The Epistemology of Mathematical Necessity

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Abstract. It seems possible to know that a mathematical claim is necessarily true by inspecting a diagrammatic proof. Yet how does this work, given that human perception seems to just (as Hume assumed) 'show us particular objects in front of us'? I draw on Peirce's account of perception to answer this question. Peirce considered mathematics as experimental a science as physics. Drawing on an example, I highlight the existence of a primitive constraint or blocking function in our thinking which we might call 'the hardness of the mathematical must'.

Keywords: Necessity, epistemology, mathematics, Hume, Peirce
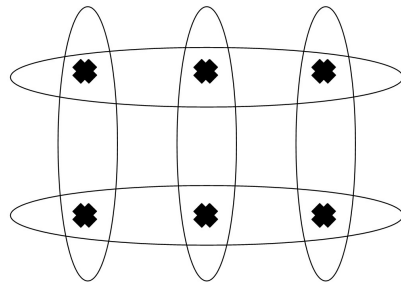
1   Introduction

Fig 1. Why $2 \times 3 = 3 \times 2$

We can come to know that a mathematical claim is true by inspecting a diagram, as in *Fig. 1*. What is being perceived here? Not just that 2×3 is 3×2, but that 2×3 must be 3×2. It is clear that trying to create an option such as $2 \times 3 = 3 \times 3$ would be futile. We could of course prove the same claim in a more stepwise, symbolic manner, but this diagram seems to give us everything we need to ascertain the necessary truth.

Yet how exactly does this work? Much mainstream analytic epistemology arguably makes such knowledge-gathering seem impossible [1], through a materialist understanding of perception, deriving ultimately from Hume's empiricism, according to which, we might say, 'experience only shows us the particular objects in front of us'. This produces a highly sceptical treatment of modality, which Hume famously used to undermine so-called causal necessity [1]. Attempts to challenge this often meet intimidating charges of anti-naturalism.

Hume's particular empiricism led him to coin a supposedly common-sense maxim, widely taken for granted today: ***"There are no necessary connections between distinct existences".*** The maxim follows from two key Humean claims about perception: i) it is *passive*—involving nothing more than 'registering' the impact of individual

physical objects on the sense organs, ii) it is *atomistic*—ideas are only distinct if fully separable in imagination. Consequently Hume banishes from his epistemology *abstract ideas*, whose determinable properties lack determination (e.g. a 'general triangle'). Allowing these would render the mind active in choosing which determinables to elide. The early modern Hume sees this denial as properly naturalistic against scholasticism. Consider *Fig. 1*. Does it display a necessary connection between distinct existences? Well, what are the 'existences'? Humeans have a number of choices here:

i) *Physical Marks:* The relevant existences are 5 oval-shaped and 6 star-shaped marks, exactly as inscribed on the page. This choice appears natural given the organisation of our visual field and—invoking Hume's separate imaginability criterion—we can imagine each shape existing alone on the page. But there *are* necessary connections between these objects in *Fig 1*. For instance one cannot remove stars from the vertical ovals without changing the number of stars in the horizontal ovals. Interpreted thus, then, Hume's maxim seems simply incorrect.

ii) *Abstract Objects:* Alternatively one might claim that *Fig.1* displays a truth about something more ideal—e.g. three '2s' and two '3s'. These are arguably not distinct, since 2 is made up of 'two ones' and 3 of 'three ones', so 2 is a proper part of 3. At this point, then, Hume might defend his maxim by stating that *Fig 1* expresses relations between ideas, not matters of fact, and only the latter is accessed through perception, whilst the former is mere semantic stipulation. But this is somewhat unsatisfying. If mathematics merely concerns a world of ideas, how are physical diagrams so startlingly effective? Furthermore, now Hume's maxim seems to beg the question. Is he not arbitrarily ruling out that we perceive existences between which necessary connections hold, effectively stating: "There are no necessary connections between distinct existences, which are those existences without necessary connections".

iii) *"Both":* One might consider combining the two views as follows: the existences are ovals and stars *and* '2s' and '3s'. However this raises tricky questions about the relationship between the physical marks and the numerical objects. If they are all separate, why include physical marks in the diagram? Why not 'cut to the chase' and just include the numerical objects? Obviously that is impossible, which points to our preferred interpretation.

iv) *"Hybrid...but not both":* Attribute to the physical marks and numbers *partial identities*. What does this mean? Just that 'twoness' is a property which may be abstracted from two star-shaped marks on the page, while precisely not being separable from them. Abstraction without separation is essential for all *structural reasoning*. This includes our example since it turns on recognising that two abstractable but not separable structures: 2×3 (two rows of three stars) and 3×2 (three columns of two stars) – are in fact identical. Structural reasoning is surely a significant part of mathematics. So what theory of perception could do justice to it?

3   A Peircean Approach

Peirce suggests we need to give separate, interlocking, accounts of: i) immediate experience of objects: the *percept*, ii) the truth of symbols derived from that experience: the *perceptual judgment*. The percept is a non-cognitive direct encounter

with some object. It is not a Humean idea, nor does it express truth-claims, it "simply knocks at the portal of my soul and stands there in the doorway" [5]. On the other hand, the perceptual judgement takes propositional (subject-predicate) form, and its interpretation opens to the community of inquiry in an endless series of judgments, each member of which is logically related to prior members. The perceptual judgement does not copy the percept, as they are too unlike one another. How do they relate? Contra Hume, percepts *cause perceptual judgements, while not being the source of their content*. Human evolution ensures that each percept causes "direct and uncontrollable interpretations". This process can and must be trained through cultivating appropriate mental *habits*, through public criticism in a common language.

So what kind of 'existence' is the mathematical percept? We should take seriously Peirce's repeated claims that mathematics is as experimental a science as physics, but the mathematician's laboratory is the diagram [5]. Again consider *Fig. 1*. First note that the mathematical percept, like all percepts, is strictly impossible to describe in words. However, when I 'got' this proof, I suddenly grasped the horizontal and vertical star-arrangements as one, as if the same 5 ovals were 'holding both together'. 'Holding together' is a metaphor, as the arrangements are not strictly *parts* of the diagram (as not separable, only abstractable). But looking at the diagram and thinking about abstracting other arrangements from it (such as three threes), I could feel myself not being able to think of them. We might call this primitive blocking or constraint, in homage to Wittgenstein, 'the hardness of the mathematical must'. With my prior mathematical training, this prompts me to an 'uncontrollable interpretation' that the proposition $2 \times 3 = 3 \times 2$ must be true. No matter how hard I try to interpret *Fig. 1* as $2 \times 3 = 3 \times 3$ – I simply cannot think that way. *(Try it yourself...)* Thus Peirce notes that although mathematics deals with a world of ideas, not material objects, its discoveries are something to which our minds are forced. My felt 'hardness' now becomes the necessity of the perceptual judgment: $2 \times 3 = 3 \times 2$.

Thus, despite many philosophers' bafflement, we do perceive necessity. I have argued that perception is in fact the *only* way we come to know necessity, as all necessary reasoning involves experimenting on diagrams to determine structural dependencies [2,3,4]. Necessary truths are abstractable from physical marks and this is not an ontological reification but an epistemic capacity. Arguably the whole concept of 'abstract object', so problematic in recent philosophy of mathematics, arises from not understanding that (contra Hume) one can abstract without separating.

4　References

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