View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by City Research Online

City Research Online

City, University of London Institutional Repository

Citation: Vlahakis, E. E. and Halikias, G. ORCID: 0000-0003-1260-1383 (2019). Distributed LQR Methods for Networks of Non-Identical Plants. In: 2018 IEEE Conference on Decision and Control (CDC). (pp. 6145-6150). IEEE. ISBN 978-1-5386-1395-5

This is the accepted version of the paper.

This version of the publication may differ from the final published version.

Permanent repository link: http://openaccess.city.ac.uk/21646/

Link to published version: http://dx.doi.org/10.1109/cdc.2018.8619185

Copyright and reuse: City Research Online aims to make research outputs of City, University of London available to a wider audience. Copyright and Moral Rights remain with the author(s) and/or copyright holders. URLs from City Research Online may be freely distributed and linked to.

City Research Online:	http://openaccess.city.ac.uk/	publications@city.ac.uk

Distributed LQR Methods for Networks of non-Identical Plants

Eleftherios E. Vlahakis[1] and George D. Halikias[2]

Abstract— **Two well-established complementary distributed linear quadratic regulator (LQR) methods applied to networks of identical plants are extended to the non-identical plant case. The first uses a top-down approach where the centralized optimal LQR controller is approximated by a distributed control scheme whose stability is guaranteed by the stability margins of LQR control. The second consists of a bottom-up approach in which optimal interactions between self-stabilizing agents are defined so as to minimize an upper bound of the global LQR criterion. In this paper, local state-feedback controllers are designed by solving model-matching type problems and mapping all the plants in the network to a target system specified a priori. Existence conditions for such schemes are established for various families of systems. The single-input case and the multi-input case relying on the controllability indices of the plants are first considered followed by an LMI approach combined with LMI regions for pole clustering. Then, the two original top-down and bottom-up methods are adapted and the stability problem for networks of non-identical plants is solved. The applicability of our approach for distributed network control is illustrated via a simple example.**

I. Introduction

Networks of systems have attracted a lot of attention of the control community in recent years. Such schemes are often referred to as multi-agent systems with each agent being represented by a dynamical system and having the ability to communicate with other counterparts within the network. The interactions established among the agents determine the network topology and define a communication pattern. The need for forming networks of systems in many cases arises from the fact that some problems might not be resolved by individual systems. Military applications, transport networks and supply chains are such paradigms which indicate that difficult tasks may be accomplished cooperatively [1]–[3]. In other cases, the topology of the network may be imposed by physical links such as in power systems where the agents take the role of power generators and the interconnections are represented by power transmission lines [4], [5].

Stability issues play key role in multi-agent systems [6], [7] where cooperative controllers should be designed to ensure stable operation for the network. In cases where networks are composed of sufficiently small number of agents, the interconnections among the systems might not be limited and fully-centralized cooperative controllers can be established applying well-known control schemes. Nevertheless, bandwidth limits as well as cost factors are main reasons to impose restrictions to network's communication capacity resulting in sparsity of interactions among the plants. Thus, distributed cooperative controllers are to be designed to solve the network stability problem. Two complementary distributed LQR methods are proposed in [8] and [9]. In the first (top-down) approach [8], the centralized optimal LQR controller is approximated by a distributed control scheme whose stability is guaranteed by the stability margins of LQR control. The second [9] consists of a bottom-up approach in which optimal interactions between self-stabilizing agents are defined so as to minimize an upper bound of the global LQR criterion. A limitation of both methods is the assumption that networks are formed by identical plants, a fact which is often unrealistic in real applications. The approach proposed in this work relaxes this assumption and therefore generalizes the approaches in [8] and [9] with only minor modifications. Rather than assuming identical models for all agents, we consider a general class of possible models which share the same structure in terms of input and state dimensions and other structural properties (e.g. controllability, controllability indices) which are identified in each case considered.

In this paper, static state-feedback controllers are proposed to solve model-matching type problems with the aim at relaxing the assumption of the repetitive pattern of the network considered in [8] and [9] in terms of the plants' model. In this respect, the systems constituting the network are assumed to be linear and non-identical with their state vector being accessible for measurement. The method is applied locally, where the model of each agent matches a target system via state-feedback control. The model of a certain target system might be specified a priori possibly posing desired local performance specifications for the agents. The conditions of such schemes to exist are examined by considering certain families of systems. Single-input plants are first investigated and then the multi-input case is analyzed taking into account the controllability indices of the plants. Plants with arbitrary number of inputs are also assumed and are converted to be controllable by one input and match a single-input target system. A Linear Matrix Inequality approach is also proposed to solve the model-matching type problem for a certain family of systems. Next, the state-feedback distributed control scheme presented in [8] and [9] are modified and then the stability problem for a network of non-identical plants that are all mapped locally to the same target system is solved. The effectiveness of the method is finally illustrated via a simple example.

*This work was not supported by any organization

[1]Eleftherios E. Vlahakis is with the Department of Electrical & Electronic Engineering, City, University of London, London, EC1V 0HB, United Kingdom eleftherios.vlahakis@city.ac.uk

[2]George D. Halikias is with the Department. of Electrical & Electronic Engineering, City, University of London, London, EC1V 0HB, United Kingdom g.halikias@city.ac.uk

The rest of the paper is organized in four sections. In the second section the problem considered in this paper is defined along with some useful definitions. The main work is analyzed in the fourth section where model-matching type problems are solved for various classes of systems. Note that in our case the definition of ”model-matching” gives us considerable flexibility as the output matrices of the mapped systems are required to be square and invertible but are otherwise arbitrary. The extension of the results presented in [8] and [9] followed by a numerical example are included in the fifth section. The sixth section presents the main conclusions of the work where a discussion of the main results and suggestions for future work are given.

II. Problem Statement

The networks considered in this paper are composed of linear agents represented by a controllable pair (A,B) and assumed to have access to their state-vector. In the remaining text, we refer to such plants as state-feedback systems highlighting the availability of their states to be used to design the control input. The outline of the setting for the cooperative control is now given. Consider network of N non-identical linear systems called agents with dynamical behaviour being described by the following differential equation:

$$\dot{x}_i = A_i x_i + B_i u_i \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ represent the state and the input vector respectively of the i^{th} agent. The matrices $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ describe its dynamics and input distribution, respectively, and are generically different for each i^{th} agent. The pairs (A_i,B_i) are assumed to be completely controllable for all $i = 1,\cdots,N$.

The network's communication scheme is described by a graph $\mathscr{G} = (\mathscr{V},\mathscr{E})$, where $\mathscr{V}$ is the set of the agents ($\mathscr{V} = \{1,\cdots,N\}$) and $\mathscr{E} \subseteq \mathscr{V}\times\mathscr{V}$ the set (i,j) representing the interconnection between agent $i \in \mathscr{V}$ and agent $j \in \mathscr{V}$. These interactions among the agents involve information exchange about their states. We assume that the graph is bidirectional, which means that if the agent i is aware of the state of the agent j, then the agent j is aware of the state of the agent i.

Let the network's state be represented by augmentation of the individual states of the agents. The augmented state-space form is given by

$$\begin{aligned} \dot{\tilde{x}} &= diag\{A_1,\cdots,A_N\}\tilde{x} + diag\{B_1,\cdots,B_N\}\tilde{u} \\ \tilde{x}_0 &= [x_1^T(0),\cdots,x_N^T(0)]^T \end{aligned} \tag{2}$$

where $\tilde{x} = Col\{x_1,\cdots,x_N\}$ and $\tilde{u} = Col\{u_1,\cdots,u_N\}$ are the augmented state and input vector of the network, respectively. In this paper, we seek cooperative controllers

$$u_i = F_i x_i + G_i v_i \tag{3}$$

which map all the agents in the network to fictitious target plants (A,B,C_i) while the augmented input vector $\tilde{u} = Col\{u_1,\cdots,u_N\}$ stabilizes the network's differential equation (2). The matrices (A,B) of the target plant are common for all agents and have the same dimensions with (A_i,B_i). Matrices C_i are square and nonsingular for all $i = 1,\cdots,N$. The target plant is defined in the next section. The state-feedback controllers $F_i x_i$ combined with input transformations G_i solve model-matching type problems locally and the transfer function of the i^{th} closed-loop system takes the following form

$$(sI - A_i - B_iF_i)^{-1}B_iG_i = C_i(sI-A)^{-1}B \tag{4}$$

The state-feedback control laws F_i and the input transformations G_i if exist, transform the original state-space form of the network (2) into the augmented state-space form given as

$$\dot{\tilde{x}} = (I\otimes A)\tilde{x} + (I\otimes B)\tilde{v} \tag{5}$$

$$\tilde{y} = \tilde{C}\tilde{x} \tag{6}$$

where $\tilde{v} = Col\{v_1,\cdots,v_N\}$ and $\tilde{C} = diag\{C_1,\cdots,C_N\}$ square and nonsingular. The symbol $\otimes$ stands for the Kronecker product. Certain families of state-feedback systems will be considered in the next section that can be mapped to the class of systems $T(\bar{A},\bar{B})$ defined by the following transfer function

$$T(\bar{A},\bar{B}) = \{\Phi(sI-\bar{A})^{-1}\bar{B}, \quad \Phi\in\mathbb{R}^{n\times n}, |\Phi|\neq 0\} \tag{7}$$

via local state-feedback control laws. Due to repetitive pattern of the fictitious network (5) top-down [8] and bottom-up [9] distributed LQR methods for network of identical plants can be used to design distributed controllers $\tilde{v}_i$ to solve the stability problem of networks formed of non-identical agents. The two original methods are now outlined.

A. Top-down Method

The distributed LQR method proposed in [8] for networks formed by identical plants is briefly presented here. Let N_L identical agents constitute a full-centralized network described by a bidirectional graph and have the ability to exchange information about their states. The state-space forms of each agent and the network are given by

$$\dot{x}_i = Ax_i + Bu_i, \qquad x_{i0} = x_i(0) \tag{8}$$

$$\dot{\tilde{x}} = (I\otimes A)\tilde{x} + (I\otimes B)\tilde{u}, \quad \tilde{x}_0 = [x_1^T(0),\cdots,x_{N_L}^T(0)]^T \tag{9}$$

Consider now performance index that couples the dynamical behavior of the individual agents chosen as

$$J(\tilde{u},\tilde{x}_0) = \int_0^\infty \sum_{i=1}^{N_L}\left(x_i^T Q_1 x_i + u_i^T R u_i + \sum_{j\neq i}^{N_L}(x_i - x_j)^T Q_2 (x_i - x_j)\right) d\tau \tag{10}$$

with $Q_1 \geq 0$, $Q_2 \geq 0$ and $R > 0$. Under the assumption that (A,B) is controllable and both pairs (A,Q_1) and (A,Q_2) are observable the solution to the following LQR problem

$$\min_{\tilde{u}} J(\tilde{u},\tilde{x}_0) \quad \text{s.t.} \quad \dot{\tilde{x}} = (I\otimes A)\tilde{x} + (I\otimes B)\tilde{u}, \quad \tilde{x}_0 \tag{11}$$

leads to the networked state-feedback gain $\tilde{K}$ with the following structure

$$\tilde{K} = \begin{bmatrix} K_1 & K_2 & \cdots & K_2 \\ K_2 & K_1 & \cdots & K_2 \\ \vdots & \ddots & \vdots & \vdots \\ K_2 & \cdots & K_2 & K_1 \end{bmatrix} \tag{12}$$

where K_1 and K_2 are functions of A, B, Q_1, Q_2, R and N_L. Due to lack of space readers are referred to [8] for detailed construction of K_1 and K_2. Exploiting the stability margins of the LQR solution a stabilizing distributed state-feedback controller is constructed according to the following theorem.

Theorem 1. *Consider a network of N identical plants with state-space given by* (8) *and topology specified by graph $\mathscr{G}$ with Laplacian matrix L and maximum vertex degree d_{max}. Consider reduced-order networked LQR problem* (11) *with $N_L = d_{max} + 1$. Let $M = \mathbb{R}^{NxN}$ reflect the structure of L and be symmetric with the following property:*

$$\lambda_i(M) > \frac{N_L}{2}, \quad \forall \lambda_i(M) \in S(M) \backslash \{0\} \tag{13}$$

and construct the state-feedback controller:

$$\hat{K} = I_N \otimes K_1 + M \otimes K_2 \tag{14}$$

Then the closed-loop system

$$I_N \otimes A + (I_N \otimes B)\hat{K} \tag{15}$$

is asymptotically stable.

B. Bottom-up Method

The distributed controller presented in [9] is shown here highlighting the technical details of the method. Networks of N identical agents are considered with state-space forms given by (8) where the input distribution matrix is assumed to have a particular structure of $\begin{bmatrix} 0 \\ B_2 \end{bmatrix}$ with B_2 being non-singular. This technical requirement can always be achieved by rotating the original coordinates of the state-space using appropriate state-space transformation obtained from singular value decomposition of the matrix B provided the latter has full-column rank. The global performance index to be minimized is given by

$$J(\tilde{u}, \tilde{x}_0) = \int_0^\infty \left(\tilde{x}^T (I_N \otimes Q_1 + L \otimes Q_2)\tilde{x} + \tilde{u}^T R \tilde{u} \right) d\tau \tag{16}$$

with $\tilde{x}$, $\tilde{x}_0$ and $\tilde{u}$ given in (2) and L representing the Laplacian matrix of the network. Note that the global performance indices for both methods are identical.

The agents are first stabilized locally via state-feedback gain K obtained by solving a typical LQR problem at node level with weighting matrices being chosen as Q_1 and R. The next step of the method is state-space transformation which diagolizes the solution P to the local algebraic Riccati equation (ARE)

$$A^T P + PA - PBR^{-1}B^T P + Q_1 = 0 \tag{17}$$

The local state-feedback gain in the new coordinates takes the form $\hat{K} = \begin{bmatrix} 0 & K_2 \end{bmatrix}$ and the distributed controller for the networked systems is constructed as

$$\hat{u}_i = \hat{K}\hat{x}_i + \Phi \hat{K} \hat{z}_i, \quad \text{for} \quad i = 1, \cdots, N \tag{18}$$

$$\text{where } \hat{z}_i = \sum_{j \in \mathscr{N}_i} (\hat{x}_i - \hat{x}_j) \tag{19}$$

with $\mathscr{N}_i$ representing the neighboring systems that agent i can interact with. The signal $\hat{z}_i$ reflects a state disagreement measure of the i^{th} agent with its neighbors. The distributed controller in the original coordinates at network level is given by

$$\hat{K} = I_N \otimes K + L \otimes \Phi K \tag{20}$$

The construction of the scaling matrix Φ involves a convex optimization problem associated with an upper bound of the global LQR criterion (16). The description of the minimization problem is omitted and readers are referred to the original paper [9] for more details.

III. Model-Matching Problems

In this section, model-matching type problems are solved for specific categories of systems via state-feedback techniques as the first stage of the solution to the stability problem of a network of non-identical plants. The systems are assumed to belong to a family of systems with common structural properties such as system size, input dimensions, controllability indices etc. The main purpose of this section is to define these properties of certain families of systems and the conditions under which there exist control laws that match all the plants of these families with a certain class of systems. This class of systems represent the target systems that the agents in the network should be mapped to via state-feedback controller of the form $u = Fx + Gv$. The target plants are represented by the following set of transfer functions

$$T(\bar{A}, \bar{B}) = \{\Phi(sI - \bar{A})^{-1}\bar{B}), \quad \Phi \in \mathbb{R}^{n \times n}, |\Phi| \neq 0\} \tag{21}$$

In the remaining text, for simplicity reasons all the families of systems are referred to as classes including the target class represented by $T(\bar{A}, \bar{B})$.

A. Single-Input Case

The first class is defined for single-input controllable plants with fixed system size. Consider a network of N single-input state-feedback plants with system size equal to n and state-space representation as

$$\dot{x}_i = A_i x_i + b_i u_i, \quad y_i = x_i \tag{22}$$

where $i = 1, \ldots, N$, $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}^n$ with (A_i, b_i) being controllable. Consider now a target system of the same class as the plants in the network with transfer function $(sI - A_d)^{-1} b_d$ and its poles placed at a desired location specified by $p = (p_1, p_2, \cdots, p_n)'$. The state-space representation of this plant is given as

$$\dot{x}_d = A_d x_d + b_d u_d, \quad y_d = x_d \tag{23}$$

According to the following theorem there are always local state-feedback laws f_i with $i = \{1, \cdots, N\}$ that map all the plants in the network to the target plant provided that the latter is chosen from the same class.

Theorem 2. *Consider a set of N linear state-feedback plants with state-space*

$$\dot{x}_i = A_i x_i + b_i u_i$$

with $i = 1, \cdots, N$ arbitrarily chosen from the class of controllable systems with single input and system size n and let a target linear plant with state-space

$$\dot{x}_d = A_d x_d + b_d u_d$$

belong in the same class of systems. Then, there are always state-feedback laws f_i such that

$$(sI - A_i - b_i f_i^{'})^{-1} b_i = \Phi_i (sI - A_d)^{-1} b_d \tag{24}$$

with $|\Phi_i| \neq 0$ for $i = 1, \cdots, N$.

To prove Theorem 2 we consider similarity transformations T_i and T_d that bring the state-space form of the i^{th} plant and the target, respectively, into controller canonical form. Additionally, under the controllability assumption, there are always unique state-feedback gains f_i that assign the poles of the i^{th} system at the target's poles' location. Since, the controller canonical form of single-input plants of the same dimension is unique, the transfer function which maps the i^{th} plant to the target's transfer function up to a nonsingular output matrix Φ_i is given by the following relationship

$$(sI - A_i - b_i f_i^{'})^{-1} b_i = \Phi_i (sI - A_d)^{-1} b_d \tag{25}$$

with $\Phi_i = T_i^{-1} T_d$ and $|\Phi_i| \neq 0$ since the similarity transformations T_i and T_d are square and non-singular.

B. Multi-Input Case with common Controllability Indices μ_j

The class of systems to be considered next consists of multi-input state-feedback plants with fixed controllability indices μ_j. Recall that $\sum_{j=1}^{m} \mu_j = n$ where m stands for the number of inputs and n the system size. Note that the controllability indices define completely the class without need for specifying input size and system size.

The following lemma describes the completeness property [10] which together with the invariance property highlights that the controllability indices $\{\mu_i\}$ constitute a set of complete invariants for the pair (A, B) under operations P, G and F.

Lemma 1. *Given (A,B) controllable, then $(P(A+BF)P^{-1}, PBG)$ will have the same controllability indices, up to reordering, for any P, F and G ($det(P) \neq 0$, $det(G) \neq 0$) of appropriate dimensions.*

The controller canonical form of a multi-input plant is now analyzed. Let the pair (A, B) be controllable with controllability indices μ_j where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. There is always similarity transformation P (see [10] how to construct matrix P) such that the pair can be reduced to controller canonical form, namely, (A_c, B_c) where

$$A_c = \bar{A}_c + \bar{B}_c A_m, \quad B_c = \bar{B}_c B_m \tag{26}$$

with $A_m \in \mathbb{R}^{m \times n}$ and $B_m \in \mathbb{R}^{m \times m}$ being free. Notice that the pair ($\bar{A}_c$, $\bar{B}_c$) is called the Brunovsky canonical form [10] and is unique (up to reordering) for the class of pairs (A_i, B_i) with common controllability indices. The structure of the matrices ($\bar{A}_c$, $\bar{B}_c$) is given as follows

$$\bar{A}_c = diag(\bar{A}_{11}, \bar{A}_{22}, \cdots, \bar{A}_{mm}), \quad \bar{B}_c = diag\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\mu_j}, \right) \tag{27}$$

with $\bar{A}_{jj} = \begin{bmatrix} 0 & \\ \vdots & I_{\mu_j - 1} \\ 0 & \\ 0 & 0 \cdots 0 \end{bmatrix} \in \mathbb{R}^{\mu_j \times \mu_j}$ and $j = 1, \cdots, m$.

Notice that the pair ($\bar{A}_c$, $\bar{B}_c$) is completely determined by the set of the m controllability indices μ_j with $\sum_1^m \mu_j = n$.

Theorem 3. *Consider a set of N linear state-feedback plants with state-space*

$$\dot{x}_i = A_i x_i + B_i u_i$$

with $i = 1, \cdots, N$ arbitrarily chosen from the class of controllable systems with multiple inputs and controllability indices μ_j with $\sum_{j=1}^{m} \mu_j = n$ and let a target linear plant with state-space

$$\dot{x}_{N+1} = A_{N+1} x_{N+1} + B_{N+1} u_{N+1}$$

belong in the same class of systems. Then, there are always state-feedback F_i and input G_i transformations such that

$$(sI - A_i - B_i F_i)^{-1} B_i G_i = \Phi_i (sI - A_{N+1})^{-1} B_{N+1} \tag{28}$$

with $\Phi_i \neq 0$ for $i = 1, \cdots, N$.

The proof of Theorem 3 follows the same rationale with the single-input case under the assumption that the plants in the set share common controllability indices and as a result have identical Brunovsky forms. The N plants can match the target system up to the output map by applying state-feedback and input transformations $u_i = F_{i,N+1} x_i + G_{i,N+1} v_i$ with the corresponding matrices (F, G) being given as

$$F_{i,N+1} = B_{m_i}^{-1} (A_{mN+1} - A_{m1}) P_i, \quad G_{i,N+1} = B_{m_i}^{-1} B_{mN+1} \tag{29}$$

The transfer function which maps the i^{th} plant to the target's transfer function up to a nonsingular output matrix Φ_i is given by the following relationship

$$(sI - A_i - B_i F_{i,N+1})^{-1} B_i G_{i,N+1} = \Phi_i (sI - A_{N+1})^{-1} B_{N+1} \tag{30}$$

with $\Phi_i = P_i^{-1} P_{N+1}$ and $\Phi_i \neq 0$ since the similarity transformations P_i and P_{N+1} are square and non-singular. Detailed proof is omitted due to lack of space.

C. Multi-Input Case with arbitrary Controllability Indices μ_j

In this point, a model-matching type problem for multi-input plants with not necessarily identical controllability indices is considered using Lemma 2 below.

Lemma 2. *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ have full-column rank and the pair (A,B) be controllable. Then there always exists $\xi \in \mathbb{R}^m$ such that the pair $(A,B\xi)$ is also controllable.*

Consider a set of N multi-input state-feedback plants (A_i, B_i) with fixed system size n and arbitrary number of inputs. Assume all pairs (A_i, B_i) are controllable and let $\xi_i \in \mathbb{R}^{m_i}$ be appropriate vector according to Lemma 2 such that (A_i, $B_i\xi_i$) is also controllable with m_i being the input dimension of the i_{th} plant. All plants in the network should be mapped to a target system chosen from the same class with realization (A_d, b_d). The target pair (A_d, b_d) is assumed to be single-input and controllable with system size n. The target plant can always be transformed to single-input, by using Lemma 2 and finding appropriate ξ_d such that the pair (A_d, $b_d\xi_d$) is also controllable. The vector $p = (p_1, p_2, \cdots, p_n)'$ represents the poles of the target plant and also determines the pole placement of the N plants in the set.

According to Theorem 2 and (25) there always exist state-feedback gains f_i and output transformations T_i that map the transformed single-input plants to the single-input target system. The corresponding transfer function that solves the exact model-matching is given by

$$T_d^{-1}T_i(sI - A_i - B_i\xi_i f_i^{'})^{-1}B_i\xi_i = (sI - A_d)^{-1}b_d \tag{31}$$

where T_d and T_i are appropriate similarity transformations that bring the target system and the i^{th} plant in the network in controller canonical form, respectively, and f_i the state-feedback gains that assign the poles of each plant at the location of the target's poles.

D. Multi-Input - LMI approach

The model-matching type problem has been solved for multi-input plants with identical controllability indices and also for multi-input plants with arbitrary controllability indices. An LMI approach to the model-matching problem is also examined without taking into consideration the controllability indices. The systems considered here are controllable state-feedback plants (A_i, B_i) with system size n and number of inputs m. Additionally, matrices B_i should have the same image $Im(B_i)$ $\forall i$ and matrices A_i are assumed to be constructed by a fixed matrix and an almost free part. The exact structure of both A_i and B_i matrices are given as

$$A_i = C + B_i\Phi_i, \quad B_i = BG_i \tag{32}$$

where $C \in \mathbb{R}^{n\times n}$ is fixed, $\Phi_i \in \mathbb{R}^{m\times n}$ is arbitrary and $G_i \in \mathbb{R}^{m\times m}$ is arbitrary and nonsingular. The class is defined by controllable state-feedback plants (A_i, B_i) with structure given by (32).

Let N plants belong in such a class of systems and form a network with known topology. Local state-feedback gains and input transformations are sought to map all the plants in the network to a target system (A_d, B_d) of the same class. The following lemma guarantees the existence of input transformations that map the set of B_i to the target matrix B_d.

Lemma 3. *Consider a set of N matrices $B_i \in \mathbb{R}^{n\times m}$ which have full-column rank. Then there is a matrix $B \in \mathbb{R}^{n\times m}$ and square and nonsingular matrices $G_i \in \mathbb{R}^{m\times m}$ such that $B_iG_i = B$ if and only if $Im\{B_1\} = Im\{B_2\} = \cdots = Im\{B_N\} = Im\{B\}$.*

The matching problem is first solved for the case of two plants and then it is generalized to the case of N plants. Consider two state-feedback plants (A_1, B_1) and (A_2, B_2) and let the pairs (A_i, B_i) for $i = \{1,2\}$ be controllable with $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$. We seek K_1 and K_2 such that

$$A_1 + B_1K_1 = A_2 + B_2K_2 \tag{33}$$

B_1 and B_2 are assumed to satisfy the conditions of Lemma 3, i.e., there exists $G \in \mathbb{R}^{m\times m}$ square and nonsingular such that $B_2 = B_1G$.

Let $K_i = Y_iX_i^{-1}$ with $Y_i \in \mathbb{R}^{m\times n}$ and $X_i \in \mathbb{R}^{n\times n}$ symmetric positive definite (s.p.f.), for $i = 1,2$. Rewriting (33) results in

$$A_1 + B_1Y_1X_1^{-1} = A_2 + B_2Y_2X_2^{-1} \tag{34}$$

Assume that there exists (s.p.d) $X \in \mathbb{R}^{n\times n}$ such that (33) gives

$$A_1 + B_1Y_1X^{-1} = A_2 + B_2Y_2X^{-1} \tag{35}$$

which then is post-multiplied by X on both sides and results in

$$A_1X + B_1Y_1 = A_2X + B_2Y_2 \tag{36}$$

The following theorem which is stated without proof due to lack of space, defines the conditions under which state-feedback gains $K_i = Y_iX^{-1}$ exist for which the model-matching problem defined earlier can be solved.

Theorem 4. *Let N state-feedback plants be completely determined by controllable pairs (A_i, B_i) with $i = 1, \cdots, N$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ and*

$$Im(B_1) = Im(B_2) = \cdots = Im(B_N)$$

Then $\exists\, X \in \mathbb{R}^{n\times n}$, $X > 0$, $Y_i \in \mathbb{R}^{m\times n}$ and $Y_j \in \mathbb{R}^{m\times n}$ for $i, j = 1, \cdots, N$ such that:

$$A_iX + B_iY_i - A_jX - B_jY_j = 0 \quad \forall i, j \in \{1, \cdots, N\} \tag{37}$$

if and only if

$$A_i \in \{C + B\Phi : C \in \mathbb{R}^{n\times n}, \Phi \in \mathbb{R}^{m\times n}\}, \text{ and } Im(B) = Im(B_i) \tag{38}$$

for $i = 1, \cdots, N$.

1) State-feedback gain synthesis: The solution of the model matching problem with state-feedback gains $K_i = Y_i X^{-1}$ (common X) is equivalent to the equalities $A_i X + B_i Y_i = A_j X + B_j Y_j$ for $i, j = 1, \cdots, N$. For numerical reasons we relax these equalities to the approximate conditions

$$\|A_i X + B_i Y_i - (A_j X + B_j Y_j)\| < \gamma \tag{39}$$

for all $i, j \in \{1, \cdots, N\}$ where γ is a small tolerance. We will use the following well-known fact.

Lemma 4. *Let $\Phi \in \mathbb{R}^{n\times n}$ be arbitrary square matrix. The following are equivalent.*

$$\|\Phi\| < \gamma \Leftrightarrow \Phi^T \Phi < \gamma^2 I \Leftrightarrow \begin{bmatrix} I & \Phi \\ \Phi^T & \gamma^2 \end{bmatrix} > 0 \tag{40}$$

In this point, it should be noticed that the solution to the model-matching problem stated in this paragraph results in arbitrary target plant not specified a priori. In this respect, the poles of the target system may be forced to lie in a confined region of the complex plane setting bounds on certain performance criteria. Such regions which ensure a minimum decay rate, a maximum undamped natural frequency and a minimum damping ratio specified by parameters (a, ρ, θ), respectively, can be defined via Linear Matrix Inequalities [11] also known as LMI regions. The approximate model-matching problem with additional constraints the placement of the poles of the n closed-loop matrices in the convex region (a, ρ, θ) can be solved as follows:

Proposition 1. *Consider a set of N controllable state-feedback plants (A_i, B_i) with structure as in* (32) *and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$. If there exist $X \in \mathbb{R}^{n\times n}$, $Y_i \in \mathbb{R}^{m\times n}$ and $Y_j \in \mathbb{R}^{m\times n}$, then the approximate model-matching problem* (39) *with additional pole placement constraints [11] can be solved if and only if the following LMI constraints*

$$X = X^T > 0 \tag{41}$$

$$\begin{bmatrix} I & A_i X + B_i Y_i - A_j X - B_j Y_j \\ * & \gamma^2 I \end{bmatrix} \geq 0 \tag{42}$$

for $i, j \in \{1, \cdots, N\}$ and $i > j$

$$2aX + \Lambda_i + \Lambda_i^T < 0,\ i \in \{1, \cdots, N\} \tag{43}$$

$$\begin{bmatrix} -rX & \Lambda_i^T \\ * & rX \end{bmatrix} < 0,\ i \in \{1, \cdots, N\} \tag{44}$$

$$\begin{bmatrix} sin\theta[\Lambda_i + \Lambda_i^T] & cos\theta[-\Lambda_i + \Lambda^T] \\ * & sin\theta[\Lambda_i + \Lambda_i^T] \end{bmatrix} < 0,\ i \in \{1, \cdots, N\} \tag{45}$$

are feasible, where $\Lambda_i = A_i X + B_i Y_i$

IV. Distributed Control Design For Networks Of Non-Identical Plants

In this section stabilizing distributed controllers

$$u_i = F_i x_i + G_i v_i \tag{46}$$

for networks of non-identical plants are constructed. The plants are assumed to be chosen from the class of linear systems with identical controllability indices μ_j with $\sum_{j=1}^m \mu_j = n$ where n and m represent the systems' dimension and the number of inputs, respectively. The state-space form of each agent is given by

$$\dot{x}_i = A_i x_i + B_i u_i, \quad x_{i0} = x_i(0) \tag{47}$$

According to Theorem 3 there exist F_i, G_i and Φ_i of appropriate dimensions for all $i = 1, \cdots, N$ such that

$$(sI - A_i - B_i F_i)^{-1} B_i G_i = \Phi_i (sI - A_{N+1})^{-1} B_{N+1} \tag{48}$$

where $\Phi_i \neq 0$ for $i = 1, \cdots, N$ and the pair (A_{N+1}, B_{N+1}) represents target plant common for all agents in the network. We write (48) in state-space form excluding the output map.

$$\dot{x}_i = (A_i + B_i F_i) x_i + B_i G_i v_i, \quad \dot{\xi} = A_{N+1}\xi + B_{N+1} v_i \tag{49}$$

Eq. (48) and (49) imply that the closed-loop states x_i of all agents scaled by Φ_i^{-1} map to the same state-vector ξ, i.e. $\Phi_i^{-1} x_i = \xi$. Thus, applying top-down or bottom-up distributed LQR methods to a network formed by N identical plants (A_{N+1}, B_{N+1}) results in stabilizing distributed state-feedback gain $\hat{K}$ which also stabilizes the closed-loop agents if post-multiplied by non-singular matrix $diag\{\Phi_1^{-1}, \cdots, \Phi_N^{-1}\}$

Theorem 5. *Consider a network of N non-identical agents with state-space given by* (47) *and topology specified by Laplacian matrix L with maximum vertex degree d_{max}. The agents share the same controllability indices and therefore according to Theorem 3 there exist F_i, G_i and Φ_i such that* (48) *holds for all $i = 1, \cdots, N$. Consider reduced-order networked LQR problem* (11) *with $N_L = d_{max} + 1$ identical plants defined by*

$$(A_{N_L+1}, B_{N_L+1}) \triangleq (\Phi_i^{-1}(A_i + B_i F_i)\Phi_i, \Phi_i^{-1} B_i G_i)$$

Let $M = \mathbb{R}^{NxN}$ be a symmetric matrix with the following property:

$$\lambda_i(M) > \frac{N_L}{2}, \quad \forall \lambda_i(M) \in S(M)\backslash\{0\} \tag{50}$$

and construct the state-feedback gain as in Theorem 1:

$$\hat{K} = I_N \otimes K_1 + M \otimes K_2 \tag{51}$$

Let N_i represent the interconnections of the i^{th} agent. Then the state-space equation

$$\dot{x}_i = [A_i + B_i(F_i + G_i K_1 \Phi_i^{-1})] x_i + B_i G_i \sum_{j\in N_i} K_2 \Phi_j^{-1} x_j \tag{52}$$

is asymptotically stable for all $i = 1, \cdots, N$.

Theorem 6. *Consider a network of N non-identical agents with state-space given by* (47) *and topology specified by Laplacian matrix L. The agents share the same controllability indices and therefore according to Theorem 3 there exist F_i, G_i and Φ_i such that* (48) *holds for all $i = 1, \cdots, N$.*

Consider state-feedback gain $K = -R^{-1}B_{N+1}^{T}P$ where P is the stabilizing solution to

$$A_{N+1}^{T}P + PA_{N+1} - PB_{N+1}R^{-1}B_{N+1}^{T}P + Q_1 = 0 \tag{53}$$

Consider fictitious network composed of N plants with realization (A_{N+1}, B_{N+1}) and same topology as the original one. Apply bottom-up method to the fictitious network and find scaling matrix Ξ such that the distributed state-feedback gain

$$\bar{K} = I_N \otimes K + L \otimes \Xi K \tag{54}$$

minimizes an upper bound of the corresponding performance index (16) *and the closed-loop matrix*

$$I_N \otimes (A_{N+1} + B_{N+1}K) + L \otimes (B_{N+1}\Xi K) \tag{55}$$

is Hurwitz. Let N_i represent the interconnections of the i^{th} agent in the original network. Then the state-space equation

$$\dot{x}_i = [A_i + B_i(F_i + G_iK\Phi_i^{-1})]x_i + B_iG_i \sum_{j \in N_i} \Xi K \Phi_j^{-1} x_j \tag{56}$$

is asymptotically stable for all $i = 1, \cdots, N$.

The proofs are omitted due to lack of space. It should be noted that Theorems 5 and 6 can be applied to all classes of systems constituting networks of non-identical plants examined in the previous section. It should also be stated that the choice of target plants plays a key role in the synthesis of the distributed scheme, since matrices Φ_i possibly selected nearly singular might result in problematic performance in cases where perturbations and inaccuracies of the models occur.

A. Example

To demonstrate the applicability of our approach, top-down method is used to solve a cooperative stabilization problem of six linear non-identical agents constituting a network and having the ability to exchange information about their states. The interactions among the agents are limited to the neighborhood of its agent where the sparsity of the communication topology is shown in Fig. 1. and the network is described by an undericted graph $\mathscr{G}$.

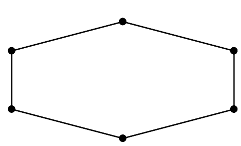

Fig. 1. Communication topology of the six agents.

Each node is modelled as a two-mass-spring system with a single input force. Two masses $m_{i,1}$ and $m_{i,2}$ are connected through a spring with spring constant $k_{i,2}$ while the mass $m_{i,1}$ is attached to a rigid wall through a spring with spring constant $k_{i,1}$. A force input u_i is applied on $m_{i,1}$ and $x_{i,1}$ and $x_{i,2}$ are the displacements of the two masses. The state-space form of each two-mass-spring system is given by

$$\begin{bmatrix} \dot{x}_{i,1} \\ \ddot{x}_{i,1} \\ \dot{x}_{i,2} \\ \ddot{x}_{i,2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_{i,1}-k_{i,2}}{m_{i,1}} & 0 & \frac{k_{i,2}}{m_{i,1}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_{i,2}}{m_{i,2}} & 0 & \frac{-k_{i,2}}{m_{i,2}} & 0 \end{bmatrix}}_{A_i} \begin{bmatrix} x_{i,1} \\ \dot{x}_{i,1} \\ x_{i,2} \\ \dot{x}_{i,2} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m_{i,1}} \\ 0 \\ 0 \end{bmatrix}}_{B_i} u_i \tag{57}$$

where the state vector is assumed to be accessible for measurement. The augmented state-space of the network is given here for simplicity

$$\begin{aligned} \dot{\tilde{x}} &= diag\{A_1, \cdots, A_6\}\tilde{x} + diag\{B_1, \cdots, B_6\}\tilde{u} \\ \tilde{x}_0 &= [x_1^T(0), \cdots, x_6^T(0)]^T \end{aligned} \tag{58}$$

The control $\tilde{u}$ objective is to stabilize each two-mass-spring subsystem cooperatively moving all the masses to zero displacement position. The first step is to specify a target system that all the agents in the network will map to via local state-feedback. The state-space form of the target system share the same structure with the agents and is given as in (57). The masses and the spring constants selected for simulations of each agent and the target are shown in Table I where the target's constants are chosen as an average of the corresponding values of the six nodes.

TABLE I
MASSES AND SPRING CONSTANTS

System	k$_{i,1}$	k$_{i,2}$	km$_{i,1}$	m$_{i,2}$
agent 1	1.50 N/m	1.00 N/m	1.10 kg	0.90 kg
agent 2	3.10 N/m	2.00 N/m	2.10 kg	1.50 kg
agent 3	0.50 N/m	1.10 N/m	1.50 kg	3.20 kg
agent 4	2.00 N/m	1.30 N/m	3.10 kg	2.10 kg
agent 5	1.70 N/m	3.10 N/m	4.10 kg	2.50 kg
agent 6	2.20 N/m	4.20 N/m	5.10 kg	4.20 kg
target $N+1$	1.83 N/m	2.12 N/m	2.83 kg	2.40 kg

Since the target system has been defined, according to Theorem 2 and the controllability of the pairs (A_i, B_i) there exist local state-feedback gains F_i and similarity transformations T_i and T_{N+1} such that (24) holds. Matrices T_i and T_{N+1} are similarity transformations that bring the state-space form of the i^{th} node and the target system, respectively, into controller canonical form while the state-feedback gains F_i place the poles of all agents at the target's location. Since the plants in the network are single-input systems F_i and T_i are unique and may be obtained from Matlab functions *place* and *canon*, respectively.

Let $\Phi_i = T_i^{-1}T_{N+1}$ for $i = 1, \cdots, 6$ then it can be proved that

$$\Phi_i^{-1}(sI - A_i - B_iF_i)^{-1}b_i = (sI - A_{N+1})^{-1}B_{N+1} \tag{59}$$

where the controllable pair (A_{N+1}, B_{N+1}) represents the target system. A stabilizing distributed controller is designed by using Theorem 5 and selecting the design matrix M to be equal to the Laplacian matrix L of the graph $\mathscr{G}$. Full-centralized networked LQR problem (11) is solved for $N_L = 2+1$ identical plants (A_{N+1}, B_{N+1}) and yields a controller matrix with structure as in (12). The stabilizing distributed controller for the network of six different two-mass-spring systems has the same structure as the corresponding graph and is constructed as follows

$$\hat{K} = diag\{F_1, \cdots, F_6\} + (I_6 \otimes K_1 + L \otimes K_2) diag\{\Phi_1^{-1}, \cdots, \Phi_6^{-1}\} \tag{60}$$

Two cases with same initial conditions are presented for different choices on the penalty imposed on the agents' states. Fig. 2 and 4 show the displacements of the two masses of each two-mass-spring system where low penalty has been put on the interaction of neighbouring states of the agents while in Fig. 3 and 5 the emphasis has been shifted on the state-information exchange.

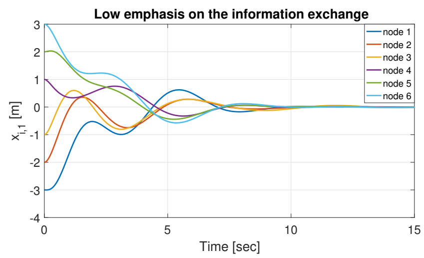

Fig. 2. Displacement of $m_{i,1}$ of six agents - low emphasis on the information exchange.

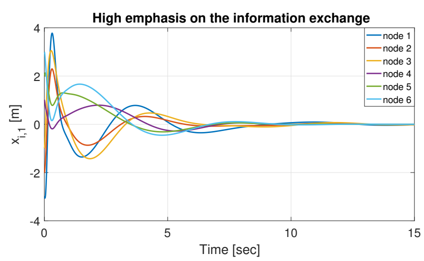

Fig. 3. Displacement of $m_{i,1}$ of six agents - high penalty on the information exchange.

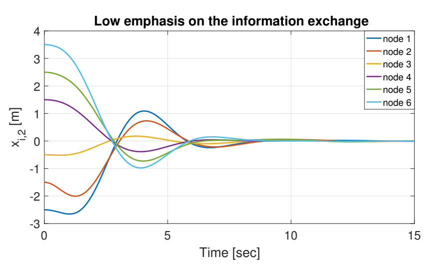

Fig. 4. Displacement of $m_{i,2}$ of six agents - low emphasis on the information exchange.

V. Conclusion

We have introduced a technique to solve stability problems for networks formed by non-identical linear plants. The first stage of the method solves model-matching problems and defines the synthesis of local state-feedback controllers which match all the plants in the network with a target

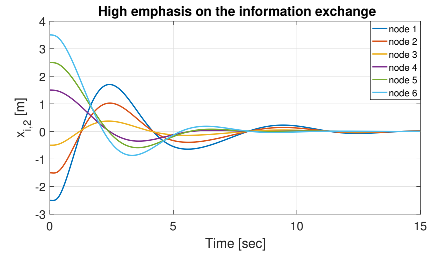

Fig. 5. Displacement of $m_{i,2}$ of six agents - high penalty on the information exchange.

plant. It was shown that the structure of the systems play a key role and thus various types of linear systems were investigated with the aim at finding the conditions under which the existence of such schemes is guaranteed. The model-matching methods proposed in this paper transform the original network to a fictitious one composed of identical closed-loop systems up to an arbitrary nonsingular output matrix. Depending on this fact, it was shown that existed distributed schemes proposed for networks of identical plants can be converted to solve stability problems of networks formed of non-identical plants. Further work is needed, however, to extend the method to be applied to a more generic class of systems and can therefore be implemented successfully in practical applications.

References

[1] G. A. Jacoby and D. J. Chang, "Towards command and control networking of cooperative autonomous robotics for military applications (CARMA)," *Can. Conf. Electr. Comput. Eng.*, pp. 815–820, 2008.

[2] M. Schwaninger and P. Vrhovec, "Supply system dynamics: Distributed control in supply chains and networks," *Cybern. Syst.*, vol. 37, no. 5, pp. 375–415, 2006.

[3] I. Tomic, E. Milonidis, and G. D. Halikias, "LQR distributed cooperative control of a formation of low-speed experimental UAVs," in *2016 UKACC Int. Conf. Control. UKACC Control 2016*, 2016.

[4] A. Bidram, F. L. Lewis, A. Davoudi, and Z. Qu, "Frequency control of electric power microgrids using distributed cooperative control of multi-agent systems," *2013 IEEE Int. Conf. Cyber Technol. Autom. Control Intell. Syst.*, pp. 223–228, 2013.

[5] M. Andreasson, H. Sandberg, D. V. Dimarogonas, and K. H. Johansson, "Distributed integral action: Stability analysis and frequency control of power systems," in *2012 IEEE 51st IEEE Conf. Decis. Control*, 2012, pp. 2077–2083. [Online]. Available: http://ieeexplore.ieee.org/document/6426463/

[6] R. Olfati-Saber, "Flocking for Multi-Agent Dynamic Systems:Algorithms and Theory," *IEEE Trans. Automat. Contr.*, vol. 51, no. 3, pp. 1–20, 2006.

[7] J. A. Fax and R. M. Murray, "Information Flow and Cooperative Control of Vehicle Formations," *IFAC Proc. Vol.*, vol. 35, no. 1, pp. 115–120, 2002.

[8] F. Borrelli and T. Keviczky, "Distributed LQR design for identical dynamically decoupled systems," *IEEE Trans. Automat. Contr.*, vol. 53, no. 8, pp. 1901–1912, 2008.

[9] P. Deshpande, P. Menon, C. Edwards, and I. Postlethwaite, "Sub-optimal distributed control law with H_{2} performance for identical dynamically coupled linear systems," *IET Control Theory Appl.*, vol. 6, no. 16, pp. 2509–2517, 2012.

[10] P. J. Antsaklis and M. A. N., *Linear Systems*. 233 Spring Street, New York, NY 10013, USA: Birkhauser Boston, 2005.

[11] M. Chilali and P. Gahinet, "H_∞ Design with Pole Place Constraints : An LMI Approach," vol. 41, no. 3, pp. 358–367, 1996.