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A NEW GENERALIZATION OF THE TRAPEZOID FORMULA FOR n-TIME DIFFERENTIABLE MAPPINGS AND APPLICATIONS

Abstract. A new generalization of the trapezoid formula for n-time differentiable mappings and applications in Numerical Analysis are given.

1. Introduction

In the recent paper [1], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following generalization of the trapezoid rule.

THEOREM 1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then we have the equality*

$$(1.1)\qquad \int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right] + \int_a^b T_n(t) f^{(n)}(t)\,dt,$$

where

$$(1.2)\qquad T_n(t) := \frac{1}{n!}\left[\frac{(b-t)^n + (-1)^n (t-a)^n}{2}\right], \qquad t \in [a, b].$$

In the same paper, the authors pointed out the following inequality which provides an approximation formula for the integral $\int_a^b f(t)\,dt$ whose error can be estimated in terms of the sup-norm of $f^{(n)}(t)$.

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COROLLARY 1. *Under the above assumptions, we have the inequality*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \le \frac{(b-a)^{n+1}}{(n+1)!}\|f^{(n)}\|_\infty \times \begin{cases} 1 & \text{if } n=2r \\ \frac{2^n-1}{2^n} & \text{if } n=2r+1. \end{cases} \tag{1.3}$$

If, in the above corollary, we consider $n = 1$, then we get the known inequality [2]

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \le \frac{1}{4}(b-a)^2\|f'\|_\infty. \tag{1.4}$$

For $n = 2$, we obtain

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) - \frac{(b-a)^2}{2}\cdot\frac{f'(a)+f'(b)}{2}\right| \le \frac{(b-a)^3}{6}\|f''\|_\infty. \tag{1.5}$$

For other recent results concerning the trapezoid formula, see the book [9] and the recent papers [1]-[8] and [10]-[11], where further references are given.

The main aim of this paper is to point out a generalization of the trapezoid rule and inequality in a different way. Applications in Numerical Analysis for quadrature formulae will also be provided. A perturbed trapezoidal type rule is presented in Section 4 in which a number of *premature* results are given that provide tighter bounds than the traditional Grüss, Chebychev and Lupaş inequalities.

2. Integral identities

We start with the following result:

THEOREM 2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \ge 1)$ is absolutely continuous on $[a, b]$. Then*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)\right] + \frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt, \tag{2.1}$$

for all $x \in [a, b]$.

Proof. The proof is by mathematical induction.

For $n = 1$, we have to prove that

$$\int_a^b f(t)\,dt = (x-a)f(a) + (b-x)f(b) + \int_a^b (x-t)f^{(1)}(t)\,dt, \tag{2.2}$$

which is straightforward as it may be seen by the integration by parts formula applied for the integral

$$\int_a^b (x-t)f^{(1)}(t)\,dt. \tag{2.3}$$

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we wish to show that

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n} \frac{1}{(k+1)!}[(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)] \\ &+ \frac{1}{(n+1)!}\int_a^b (x-t)^{n+1}f^{(n+1)}(t)\,dt.\end{aligned} \tag{2.4}$$

For this purpose, we apply formula (2.2) for the mapping $g(t) := (x-t)^n f^{(n)}(t)$, which is absolutely continuous on $[a,b]$, and then, we can write

$$\begin{aligned}\int_a^b (x-t)^n f^{(n)}(t)\,dt &= (x-a)(x-a)^n f^{(n)}(a) + (b-x)(x-b)^n f^{(n)}(b) \\ &+ \int_a^b (x-t)\frac{d}{dt}[(x-t)^n f^{(n)}(t)]\,dt \\ &= \int_a^b (x-t)[-n(x-t)^{n-1}f^{(n)}(t) + (x-t)^n f^{(n+1)}(t)]\,dt \\ &+ (x-a)^{n+1}f^{(n)}(a) + (-1)^n(b-x)^{n+1}f^{(n)}(b) \\ &= -n\int_a^b (x-t)^n f^{(n)}(t)\,dt + \int_a^b (x-t)^{n+1}f^{(n+1)}(t)\,dt \\ &+ (x-a)^{n+1}f^{(n)}(a) + (-1)^n(b-x)^{n+1}f^{(n+1)}(t)\,dt.\end{aligned} \tag{2.5}$$

From (24) we can get

$$\begin{aligned}&\int_a^b (x-t)^n f^{(n)}(t)\,dt\\ &= \frac{1}{n+1}\int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt\\ &\quad+\frac{1}{n+1}[(x-a)^{n+1}f^{(n)}(a)+(-1)^n(b-x)^{n+1}f^{(n)}(b)].\end{aligned}$$

Now, using the induction hypothesis, we have

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)]\\ &\quad+\frac{1}{n!}\left[\frac{1}{n+1}\int_a^b (x-t)^{n+1}f^{(n+1)}(t)\,dt\right.\\ &\quad\left.+\frac{1}{n+1}[(x-a)^{n+1}f^{(n)}(a)+(b-x)^{n+1}f^{(n)}(b)]\right]\\ &=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)]\\ &\quad+\frac{1}{(n+1)!}\int_a^b (x-t)^{n+1}f^{(n+1)}(t)\,dt\end{aligned}$$

and the identity (2.4) is proved. This completes the proof. ∎

The following corollary is useful in practice.

COROLLARY 2. *With the above assumptions for f and R, we have the particular identities* (*which can also be obtained by using Taylor's formula with the integral remainder*)

$$\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}(b-a)^{k+1}f^{(k)}(b)+\frac{(-1)^n}{n!}\int_a^b (t-a)^n f^{(n)}(t)\,dt, \tag{2.6}$$

$$\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}(b-a)^{k+1}f^{(k)}(a)+\frac{1}{n!}\int_a^b (b-t)^n f^{(n)}(t)\,dt, \tag{2.7}$$

and the identity (*see also* [11])

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}[f^{(k)}(a)+(-1)^k f^{(k)}(b)]\\ &\quad+\frac{(-1)^n}{n!}\int_a^b\left(t-\frac{a+b}{2}\right)^n f^{(n)}(t)\,dt.\end{aligned}\tag{2.8}$$

REMARK 1. a) *For $n = 1$, we get the identity* (2.2) *which is a generalization of the trapezoid rule.*

i) *For $x = a$ in* (2.2), *we capture the "right rectangle rule"*

$$\int_a^b f(t)\,dt = (b-a)f(b) - \int_a^b (t-a)f'(t)\,dt.$$

ii) *For $x = b$ in* (2.2), *we obtain the "left rectangle rule"*

$$\int_a^b f(t)\,dt = (b-a)f(a) - \int_a^b (b-t)f'(t)\,dt. \tag{2.9}$$

iii) *Finally, for $x = \frac{a+b}{2}$, we get* [2]

$$\int_a^b f(t)\,dt = \frac{f(a)+f(b)}{2}(b-a) - \int_a^b \left(t - \frac{a+b}{2}\right) f'(t)\,dt \tag{2.10}$$

which is the "trapezoid rule".

b) *For $n = 2$, we get the identity:*

$$\begin{aligned}\int_a^b f(t)\,dt &= (x-a)f(a) + (b-x)f(b) \\ &\quad + \frac{1}{2}[(x-a)^2 f'(a) - (b-x)^2 f'(b)] + \frac{1}{2}\int_a^b (x-t)^2 f''(t)\,dt.\end{aligned} \tag{2.11}$$

i) *If in* (2.11) *we choose $x = b$, then we obtain the "perturbed left rectangle rule"*

$$\int_a^b f(t)\,dt = (b-a)f(a) + \frac{1}{2}(b-a)^2 f'(a) + \frac{1}{2}\int_a^b (t-a)^2 f''(t)\,dt, \tag{2.12}$$

which can also be obtained by using Taylor's formula with the integral remainder.

ii) *If in* (2.11) *we choose $x = a$, we can write the "perturbed right rectangle rule"*

$$\int_a^b f(t)\,dt = (b-a)f(b) - \frac{1}{2}(b-a)^2 f'(b) + \frac{1}{2}\int_a^b (t-b)^2 f''(t)\,dt. \tag{2.13}$$

iii) *Finally, for $x = \frac{a+b}{2}$, we capture the "perturbed trapezoid rule"* [11]

$$\int_a^b f(t)\,dt = \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{8}(f'(a)-f'(b)) + \frac{1}{2}\int_a^b \left(t-\frac{a+b}{2}\right)^2 f''(t)\,dt. \tag{2.14}$$

3. Integral inequalities

Using the integral representation of Theorem 1, we can prove the following inequality

THEOREM 3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$. Then*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right| \tag{3.1}$$

$$\leq \begin{cases} \dfrac{\|f^{(n)}\|_\infty}{(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right] & \text{if } f^{(n)} \in L_\infty[a,b]; \\ \dfrac{\|f^{(n)}\|_p}{n!}\left[\dfrac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}} & \text{if } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1 \\ & \text{and } f^{(n)} \in L_p[a,b]; \\ \dfrac{\|f^{(n)}\|_1}{n!}\left[\dfrac{1}{2}(b-a)+\left|x-\dfrac{a+b}{2}\right|\right]^n \end{cases}$$

for all $x \in [a,b]$.

Proof. Using the representation (2.1) and the properties of the modulus, we have

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq \frac{1}{n!}\int_a^b |x-t|^n |f^{(n)}(t)|\,dt =: R.$$

Observe that

$$\begin{aligned} R &\leq \left[\frac{1}{n!}\int_a^b |x-t|^n\,dt\right]\|f^{(n)}\|_\infty \\ &= \frac{\|f^{(n)}\|_\infty}{n!}\left[\int_a^x (x-t)^n\,dt + \int_x^b (t-x)^n\,dt\right] \\ &= \frac{\|f^{(n)}\|_\infty}{n!}\left[\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n+1}\right] \end{aligned}$$

and the first inequality in (3.1) is proved.

Using Hölder's integral inequality, we also have

$$R \le \frac{1}{n!}\Big(\int_a^b |f^{(n)}(t)|^p\,dt\Big)^{\frac{1}{p}}\Big(\int_a^b |x-t|^{nq}\,dt\Big)^{\frac{1}{q}}$$
$$= \frac{1}{n!}\|f^{(n)}\|_p\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}},$$

which proves the second inequality in (3.1).

Finally, let us observe that

$$R \le \frac{1}{n!}\sup_{t\in[a,b]}|x-t|^n\int_a^b |f^{(n)}(t)|\,dt$$
$$= \frac{1}{n!}[\sup_{t\in[a,b]}|x-t|]^n\|f^{(n)}\|_1$$
$$= \frac{1}{n!}[\max(x-a,b-x)]^n\|f^{(n)}\|_1$$
$$= \frac{1}{n!}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^n\|f^{(n)}\|_1$$

and the theorem is completely proved. ■

The following corollary is useful in practice.

COROLLARY 3. *With the above assumptions for f and n, we have the particular inequalities*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}(b-a)^{k+1}f^{(k)}(b)\right|$$
$$\le M := \begin{cases} \dfrac{\|f^{(n)}\|_\infty}{(n+1)!}(b-a)^{n+1} & \text{if } f^{(n)}\in L_\infty[a,b];\\[2mm] \dfrac{\|f^{(n)}\|_p}{n!}\dfrac{(b-a)^{n+\frac{1}{q}}}{(nq+1)^{1/q}} & \text{if } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1\\ & \text{and } f^{(n)}\in L_p[a,b];\\[2mm] \dfrac{\|f^{(n)}\|_1}{n!}(b-a)^n, & \end{cases}$$

and

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}(b-a)^{k+1}f^{(k)}(b)\right|\le M$$

and (*see also* [11])

$$(3.2)\quad \left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}[f^{(k)}(a)+(-1)^k f^{(k)}(b)]\right|$$
$$\leq\begin{cases}\dfrac{\|f^{(n)}\|_\infty}{2^n(n+1)!}(b-a)^{n+1} & \text{if } f^{(n)}\in L_\infty[a,b];\\ \dfrac{\|f^{(n)}\|_p}{2^n n!(nq+1)^{1/q}}(b-a)^{n+\frac{1}{q}} & \text{if } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1\\ & \text{and } f^{(n)}\in L_p[a,b];\\ \dfrac{\|f^{(n)}\|_1}{2^n n!}(b-a)^n; & \end{cases}$$

respectively.

REMARK 2. *If we put $n=1$ in (3.1), we capture the inequality*

$$(3.3)\quad \left|\int_a^b f(t)\,dt-(x-a)f(a)-(b-x)f(b)\right|$$
$$\leq\begin{cases}\left[\dfrac{1}{4}(b-a)^2+\left(x-\dfrac{a+b}{2}\right)^2\right]\|f^{(1)}\|_\infty & \text{if } f'\in L_\infty[a,b];\\ \|f'\|_p\left[\dfrac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} & \text{if } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1\\ & \text{and } f'\in L_p[a,b];\\ \left[\dfrac{1}{2}(b-a)+\left|x-\dfrac{a+b}{2}\right|\right]\|f'\|_1; & \end{cases}$$

for all $x\in[a,b]$, and, in particular,

a) *the "left rectangle" inequality*

$$\left|\int_a^b f(t)\,dt-(b-a)f(a)\right|\leq\begin{cases}\dfrac{\|f'\|_\infty}{2}(b-a)^2 & \text{if } f'\in L_\infty[a,b];\\ \dfrac{\|f'\|_p}{(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}} & \text{if } f'\in L_p[a,b];\\ \|f'\|_1(b-a). & \end{cases}$$

b) *the "right rectangle" inequality*

$$\left|\int_a^b f(t)\,dt-(b-a)f(b)\right|\leq\begin{cases}\dfrac{\|f'\|_\infty}{2}(b-a)^2 & \text{if } f'\in L_\infty[a,b];\\ \dfrac{\|f'\|_p}{(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}} & \text{if } f'\in L_p[a,b];\\ \|f'\|_1(b-a). & \end{cases}$$

c) *the "trapezoid" inequality*

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \tag{3.4}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{4}(b-a)^2 & \text{if } f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{2(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a,b]; \\ \frac{\|f'\|_1}{2}(b-a). \end{cases}$$

REMARK 3. *If we put $n = 2$ in (3.1), we get the inequality*

$$\left|\int_a^b f(t)\,dt - (x-a)f(a) - (b-x)f(b) - \frac{1}{2}[(x-a)^2 f'(a) - (b-x)^2 f'(b)]\right| \tag{3.5}$$
$$\leq \begin{cases} \frac{\|f''\|_\infty}{6}[(b-a)^3 + (b-x)^3] & \text{if } f'' \in L_\infty[a,b]; \\ \frac{\|f''\|_p}{2}\left[\frac{(x-a)^{2q+1} + (b-x)^{2q+1}}{2q+1}\right]^{\frac{1}{q}} & \text{if } p > 1,\ \frac{1}{p}+\frac{1}{q} = 1; \text{ and } f'' \in L_p[a,b]; \\ \frac{\|f''\|_1}{2}\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2; \end{cases}$$

for all $x \in [a,b]$, and, in particular:

a) *the "perturbed left rectangle" inequality*

$$\left|\int_a^b f(t)\,dt - (b-a)f(a) - \frac{1}{2}(b-a)^2 f'(a)\right| \tag{3.6}$$
$$\leq M_2 := \begin{cases} \frac{\|f''\|_\infty}{6}(b-a)^3 & \text{if } f'' \in L_\infty[a,b]; \\ \frac{\|f''\|_p}{2(2q+1)^{1/q}}(b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a,b]; \\ \frac{\|f''\|_1}{2}(b-a)^2; \end{cases}$$

b) *the "perturbed right rectangle" inequality*

$$\left|\int_a^b f(t)\,dt - (b-a)f(b) + \frac{1}{2}(b-a)^2 f'(b)\right| \leq M_2 \tag{3.7}$$

c) *the "perturbed trapezoid" inequality*

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{8}\left(f'(b)-f'(a)\right)\right| \tag{3.8}$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{24}(b-a)^3 & \text{if } f'' \in L_\infty[a,b]; \\ \dfrac{\|f''\|_p}{8(2q+1)^{1/q}}(b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a,b]; \\ \dfrac{\|f''\|_1}{8}(b-a)^2. \end{cases}$$

4. A perturbed version

A *premature* Grüss inequality is embodied in the following lemma (see papers [12] or [14] for a proof).

LEMMA 1. *Let f, g be integrable functions defined on $[a,b]$ and let $d \leq g(t) \leq D$. Then*

$$|T(f,g)| \leq \frac{D-d}{2}[T(f,f)]^{\frac{1}{2}}, \tag{4.1}$$

where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt.$$

Using the above lemma, the following result may be stated.

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative $f^{(n-1)}$, $n \geq 1$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \leq f^{(n)}(t) \leq \Gamma$ a.e on $[a,b]$. Then, the following inequality holds*

$$\begin{aligned} |P_T(x)| := \Bigg| &\int_a^b f(t)\,dt \\ &- \sum_{k=0}^{n-1}\left(\frac{1}{(k+1)!} \times \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right) \\ &- \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!}\left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right]\Bigg| \\ \leq\ & \frac{\Gamma-\gamma}{2} \cdot \frac{1}{n!} I(x,n) \\ \leq\ & \frac{\Gamma-\gamma}{2} \cdot \frac{n}{(n+1)!} \cdot \frac{(b-a)^{n+1}}{\sqrt{2n+1}}, \end{aligned} \tag{4.2}$$

where

$$(4.3)\quad I(x,n) = \frac{1}{(n+1)\sqrt{2n+1}}\{n^2(b-a)[(x-a)^{2n+1}+(b-x)^{2n+1}] \\ +(2n+1)(x-a)(b-x)[(x-a)^n-(x-b)^n]^2\}^{\frac{1}{2}}.$$

Proof. Applying the premature Grüss result (4.1) on $(x-t)^n$ and $f^{(n)}(t)$, we have

$$\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^n\,dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right|$$
$$\leq \frac{\Gamma-\gamma}{2}\left\{\frac{1}{b-a}\int_a^b (x-t)^{2n}\,dt - \left[\frac{1}{b-a}\int_a^b (x-t)^n\,dt\right]^2\right\}^{\frac{1}{2}}.$$

Therefore,

$$\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt \right.$$
$$\left. -\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)(b-a)} \cdot \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right|$$
$$\leq \frac{\Gamma-\gamma}{2}\left\{\frac{(x-a)^{2n+1}+(b-x)^{2n+1}}{(2n+1)(b-a)} - \left[\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(b-a)(n+1)}\right]^2\right\}^{\frac{1}{2}}.$$

We get further simplification of the above result by multiplying throughout by $\frac{b-a}{n!}$. This gives

$$(4.4)\qquad \left|\frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt \right.$$
$$\left. -\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!} \cdot \left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\right|$$
$$\leq \frac{\Gamma-\gamma}{2} \cdot \frac{1}{n!} J(x,n),$$

where

$$(4.5)\quad J^2(x,n) = \frac{1}{(2n+1)(n+1)^2}\{(n+1)^2(A+B)(A^{2n+1}+B^{2n+1}) \\ -(2n+1)(A^{n+1}+(-1)^nB^{n+1})^2\}$$

with $A = x-a$, $B = b-x$.

Now, from (4.5),

$$(2n+1)(n+1)^2J^2(x,n)$$
$$= n^2(A+B)(A^{2n+1}+B^{2n+1})$$
$$+(2n+1)[(A+B)(A^{2n+1}+B^{2n+1}) - (A^{n+1}+(-1)^nB^{n+1})^2]$$

$$
\begin{aligned}
&= n^2(A+B)(A^{2n+1}+B^{2n+1}) \\
&\quad +(2n+1)[AB(A^{2n}+B^{2n}) - 2A^{n+1}\cdot(-1)^n B^{n+1}] \\
&= n^2(A+B)[A^{2n+1}+B^{2n+1}] + (2n+1)AB[A^n - (-B)^n]^2
\end{aligned}
$$

Now, substitution of $A = x-a$, $B = b-x$ and the fact that $A+B = b-a$ gives $I(x,n) = \frac{J(x,n)}{(n+1)\sqrt{2n+1}}$, as presented in (4.3). Substitution of identity (2.1) into (4.4) gives (4.2) and thus the first part of the theorem is proved.

The upper bound is obtained by taking either $I(a,n)$ or $I(b,n)$ since $I(x,n)$ is convex. Hence the theorem is completely proved. ■

COROLLARY 4. *Let the conditions of Theorem 4 hold. Then the following result holds*

$$
\begin{aligned}
(4.6)\quad &\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}[f^{(k)}(a)+(-1)^k f^{(k)}(b)]\right. \\
&\quad \left. -\left(\frac{b-a}{2}\right)^n \frac{[1+(-1)^n]}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\right| \\
&\le \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}\left(\frac{b-a}{2}\right)^{n+1}\cdot\frac{1}{\sqrt{2n+1}}\cdot\begin{cases}\frac{2n}{n+1}, & n \text{ even} \\ 2, & n \text{ odd.}\end{cases}
\end{aligned}
$$

Proof. Taking $x = \frac{a+b}{2}$ in (4.2) gives (4.2), where

$$
I\left(\frac{a+b}{2},n\right) = \frac{1}{(n+1)\sqrt{2n+1}}\left(\frac{b-a}{2}\right)^{n+1}\{4n^2+(2n+1)[1+(-1)^n]^2\}^{\frac{1}{2}}.
$$

Examining the above expression for n even or n odd readily gives the result (4.6). ■

REMARK 4. *For n even, the third term in the modulus sign vanishes and thus there is no perturbation to the trapezoidal rule* (4.6).

THEOREM 5. *Let the conditions of Theorem 4 be satisfied. Further, suppose that $f^{(n)}$ is differentiable and be such that*

$$
\|f^{(n+1)}\|_\infty := \sup_{t\in[a,b]} |f^{n+1}(t)| < \infty.
$$

Then

$$
(4.7)\qquad |P_T(x)| \le \frac{b-a}{\sqrt{12}}\|f^{(n+1)}\|_\infty \cdot \frac{1}{n!}I(x,n),
$$

where $P_T(x)$ is the perturbed trapezoidal type rule given by the left hand side of (4.2) *and $I(x,n)$ is as given by* (4.3).

Proof. Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous and f', g' be bounded. Then Chebychev's inequality holds (see [13, p. 207])

$$|T(f,g)| \le \frac{(b-a)^2}{12} \sup_{t\in[a,b]} |f'(t)| \cdot \sup_{t\in[a,b]} |g'(t)|.$$

In [14] Matić, Pečarić and Ujević, using a *premature* Grüss type argument, proved that

$$|T(f,g)| \le \frac{(b-a)}{\sqrt{12}} \sup_{t\in[a,b]} |g'(t)| \sqrt{T(f,f)}. \tag{4.8}$$

Thus, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $(x-t)^n$ with f in (4.8) readily produces (4.7) where $I(x,n)$ is as given by (4.3). ∎

THEOREM 6. *Let the conditions of Theorem 4 be satisfied. Further, suppose that $f^{(n)}$ is locally absolutely continuous on (a,b) and let $f^{(n+1)} \in L_2(a,b)$. Then*

$$|P_T(x)| \le \frac{b-a}{\pi} \|f^{(n+1)}\|_2 \cdot \frac{1}{n!} I(x,n), \tag{4.9}$$

where $P_T(x)$ is the perturbed trapezoidal type rule given by the left hand side of (4.2) *and $I(x,n)$ is as given in* (4.3).

Proof. The following result was obtained by Lupaş (see [13, p. 210]). For $f, g : (a,b) \to \mathbb{R}$ being locally absolutely continuous on (a,b) and $f', g' \in L_2(a,b)$, then

$$|T(f,g)| \le \frac{(b-a)^2}{\pi^2} \|f'\|_2 \|g'\|_2,$$

where

$$\|h\|_2 := \left(\frac{1}{b-a}\int_a^b |h(t)|^2\right)^{\frac{1}{2}} \quad \text{for} \quad h \in L_2(a,b).$$

In [14] Matić, Pečarić and Ujević further show that

$$|T(f,g)| \le \frac{(b-a)}{\pi} \|g'\|_2 \sqrt{T(f,f)}. \tag{4.10}$$

Now, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $(x-t)^n$ with f in (4.10) gives (4.9), where $I(x,n)$ is found in (4.3). ∎

REMARK 5. *Results* (4.7) *and* (4.9) *are not readily comparable to that obtained in Theorem* 4 *since the bound now involves the behaviour of $f^{(n+1)}(\cdot)$ rather than $f^{(n)}(\cdot)$.*

5. Application in numerical integration

Consider the partition $I_m : a = x_0 < x_1 < ... < x_{m-1} < x_m = b$ of the interval $[a, b]$ and the intermediate points $\xi = (\xi_0, ..., \xi_{m-1})$, where $\xi_j \in [x_j, x_{j+1}]$ $(j = 0, ..., m-1)$. Put $h_j := x_{j+1} - x_j$ and $\vartheta(h) = \max\{h_j | j = 0, ..., m-1\}$.

In [1], the authors considered the following generalization of the trapezoid formula

$$T_{m,n}(f,I_m) := \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{h_j^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(x_j)+(-1)^k f^{(k)}(x_{j+1})}{2}\right] \tag{5.1}$$

and proved the following theorem:

THEOREM 7. *Let $f:[a,b]\to\mathbb{R}$ be such that it's derivative $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have*

$$\int_a^b f(t)\,dt = T_{m,n}(f,I_m)+R_{m,n}(f,I_m), \tag{5.2}$$

where the reminder $R_{m,n}(f,I_m)$ satisfies the estimate

$$|R_{m,n}(f,I_m)| \le \frac{C_n}{(n+1)!}\|f^{(n)}\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}, \tag{5.3}$$

and

$$C_n := \begin{cases} 1 & \text{if } n=2r \\ \dfrac{2^{2r+1}-1}{2^{2r+1}} & \text{if } n=2r+1. \end{cases}$$

Now, let us define the even more generalized quadrature formula

$$\widetilde{T}_{m,n}(f,\xi,I_m) := \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{1}{(k+1)!}[(\xi_j-x_j)^{k+1}f^{(k)}(x_j)$$
$$+(-1)^k(x_{j+1}-\xi_j)^{k+1}f^{(k)}(x_{j+1})],$$

where x_j,ξ_j $(j=0,\dots,m-1)$ are as above.

The following theorem holds.

THEOREM 8. *Let f be as in Theorem 7. Then we have the formula*

$$\int_a^b f(t)\,dt = \widetilde{T}_{m,n}(f,\xi,I_m)+\widetilde{R}_{m,n}(f,\xi,I_m), \tag{5.4}$$

where the reminder satisfies the estimate

$$|\widetilde{R}_{m,n}(f,\xi,I_m)| \tag{5.5}$$
$$:= \begin{cases} \dfrac{1}{(n+1)!}\|f^{(n)}\|_\infty \sum_{j=0}^{m-1}[(\xi_j-x_j)^{n+1}+(x_{j+1}-\xi_j)^{n+1}], \\ \dfrac{1}{n!(nq+1)^{1/q}}\|f^{(n)}\|_p\left[\sum_{j=0}^{m-1}(\xi_j-x_j)^{nq+1}+\sum_{j=0}^{m-1}(x_{j+1}-\xi_j)^{nq+1}\right]^{\frac{1}{q}}, \\ \dfrac{1}{n!}\|f^{(n)}\|_1\left[\dfrac{1}{2}\vartheta(h)+\max_{j=0,\dots,m-1}\left|\xi_j-\dfrac{x_j+x_{j+1}}{2}\right|\right]^n. \end{cases}$$

Proof. Apply the inequality (3.1) on the subinterval $[x_j, x_{j+1}]$ to get

$$\Big| \int_{x_j}^{x_{j+1}} f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \times [(\xi_j - x_j)^{k+1} f^{(k)}(x_j) + (-1)^k (x_{j+1} - \xi_j)^{k+1} f^{(k)}(x_{j+1})] \Big|$$

$$\leq \begin{cases} \frac{1}{(n+1)!} \sup\limits_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}], \\ \frac{1}{n!} \Big(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \Big)^{\frac{1}{p}} \Big[\frac{(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}}{nq+1} \Big]^{\frac{1}{q}}, \\ \frac{1}{n!} \Big(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \Big) \Big[\frac{1}{2} h_j + |\xi_j - \frac{x_j + x_{j+1}}{2}| \Big]^n. \end{cases}$$

Summing over j from 0 to $m-1$ and using the generalized triangle inequality, we have

$$|\widetilde{R}_{m,n}(f, \xi, I_m)|$$
$$\leq \Big| \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \times [(\xi_j - x_j)^{k+1} f^{(k)}(x_j) + (-1)^k (x_{j+1} - \xi_j)^{k+1} f^{(k)}(x_{j+1})] \Big|$$

$$:= \begin{cases} \frac{1}{(n+1)!} \sum\limits_{j=0}^{m-1} \sup\limits_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}], \\ \frac{1}{n!} \sum\limits_{j=0}^{m-1} \Big(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \Big)^{\frac{1}{p}} \Big[\frac{(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}}{nq+1} \Big]^{\frac{1}{q}}, \\ \frac{1}{n!} \sum\limits_{j=0}^{m-1} \Big(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \Big) \Big[\frac{1}{2} h_j + \Big|\xi_j - \frac{x_j - x_{j+1}}{2}\Big| \Big]^n. \end{cases}$$

Since $\sup\limits_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| \leq \|f^{(n)}\|_\infty$, the first inequality is obvious.

Using the discrete Hölder inequality, we have

$$\frac{1}{(nq+1)^{1/q}} \sum_{j=0}^{m-1} \Big(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \Big)^{\frac{1}{p}} [(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}]^{\frac{1}{q}}$$
$$\leq \frac{1}{(nq+1)^{1/q}} \Big[\sum_{j=0}^{m-1} \Big[\Big(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \Big)^{\frac{1}{p}} \Big]^p \Big]^{\frac{1}{p}}$$

$$\times\Big[\sum_{j=0}^{m-1}[[(\xi_j-x_j)^{nq+1}+(x_{j+1}-\xi_j)^{nq+1}]^{\frac{1}{q}}]^q\Big]^{\frac{1}{q}}$$

$$=\frac{1}{(nq+1)^{1/q}}\|f^{(n)}\|_p\Big[\sum_{j=0}^{m-1}(\xi_j-x_j)^{nq+1}+\sum_{j=0}^{m-1}(x_{j+1}-\xi_j)^{nq+1}\Big]^{\frac{1}{q}}$$

and the second inequality in (5.5) is proved.

Finally, let us observe that

$$\frac{1}{n!}\sum_{j=0}^{m-1}\Big(\int_{x_j}^{x_{j+1}}|f^{(n)}(s)|ds\Big)\left[\frac{1}{2}h_j+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n\le$$

$$\le\max_{j=0,\dots,m-1}\left[\frac{1}{2}h_j+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n\sum_{j=0}^{m-1}\Big(\int_{x_j}^{x_{j+1}}|f^{(n)}(s)|ds\Big)$$

$$\le\left[\frac{1}{2}h_j+\max_{j=0,\dots,m-1}\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n\|f^{(n)}\|_1$$

and the last part of (5.5) is proved. ∎

REMARK 6. *Since* $(x-a)^\alpha+(b-x)^\alpha\le(b-a)^\alpha$ *for* $\alpha\ge1$, $x\in[a,b]$, *then we can remark that the first branch of* (5.5) *can be bounded by*

$$\frac{1}{(n+1)!}\|f^{(n)}\|_\infty\sum_{j=0}^{m-1}h_j^{n+1}. \tag{5.6}$$

The second branch can be upper bounded by

$$\frac{1}{n!(nq+1)^{1/q}}\|f^{(n)}\|_p\Big[\sum_{j=0}^{m-1}h_j^{nq+1}\Big]^{\frac{1}{q}} \tag{5.7}$$

and finally, the last branch in (5.5) *can be upper bounded by*

$$\frac{1}{n!}[\vartheta(h)]^n\|f^{(n)}\|_1. \tag{5.8}$$

Note that all the bounds provided by (5.6)–(5.8) *are uniform bounds for* $\widetilde{R}_{m,n}(f,\xi,I_m)$ *in terms of the intermediate points* ξ.

The last inequality we can get from (5.5) is that one for which we have $\xi_j=\frac{x_j+x_{j+1}}{2}$. Consequently, we can state the following corollary (see also [11]):

COROLLARY 5. *Let* f *be as in Theorem* 8. *Then we have the formula*

$$\int_a^b f(t)\,dt=\widetilde{T}_{m,n}(f,I_m)+\widetilde{R}_{m,n}(f,I_m), \tag{5.9}$$

where

$$\widetilde{T}_{m,n}(f,I_m)=\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{1}{2^{k+1}(k+1)!}[f^{(k)}(x_j)+(-1)^k f^{(k)}(x_{j+1})]h_j^{n+1} \tag{5.10}$$

and the remainder $\widetilde{R}$ *satisfies the estimate*

$$|\widetilde{R}_{m,n}(f,I_m)|\le\begin{cases}\dfrac{1}{2^n(n+1)!}\|f^{(n)}\|_\infty\displaystyle\sum_{j=0}^{m-1}h_j^{n+1},\\[2mm] \dfrac{1}{2^n n!(nq+1)^{1/q}}\|f^{(n)}\|_p\Big[\displaystyle\sum_{j=0}^{m-1}h_j^{n+1}\Big]^{\frac{1}{q}},\\[2mm] \dfrac{1}{2^n n!}[\vartheta(h)]^n\|f^{(n)}\|_1.\end{cases}$$

REMARK 7. *Similar results can be stated by using the "perturbed" versions embodied in Theorems* 4, 5 *and* 6, *but we omit the details.*

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