb{R}^n$, $n > 4$, and any fixed point $x_0 \in \mathbb{R}^n$. By Lemma 2, the number of non-overlapping, open (closed) balls generating shadow at x_0 is not less than n. Suppose there exist such n non-overlapping, open (closed) balls $\{B_i(r) : i = \overline{1,n}\} \subset \mathbb{R}^n$ with the same radii equal to r that generate shadow at x_0. Then let us draw an $(n-1)$-dimensional plane H through the point x_0 and the centers of balls B_1, …, B_{n-1}. The intersection of the hyperplane H ($(n-1)$-dimensional plane) and balls B_1, …, B_{n-1} gives $(n-1)$-dimensional balls with the same radii that are equal to r. By the assumption these balls do not generate shadow at point x_0 in the hyperplane H. Thus, there exists a straight line L in H passing through x_0 and not intersecting any of the $(n-1)$-dimensional balls. Then, the two-dimensional plane P passing through the straight line L perpendicular to the hyperplane H in $\mathbb{R}^n$ does not intersect any of the initial n-dimensional balls $B_1, \ldots, B_{n-1}$. We claim that the ball B_n does not overlap the two-dimensional plane P in the space $\mathbb{R}^n$. Indeed, by Lemma 2, one disk does not generate shadow at x_0 in the two-dimensional plane. Thus, in the space $\mathbb{R}^n$ any n non-overlapping, open (closed) balls with equal radii not containing a fixed point of the space do not generate shadow at the point. □

Combining Lemma 4 and Theorem 1, we get the solution of Problem 1, as $n \geq 3$.

Theorem 2. *Let $n \geq 3$; then $m(n) = n + 1$ for any fixed point $x \in \mathbb{R}^n$.*

Remark 1. Let $n = 2$; then $m(n) = 2$ for any fixed point $x \in \mathbb{R}^2$.

Proof. Let us consider a circle with center at a fixed point of the plane and two pairwise disjoint, open (closed) disks in the plane with centers on a circle and radii r_1, r_2 less than the circle radius generating shadow in the circle center. It is obvious that $r_1 \neq r_2$. Let $r_1 > r_2$ for definiteness. Let us apply homothety to each ball with respective coefficient of homothety $k_i = r_1/r_i$, $i = 1, 2$. Since $k_1 < k_2$, we conclude that obtained balls are pairwise disjoint by Lemma 3, do not contain the sphere center and generate shadow at the sphere center by the constructions. The number of disks is minimal by Lemma 2. □

Remark 2. None n pairwise disjoint, open (closed) balls in $\mathbb{R}^n$ centered on a sphere S^{n-1} and not containing the sphere center generate shadow at the sphere center.

Proof. This result particular solves Khudaiberganov's shadow problem and is a generalization of well known result for $\mathbb{R}^3$ (see [3], [11]). But, since it is not used in the proof of Theorem 2, we can prove it as follows.

Suppose n pairwise disjoint, open (closed) balls $\{B_i : i = 1, \ldots n\} \subset \mathbb{R}^n$ centered on a sphere S^{n-1} with radii $r_1 \geq \ldots \geq r_i \geq r_{i+1} \geq \ldots \geq r_n$ generate shadow at the sphere center. Let us apply homothety to each ball with respective coefficient of homothety $k_i = r_1/r_i$, $i = 1, \ldots n$. Thus, the obtained system consists of n balls with the same radii that are equal to r_1. Since $k_1 \leq \ldots \leq k_i \leq k_{i+1} \leq \ldots \leq k_n$, new balls are pairwise disjoint by Lemma 3, do not contain the sphere center and generate shadow at the sphere center by the constructions. But this contradicts Theorem 2. □

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Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on May 14, 2018.

O UKŁADZIE KUL Z RÓWNYMI PROMIENIAMI GENERUJĄCYCH CIEŃ W PUNKCIE

S t r e s z c z e n i e

Problemy związane z ustaleniem minimalnej liczby kul generujących cień w ustalonym punkcie w wielowymiarowej przestrzeni euklidesowej $\mathbb{R}^n$ są rozpatrywane przyjmując, że każda prosta przechodząca przez dany punkt przecina jedną z kul układu. Wyznaczona jest minimalna liczba parami rozłącznych kul o równych promieniach w $\mathbb{R}^n$, ktre nie zawierają ustalonego punktu przestrzeni i generują cień w tym punkcie.

Słowa kluczowe: zbiór wypukły, problem cienia, układ kul, sfera, wielowymiarowa przestrzeń euklidesowa

DOI: 10.26485/0459-6854/2018/68.2/9 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 85–94

Dedicated to the memory of
Professor Yurii B. Zelinskii

Mariia V. Stefanchuk

GENERALIZATION OF THE CONCEPT OF CONVEXITY IN A HYPERCOMPLEX SPACE

Summary

Extremal elements and a h-hull of sets in the n-dimensional hypercomplex space $\mathbb{H}^n$ are investigated. The class of $\mathbb{H}$-quasiconvex sets including strongly hypercomplexly convex sets and closed relatively to intersections is introduced. Some results concerning multivalued functions in the complex space were generalized into the n-dimensional hypercomplex space: there was proved the hypercomplex analogue of the Fenchel-Moreau theorem and some properties of functions that are conjugate to functions $f\colon \mathbb{H}^n \setminus \Theta \longrightarrow \mathbb{H}$.

Keywords and phrases: hypercomplexly convex set, h-hull of a set, h-extremal point, h-extremal ray, $\mathbb{H}$-quasiconvex set, linearly convex function, conjugate function

1. Introduction

The natural analogue of complex analysis is a hypercomplex analysis. Therefore, there is a need to transfer a series of results of a convex analysis known in n-dimensional real and complex spaces, on the n-dimensional hypercomplex space $\mathbb{H}^n$, $n \in N$, which is a direct product of n-copies of the body of quaternions $\mathbb{H}$ [1]. G. Mkrtchyan worked on these problems [2, 3]. He introduced the concepts of hypercomplexly convex, strongly hypercomplexly convex sets and transfered a series of results of linearly convex analysis on hypercomplex space $\mathbb{H}^n$. Yu. Zelinskii [4] and his students (M. Tkachuk, T. Osipchuk, B. Klishchuk) continued to develop this direction.

Let $E \subset \mathbb{H}^n$ be an arbitrary set of the space $\mathbb{H}^n$ containing the origin of coordinates $\Theta = (0, 0, \ldots, 0)$. We put $x = (x_1, x_2, \ldots, x_n)$, $h = (h_1, h_2, \ldots, h_n)$, $\langle x, h\rangle = x_1h_1 + x_2h_2 + \cdots + x_nh_n$. The set $E^* = \{h | \langle x, h\rangle \neq 1,\ \forall x \in E\}$ is called the conjugate set to the set E [2].

A hyperplane is called a set $L \subset \mathbb{H}^n$ that satisfies one of the conditions $\langle x, a\rangle = w$, $\langle x - x_0, a\rangle = 0$, where x is an arbitrary point of the set L, x_0 is a fixed vector, w is a fixed scalar with $\mathbb{H}$, and a is a fixed covector. We call the covector a a normal. Accordingly, affine we will call only the functions of the species $l(x) = \langle x, a\rangle + b$, $b \in \mathbb{H}$.

Definition 1 [2]. The set $E \subset \mathbb{H}^n$ is called a hypercomplexly convex if for any point $x_0 \in \mathbb{H}^n \setminus E$ there exists a hyperplane that passes through the point x_0 and does not intersect E.

Definition 2 [2]. The set $E \subset \mathbb{H}^n$ is called a strongly hypercomplexly convex if its arbitrary intersection with the hypercomplex straight line γ is acyclic, that is $\widetilde{H}^i(\gamma \cap E) = 0$, $\forall i \geq 0$, where $\widetilde{H}^i(\gamma \cap E)$ is a consolidated group of Aleksandrov-Cech cohomology sets $\gamma \cap E$ with coefficients in the set of integers.

2. Extremal elements

Let $E \subset \mathbb{H}$ be an arbitrary set. The complement to the union of the unbounded components of the set $\mathbb{H} \setminus E$ is called the h-combination of the points of the set E and is denoted by $[E]$. If E is an arbitrary set in the space $\mathbb{H}^n$, $n > 1$, then we say that the point x belongs to the h-combination of points from E if there exists an intersection of the set E with a hypercomplex straight line γ such that $x \in [E \cap \gamma]$. The set of such points with $\mathbb{H}^n$ is called the h-combination of the points E and denoted $[E]$; the m-multiple h-combination is determined by the induction $[E]^m = [[E]^{m-1}]$ [4].

Definition 3 [2]. The set $\widehat{E} = \cap_\pi \pi^{-1}[\pi(E)]$ is called the h-hull of the set $E \subset \mathbb{H}^n$, where $\pi \colon \mathbb{H}^n \longrightarrow \lambda$ — all possible linear projections of the set on the hypercomplex straight lines, $[\pi(E)]$ is the h-combination of the points of the set $\pi(E)$, and $\pi^{-1}[\pi(E)] = \{x \in \mathbb{H}^n |\ \pi(x) \in \pi(E)\}$ is its complete preimage.

The following theorem [5] asserts that for an arbitrary set of the space $\mathbb{H}^n$ the set of points of its h-hull coincides with the h-combination of the points of this set.

Theorem 1. *If the set $E \subset \mathbb{H}^n$ is an h-hull, then $E = [E]$.*

Proof. Let $x \in [\lambda \cap E]$ for some hypercomplex plane λ. Then, the inclusion $\pi(x) \in [\pi(\lambda \cap E)]$ for all projections π is obviously true, since the restriction of any projection π to each straight line is either homeomorphism or projection into a point. $\square$

Definition 4 [2]. The h-interval with center at the point x of radius r is the intersection of an open ball of radius r with center at the point x with a hypercomplex straight line, which passes through the point x.

Definition 5 [2]. A point $x \in E \subset \mathbb{H}^n$ is called the h-extremal point of the set E if E has no h-intervals containing x.

We extend the Klee's theorem of a convex analysis [6] to a hypercomplex case.

Definition 6 [5]. The h-ray is called a closed unbounded acyclic subset of a hypercomplex straight line with a non-empty boundary.

Definition 7 [5]. The extremal h-ray of the set $E \subset \mathbb{H}^n$ is called the h-ray H belonging to the set E if the set $E \setminus \mathbb{H}$ is hypercomplexly convex and each point of the boundary of the ray H will be an h-extremal point for the set E. (This is equivalent to that no point of the ray H will be internal to the arbitrary h-interval that belongs to the set E and has at least one point outside H).

For the set $E \subset \mathbb{H}^n$ we denote: $\operatorname{hext} E$ is the set of its h-extremal points, $\operatorname{rhext} E$ is the set of h-extremal rays, $\operatorname{hconv} E$ is the h-hull of the set E.

Lemma 1. *Let $E \subset \mathbb{H}^n$ be a closed strongly hypercomplexly convex body ($\operatorname{int} E \neq \emptyset$) with a non-empty strongly hypercomplexly convex boundary ∂E, then E has the form $E = E_1 \times \mathbb{H}^{n-1}$, where E_1 is an acyclic subset of straight line $\mathbb{H}$ with non-empty interior relative to this straight line.*

Proof. Since the boundary ∂E is strongly hypercomplexly convex, then for an arbitrary point $x \in \operatorname{int} E$ there exists a hyperplane that does not intersect ∂E. Therefore, the set E contains a hyperplane. Consequently, by theorem 3 [4], the set E can be depicted in the form of Cartesian product $E = E_1 \times \mathbb{H}^{n-1}$. The set E_1 will be acyclic, because there are intersections E be hypercomplex straight lines that are homeomorphic to E_1. $\square$

Definition 8. An affine subset L is called a tangent to the set E if $L \cap \overline{E} \subset \partial E$, $L \cap \overline{E} \neq \emptyset$.

Lemma 2. *If $E \subset \mathbb{H}^n$ is a strongly hypercomplexly convex closed set and L is its tangent hypercomplex straight line, then* $\operatorname{hext}(E \cap L) = (\operatorname{hext} E) \cap L$.

Proof. Since the inclusion of sets $E \cap L \subset E$ is fair, then by the definition of h-extremal points we have $\operatorname{hext}(E \cap L) \supset (\operatorname{hext} E) \cap L$. Let $x \in \operatorname{hext}(E \cap L)$. Then, inclusion $x \in [K] \setminus K$, where $K \subset E$, can not be performed, because otherwise $K \subset E \cap L$ (since $x \in L$ and L is a hypercomplex straight line, tangent to E). This contradicts the fact that $x \in \operatorname{hext}(E \cap L)$. Consequently, the inverse inclusion of $\operatorname{hext}(E \cap L) \subset (\operatorname{hext} E) \cap L$ is correct and the lemma is proved. $\square$

Remark 1. Analogically, we can prove the equality $\operatorname{rhext}(E \cap L) = (\operatorname{rhext} E) \cap L$ for h-extremal rays.

Theorem 2. *Each closed strongly hypercomplexly convex set $E \subset \mathbb{H}^n$, which does not contain a hypercomplex straight line, will be the h-hull of its h-extremal points and h-extremal rays $E = \operatorname{hconv}(\operatorname{hext} E \cup \operatorname{rhext} E)$.*

Proof. The proof is carried out by induction according to the hypercomplex dimension of the set E. For $\dim_{\mathbb{H}} E = 0$ and $\dim_{\mathbb{H}} E = 1$, the theorem is obvious. Assume that the theorem is valid for all hypercomplex dimensions of the set E, which are less than m $(1 < m \leq n)$. Let us prove it for $\dim_{\mathbb{H}} E = m$.

By the condition of the theorem, the set E does not contain a hypercomplex straight line, therefore it can not coincide neither with its affine hull, nor with the Cartesian product $E_1 \times \mathbb{H}^{n-1}$. Therefore, it follows from lemma 1 that the non-empty boundary ∂E will not be strongly hypercomplexly convex set.

By the definition of a strong hypercomplex convexity, the intersection of the set E with an arbitrary hypercomplex straight line will also be strongly hypercomplexly convex. Let x be an arbitrary point of the set E. If x belongs to a certain tangent straight line L to E, then by the hypothesis of induction we have the inclusion

$$x \in \operatorname{hconv}((\operatorname{hext} E \cap L) \cup \operatorname{rhext}(E \cap L)).$$

If there are points of the set E, through which there is no hypercomplex straight line tangent to E, then there is a point $x \in \operatorname{int} E$.

In this case, we draw a hypercomplex straight line l through the point x. The intersection of $l \cap E$ is a strongly hypercomplexly convex set and does not coincide with l. Therefore, $x \notin [\partial(l \cap E)]$. Now let y be an arbitrary point of the boundary of intersection $\partial(l \cap E)$. Taking into account the strong hypercomplex convexity through the point y, one can draw a straight line T tangent to the set E. By the hypothesis of induction, we obtain $y \in \operatorname{hconv}((\operatorname{hext} E \cap T) \cup \operatorname{rhext}(E \cap T))$. We note that this is fair for every point $y \in \partial(l \cap E)$. Then, taking into account the lemma 2 and the remark 1, we obtain $x \in \operatorname{hconv}(\operatorname{hext} E \cup \operatorname{rhext} E)$. As a result of arbitrariness of choice of the point x we obtain the inclusion $E \subset \operatorname{hconv}(\operatorname{hext} E \cup \operatorname{rhext} E)$. The inverse inclusion is trivial. The theorem is proved. □

3. $\mathbb{H}$-quasiconvex sets

The class of strongly hypercomplexly convex sets is non-closed relatively to the intersection [3]. Therefore, the main axiom of the convexity is not fulfilled: the intersection of any number of convex sets must be convex. We denote the class of sets, which includes strongly hypercomplexly convex sets and is closed relatively to intersections.

Definition 9 [5]. A hypercomplexly convex set $E \subset \mathbb{H}^n$ is called $\mathbb{H}$-quasiconvex set if its intersection with an arbitrary hypercomplex straight line γ does not contain a three-dimensional cocycle, i.e. $\mathbb{H}^3(\gamma \cap E) = 0$.

It is obvious that the class of $\mathbb{H}$-quasiconvex sets includes a strongly hypercomplexly convex domains and compacts.

Let us show the closure of a class of $\mathbb{H}$-quasiconvex sets in the sense that the intersection of an arbitrary family of compact $\mathbb{H}$-quasiconvex sets will be an $\mathbb{H}$-quasiconvex set.

Theorem 3. *The intersection of an arbitrary family of $\mathbb{H}$-quasiconvex compacts will be an $\mathbb{H}$-quasiconvex compact.*

Proof. It is enough to do the proof for two compacts. Let $K_1,\ K_2$ be two arbitrary $\mathbb{H}$-quasiconvex compacts, γ is an arbitrary hypercomplex straight line that intersects the set $K_1 \cap K_2$. We use the exact cohomological sequence of Mayer-Vietoris [7]

$$H^3(\gamma \cap K_1) \oplus H^3(\gamma \cap K_2) \to$$
$$\to H^3(\gamma \cap K_1 \cap K_2) \to H^4(\gamma \cap (K_1 \cup K_2)).$$

Since the compacts K_1 and K_2 are $\mathbb{H}$-quasiconvex, then $H^3(\gamma \cap K_1) = 0$ and $H^3(\gamma \cap K_2) = 0$. Therefore

$$H^3(\gamma \cap K_1) \oplus H^3(\gamma \cap K_2) = 0.$$

On the other hand, a compact intersection

$$\gamma \cap (K_1 \cup K_2) = (\gamma \cap K_1) \cup (\gamma \cap K_2)$$

can not hold the entire hypercomplex straight line γ, which is a four-dimensional real manifold, therefore $H^4(\gamma \cap (K_1 \cup K_2)) = 0$.

From the accuracy of the cohomological sequence it follows that $H^3(\gamma \cap K_1 \cap K_2) = 0$. This is equivalent to the assertion, that the intersection of the set $K_1 \cap K_2$ with an arbitrary hypercomplex straight line does not contain a three-dimensional cocycle. From the previous follows the $\mathbb{H}$-quasiconvexity of the compact $K_1 \cap K_2$. The theorem is proved. □

4. Linearly convex functions

Definition 10 [8]. The function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called multivalued if the image of the point $x \in \mathbb{H}^n$ is a set of $f(x) \in \mathbb{H}$.

The domain of definition of such a function will be denoted by $E_f := \{x \in \mathbb{H}^n : y \in \mathbb{H},\ y = f(x)\}$.

Definition 11. The function $l\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called affine if its graph is a hyperplane.

Definition 12 [8, 9]. A multivalued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called a linearly convex if there exists an affine function $l\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ for an arbitrary pair of points $(x_0, y_0) \in$

$(\mathbb{H}^n \times \mathbb{H}) \setminus \Gamma(f)$ such that $y_0 = l(x_0)$ and $\Gamma(l) \cap \Gamma(f) = \emptyset$ for all $x \in \mathbb{H}^n$, where the graphs of functions l and f, respectively, are denoted by $\Gamma(l)$ and $\Gamma(f)$.

Definition 13. A linearly concave function is called a multivalued function f for which the function $\varphi = \mathbb{H} \setminus f$ is linearly convex.

This means that $\mathbb{H}^{n+1} \setminus \Gamma(f)$ is a graph of a linearly convex function, i.e. through each point $(x_0, y_0) \in \Gamma(f)$ the graph of the affine function passes, which is completely contained in $\Gamma(f)$.

Definition 14 [8, 9]. A multivalued affine function is called a function that is linearly convex and linearly concave simultaneously, and for which there is at least one point $x \in \mathbb{H}^n$, in which each of the sets $(f(x) \cap \mathbb{H})$ and $(\mathbb{H} \setminus f(x))$ is non-empty.

The definition of a linearly convex function can be extended to multivalued functions that take values in an expanded hypercomplex plane $\mathbb{H}^o = \mathbb{H} \cup (\infty)$, compacted by one point.

Here are some examples of linearly convex functions.

Definition 15. A function

$$W_E(y) = \mathbb{H}^o \setminus \cup_{x \in E} \langle x, y \rangle$$

is called the reference function of the set $E \subset \mathbb{H}^n$.

Definition 16. If $E \subset \mathbb{H}^n$ is a linearly convex set, then the function

$$\delta_E(x) = \begin{cases} 0, & \text{if } x \in E, \\ \infty, & \text{if } x \notin E, \end{cases}$$

is called its indicator function.

It is easy to verify that the reference and indicator functions are linearly convex.

Theorem 5. *If f_α, $\alpha \in \mathrm{A}$, is a family of linearly convex functions, where A is an arbitrary set of indices, then the function $f = \cap_{\alpha \in A} f_\alpha$ is linearly convex.*

Proof. We have $\Gamma(f) = \cap_{\alpha \in A} \Gamma(f_\alpha)$. Let us take an arbitrary point

$$(x_0, y_0) \in (\mathbb{H}^n \times \mathbb{H}) \setminus \Gamma(f) = (\mathbb{H}^n \times \mathbb{H}) \setminus \cap_{\alpha \in A} \Gamma(f_\alpha).$$

Then

$$(x_0, y_0) \in (\mathbb{H}^n \times \mathbb{H}) \setminus \Gamma(f_\alpha)$$

for some α, and therefore there is an affine function $l \colon \mathbb{H}^n \longrightarrow \mathbb{H}$ whose graph does not intersect $\Gamma(f_\alpha)$. Therefore, it does not intersect $\Gamma(f)$. Consequently, the function f is linearly convex. The theorem is proved. $\square$

5. Conjugate functions

Definition 17. A function conjugated to f is called a function given by the equality

$$f^*(y) = \mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - f(x)). \tag{1}$$

From the definition of conjugate function follows a hypercomplex analogue of Jung-Fenhel's inequality [10]:

$$\langle x, y\rangle \notin f(x) + f^*(y). \tag{2}$$

The correlation (2) can be rewritten in the form

$$\langle x, y\rangle \in \mathbb{H} \setminus (f(x) + f^*(y)),$$

or

$$f(x) \cap (\langle x, y\rangle - f^*(y)) = \emptyset$$

with all $x \in \mathbb{H}^n$, $y \in \mathbb{H}^n$.

We find a function conjugate to a function f^*:

$$f^{**}(x) = (f^*)^*(x) = \mathbb{H}^o \setminus \cup_y(\langle x, y\rangle - f^*(y)).$$

Example 1. Conjugate with a multivalued affine function $f(x) = \langle x, y_0\rangle + f(\Theta)$, where $f(\Theta) \subset \mathbb{H}$ is the set which is the image of the point $\Theta = (0, 0, ..., 0) \in \mathbb{H}^n$, is the function

$$f^*(y) = \mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - \langle x, y_0\rangle - f(\Theta)) = \mathbb{H}^o \setminus \cup_x(\langle x, y - y_0\rangle - f(\Theta)) =$$
$$= \begin{cases} \mathbb{H}^o \setminus (-f(\Theta)), & \text{if } y = y_0, \\ \infty, & \text{if } y \neq y_0. \end{cases}$$

Example 2. Let $E \subset \mathbb{H}^n$, $\mathbb{H}^n \setminus E \neq \emptyset$, $f(x) = \delta_E(x)$. Then

$$f^*(y) = \mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - \delta_E(x)) = \mathbb{H}^o \setminus \cup_{x \subset E}\langle x, y\rangle,$$

that is, conjugate with the indicator function of its own subset E will be the reference function of this set.

Theorem 6. *For each multivalued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ the inclusion $f \subset f^{**}$ is valid.*

Proof. Let us take an arbitrary pair of points

$$x = (x_1, \ldots, x_n) \in \mathbb{H}^n, \quad y = (y_1, \ldots, y_n) \in \mathbb{H}^n.$$

We obtain from the inequality 2

$$\langle x, y\rangle - f^*(y) \cap f(x) = \emptyset,\ \langle x, y\rangle - f^*(y) \subset \mathbb{H}^o \setminus f(x),$$

i.e.

$$\mathbb{H}^o \setminus (\langle x, y\rangle - f^*(y)) \supset f(x).$$

Taking in the last inclusion the intersection of all $y \in \mathbb{H}^n$, we will obtain such inclusions

$$\cap_y[\mathbb{H}^o \setminus (\langle x, y\rangle - f^*(y))] \supset f(x),$$
$$\mathbb{H}^o \setminus \cup_y(\langle x, y\rangle - f^*(y)) \supset f(x),\ f \subset f^{**}.$$

The theorem is proved. □

Definition 18. A multivalued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called an open (respectively, closed or compact) function when its graph is open (respectively, closed or compact) set in $\mathbb{H}^{n+1}$.

Theorem 7. *Let $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ be a multivalued function. Then the function f^* conjugate to it is linearly convex. If f is open then f^* is closed.*

Proof. The value of the conjugate function can be written as

$$f^*(y) = \cap_x(\mathbb{H}^o \setminus (\langle x, y\rangle - f(x))).$$

For a fixed x the function $y \mapsto \mathbb{H}^o \setminus (\langle x, y\rangle - f(x))$ is a multivalued affine function in y, and therefore it can be presented in the form

$$y \mapsto \langle x, y\rangle + [\mathbb{H}^0 \setminus (-f(x))]. \tag{3}$$

The function f^* is the intersection of linearly convex functions of the form (3), and hence by the Theorem 5 f^* is a linearly convex function. Moreover, if f is open, then each of the functions (3) is closed, and therefore f^* is also closed. The theorem is proved. □

The following theorem is a hypercomplex analogue of the Fenhel-Moro theorem.

Theorem 8. *Let the multivalued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ be such that $\mathbb{H} \setminus f(x) \neq \emptyset$ for all $x \in \mathbb{H}^n$. Then $f^{**} = f$ if and only if when f is linearly convex.*

Proof. We shall show that the equality $f^{**} = f$ is equivalent to the linear convexity of the function f.

If $f^{**} = f$, then, according to the Theorem 7, a function conjugate to an arbitrary function will be linearly convex. If $f(\mathbb{H}^n) \equiv \infty$, then the equality $f^{**} = f$ is obtained from formulas 1 and 2. We have $f^*(y) = \mathbb{H}$ for all $y \in \mathbb{H}^{n*}$ and $f^{**} = \infty$. Since $f \subset f^{**}$ by Theorem 6, it suffices to show that the inverse inclusion $f \supseteq f^{**}$ is valid for a linearly convex function.

Let there be inequality $f(x_0) \neq f^{**}(x_0)$ at some point x_0. Then there is an affine function $l(x) = \langle x, y_0\rangle + \alpha$, such that $\Gamma(l) \cap \Gamma(f) = \emptyset$ and $w_0 = \langle x_0, y_0\rangle + \alpha$, where $w_0 \in f^{**}(x_0) \setminus f(x_0)$. Then

$$f^*(y_0) = \mathbb{H}^o \setminus \cup_x(\langle x, y_0\rangle - f(x)) = \cap_x[\mathbb{H}^o \setminus (\langle x, y_0\rangle - f(x))] \supsetneq (-\alpha),$$

because $[\langle x, y_0\rangle - f(x)] \neq -\alpha$ for all $x \in \mathbb{H}^n$. For the function f^{**} valid is an inclusion

$$f^{**}(x_0) = \cap_y[\mathbb{H}^o \setminus (\langle x_0, y\rangle - f^*(y))] \subset$$

$$\subset \mathbb{H}^o \setminus (\langle x_0, y_0\rangle - f^*(y_0)) \subset \mathbb{H}^o \setminus (\langle x_0, y_0\rangle + \alpha) = \mathbb{H}^o \setminus w_0.$$

Therefore, $w_0 \notin f^{**}(x_0)$, which contradicts the choice of the point $w_0 \in f^{**}(x_0) \setminus f(x_0)$. The theorem is proved. □

Definition 19. Let $f_\alpha\colon \mathbb{H}^n \longrightarrow \mathbb{H}, \alpha \in A$, be multivalued functions. The function $(\cup_\alpha f_\alpha)(x) := \cup_\alpha f_\alpha(x)$ we call the union of functions f_α, and the function $(\cap_\alpha f_\alpha)(x) := \cap_\alpha f_\alpha(x)$ we call their intersection.

For the conjugate functions, there is the theorem of duality.

Theorem 9. *Let $f_\alpha\colon \mathbb{H}^n \longrightarrow \mathbb{H}, \alpha \in \mathrm{A}$, be multivalued functions. Then equality holds*

$$(\cup_\alpha f_\alpha)^* = \cap_\alpha f_\alpha^*.$$

Proof. From expression 1 we obtain for conjugate functions

$$(\cup_\alpha f_\alpha)^*(y) = \mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - \cup_\alpha f_\alpha(x)) =$$
$$= \mathbb{H}^o \setminus \cup_x \cup_\alpha (\langle x, y\rangle - f_\alpha(x)) = \mathbb{H}^o \setminus \cup_\alpha \cup_x (\langle x, y\rangle - f_\alpha(x)) =$$
$$= \cap_\alpha(\mathbb{H}^o \setminus \cup_x(\langle x, y\rangle - f_\alpha(x))) = \cap_\alpha f_\alpha^*(y).$$

The theorem is proved. □

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on April 16, 2018.

UOGÓLNIENIE IDEI WYPUKŁOŚCI NA PRZESTRZENIE HIPERZESPOLONE

S t r e s z c z e n i e

Badamy ekstremalne elementy i h-otoczki zbiorów z n-wymiarowej przestrzeni hiperzespolonej $\mathbb{H}^n$. Wprowadzana jest klasa zbiorów $\mathbb{H}$-quasi-wypukłych włączając zbiory silnie hiperzespolenie wypukłe, domknięte w odniesieniu do przecięć. Pewne wyniki dotyczące funkcji wielowartościowych w przestrzeniach zespolonych są uogólnione na przestrzenie hiperzespolone. Dotyczy to twierdzenia Fenchela-Moreau i pewnych własności funkcji sprzężonych do funkcji $f\colon \mathbb{H}^n \setminus \Theta \longrightarrow \mathbb{H}$.

Słowa kluczowe: zbiór hiperzespolenie wypukły, h-otoczka zbioru, punkt h-ekstremalny, zbiór $\mathbb{H}$-guasi-wypukły, funkcja liniowo wypukła, funkcja sprzężona

DOI: 10.26485/0459-6854/2018/68.2/10 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 95–108

In memory of Professor Yurii Zelinskii

Anna Futa and Dariusz Partyka

THE SCHWARZ TYPE INEQUALITY FOR HARMONIC FUNCTIONS OF THE UNIT DISC SATISFYING A SECTORIAL CONDITION

Summary

Let T_1, T_2 and T_3 be closed arcs contained in the unit circle $\mathbb{T}$ with the same length $2\pi/3$ and covering $\mathbb{T}$. In the paper [3] D. Partyka and J. Zając obtained the sharp estimation of the module $|F(z)|$ for $z \in \mathbb{D}$ where $\mathbb{D}$ is the unit disc and F is a complex-valued harmonic function of $\mathbb{D}$ into itself satisfying the following sectorial condition: For each $k \in \{1, 2, 3\}$ and for almost every $z \in T_k$ the radial limit of the function F at the point z belongs to the angular sector determined by the convex hull spanned by the origin and arc T_k. In this article a more general situation is considered where the three arcs are replaced by a finite collection $T_1, T_2, \ldots, T_n$ of closed arcs contained in $\mathbb{T}$ with positive length, total length 2π and covering $\mathbb{T}$.

Keywords and phrases: harmonic functions, Harmonic mappings, Poisson integral, Schwarz Lemma

1. Introduction

Throughout the paper we always assume that all topological notions and operations are understood in the complex plane $\mathrm{E}(\mathbb{C}) := (\mathbb{C}, \rho_e)$, where ρ_e is the standard euclidean metric. We will use the notations $\mathrm{cl}(A)$ and $\mathrm{fr}(A)$ for the closure and boundary of a set $A \subset \mathbb{C}$ in $\mathrm{E}(\mathbb{C})$, respectively. By $\mathrm{Har}(\Omega)$ we denote the class of all complex-valued harmonic functions in a domain Ω, i.e., the class of all twice continuously differentiable functions F in Ω satisfying the Laplace equation

$$\frac{\partial^2 F(z)}{\partial x^2}+\frac{\partial^2 F(z)}{\partial y^2}=0, \quad z=x+\mathrm{i}y\in\Omega.$$

The sets $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ and $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$ are the unit disc and unit circle, respectively. The standard measure of a Lebesgue measurable set $A\subset\mathbb{T}$ will be denoted by $|A|_1$. In particular, if A is an arc then $|A|_1$ means its length. Set $\mathbb{Z}_{p,q}:=\{k\in\mathbb{Z}:p\le k\le q\}$ for any $p,q\in\mathbb{Z}$.

Definition 1.1. For every $n\in\mathbb{N}$ a sequence $\mathbb{Z}_{1,n}\ni k\mapsto T_k\subset\mathbb{T}$ is said to be a *partition of the unit circle* provided T_k is a closed arc of length $|T_k|_1>0$ for $k\in\mathbb{Z}_{1,n}$ as well as

$$\bigcup_{k=1}^{n}T_k=\mathbb{T} \quad\text{and}\quad \sum_{k=1}^{n}|T_k|_1=2\pi. \tag{1.1}$$

For any function $F:\mathbb{D}\to\mathbb{C}$ and $z\in\mathbb{T}$ we define the set $F^{**}(z)$ of all $w\in\mathbb{C}$ such that there exists a sequence $\mathbb{N}\ni n\mapsto r_n\in[0;1)$ satisfying the equalities

$$\lim_{n\to+\infty}r_n=1 \quad\text{and}\quad \lim_{n\to+\infty}F(r_nz)=w.$$

Definition 1.2. By the *sectorial boundary normalization given by a partition* $\mathbb{Z}_{1,n}\ni k\mapsto T_k\subset\mathbb{T}$ *of the unit circle* we mean the class $\mathcal{N}(T_1,T_2,\ldots,T_n)$ of all functions $F:\mathbb{D}\to\mathbb{D}$ such that for every $k\in\mathbb{Z}_{1,n}$ and almost every (a.e. in abbr.) $z\in T_k$,

$$F^{**}(z)\subset D_k:=\{ru:0\le r\le 1\,,\,u\in T_k\}=\operatorname{conv}(T_k\cup\{0\}). \tag{1.2}$$

Given $n\in\mathbb{N}$ and a partition $\mathbb{Z}_{1,n}\ni k\mapsto T_k\subset\mathbb{T}$ of the unit circle we will study the Schwarz type inequality for the class

$$\mathcal{F}:=\operatorname{Har}(\mathbb{D})\cap\mathcal{N}(T_1,T_2,\ldots,T_n).$$

If $n\le 2$ then we have a trivial sharp estimation $|F(z)|\le 1$ for $F\in\mathcal{F}$ and $z\in\mathbb{D}$, where the equality is attained for a constant function. Therefore, from now on we always assume that $n\ge 3$.

In Section 2 we prove a few useful properties of the class $\mathcal{F}$. Most essential here is Theorem 2.3. We use it to show in Section 3 Theorem 3.1, which is our main result. Then we apply the last theorem in specific cases; cf. Examples 3.4 and 3.5. In particular, we derive the estimation (3.13), obtained by D. Partyka and J. Zając in [3, Corollary 2.2]. Thus the estimation (3.1), valid for an arbitrary partition of $\mathbb{T}$, generalizes the one (3.13), which holds only in the case where $n=3$ and the arcs T_1, T_2 and T_3 have the same length. Note that the estimation (3.12) is a directional improvement of the radial one (3.13). In Example 3.5 we study a general case of an arbitrary partition of the unit circle. As a result, we derive reasonable estimations (3.23) and (3.24), which depend on the largest length among the ones $|T_k|_1$ for $k\in\mathbb{Z}_{1,n}$.

2. Auxiliary results

Let $\mathrm{P}[f]$ stand for the Poisson integral of an integrable function $f:\mathbb{T}\to\mathbb{C}$, i.e., $\mathrm{P}[f]:\mathbb{D}\to\mathbb{C}$ is the function given by the following formula

$$\mathrm{P}[f](z) := \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\frac{1-|z|^2}{|u-z|^2}|\mathrm{d}u| = \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\,\mathrm{Re}\,\frac{u+z}{u-z}|\mathrm{d}u|,\ z\in\mathbb{D}. \tag{2.1}$$

The Poisson integral provides the unique solution to the Dirichlet problem in the unit disc $\mathbb{D}$ provided that the boundary function f is continuous. It means that $\mathrm{P}[f]$ is a harmonic function in $\mathbb{D}$, which has a continuous extension to the closed disc $\mathrm{cl}(\mathbb{D})$ and its boundary values function coincides with f. For any function $F:\mathbb{D}\to\mathbb{C}$ we define the radial limit function of F by the formula

$$\mathbb{T}\ni z\mapsto F^*(z) := \begin{cases} \lim\limits_{r\to 1^-} F(rz), & \text{if the limit exists,}\\ 0, & \text{otherwise.}\end{cases}$$

Since a real-valued harmonic and bounded function in $\mathbb{D}$ has the radial limit for a.e. point of $\mathbb{T}$ (see e.g. [2, Cor. 1, Sect. 1.2]), it follows that $F^* = (\mathrm{Re}\,F)^* + \mathrm{i}(\mathrm{Im}\,F)^*$ almost everywhere on $\mathbb{T}$ provided $F\in\mathrm{Har}(\mathbb{D})$ is bounded in $\mathbb{D}$. Therefore,

$$F^{**}(z) = \{F^*(z)\}\quad\text{for every } F\in\mathcal{F} \text{ and a.e. } z\in\mathbb{T}. \tag{2.2}$$

In particular, for each function $F:\mathbb{D}\to\mathbb{D}$, $F\in\mathcal{F}$ if and only if $F\in\mathrm{Har}(\mathbb{D})$ and $F^*(z)\in D_k$ for $k\in\mathbb{Z}_{1,n}$ and a.e. $z\in T_k$. From the property (2.2) it follows that for each $F\in\mathcal{F}$ the sequence $\mathbb{N}\ni m\mapsto f_m$, where

$$\mathbb{T}\ni u\mapsto f_m(u) := F((1-\tfrac{1}{m})u),\quad m\in\mathbb{N},$$

is convergent to F^* almost everywhere on $\mathbb{T}$. Then applying the dominated convergence theorem we see that for every $z\in\mathbb{D}$,

$$F((1-\tfrac{1}{m})z) = \mathrm{P}[f_m](z)\to\mathrm{P}[F^*](z)\quad\text{as } m\to+\infty,$$

which yields

$$F = \mathrm{P}[F^*],\quad F\in\mathcal{F}. \tag{2.3}$$

Let χ_I be the characteristic function of a set $I\in\mathbb{T}$, i.e., $\chi_I(t) := 1$ for $t\in I$ and $\chi_I(t) := 0$ for $t\in\mathbb{T}\setminus I$.

Lemma 2.1. *For all $F\in\mathcal{F}$ and $z\in\mathbb{D}$ there exists a sequence $\mathbb{Z}_{1,n}\ni k\mapsto c_k\in D_k$ such that the following equality holds*

$$F(z) = \sum_{k=1}^{n} c_k\,\mathrm{P}[\chi_{T_k}](z). \tag{2.4}$$

Proof. Fix $F\in\mathcal{F}$ and $z\in\mathbb{D}$. Since $|T_k|_1>0$ for $k\in\mathbb{Z}_{1,n}$, it follows that

$$0<p_k := \mathrm{P}[\chi_{T_k}](z) < 1,\quad k\in\mathbb{Z}_{1,n}. \tag{2.5}$$

By (1.2) each sector D_k, $k \in \mathbb{Z}_{1,n}$, is closed and convex. Moreover, from (1.2) and (2.2) we see that $F^*(z) \in D_k$ for $k \in \mathbb{Z}_{1,n}$ and a.e. $z \in T_k$. Then applying the integral mean value theorem for complex-valued functions we deduce from (2.5) that

$$c_k := P\Big[\frac{1}{p_k} \cdot F^* \cdot \chi_{T_k}\Big](z) \in D_k, \quad k \in \mathbb{Z}_{1,n}.$$

Hence and by (2.3),

$$\begin{aligned} F(z) = \mathrm{P}[F^*](z) = \mathrm{P}\Big[\sum_{k=1}^{n} F^* \cdot \chi_{T_k}\Big](z) &= \sum_{k=1}^{n} \mathrm{P}\Big[F^* \cdot \chi_{T_k}\Big](z) \\ &= \sum_{k=1}^{n} p_k \, \mathrm{P}\Big[\frac{1}{p_k} \cdot F^* \cdot \chi_{T_k}\Big](z) = \sum_{k=1}^{n} p_k c_k, \end{aligned}$$

which implies the equality (2.4). □

Lemma 2.2. *For every sequence* $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$,

$$F := \sum_{k=1}^{n} c_k \, \mathrm{P}[\chi_{T_k}] \in \mathcal{F}. \tag{2.6}$$

Proof. Given a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ consider the function F defined by the formula (2.6). Since $\mathrm{P}[\chi_{T_k}] \in \mathrm{Har}(\mathbb{D})$ for $k \in \mathbb{Z}_{1,n}$, we see that $F \in \mathrm{Har}(\mathbb{D})$. Furthermore, for each $z \in \mathbb{D}$,

$$\sum_{k=1}^{n} \mathrm{P}[\chi_{T_k}](z) = \mathrm{P}\Big[\sum_{k=1}^{n} \chi_{T_k}\Big](z) = \mathrm{P}[\chi_{\mathbb{T}}](z) = 1,$$

whence

$$|F(z)| \le \sum_{k=1}^{n} |c_k| \, \mathrm{P}[\chi_{T_k}](z) \le \sum_{k=1}^{n} \mathrm{P}[\chi_{T_k}](z) = 1.$$

By the definition of the function F we have

$$F^*(z) = \sum_{k=1}^{n} c_k \chi_{T_k}(z), \quad z \in \mathbb{T} \setminus E, \tag{2.7}$$

where E is the set of all $u \in \mathbb{T}$ such that u is an endpoint of a certain arc among the arcs T_k for $k \in \mathbb{Z}_{1,n}$.

Assume that $|F(z_0)| = 1$ for some $z_0 \in \mathbb{D}$. By the maximum modulus principle for complex-valued harmonic functions (cf. [1, Corollary 1.11, p. 8]) there exists $w \in \mathbb{T}$ such that $F(z) = w$ for $z \in \mathbb{D}$, and so $F^*(z) = w$ for $z \in \mathbb{T}$. By (2.7), $F^*(z) = c_k$ for $k \in \mathbb{Z}_{1,n}$ and $z \in T_k \setminus E$. Therefore $w = c_k \in D_k$ for $k \in \mathbb{Z}_{1,n}$, and so $w \in D_1 \cap D_2 \cap D_3 = \{0\}$. Hence $w = 0$, which contradicts the equality $|w| = 1$. Thus $F(z) < 1$ for $z \in \mathbb{D}$, and so $F : \mathbb{D} \to \mathbb{D}$. Furthermore, from (2.7) it follows that for all $k \in \mathbb{Z}_{1,n}$ and $z \in T_k \setminus E$, $F^*(z) = c_k \in D_k$. Thus $F \in \mathcal{N}(T_1, T_2, \ldots, T_n)$, which implies (2.6). □

Theorem 2.3. *For every compact set $K \subset \mathbb{D}$ there exist a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ and $z_K \in \mathrm{fr}(K)$ such that*

$$F_K := \sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}] \in \mathcal{F} \tag{2.8}$$

and

$$|F(z)| \le |F_K(z_K)| = \Big|\sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}](z_K)\Big|, \quad F \in \mathcal{F}\,,\, z \in K. \tag{2.9}$$

In particular,

$$\max(\{|F(z)| : F \in \mathcal{F}\,,\, z \in K\}) = |F_K(z_K)|. \tag{2.10}$$

Proof. Fix a compact set $K \subset \mathbb{D}$. Since $F(K) \subset F(\mathbb{D}) \subset \mathbb{D}$ for $F \in \mathcal{F}$,

$$M_K := \sup(\{|F(z)| : F \in \mathcal{F}\,,\, z \in K\}) \le 1. \tag{2.11}$$

Hence, there exist sequences $\mathbb{N} \ni m \mapsto F_m \in \mathcal{F}$ and $\mathbb{N} \ni m \mapsto z_m \in K$ such that

$$\lim_{m\to+\infty} |F_m(z_m)| = M_K. \tag{2.12}$$

From Lemma 2.1 it follows that for each $m \in \mathbb{N}$ there exists a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_{m,k} \in D_k$ such that

$$F_m(z_m) = \sum_{k=1}^{n} c_{m,k} \,\mathrm{P}[\chi_{T_k}](z_m). \tag{2.13}$$

Since the set D_k is compact for $k \in \mathbb{Z}_{1,n}$ we see, using the standard technique of choosing a convergent subsequence from a sequence in a compact set, that there exists an increasing sequence $\mathbb{N} \ni l \mapsto m_l \in \mathbb{N}$, a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ and $z'_K \in K$ such that

$$c_{m_l,k} \to c_k \quad \text{as } l \to +\infty \quad \text{for } k \in \mathbb{Z}_{1,n} \tag{2.14}$$

and

$$z_{m_l} \to z'_K \quad \text{as } l \to +\infty. \tag{2.15}$$

By Lemma 2.2, the property (2.8) holds. From (2.13) we conclude that for every $m \in \mathbb{N}$,

$$\begin{aligned} |F_K(z_m) - F_m(z_m)| &= \Big|\sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}](z_m) - \sum_{k=1}^{n} c_{m,k} \,\mathrm{P}[\chi_{T_k}](z_m)\Big| \\ &\le \sum_{k=1}^{n} |c_k - c_{m,k}| \,\mathrm{P}[\chi_{T_k}](z_m) \\ &\le \sum_{k=1}^{n} |c_k - c_{m,k}|, \end{aligned}$$

which together with (2.14) leads to

$$\lim_{l\to+\infty} |F_K(z_{m_l}) - F_{m_l}(z_{m_l})| = 0. \tag{2.16}$$

Since $|c_k| \le 1$ for $k \in \mathbb{Z}_{1,n}$, it follows that

$$\begin{aligned}|F_K(z'_K) - F_K(z_m)| &\le \Big|\sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}](z'_K) - \sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}](z_m)\Big| \\ &\le \sum_{k=1}^{n} |c_k| \cdot |\,\mathrm{P}[\chi_{T_k}](z'_K) - \mathrm{P}[\chi_{T_k}](z_m)| \\ &\le \sum_{k=1}^{n} |\,\mathrm{P}[\chi_{T_k}](z'_K) - \mathrm{P}[\chi_{T_k}](z_m)|, \quad m \in \mathbb{N}.\end{aligned}$$

This together with (2.15) yields

$$\lim_{l\to+\infty} |F_K(z'_K) - F_K(z_{m_l})| = 0. \tag{2.17}$$

Since for every $l \in \mathbb{N}$,

$$|F_K(z'_K) - F_{m_l}(z_{m_l})| \le |F_K(z'_K) - F_K(z_{m_l})| + |F_K(z_{m_l}) - F_{m_l}(z_{m_l})|,$$

we deduce from (2.17) and (2.16) that

$$\lim_{l\to+\infty} |F_{m_l}(z_{m_l})| = |F_K(z'_K)|.$$

Hence and by (2.12), $|F_K(z'_K)| = M_K$. Since $F_K \in \mathrm{Har}(\mathbb{D})$, the maximum modulus principle for complex-valued harmonic function (cf. [1, Corollary 1.11, p. 8]) implies that there exists $z_K \in \mathrm{fr}(K)$ such that $|F_K(z)| \le |F_K(z_K)|$ for $z \in K$. In particular, $M_K = |F_K(z'_K)| \le |F_K(z_K)|$. On the other hand, by (2.8) and (2.11), $|F_K(z_K)| \le M_K$. Eventually, $|F_K(z_K)| = M_K$. This implies (2.10), and thereby, the inequality (2.9) holds, which is the desired conclusion. □

3. Estimations

As an application of Theorem 2.3 we shall prove the following result.

Theorem 3.1. *For every $z \in \mathbb{D}$ the following inequality holds*

$$|F(z)| \le 1 - (n - S)p(z), \quad F \in \mathcal{F}, \tag{3.1}$$

where

$$S := \sup\Big(\Big\{\mathrm{Re}\Big(\overline{u}\sum_{k=1}^{n} v_k\Big) : u \in \mathbb{T}\,,\ \mathbb{Z}_{1,n} \ni k \mapsto v_k \in D_k\Big\}\Big) \tag{3.2}$$

and

$$p(z) := \min(\{\mathrm{P}[\chi_{T_k}](z) : k \in \mathbb{Z}_{1,n}\}). \tag{3.3}$$

Proof. It is clear that $K := \{z\}$ is a compact set for a given $z \in \mathbb{D}$. By Theorem 2.3 there exists a sequence $\mathbb{Z}_{1,n} \ni k \mapsto c_k \in D_k$ such that

$$F_K := \sum_{k=1}^{n} c_k \,\mathrm{P}[\chi_{T_k}] \in \mathcal{F}$$

and

$$|F(z)| \le |F_K(z)|, \quad F \in \mathcal{F}. \tag{3.4}$$

Setting $u := F_K(z)/|F_K(z)|$ if $F_K(z) \neq 0$ and $u := 1$ if $F_K(z) = 0$, we see that $u \in \mathbb{T}$ and $F_K(z) = u|F_K(z)|$. Hence

$$|F_K(z)| = \overline{u}F_K(z) = \mathrm{Re}(\overline{u}F_K(z)) = \mathrm{Re}\Big(\overline{u}\sum_{k=1}^{n} c_k p_k\Big) = \sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k)p_k, \tag{3.5}$$

where $p_k := \mathrm{P}[\chi_{T_k}](z)$ for $k \in \mathbb{Z}_{1,n}$. Since

$$\sum_{k=1}^{n} p_k = 1 \quad \text{and} \quad \mathrm{Re}(\overline{u}c_k) \le M := \max(\{\mathrm{Re}(\overline{u}c_l) : l \in \mathbb{Z}_{1,n}\}) \le 1, \quad k \in \mathbb{Z}_{1,n},$$

we deduce from the formula (3.3) that

$$\begin{aligned}
\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k)p_k &= \sum_{k=1}^{n}(\mathrm{Re}(\overline{u}c_k) - M + M)p_k \\
&= M\sum_{k=1}^{n} p_k + \sum_{k=1}^{n}(\mathrm{Re}(\overline{u}c_k) - M)p_k \\
&\le M\sum_{k=1}^{n} p_k + \sum_{k=1}^{n}(\mathrm{Re}(\overline{u}c_k) - M)p(z) \\
&= M\sum_{k=1}^{n}(p_k - p(z)) + p(z)\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k) \\
&\le \sum_{k=1}^{n}(p_k - p(z)) + p(z)\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k) \\
&= 1 - np(z) + p(z)\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k).
\end{aligned}$$

This together with (3.5) and (3.2) yields

$$\begin{aligned}
|F_K(z)| &\le 1 - np(z) + p(z)\sum_{k=1}^{n}\mathrm{Re}(\overline{u}c_k) \\
&\le 1 - np(z) + p(z)S \\
&= 1 - (n - S)p(z).
\end{aligned}$$

Hence and by (3.4) we obtain the estimation (3.1), which proves the theorem. □

The estimation (3.1) is useful provided we can estimate $p(z)$ from below and S from above. The first task is easy and depends on the following quantity

$$\delta := \frac{1}{2}\min(\{|T_k|_1 : k \in \mathbb{Z}_{1,n}\}). \tag{3.6}$$

Lemma 3.2. *For every $\alpha \in (0;\pi/2]$ the following estimation holds*

$$\mathrm{P}[\chi_{I_\alpha}](z) \geq \mathrm{P}[\chi_{I_\alpha}](|z|) = \frac{2}{\pi}\arctan\left(\frac{\sin(\alpha)}{|z|+\cos(\alpha)}\right) - \frac{\alpha}{\pi}, \quad z \in \mathbb{D}, \tag{3.7}$$

where $I_\alpha := \{\mathrm{e}^{\mathrm{i}t} : |t-\pi| \leq \alpha\}$.

Proof. Given $\alpha \in (0;\pi/2]$ we see that $e_1 := \mathrm{e}^{\mathrm{i}(\pi-\alpha)} = -\mathrm{e}^{-\mathrm{i}\alpha}$ and $e_2 := \mathrm{e}^{\mathrm{i}(\pi+\alpha)} = -\mathrm{e}^{\mathrm{i}\alpha}$ are the endpoints of the arc I_α. Let $z \in \mathbb{D}$ be arbitrarily fixed. Since $I_\alpha \subset \Omega_z := \mathbb{C} \setminus \{z+t : t > 0\}$, the function $\Omega_z \ni \zeta \mapsto \log(z-\zeta)$ is holomorphic and

$$\frac{\mathrm{d}}{\mathrm{d}t}\log(z-\mathrm{e}^{\mathrm{i}t}) = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-z}, \quad t \in [\pi-\alpha;\pi+\alpha].$$

Here we understand the function log as the inverse of the function $\exp_{|\Omega}$, where $\Omega := \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < \pi\}$. By (2.1) we have

$$\begin{aligned}
\mathrm{P}[\chi_{I_\alpha}](z) &= \frac{1}{2\pi}\int_{\mathbb{T}} \chi_{I_\alpha}(u)\operatorname{Re}\frac{u+z}{u-z}|\mathrm{d}u| \\
&= \frac{1}{2\pi}\int_{\pi-\alpha}^{\pi+\alpha} \operatorname{Re}\frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{\pi-\alpha}^{\pi+\alpha} \operatorname{Re}\Big(\frac{2\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-z}-1\Big)\mathrm{d}t \\
&= \frac{1}{\pi}\int_{\pi-\alpha}^{\pi+\alpha} \operatorname{Im}\Big(\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}-z}\Big)\mathrm{d}t - \frac{\alpha}{\pi} \\
&= \frac{1}{\pi}\int_{\pi-\alpha}^{\pi+\alpha} \operatorname{Im}\frac{\mathrm{d}}{\mathrm{d}t}\log(z-\mathrm{e}^{\mathrm{i}t})\mathrm{d}t - \frac{\alpha}{\pi} \\
&= \frac{1}{\pi}\operatorname{Im}\big[\log(z-e_2)-\log(z-e_1)\big] - \frac{\alpha}{\pi}.
\end{aligned}$$

Therefore, for an arbitrarily fixed $r \in [0;1)$,

$$\mathrm{P}[\chi_{I_\alpha}](r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\pi}\operatorname{Im}\big[\log(r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}) - \log(r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha})\big] - \frac{\alpha}{\pi}, \quad \theta \in \mathbb{R}. \tag{3.8}$$

Consequently,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{P}[\chi_{I_\alpha}](r\mathrm{e}^{\mathrm{i}\theta}) &= \frac{1}{\pi}\operatorname{Im}\Big[\frac{\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}} - \frac{\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha}}\Big] \\
&= \frac{r}{\pi}\operatorname{Im}\Big[\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}(-\mathrm{e}^{\mathrm{i}\alpha}+\mathrm{e}^{-\mathrm{i}\alpha})}{(r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha})(r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha})}\Big] \\
&= \frac{2r\sin(\alpha)}{\pi}\,\frac{\operatorname{Im}[\mathrm{e}^{\mathrm{i}\theta}(r\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha})(r\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha})]}{|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}|^2|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha}|^2} \\
&= \frac{2r\sin(\alpha)}{\pi}\,\frac{\operatorname{Im}[r^2\mathrm{e}^{-\mathrm{i}\theta}+r\mathrm{e}^{-\mathrm{i}\alpha}+r\mathrm{e}^{\mathrm{i}\alpha}+\mathrm{e}^{\mathrm{i}\theta}]}{|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}|^2|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha}|^2} \\
&= \frac{2r(1-r^2)\sin(\alpha)\sin(\theta)}{\pi|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\alpha}|^2|r\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\alpha}|^2}, \quad \theta \in \mathbb{R}.
\end{aligned}$$

Combining this with (3.8) we derive the estimation (3.7), which proves the lemma. □

Corollary 3.3. *The following estimation holds*

$$p(z) \geq \mathrm{P}[\chi_{I_\delta}](|z|) = \frac{2}{\pi}\arctan\left(\frac{\sin(\delta)}{|z| + \cos(\delta)}\right) - \frac{\delta}{\pi}, \quad z \in \mathbb{D}, \tag{3.9}$$

where $p(z)$ and δ are defined by the formulas (3.3) *and* (3.6), *respectively.*

Proof. Let $\mathbb{Z}_{1,n} \ni k \mapsto a_k \in \mathbb{T}$ be the sequence of midpoints of the partition $\mathbb{Z}_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$, i.e.,

$$T_k := \{a_k \mathrm{e}^{\mathrm{i}t} : |t| \leq \alpha_k\}, \tag{3.10}$$

where $\alpha_k := \frac{1}{2}|T_k|_1$ for $k \in \mathbb{Z}_{1,n}$. Hence and by (3.6) we obtain $I_\delta \subset I_{\alpha_k}$ for $k \in \mathbb{Z}_{1,n}$, where $I_\alpha := \{\mathrm{e}^{\mathrm{i}t} : |t - \pi| \leq \alpha\}$ for $\alpha \in (0;\pi]$. Then applying the formula (2.1) we see that for an arbitrarily fixed $z \in \mathbb{D}$,

$$\mathrm{P}[\chi_{I_{\alpha_k}}](|z|) = \mathrm{P}[\chi_{I_\delta}](|z|) + \mathrm{P}[\chi_{I_{\alpha_k}\setminus I_\delta}](|z|) \geq \mathrm{P}[\chi_{I_\delta}](|z|), \quad k \in \mathbb{Z}_{1,n}.$$

Therefore

$$\min(\{\mathrm{P}[\chi_{I_{\alpha_k}}](|z|) : k \in \mathbb{Z}_{1,n}\}) = \mathrm{P}[\chi_{I_\delta}](|z|), \tag{3.11}$$

because $\delta = \alpha_{k'}$ for some $k' \in \mathbb{Z}_{1,n}$. Fix $k \in \mathbb{Z}_{1,n}$. Using the rotation mapping $\mathbb{C} \ni \zeta \mapsto \varphi(\zeta) := -a_k^{-1}\zeta$ we have $\varphi(T_k) = I_{\alpha_k}$. Then integrating by substitution we deduce from the formula (2.1) that

$$\mathrm{P}[\chi_{T_k}](z) = \mathrm{P}[\chi_{\varphi(T_k)}](\varphi(z)) = \mathrm{P}[\chi_{I_{\alpha_k}}](\varphi(z)).$$

On the other hand, by Lemma 3.2,

$$\mathrm{P}[\chi_{I_{\alpha_k}}](\varphi(z)) \geq \mathrm{P}[\chi_{I_{\alpha_k}}](|\varphi(z)|) = \mathrm{P}[\chi_{I_{\alpha_k}}](|z|).$$

Thus

$$\mathrm{P}[\chi_{T_k}](z) \geq \mathrm{P}[\chi_{I_{\alpha_k}}](|z|), \quad k \in \mathbb{Z}_{1,n}.$$

Combining this with (3.3) and (3.11) we derive the estimation (3.9), which completes the proof. □

A more difficult problem is to estimate from above the quantity S given by the formula (3.2). It will be studied elsewhere. Now we present two examples.

Example 3.4. Suppose that $\mathbb{Z}_{1,3} \ni k \mapsto T_k \subset \mathbb{T}$ is a partition of $\mathbb{T}$ such that $|T_1|_1 = |T_2|_1 = |T_3|_1$. As in the proof of [3, Theorem 2.1] we can show that $S \leq 2$. Hence and by Theorem 3.1 we obtain

$$|F(z)| \leq 1 - p(z) = 1 - \min(\{\mathrm{P}[\chi_{T_k}](z) : k \in \mathbb{Z}_{1,3}\}), \quad F \in \mathcal{F},\, z \in \mathbb{D}. \tag{3.12}$$

Corollary 3.3 now implies the estimation

$$|F(z)| \leq \frac{4}{3} - \frac{2}{\pi}\arctan\left(\frac{\sqrt{3}}{1 + 2|z|}\right), \quad F \in \mathcal{F},\, z \in \mathbb{D}; \tag{3.13}$$

cf. [3, Corollary 2.2]. Therefore, the estimation (3.12) is a directional type enhancement of the radial one (3.13) for the class $\mathcal{F}$.

Example 3.5. Suppose that $\mathbb{Z}_{1,n} \ni k \mapsto T_k \subset \mathbb{T}$ is a partition of $\mathbb{T}$ such that

$$\Delta := \max(\{|T_k|_1 : k \in \mathbb{Z}_{1,n}\}) \leq \frac{\pi}{2}. \tag{3.14}$$

Then

$$N := \operatorname{Ent}\Big(\frac{\pi}{2\Delta}\Big) \geq 1. \tag{3.15}$$

Fix $u \in \mathbb{T}$ and a sequence $\mathbb{Z}_{1,n} \ni k \mapsto v_k \in D_k$. There exist a bijective function σ of the set $\mathbb{Z}_{1,n}$ onto itself and an increasing sequence $\mathbb{Z}_{1,n} \ni k \mapsto \alpha_k \in \mathbb{R}$ such that $\alpha_n = 2\pi + \alpha_0$, $u \in T_{\sigma(1)}$ and

$$T_{\sigma(k)} = \{\mathrm{e}^{\mathrm{i}t} : \alpha_{k-1} \leq t \leq \alpha_k\}, \quad k \in \mathbb{Z}_{1,n}.$$

Hence there exist $\theta \in [\alpha_0; \alpha_1]$ and a sequence $\mathbb{Z}_{1,n} \ni k \mapsto (r_k, \theta_k) \in [0;1] \times \mathbb{R}$ such that $u = \mathrm{e}^{\mathrm{i}\theta}$, $v_k = r_k \mathrm{e}^{\mathrm{i}\theta_k}$ for $k \in \mathbb{Z}_{1,n}$ and

$$\alpha_{k-1} \leq \theta_{\sigma(k)} \leq \alpha_k, \quad k \in \mathbb{Z}_{1,n}. \tag{3.16}$$

Since for each $k \in \mathbb{Z}_{1,n}$,

$$\operatorname{Re}(\overline{u}v_k) = \operatorname{Re}\Big(r_k \mathrm{e}^{\mathrm{i}\theta_k}\mathrm{e}^{-\mathrm{i}\theta}\Big) = \operatorname{Re}\Big(r_k \mathrm{e}^{\mathrm{i}(\theta_k - \theta)}\Big) = r_k \cos(\theta_k - \theta),$$

we conclude that

$$\operatorname{Re}(\overline{u}v_k) \leq \max(\{0\,, \cos(\theta_k - \theta)\}), \quad k \in \mathbb{Z}_{1,n}. \tag{3.17}$$

From (3.14) it follows that

$$\alpha_j - \alpha_i = \sum_{l=i+1}^{j} (\alpha_l - \alpha_{l-1}) \leq (j-i)\Delta, \quad i, j \in \mathbb{Z}_{0,n}\,,\ i < j. \tag{3.18}$$

Setting

$$p := \min(\{k \in \mathbb{Z}_{1,n} : \alpha_k \geq \frac{\pi}{2} + \theta\}) \quad \text{and} \quad q := \max(\{k \in \mathbb{Z}_{1,n} : \alpha_k < \frac{3\pi}{2} + \theta\})$$

we conclude from (3.15) and (3.18) that

$$N\Delta \leq \frac{\pi}{2} \leq \alpha_p - \theta \leq \alpha_p - \alpha_0 \leq p\Delta$$

as well as

$$N\Delta \leq \frac{\pi}{2} = \alpha_q + \frac{\pi}{2} - \alpha_q < 2\pi + \theta - \alpha_q \leq \alpha_n - \alpha_q + \alpha_1 - \alpha_0 \leq (n - q + 1)\Delta.$$

Therefore $N \leq p$ and $q + N \leq n$. Given $k \in \mathbb{Z}_{1,n}$ the following four cases can appear. If $p + 1 - N \leq k \leq p$ then by (3.16) and (3.18),

$$\frac{\pi}{2} + \theta - \theta_{\sigma(k)} \leq \alpha_p - \alpha_{k-1} \leq (p + 1 - k)\Delta \leq N\Delta \leq \frac{\pi}{2}$$

as well as

$$\frac{\pi}{2} + \theta - \theta_{\sigma(k)} > \alpha_{p-1} - \alpha_p \geq -\Delta \geq -\frac{\pi}{2},$$

which gives

$$\cos(\theta_{\sigma(k)} - \theta) = \sin(\pi/2 + \theta - \theta_{\sigma(k)}) \le \sin((p+1-k)\Delta).$$

Hence and by (3.17) we obtain

$$\mathrm{Re}(\overline{u} v_{\sigma(k)}) \le \sin((p+1-k)\Delta), \quad k \in \mathbb{Z}_{p+1-N,p}\,. \tag{3.19}$$

If $p+1 \le k \le q$ then by (3.16),

$$\frac{\pi}{2} + \theta \le \alpha_{k-1} \le \theta_{\sigma(k)} \le \alpha_k < \frac{3\pi}{2} + \theta,$$

and so $\cos(\theta_{\sigma(k)} - \theta) \le 0$. This together with (3.17) leads to

$$\mathrm{Re}(\overline{u} v_{\sigma(k)}) \le 0, \quad k \in \mathbb{Z}_{p+1,q}\,. \tag{3.20}$$

If $q+1 \le k \le q+N$ then by (3.16) and (3.18),

$$\theta_{\sigma(k)} - \frac{3\pi}{2} - \theta \le \alpha_k - \frac{3\pi}{2} - \theta < \alpha_k - \alpha_q \le (k-q)\Delta \le N\Delta \le \frac{\pi}{2}$$

as well as

$$\theta_{\sigma(k)} - \frac{3\pi}{2} - \theta \ge \alpha_q - \alpha_{q+1} \ge -\Delta \ge -\frac{\pi}{2},$$

and consequently,

$$\cos(\theta_{\sigma(k)} - \theta) = \sin(\theta_{\sigma(k)} - 3\pi/2 - \theta) \le \sin((k-q)\Delta).$$

Hence and by (3.17) we obtain

$$\mathrm{Re}(\overline{u} v_{\sigma(k)}) \le \sin((k-q)\Delta), \quad k \in \mathbb{Z}_{q+1,q+N}. \tag{3.21}$$

If $1 \le k \le p-N$ or $q+N+1 \le k \le n$, then clearly $\mathrm{Re}(\overline{u} v_{\sigma(k)}) \le 1$. Combining this with (3.19), (3.20) and (3.21) we see that

$$\begin{aligned}\sum_{k=1}^{n} \mathrm{Re}(\overline{u} v_{\sigma(k)}) &\le \sum_{k=p+1-N}^{p} \sin((p+1-k)\Delta) + \sum_{k=q+1}^{q+N} \sin((k-q)\Delta) \\ &\quad + (p-N) + (n-q-N) \\ &= 2\sum_{k=1}^{N} \sin(k\Delta) + n - 2N - (q-p).\end{aligned} \tag{3.22}$$

Since $\pi < \alpha_{q+1} - \alpha_{p-1} \le (q-p+2)\Delta$, we deduce from (3.15) that $2N \le q-p+1$. Combining this with (3.22) we get

$$\begin{aligned}\sum_{k=1}^{n} \mathrm{Re}(\overline{u} v_{\sigma(k)}) &\le 2\sum_{k=1}^{N} \sin(k\Delta) + n - 2N - (2N-1) \\ &= n + 1 - 4N + 2\frac{\sin\left(\frac{(N+1)\Delta}{2}\right)\sin\left(\frac{N\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2}\right)}.\end{aligned}$$

Hence and by (3.2),

$$S \leq n + 1 - 4N + 2\frac{\sin\left(\frac{(N+1)\Delta}{2}\right)\sin\left(\frac{N\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2}\right)}.$$

Theorem 3.1 now shows that

$$|F(z)| \leq 1 - \left(4N - 1 - 2\frac{\sin\left(\frac{(N+1)\Delta}{2}\right)\sin\left(\frac{N\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2}\right)}\right)p(z), \quad F \in \mathcal{F}, z \in \mathbb{D}, \tag{3.23}$$

where N and $p(z)$ are defined by (3.15) and (3.3), respectively. Applying now Corollary 3.3 we derive from (3.23) the following estimation of radial type

$$|F(z)| \leq 1 - \left(4N - 1 - 2\frac{\sin\left(\frac{(N+1)\Delta}{2}\right)\sin\left(\frac{N\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2}\right)}\right)\mathrm{P}[\chi_{I_\delta}](|z|),$$
$$F \in \mathcal{F}, z \in \mathbb{D}, \tag{3.24}$$

where δ is given by the formula (3.6).

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Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on April 16, 2018.

NIERÓWNOŚCI TYPU SCHWARZA DLA FUNKCJI HARMONICZNYCH W KOLE JEDNOSTKOWYM SPEŁNIAJĄCYCH PEWIEN WARUNEK SEKTOROWY

S t r e s z c z e n i e

Niech T_1, T_2 i T_3 będą łukami domkniętymi, zawartymi w okręgu jednostkowym $\mathbb{T}$, o tej samej dlugości $2\pi/3$ i pokrywającymi $\mathbb{T}$. W pracy [3] D. Partyka and J. Zając otrzymali dokładne oszacowanie modułu $|F(z)|$ dla $z \in \mathbb{D}$, gdzie $\mathbb{D}$ jest kołem jednostkowym, zaś F jest funkcją harmoniczną o wartościach zespolonych koła $\mathbb{D}$ w siebie, spełniających następujący warunek sektorowy: dla każdego $k \in \{1, 2, 3\}$ i prawie każdego $z \in T_k$ granica radialna funkcji F w punkcie z należy do sektora kątowego będącego otoczką wypukłą zbioru $\{0\} \cup T_k$. W tym artykule rozważamy ogólniejszy przypadek, gdzie trzy łuki są zastąpione przez skończony układ łuków domkniętych $T_1, T_2, \ldots, T_n$ zawartych w $\mathbb{T}$, o dodatniej długości, całkowitej długości 2π i pokrywających $\mathbb{T}$.

Słowa kluczowe: całka Poissona, funkcje harmoniczne, lemat Schwarza, odwzorowania harmoniczne

DOI: 10.26485/0459-6854/2018/68.2/11 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 2

pp. 109–122

Dedicated to the memory of
Professor Yurii B. Zelinskii

Osamu Suzuki, Julian Ławrynowicz, Małgorzata Nowak-Kępczyk,
and Mariusz Zubert

SOME GEOMETRICAL ASPECTS OF BINARY, TERNARY, QUATERNARY, QUINARY AND SENARY STRUCTURES IN PHYSICS

Summary

It is observed that quinary and senary structures like in pentacene and several other polymers may be composed from binary and ternary structures in the sense of differential-equational and geometrical description. In the case of pentacene its leaves are attached to the silicon background and have the form of five connected carbon-hydrogen hexagons; in total they do not form the precisely planar structure but a slightly wavy structure which minimizes total energy. In the case of a quinary structure the leaves form solitary, nearly periodical zigzags and meanders.

Keywords and phrases: finite-dimensional algebras, associative rings and algebras, binary physical structure, ternary physical structure, quinary physical structure, senary physical structure, pentacene, polymer

Contents and introduction

1. Quinary and senary structures in pentacene and several other polymers
2. The role of total energy maxima for the infrared and Raman activity energy spectra
3. Decomposition of a quinary structure to binary structures
4. Decomposition of a senary structure to ternary structures

5. Slightly wavy behaviour of the system of hexagons in a pentacene leaf
6. Zigzag or meander soliton behaviour of a twisted structure of pentagons in a pentacene leaf. The sine-like case
7. An analogue of a pentacene structure in the cosine-like case
8. Pentacene as a foliated manifold with a soliton behaviour of leaves

This paper in some sense is a continuation of the paper [3] by E. Z. Frątczak, J. Ławrynowicz, M. Nowak-Kępczyk, H. Polatoglou, L. Wojtczak and [7] by J. Ławrynowicz, M. Nowak-Kępczyk, and O. Suzuki.

1. Quinary and senary structures in pentacene and several other polymers

When looking at formulae for chromophore P in the elongated form (A, B) and the cyclic form (C, D): A,C – Pr, B,D–Pfr, where Pr and Pfr are red-absorbing (infra-red-absorbing) forms of phytochrome, respectively (Fig. 1, cf. [1, 2, 6, 10]) we can see a quinary structure together with a senary structure.

In the case of pentacene in the usual form $C_{22}H_{14}$ (Fig. 2) where C stands for the carbon atom, H for hydrogen atom, a thin film of pentacene forms almost co-planar leaves consisting of five pentagons with each pair having one side of the vertices of hexagons in common, attached as the whole structure to the silicone SiO_2 substrate. An example of a quinary structure in principle possible for pentacene is shown in Fig. 3. There are two basic forms of position of pentacene leaves with respect to substrate, as shown in the figure.

2. The role of total energy maxima for the infrared and Raman activity energy spectra

When changing the wave number we meet two sharp energy maxima (Fig. 4) which may serve for the corresponding nanomolecule as the nanomotor where the original energy structure is changed to a quinary structure (cf. J. -P. Sauvage, Sir J. Fraser Stoddart, and B. L. Feringa [11]), more precisely, for a structure of leaves corresponding to Fig. 3.

Looking more carefully, when changing the wave number in both infrared and Raman activity energy spectra (Figs 5 and 6) we meet again two sharp maxima which may serve for the corresponding nanomolecule as nanomotors where the original senary structure has changed into a quinary structure according to the formulae [3]:

$$x\mathrm{C}_{22}\mathrm{H}_{14} + z\mathrm{H}_2 \leftrightarrow y\mathrm{C}_\xi\mathrm{H}_\eta, \tag{1}$$

where x, y, ξ, η are positive integers and z is an integer.

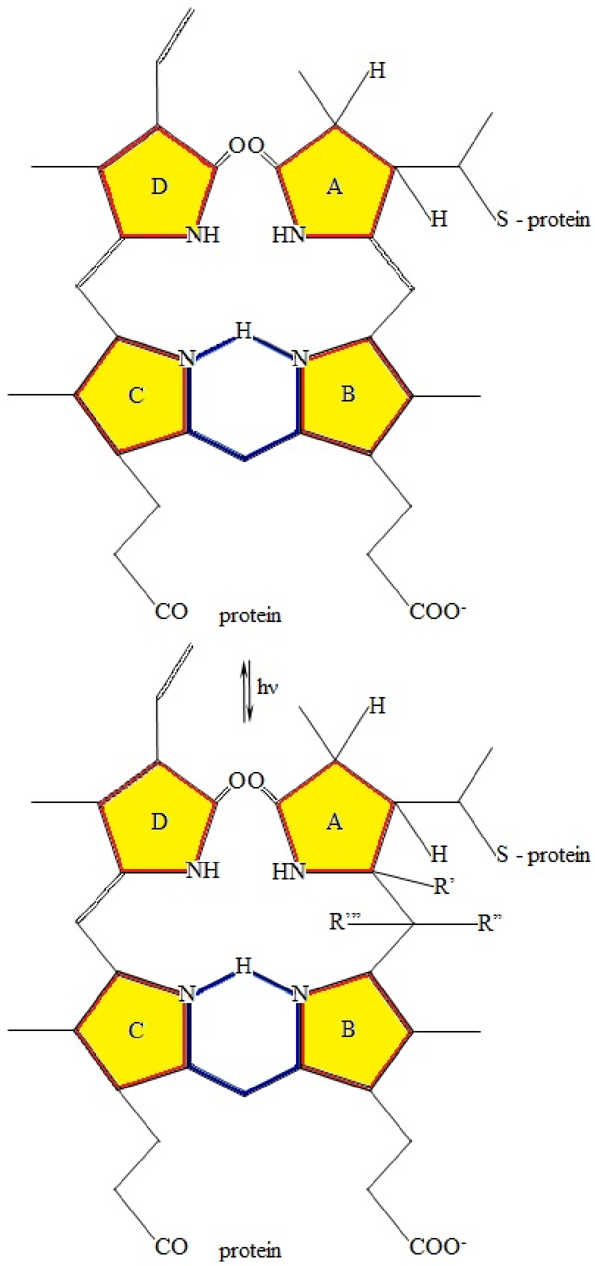

Fig. 1. Structural formula for chromophore P.

More precisely, in the case of leaves consisting of six pentagons, with help of the urn model of Gaveau and Schulmann [5] (cf. also [4]) it is possible to calculate the probability of the occurence of the transformation (1) for definite (ξ, η). For our calculations we take $\xi = 22$, $\eta = 16$,

$$c \equiv 11\frac{\eta}{\xi} - 7 = 0. \tag{2}$$

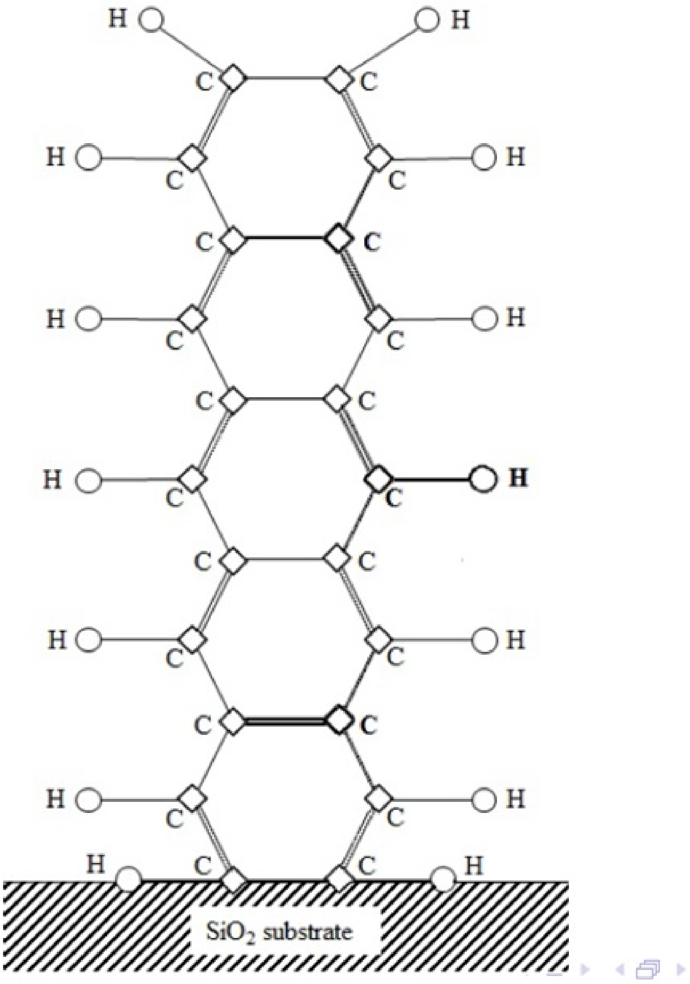

Fig. 2. The pentacene molecule $C_{22}H_{14}$ in the usual form.

3. Decomposition of a quinary structure to binary structures

The carbon atom C has four 3-hands of electrons whereas the hydrogen atom H has one hand. Therefore the corresponding binary extension leading to a polymer may be proposed as shown in Fig. 7 (cf. [13]).

4. Decomposition of a senary structure to ternary structures

In analogy to the previous Section the corresponding ternary extension leading to a polymer may be proposed as shown in Fig. 8.

It is possible (cf. Section 1) to have polymer involving both pentagons and hexagons, for instance five pentagons and one hexagon with carbon atoms in the edges. At the moment we are leaving aside the question of composing it from the binaries only or the ternaries only (cf. [8, 9]).

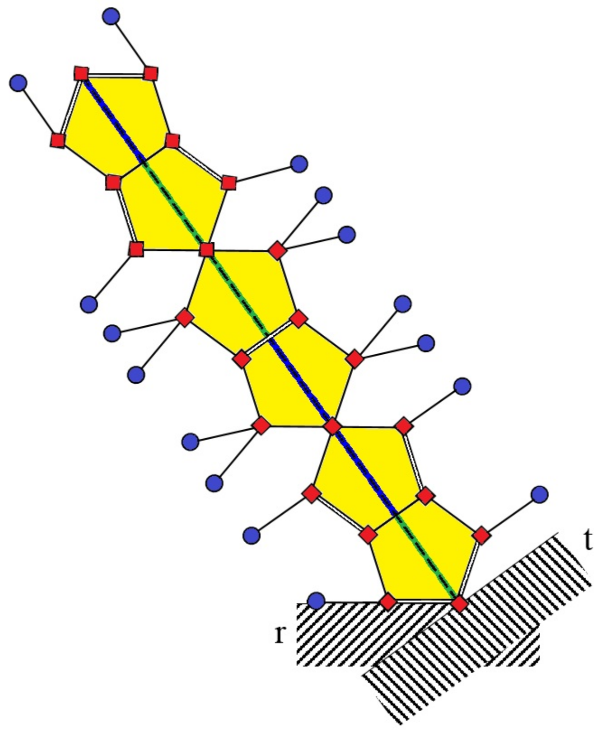

Fig. 3. A candidate for the pentacene molecule $C_{22}H_{16}$ in the form of six pentagons (c.f. the next Section).

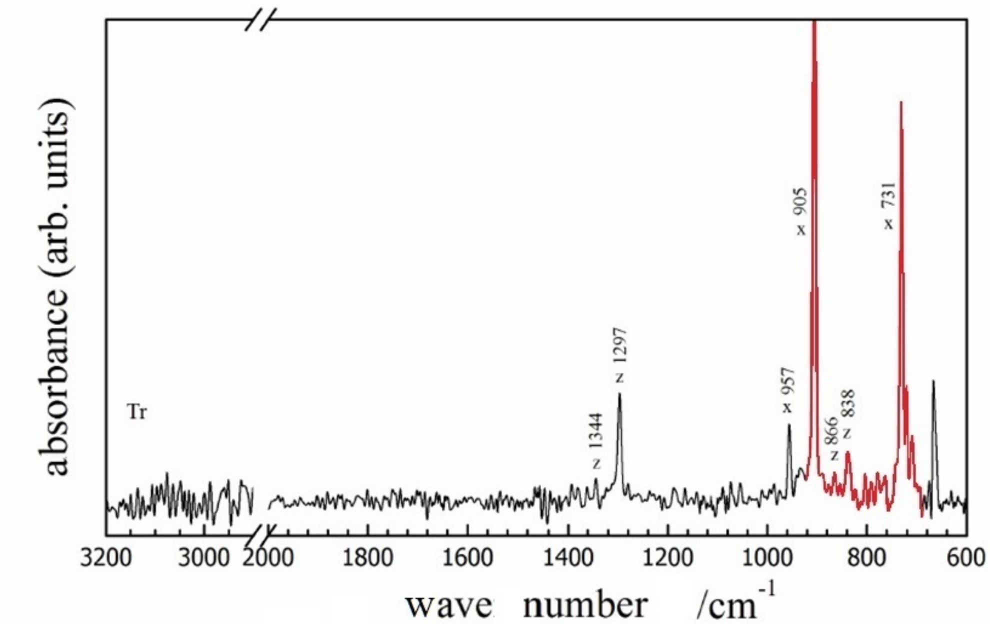

Fig. 4. The total energy (absorbance) maxima.

5. Slightly wavy behaviour of the system of hexagons in a pentacene leaf

The distance between the usual pentacene $C_{22}H_{14}$ leaves (having five carbon-atomic hexagons) amount ca. at $d \approx 1.6$ nm. It appears that the leaves of pentacene are not

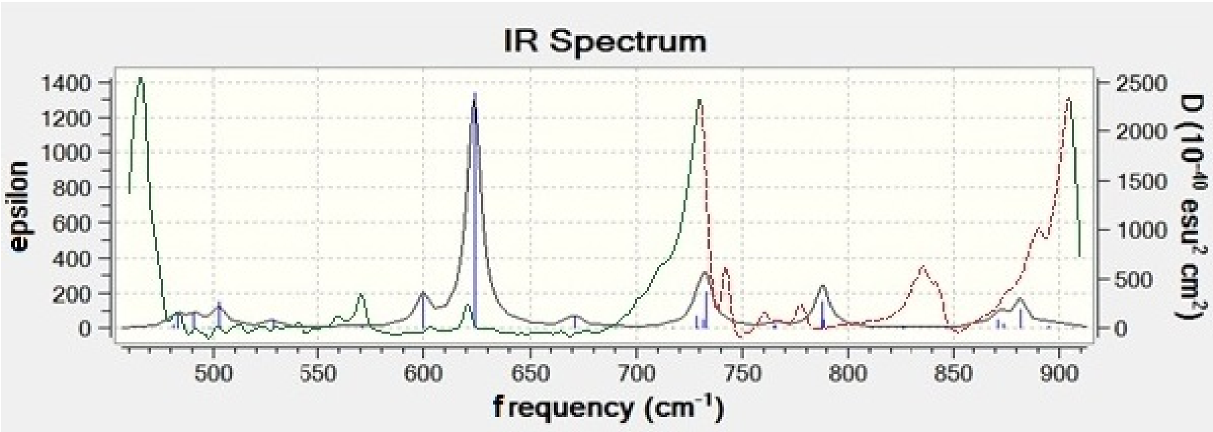

Fig. 5. Infrared activity spectrum.

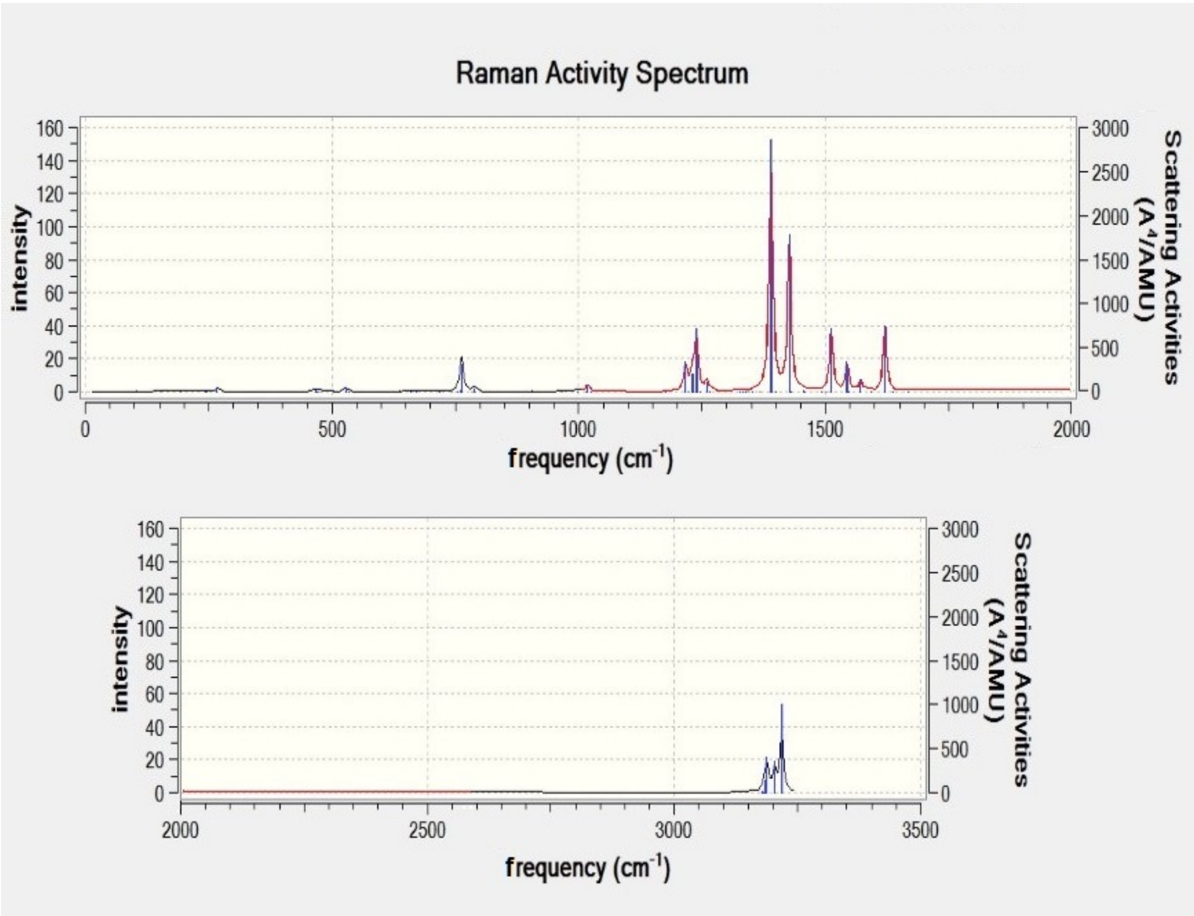

Fig. 6. Raman activity spectrum.

precisely co-planar; they meet optimal global energetic conditions when they have a slightly wavy behaviour:

Theorem 1. *A section of the pentacene* $C_{22}H_{14}$ *orthogonal to the silicone* SiO_2

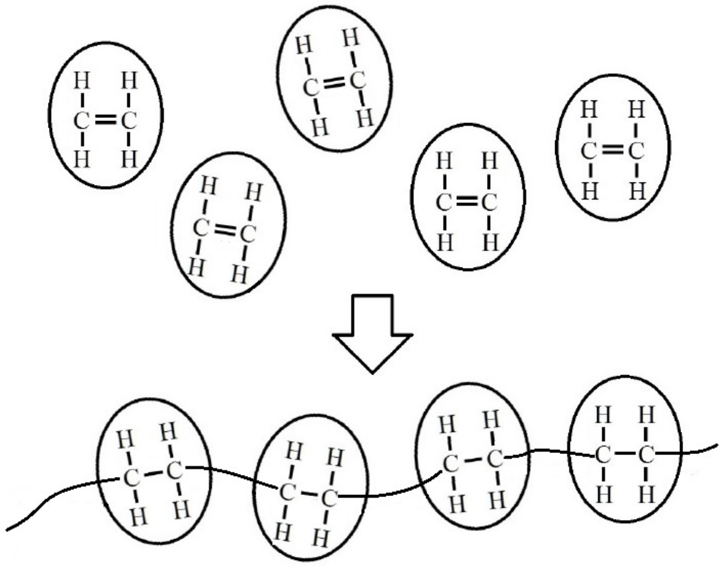

Fig. 7. The binary extension type leading to a polymer related to C and H.

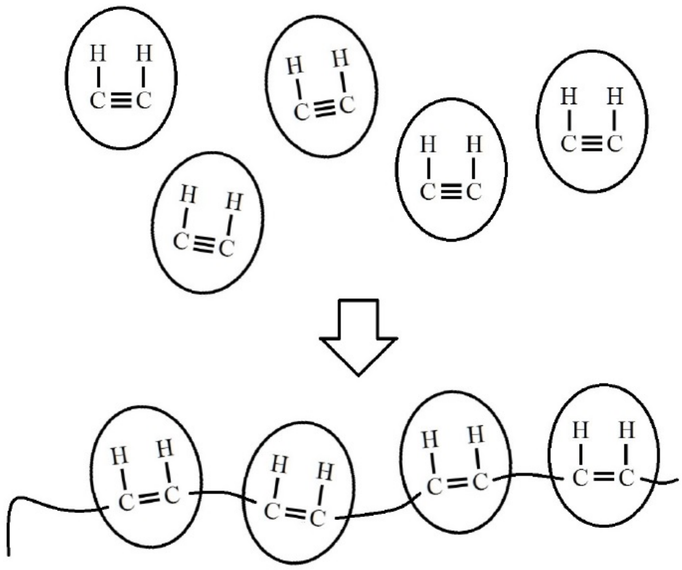

Fig. 8. The ternary extension type leading to a polymer related to C and H.

substrate is a sine-like soliton curve with maxima at $\delta \in (0.013\,\mathrm{nm};\, 0.014\,\mathrm{nm})$ (cf. Figs 2, 8, 9).

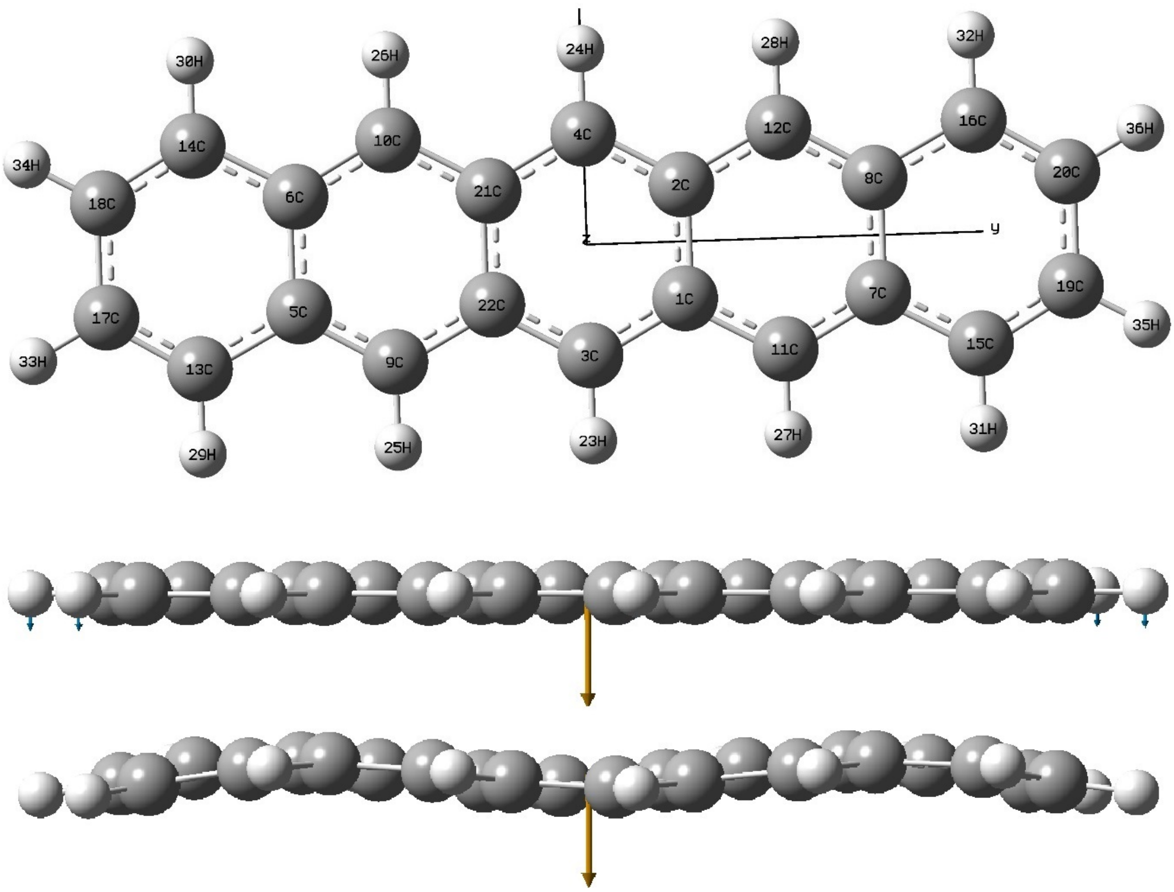

Fig. 9. The pentacene $C_{22}H_{14}$ molecule leaf with the sine-like soliton sections.

6. Zigzag or meander soliton behaviour of a twisted structure of pentagons in a pentacene leaf. The sine-like case

In this case again, the distance between the consecutive leaves of the modified pentacene (leaves being six carbon-atomic pentagons) $C_\xi H_\eta$, in our case $C_{22}H_{16}$ (Figs 3 and 9) amounts at d=1.6 nm. We suppose that the leaves of pentacene are far from being co-planar; they meet optimal energetic conditions when they form solitary zigzags and meanders.

If we concentrate on the sine-like case (Figs 11 and 12) we get:

Theorem 2. A section of the pentacene $C_{22}H_{16}$ *leaf orthogonal to the silicone* SiO_2 *substrate, in the sine-like case is a sine-like soliton curve with maxima at* $h \in (0.139$ nm, 0.140 nm) (cf. Figs 3, 9, 10, 11, and 12).

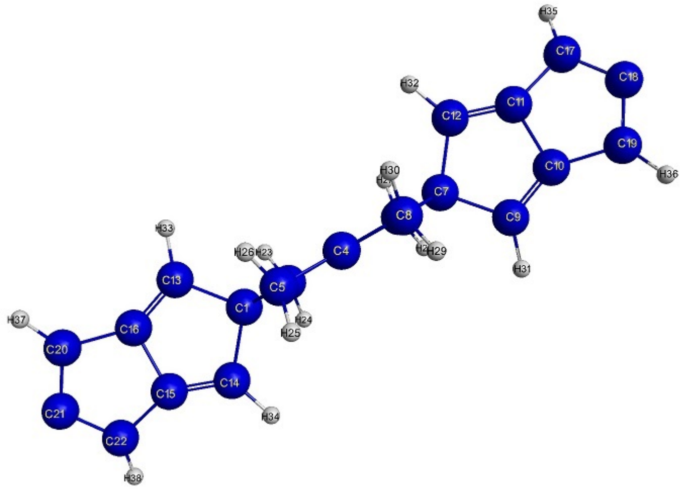

Fig. 10. Some candidate $C_{22}H_{16}$ for a pentacene molecule in the form of six pentagons. A twisted sine-like soliton structure.

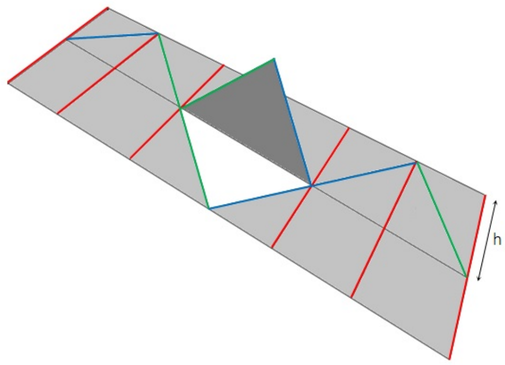

Fig. 11. The pentacene molecule structure in the form of six pentagons. Form in the sine case. The right screw-twisted structure.

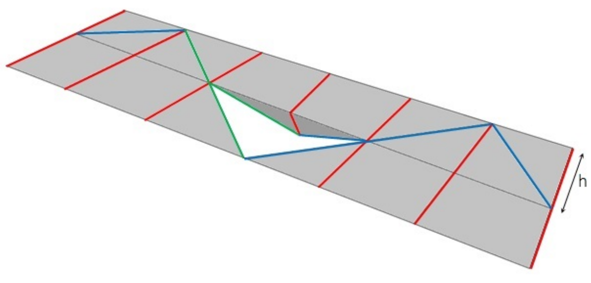

Fig. 12. The pentacene molecule structure, the left cosine twisted case.

7. Zigzag or meander soliton behaviour of a twisted structure of pentagons in a pentacene leaf. The cosine-like case

In this case again, the distance between the consecutive leaves of the modified pentacene (leaves being carbon atomic pentagons) $C_\xi H_\eta$, in our case $C_{22}H_{16}$ (Figs 3 and 10) amounts at d≈1.6 nm. It appears that the leaves on the pentacene are far from being co-planar: they have optimal global energetic conditions having the form of soliton zigzags or meanders. We are concentrated on the cosine-like case (Figs 14 nad 15) and arrive at:

Theorem 3. *A section of the pentacene* $C_{22}H_{16}$ leaf *orthogonal to the silicone* SiO_2 *substrate, in the cosine-like case is a cosine-like soliton curve with maxima at* $h \in$ (0.139 nm, 0.140 nm) (cf. Figs 3, 11, 14 and 15).

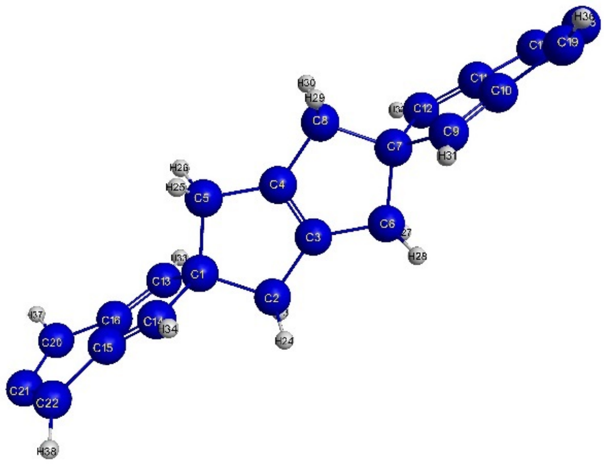

Fig. 13. Some candidate $C_{22}H_{16}$ for a pentacene molecule in the form of six pentagons. Another twisted cosine-like structure.

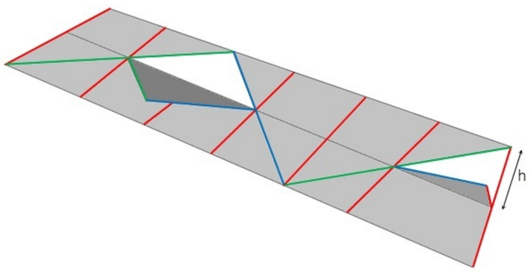

Fig. 14. The pentacene structure in the right-cosine twisted case.

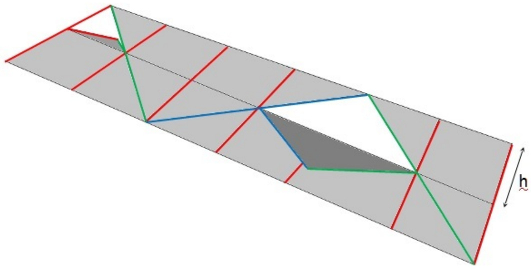

Fig. 15. The pentacene structure in the left-cosine twisted case.

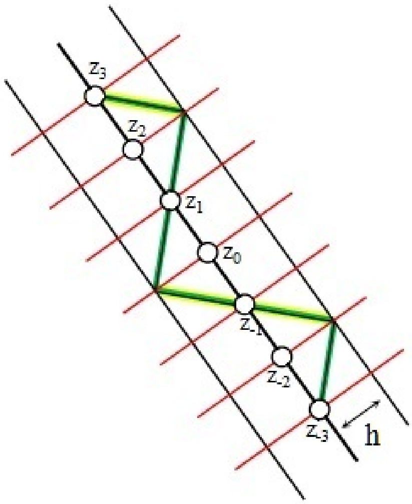

Fig. 16. A section of the pentacene $C_{22}H_{16}$ leaf orthogonal to the silicone SiO_2 substrate, in the sine-like twisted case.

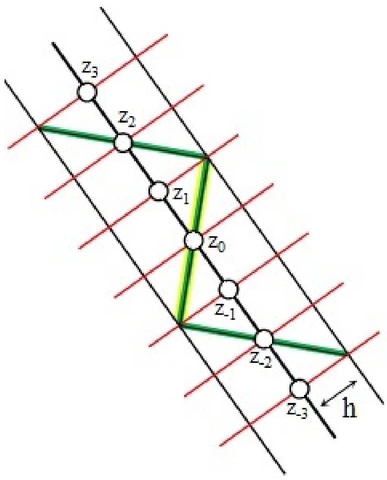

Fig. 17. A section of the pentacene $C_{22}H_{16}$ leaf orthogonal to the silicone SiO_2 substrate, in the cosine-like twisted case.

8. Conclusions. Pentacene as a foliated manifold with a soliton behaviour of leaves

Summing up we may consider pentacene, both in the form $C_{22}H_{14}$ and $C_{22}H_{16}$ as a foliated manifold with a soliton behaviour. In the senary case the system is quite close to a system of parallel planes because of considerable difference between distance d≈1.6nm and the maximal deviation δ=0.013nm of surfaces forming the system of leaves.

The whole configuration may have several mathematical and physical properties worth further investigation.

Acknowlegements

The authors thank Professor Leszek Wojtczak and Doctor Ewelina Z. Frątczak for stimulating and helpful discussion.

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on May 14, 2016.

GEOMETRYCZNE ASPEKTY BINARNYCH, TERNARNYCH, KWATERNARNYCH I SENARNYCH STRUKTUR W FIZYCE

S t r e s z c z e n i e

Obserwujemy, że struktury kwinarne i senarne, zarówno w przypadku pentacenu, jak i innych polimerów, można utworzyć ze struktur binarnych i senarnych w sensie równań różniczkowych i opisu geometrycznego. Liście pentacenu umieszczone na silikonowym podłożu mają postać pięciu połączonych węglowo-wodorowych sześciokątów; w całości nie tworzą dokładnie struktury planarnej lecz lekko falującą, która minimalizuje energię całkowitą. W przypadku struktury kwinarnej liście tworzą odosobnione, niemal periodyczne zygzaki i meandry.

Słowa kluczowe: algebry skończenie wymiarowe, pierścienie i algebry łączne, binarne struktury fizyczne, ternarne struktury fizyczne, kwinarne struktury fizyczne, senarne struktury fizyczne, pentacen, polimer

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