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Research Article

Adams Predictor-Corrector Systems for Solving Fuzzy Differential Equations

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A predictor-corrector algorithm and an improved predictor-corrector (IPC) algorithm based on Adams method are proposed to solve first-order differential equations with fuzzy initial condition. These algorithms are generated by updating the Adams predictor-corrector method and their convergence is also analyzed. Finally, the proposed methods are illustrated by solving an example.

1. Introduction

Fuzzy differential equations (FDEs), which are utilized for the purpose of the modeling problems in science and engineering, have been studied by many researchers. Most of the practical problems require the solutions of fuzzy differential equations (FDEs) which are satisfied with fuzzy initial conditions; therefore a fuzzy initial problem occurs and should be solved. However, for the vast majority of fuzzy initial value problems, their exact solutions are difficult to be obtained. Thus it is necessary to consider their numerical methods.

The concept of a fuzzy derivative was first introduced by Chang and Zadeh [1]; it was followed up by Dubois and Prade [2] who used the extension principle in their approach. Other fuzzy derivative concepts have been proposed by Puri and Ralescu [3] and Goetschel Jr. and Voxman [4] as an extension of the Hukuhara derivative of multivalued functions. In the past decades, many works have been appeared on the aspects of theories and applications on fuzzy differential equations; see [5–12]. The notation of fuzzy differential equation was initially introduced by Kandel and Byatt [13, 14] and later they applied the concept of fuzzy differential equation to the analysis of fuzzy dynamical problems [7, 15]. A thorough theoretical research of fuzzy Cauchy problems was given by Kaleva [16, 17], Wu and Song [18], Ouyang and Wu [19], Kim and Sakthivel [20], and M. D. Wu [21]. A generalization of fuzzy differential equation was given by Aubin [22, 23], Baĭdosov [6], Kloeden [24], and Colombo and Křivan [25].

For a fuzzy Cauchy problem

$$\begin{aligned} y'(t) &= f(t, y), \quad t_0 \le t \le T, \\ y(t_0) &= \alpha_0, \end{aligned} \tag{1}$$

in 1999, Friedman et al. [26] firstly treated it and obtained its numerical solution by Euler method. In recent years, some researchers such as Abbasbandy and Allahviranloo applied the Taylor series method, the Runge-Kutta method, and the linear multistep method to solve fuzzy differential equations [5, 20, 27–32]. They proposed some numerical methods and discussed the convergence and stability of their methods under the fuzzy numbers background. However, their methods always have some of low convergence order.

In this paper, based on Adams-Bashforth four-step method and Adams-Moulton three-step method, two Adams predictor-corrector algorithms are proposed to solve fuzzy initial problems. The convergence of the proposed methods is also presented in detail. Finally, an example is given to illustrate our methods. The structure of this paper is organized as follows.

In Section 2, some basic definitions and results are recalled. An explicit Adams-Bashforth method and an implicit Adams-Moulton method for solving FDEs are mentioned in Section 3. The predictor-corrector method and the improved predictor-corrector (IPC) systems algorithm are introduced in Section 4. The convergence of the proposed methods is discussed in Section 5. An illustrating example is given in Section 6 and the conclusion is drawn in Section 7.

2. Preliminaries

2.1. Fuzzy Numbers

Definition 1 (see [1]). A fuzzy number is a fuzzy set like $u : R \to I = [0, 1]$ which satisfies the following:

(1) u is upper semicontinuous;

(2) u is fuzzy convex; that is, $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$, for all $x, y \in R, \lambda \in [0, 1]$;

(3) u is normal; that is, there exists $x_0 \in R$ such that $u(x_0) = 1$;

(4) $\operatorname{supp} u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure $\operatorname{cl}(\operatorname{supp} u)$ is compact.

Let E^1 be the set of all fuzzy numbers on R.

Definition 2 (see [2]). A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfies the following requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function;

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function;

(3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Let I be a real interval. A mapping $y : I \to E^1$ is called a fuzzy process and its α-level set is denoted by

$$[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)], \quad t \in I, \ \alpha \in (0, 1]. \tag{2}$$

Definition 3. A triangular fuzzy number is a fuzzy set u in E^1 that is characterized by an ordered triple $(x_l, x_c, x_r) \in R^3$ with $x_l \leq x_c \leq x_r$ such that $[u]^0 = [x_l, x_r]$ and $[u]^1 = \{x_c\}$.

The α-level set of a triangular fuzzy number u is given by

$$[u]^\alpha = [x_c - (1 - \alpha)(x_c - x_l), x_c + (1 - \alpha)(x_r - x_c)], \tag{3}$$

for any $\alpha \in I$.

Definition 4 (see [9]). The supremum metric d_∞ on E^1 is defined by

$$d_\infty(u, v) = \sup\{d_H\{[u]^\alpha, [v]^\alpha\} : \alpha \in I\}. \tag{4}$$

With the spermium metric, the space (E^1, d_∞) is a complete metric space.

Definition 5 (see [9]). A mapping $F : T \to E^1$ is Hukuhara differentiable at $t_0 \in T \subset R$ if for some $h_0 > 0$ the Hukuhara difference

$$F(t_0 + \Delta t) \sim_h F(t_0), \quad F(t_0) \sim_h F(t_0 - \Delta t) \tag{5}$$

exists in E^1, for all $0 < \Delta t < h_0$, and if there exists an $F'(t_0) \in E^1$ such that

$$\begin{gathered} \lim_{h \to 0+} d_\infty\left(F(t_0 + \Delta t) \sim_h \frac{F(t_0)}{\Delta t}, F'(t_0)\right) = 0, \\ \lim_{h \to 0+} d_\infty\left(F(t_0) \sim_h \frac{F(t_0 - \Delta t)}{\Delta t}, F'(t_0)\right) = 0. \end{gathered} \tag{6}$$

The fuzzy set $F'(t_0)$ is called the Hukuhara derivative of F at t_0.

Recall that $u \sim_h v = w \in E^1$ is defined on α-level sets, where $[u]^\alpha \sim_h [v]^\alpha = [w]^\alpha$, for all $\alpha \in I$. By the definition of the metric d_∞, all the α-level set mappings $[F(\cdot)]^\alpha$ are Hukuhara differentiable at t_0 with Hukuhara derivative $[F'(t_0)]^\alpha$ for each $\alpha \in I$ when $F : T \to E^1$ is Hukuhara differentiable at t_0 with Hukuhara derivative $F'(t_0)$.

Remark 6. If $F : T \to E^1$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over $[0, 1]$, then

$$F(t) = F(t_0) + \int_{t_0}^{t} F'(s)\, ds, \tag{7}$$

for all $0 \leq t_0 \leq t \leq 1$.

Definition 7. A mapping $y : I \to E^1$ is called a fuzzy process. We designate

$$[y(t)]^\alpha = [y_1^\alpha(t), y_2^\alpha(t)], \quad t \in I, \ 0 \leq \alpha \leq 1. \tag{8}$$

The Seikkala derivative $y'(t)$ of a fuzzy process y is defined by

$$[y'(t)]^\alpha = [(y_1^\alpha)'(t), (y_2^\alpha)'(t)], \quad 0 \leq \alpha \leq 1, \tag{9}$$

provided that this equation in fact defines a fuzzy number $y'(t) \in E^1$.

Remark 8. If $y : I \to E^1$ is Seikkala differentiable and its Seikkala derivative y' is integrable over $[0, 1]$, then

$$y(t) = y(t_0) + \int_{t_0}^{t} y'(s)\, ds, \tag{10}$$

for all $t_0, t \in I$.

2.2. A Fuzzy Cauchy Problem. Consider the first-order fuzzy differential equation $y' = f(t, y)$, where y is a fuzzy function of t, $f(t, y)$ is a a fuzzy function of the crisp variable t and the fuzzy variable y, and y' is the Hukuhara or Seikkala fuzzy derivative of y. Given an initial value $y(t_0) = \alpha_0$, we can define a first-order fuzzy Cauchy problem as follows:

$$\begin{gathered} y'(t) = f(t, y), \quad t_0 \leq t \leq T, \\ y(t_0) = y_0. \end{gathered} \tag{11}$$

The existence theorem is obtained for the Cauchy problem (11).

Let $[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)]$; if $y(t)$ is Hukuhara differentiable then $[y'(t)]^\alpha = [(\underline{y}^\alpha(t))', (\overline{y}^\alpha(t))']$. So (11) translates into the following system of ODEs:

$$\begin{aligned}(\underline{y}^\alpha(t))' &= \underline{f}^\alpha(t, \underline{y}^\alpha(t), \overline{y}^\alpha(t)),\\ (\overline{y}^\alpha(t))' &= \overline{f}^\alpha(t, \underline{y}^\alpha(t), \overline{y}^\alpha(t)),\\ \underline{y}^\alpha(t_0) &= \underline{y}_0^\alpha,\\ \overline{y}^\alpha(t_0) &= \overline{y}_0^\alpha.\end{aligned} \tag{12}$$

Theorem 9 (see [33]). *Let one consider the FCP* (11) *where* $f : [t_0, T] \times E^1 \to E^1$ *is such that*

(i) $[f(t, y)]^\alpha = [\underline{f}^\alpha(t, \underline{y}^\alpha(t), \overline{y}^\alpha(t)), \overline{f}^\alpha(t, \underline{y}^\alpha(t), \overline{y}^\alpha(t))]$;

(ii) $\underline{f}^\alpha$ *and* $\overline{f}^\alpha$ *are equicontinuous and uniformly bounded on any bounded set;*

(iii) $\underline{f}^\alpha$ *and* $\overline{f}^\alpha$ *satisfy the Lipschitz conditions.*

Then the FCP (11) *and the system of ODEs* (12) *are equivalent.*

2.3. Interpolation for Fuzzy Numbers. The problem of interpolation for fuzzy sets is as follows.

Suppose that at various time instant t information $f(t)$ is presented as fuzzy set. The aim is to approximate the function $f(t)$, for all t in the domain of f. Let $t_0 < t_1 < \cdots < t_n$ be $n+1$ distinct points in R and let $u_0, u_1, \ldots, u_n$ be $n+1$ fuzzy sets in E^1. A fuzzy polynomial interpolation of the data is a fuzzy-value function $F : R \to E^1$ satisfying the following conditions:

(1) $f(t_i) = u_i$, for all $i = 0, 1, \ldots, n$;

(2) f is continuous;

(3) if the data is crisp, then the interpolation f is a crisp polynomial.

A function f fulfilling these conditions may be constructed as follows. Let $C_\alpha^i = [u_i]^\alpha$, for any $\alpha \in [0, 1]$, $i = 0, 1, \ldots, n$. For each $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$, the unique polynomial of degree $\le n$ is denoted by P_X such that

$$P_X(t_i) = x_i, \quad i = 0, 1, \ldots, n,$$
$$P_X(t) = \sum_{i=0}^{n} x_i \left(\prod_{i \ne j} \frac{t - t_j}{t_i - t_j} \right). \tag{13}$$

Finally, for all $t, \xi \in R$ are defined by $f(t) \in E^1$ such that

$$f(t)(\xi) = \sup\{\alpha \in [0, 1] : \exists X \in C_\alpha^0 \times C_\alpha^1 \times \cdots \times C_\alpha^n, P_X(t) = \xi\}. \tag{14}$$

The interpolation polynomial can be written level setwise as

$$[f(t)]^\alpha = \{y \in R : y = P_X(t), x \in C_\alpha^i, i = 0, 1, \ldots, n\}, \tag{15}$$

for $0 \le \alpha \le 1$.

Theorem 10 (see [34]). *Let* (t_i, u_i), $i = 0, 1, \ldots, n$ *be the observed data and suppose that each of the* $u_i = (u_i^l, u_i^c, u_i^r)$ *is an element of* E^1. *Then for each* $t \in [t_0, t_n]$, $f(t) = (f^l(t), f^c(t), f^r(t)) \in E^1$,

$$\begin{aligned} f^l(t) &= \sum_{l_i(t)\ge 0} l_i(t) u_i^l + \sum_{l_i(t)<0} l_i(t) u_i^r,\\ f^c(t) &= \sum_{i=0}^{n} l_i(t) u_i^c,\\ f^r(t) &= \sum_{l_i(t)\ge 0} l_i(t) u_i^r + \sum_{l_i(t)<0} l_i(t) u_i^l,\end{aligned} \tag{16}$$

where $l_i(t) = \prod_{i \ne j}((t - t_j)/(t_i - t_j))$, $i = 0, 1, \ldots, n$.

3. Adams Method

3.1. Adams-Bashforth Method. Now we are going to solve fuzzy initial problem $y'(t) = f(t, y)$ by Adams-Bashforth four-step method. Let the fuzzy initial values be $\tilde{y}(t_{i-3})$, $\tilde{y}(t_{i-2})$, $\tilde{y}(t_{i-1})$, $\tilde{y}(t_i)$, that is,

$$\begin{aligned}&\tilde{f}(t_{i-3}, y(t_{i-3})), \quad \tilde{f}(t_{i-2}, y(t_{i-2})),\\ &\tilde{f}(t_{i-1}, y(t_{i-1})), \quad \tilde{f}(t_i, y(t_i)),\end{aligned} \tag{17}$$

which are triangular fuzzy numbers and are shown by

$$\begin{aligned}&\{f^l(t_{i-3}, y(t_{i-3})), f^c(t_{i-3}, y(t_{i-3})), f^r(t_{i-3}, y(t_{i-3}))\},\\ &\{f^l(t_{i-2}, y(t_{i-2})), f^c(t_{i-2}, y(t_{i-2})), f^r(t_{i-2}, y(t_{i-2}))\},\\ &\{f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))\},\\ &\{f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))\},\end{aligned} \tag{18}$$

also

$$\tilde{y}(t_{i+1}) = \tilde{y}(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\, dt. \tag{19}$$

By fuzzy interpolation of $\tilde{f}(t_{i-3}, y(t_{i-3}))$, $\tilde{f}(t_{i-2}, y(t_{i-2}))$, $\tilde{f}(t_{i-1}, y(t_{i-1}))$, $\tilde{f}(t_i, y(t_i))$, we have

$$\begin{aligned} f^l(t, y(t)) &= \sum_{l_j(t)\ge 0} l_j(t) f^l(t_j, y(t_j)) + \sum_{l_j(t)<0} l_j(t) f^r(t_j, y(t_j)),\\ f^c(t) &= \sum_{j=i-3}^{i} l_j(t) f^c(t_j, y(t_j)),\\ f^r(t, y(t)) &= \sum_{l_j(t)\ge 0} l_j(t) f^r(t_j, y(t_j)) + \sum_{l_j(t)<0} l_j(t) f^l(t_j, y(t_j)),\end{aligned} \tag{20}$$

for $t_{i-1} \le t \le t_i$,

$$\begin{aligned} l_{i-3}(t) &= \frac{(t-t_{i-2})(t-t_{i-1})(t-t_i)}{(t_{i-3}-t_{i-1})(t_{i-3}-t_{i-1})(t_{i-3}-t_i)} \ge 0, \\ l_{i-2}(t) &= \frac{(t-t_{i-3})(t-t_{i-1})(t-t_i)}{(t_{i-2}-t_{i-3})(t_{i-2}-t_{i-1})(t_{i-2}-t_i)} \le 0, \\ l_{i-1}(t) &= \frac{(t-t_{i-3})(t-t_{i-2})(t-t_i)}{(t_{i-1}-t_{i-3})(t_{i-1}-t_{i-2})(t_{i-1}-t_i)} \ge 0, \\ l_i(t) &= \frac{(t-t_{i-3})(t-t_{i-2})(t-t_{i-1})}{(t_i-t_{i-3})(t_i-t_{i-2})(t_i-t_{i-1})} \ge 0, \end{aligned} \tag{21}$$

therefore the following results will be obtained:

$$\begin{aligned} f^l(t,y(t)) &= l_{i-3}(t) f^l(t_{i-3},y(t_{i-3})) \\ &\quad + l_{i-2}(t) f^r(t_{i-2},y(t_{i-2})) \\ &\quad + l_{i-1}(t) f^l(t_{i-1},y(t_{i-1})) \\ &\quad + l_i(t) f^l(t_i,y(t_i)), \\ f^c(t,y(t)) &= l_{i-3}(t) f^c(t_{i-3},y(t_{i-3})) \\ &\quad + l_{i-2}(t) f^c(t_{i-2},y(t_{i-2})) \\ &\quad + l_{i-1}(t) f^c(t_{i-1},y(t_{i-1})) \\ &\quad + l_i(t) f^c(t_i,y(t_i)), \\ f^r(t,y(t)) &= l_{i-3}(t) f^r(t_{i-3},y(t_{i-3})) \\ &\quad + l_{i-2}(t) f^l(t_{i-2},y(t_{i-2})) \\ &\quad + l_{i-1}(t) f^r(t_{i-1},y(t_{i-1})) \\ &\quad + l_i(t) f^r(t_i,y(t_i)). \end{aligned} \tag{22}$$

From (3) and (19) it follows that

$$\tilde{y}^\alpha(t_{i+1}) = \left[\underline{y}^\alpha(t_{i+1}), \overline{y}^\alpha(t_{i+1})\right], \tag{23}$$

where

$$\begin{aligned} &\underline{y}^\alpha(t_{i+1}) \\ &\quad = \underline{y}^\alpha(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha f^c(t,y(t)) \\ &\qquad + (1-\alpha) f^l(t,y(t))\} \, dt, \\ &\overline{y}^\alpha(t_{i+1}) \\ &\quad = \overline{y}^\alpha(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha f^c(t,y(t)) \\ &\qquad + (1-\alpha) f^r(t,y(t))\} \, dt. \end{aligned} \tag{24}$$

If (22) are situated in (24), we have

$$\begin{aligned} \underline{y}^\alpha(t_{i+1}) &= \underline{y}^\alpha(t_i) \\ &\quad + \int_{t_i}^{t_{i+1}} \{\alpha (l_{i-3}(t) f^c(t_{i-3},y(t_{i-3})) \\ &\qquad + l_{i-2}(t) f^c(t_{i-2},y(t_{i-2})) \\ &\qquad + l_{i-1}(t) f^c(t_{i-1},y(t_{i-1})) \\ &\qquad + l_i(t) f^c(t_i,y(t_i))) \\ &\qquad + (1-\alpha)(l_{i-3}(t) f^l(t_{i-3},y(t_{i-3})) \\ &\qquad + l_{i-2}(t) f^r(t_{i-2},y(t_{i-2})) \\ &\qquad + l_{i-1}(t) f^l(t_{i-1},y(t_{i-1})) \\ &\qquad + l_i(t) f^l(t_i,y(t_i)))\} \, dt, \\ \overline{y}^\alpha(t_{i+1}) &= \overline{y}^\alpha(t_i) \\ &\quad + \int_{t_i}^{t_{i+1}} \{\alpha (l_{i-3}(t) f^c(t_{i-3},y(t_{i-3})) \\ &\qquad + l_{i-2}(t) f^c(t_{i-2},y(t_{i-2})) \\ &\qquad + l_{i-1}(t) f^c(t_{i-1},y(t_{i-1})) \\ &\qquad + l_i(t) f^c(t_i,y(t_i))) \\ &\qquad + (1-\alpha)(l_{i-3}(t) f^r(t_{i-3},y(t_{i-3})) \\ &\qquad + l_{i-2}(t) f^l(t_{i-2},y(t_{i-2})) \\ &\qquad + l_{i-1}(t) f^r(t_{i-1},y(t_{i-1})) \\ &\qquad + l_i(t) f^r(t_i,y(t_i)))\} \, dt. \end{aligned} \tag{25}$$

The following results will be obtained by integration:

$$\begin{aligned} \underline{y}^\alpha(t_{i+1}) &= \underline{y}^\alpha(t_i) + \frac{h}{24} \\ &\quad \times \{-9[\alpha f^c(t_{i-3},y(t_{i-3})) \\ &\qquad + (1-\alpha) f^r(t_{i-3},y(t_{i-3}))] \\ &\quad + 37[\alpha f^c(t_{i-2},y(t_{i-2})) \\ &\qquad + (1-\alpha) f^l(t_{i-2},y(t_{i-2}))] \end{aligned}$$

$$
\begin{aligned}
&- 59 \left[\alpha f^{c} \left(t_{i-1}, y \left(t_{i-1}\right)\right) \right.\\
&\qquad \left. + (1-\alpha) f^{r} \left(t_{i-1}, y \left(t_{i-1}\right)\right)\right]\\
&+ 55 \left[\alpha f^{c} \left(t_{i}, y \left(t_{i}\right)\right)\right.\\
&\qquad \left.\left. + (1-\alpha) f^{l} \left(t_{i}, y \left(t_{i}\right)\right)\right]\right\},\\
\overline{y}^{\alpha} \left(t_{i+1}\right) = \overline{y}^{\alpha} \left(t_{i}\right) + \frac{h}{24}
&\times \left\{-9 \left[\alpha f^{c} \left(t_{i-3}, y \left(t_{i-3}\right)\right)\right.\right.\\
&\qquad \left. + (1-\alpha) f^{l} \left(t_{i-3}, y \left(t_{i-3}\right)\right)\right]\\
&+ 37 \left[\alpha f^{c} \left(t_{i-2}, y \left(t_{i-2}\right)\right)\right.\\
&\qquad \left. + (1-\alpha) f^{r} \left(t_{i-2}, y \left(t_{i-2}\right)\right)\right]\\
&- 59 \left[\alpha f^{c} \left(t_{i-1}, y \left(t_{i-1}\right)\right)\right.\\
&\qquad \left. + (1-\alpha) f^{l} \left(t_{i-1}, y \left(t_{i-1}\right)\right)\right]\\
&+ 55 \left[\alpha f^{c} \left(t_{i}, y \left(t_{i}\right)\right)\right.\\
&\qquad \left.\left. + (1-\alpha) f^{r} \left(t_{i}, y \left(t_{i}\right)\right)\right]\right\}.
\end{aligned} \tag{26}
$$

Thus

$$
\begin{aligned}
&\underline{y}^{\alpha} \left(t_{i+1}\right)\\
&\quad = \underline{y}^{\alpha} \left(t_{i}\right) + \frac{h}{24} \left[-9\overline{f}^{\alpha} \left(t_{i-3}, y \left(t_{i-3}\right)\right) + 37\underline{f}^{\alpha} \left(t_{i-2}, y \left(t_{i-2}\right)\right)\right.\\
&\qquad \left. - 59\overline{f}^{\alpha} \left(t_{i-1}, y \left(t_{i-1}\right)\right) + 55\underline{f}^{\alpha} \left(t_{i}, y \left(t_{i}\right)\right)\right],
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&\overline{y}^{\alpha} \left(t_{i+1}\right)\\
&\quad = \overline{y}^{\alpha} \left(t_{i}\right) + \frac{h}{24} \left[\underline{f}^{\alpha} \left(t_{i-3}, y \left(t_{i-3}\right)\right) + 37\overline{f}^{\alpha} \left(t_{i-2}, y \left(t_{i-2}\right)\right)\right.\\
&\qquad \left. - 59\underline{f}^{\alpha} \left(t_{i-1}, y \left(t_{i-1}\right)\right) + 55\overline{f}^{\alpha} \left(t_{i}, y \left(t_{i}\right)\right)\right].
\end{aligned} \tag{28}
$$

Therefore the four-step Adams-Bashforth method for solving fuzzy initial problems is obtained as follows:

$$
\begin{aligned}
&\underline{y}^{\alpha} \left(t_{i+1}\right)\\
&\quad = \underline{y}^{\alpha} \left(t_{i}\right) + \frac{h}{24} \left[-9\overline{f}^{\alpha} \left(t_{i-3}, y \left(t_{i-3}\right)\right) + 37\underline{f}^{\alpha} \left(t_{i-2}, y \left(t_{i-2}\right)\right)\right.\\
&\qquad \left. - 59\overline{f}^{\alpha} \left(t_{i-1}, y \left(t_{i-1}\right)\right) + 55\underline{f}^{\alpha} \left(t_{i}, y \left(t_{i}\right)\right)\right],\\
&\overline{y}^{\alpha} \left(t_{i+1}\right)\\
&\quad = \overline{y}^{\alpha} \left(t_{i}\right) + \frac{h}{24} \left[-9\underline{f}^{\alpha} \left(t_{i-3}, y \left(t_{i-3}\right)\right) + 37\overline{f}^{\alpha} \left(t_{i-2}, y \left(t_{i-2}\right)\right)\right.\\
&\qquad \left. - 59\underline{f}^{\alpha} \left(t_{i-1}, y \left(t_{i-1}\right)\right) + 55\overline{f}^{\alpha} \left(t_{i}, y \left(t_{i}\right)\right)\right],\\
&\underline{y}^{\alpha} \left(t_{i-3}\right) = \alpha_{0}, \qquad \underline{y}^{\alpha} \left(t_{i-2}\right) = \alpha_{1},\\
&\underline{y}^{\alpha} \left(t_{i-1}\right) = \alpha_{2}, \qquad \underline{y}^{\alpha} \left(t_{i}\right) = \alpha_{3},\\
&\overline{y}^{\alpha} \left(t_{i-3}\right) = \alpha_{4}, \qquad \overline{y}^{\alpha} \left(t_{i-2}\right) = \alpha_{5},\\
&\overline{y}^{\alpha} \left(t_{i-1}\right) = \alpha_{6}, \qquad \overline{y}^{\alpha} \left(t_{i}\right) = \alpha_{7}\\
&\qquad i = 3, 4, \ldots, N-1.
\end{aligned} \tag{29}
$$

3.2. Adams-Moulton Method. From [30], the Adams-Moulton three-step method to solve fuzzy initial problem $y'(t) = f(t, y)$ is as follows:

$$
\begin{aligned}
\underline{y}^{\alpha} \left(t_{i+1}\right) &= \underline{y}^{\alpha} \left(t_{i}\right)\\
&+ \frac{h}{24} \left[\underline{f}^{\alpha} \left(t_{i-2}, y \left(t_{i-2}\right)\right)\right.\\
&\qquad - 5\overline{f}^{\alpha} \left(t_{i-1}, y \left(t_{i-1}\right)\right)\\
&\qquad \left. + 19\underline{f}^{\alpha} \left(t_{i}, y \left(t_{i}\right)\right) + 9\underline{f}^{\alpha} \left(t_{i+1}, y \left(t_{i+1}\right)\right)\right],\\
\overline{y}^{\alpha} \left(t_{i+1}\right) &= \overline{y}^{\alpha} \left(t_{i-1}\right)\\
&+ \frac{h}{24} \left[\overline{f}^{\alpha} \left(t_{i-2}, y \left(t_{i-2}\right)\right)\right.\\
&\qquad - 5\underline{f}^{\alpha} \left(t_{i-1}, y \left(t_{i-1}\right)\right)\\
&\qquad \left. + 19\overline{f}^{\alpha} \left(t_{i}, y \left(t_{i}\right)\right) + 9\overline{f}^{\alpha} \left(t_{i+1}, y \left(t_{i+1}\right)\right)\right],\\
\underline{y}^{\alpha} \left(t_{i-2}\right) &= \alpha_{0}, \quad \underline{y}^{\alpha} \left(t_{i-1}\right) = \alpha_{1}, \quad \underline{y}^{\alpha} \left(t_{i}\right) = \alpha_{2},\\
\overline{y}^{\alpha} \left(t_{i-2}\right) &= \alpha_{3}, \quad \overline{y}^{\alpha} \left(t_{i-1}\right) = \alpha_{4}, \quad \overline{y}^{\alpha} \left(t_{i}\right) = \alpha_{5}\\
&\qquad i = 2, 3, \ldots, N-1.
\end{aligned} \tag{30}
$$

4. Predictor-Corrector Method

4.1. Adams Predictor-Corrector Method. The following algorithm is based on Adams-Bashforth four-step method as a predictor and also an iteration of Adams-Moulton three-step method as a corrector.

Algorithm 11 (predictor-corrector four-step method). To approximate the solution of the following fuzzy initial value problem:

$$
\begin{aligned}
&y'(t) = f(t, y), \quad t_{0} \leq t \leq T,\\
&\underline{y}^{\alpha} \left(t_{0}\right) = \alpha_{0}, \qquad \underline{y}^{\alpha} \left(t_{1}\right) = \alpha_{1},\\
&\underline{y}^{\alpha} \left(t_{2}\right) = \alpha_{2}, \qquad \underline{y}^{\alpha} \left(t_{3}\right) = \alpha_{3},
\end{aligned}
$$

$$\overline{y}^{\alpha}(t_0)=\alpha_4,\quad \overline{y}^{\alpha}(t_1)=\alpha_5,$$
$$\overline{y}^{\alpha}(t_2)=\alpha_6,\quad \overline{y}^{\alpha}(t_3)=\alpha_7, \tag{31}$$

positive integer N is chosen.

Step 1. Let $h=(T-t_0)/N$,

$$\begin{aligned}&\underline{w}^{\alpha}(t_0)=\alpha_0,\quad \underline{w}^{\alpha}(t_1)=\alpha_1,\quad \underline{w}^{\alpha}(t_2)=\alpha_2,\\&\underline{w}^{\alpha}(t_3)=\alpha_3,\\&\overline{w}^{\alpha}(t_0)=\alpha_4,\quad \overline{w}^{\alpha}(t_1)=\alpha_5,\quad \overline{w}^{\alpha}(t_2)=\alpha_6,\\&\overline{w}^{\alpha}(t_3)=\alpha_7.\end{aligned} \tag{32}$$

Step 2. Let $i=3$.

Step 3. Let

$$\begin{aligned}\underline{w}^{(0)\alpha}(t_{i+1})&=\underline{w}^{\alpha}(t_i)\\&\quad+\frac{h}{24}\left[-9\overline{f}^{\alpha}(t_{i-3},w(t_{i-3}))\right.\\&\quad+37\underline{f}^{\alpha}(t_{i-2},w(t_{i-2}))\\&\quad-59\overline{f}^{\alpha}(t_{i-1},w(t_{i-1}))\\&\quad\left.+55\underline{f}^{\alpha}(t_i,w(t_i))\right],\\\overline{w}^{(0)\alpha}(t_{i+1})&=\overline{w}^{\alpha}(t_i)\\&\quad+\frac{h}{24}\left[-9\underline{f}^{\alpha}(t_{i-3},w(t_{i-3}))\right.\\&\quad+37\overline{f}^{\alpha}(t_{i-2},w(t_{i-2}))\\&\quad-59\underline{f}^{\alpha}(t_{i-1},w(t_{i-1}))\\&\quad\left.+55\overline{f}^{\alpha}(t_i,w(t_i))\right].\end{aligned} \tag{33}$$

Step 4. Let $t_{i+1}=t_0+ih$.

Step 5. Let

$$\begin{aligned}\underline{w}^{\alpha}(t_{i+1})&=\underline{w}^{\alpha}(t_i)\\&\quad+\frac{h}{24}\left[\underline{f}^{\alpha}(t_{i-2},w(t_{i-2}))\right.\\&\quad-5\overline{f}^{\alpha}(t_{i-1},w(t_{i-1}))\\&\quad+19\underline{f}^{\alpha}(t_i,w(t_i))\\&\quad\left.+9\underline{f}^{\alpha}\left(t_{i+1},\underline{w}^{(0)}(t_{i+1})\right)\right],\\\overline{w}^{\alpha}(t_{i+1})&=\overline{w}^{\alpha}(t_{i-1})\\&\quad+\frac{h}{24}\left[\overline{f}^{\alpha}(t_{i-2},w(t_{i-2}))\right.\\&\quad-5\underline{f}^{\alpha}(t_{i-1},w(t_{i-1}))\\&\quad+19\overline{f}^{\alpha}(t_i,w(t_i))\\&\quad\left.+9\overline{f}^{\alpha}\left(t_{i+1},\overline{w}^{(0)}(t_{i+1})\right)\right].\end{aligned} \tag{34}$$

Step 6. $i=i+1$.

Step 7. If $i\le N-1$ goto Step 3.

Step 8. Algorithm is completed and $(\underline{w}^{\alpha}(t_i),\overline{w}^{\alpha}(t_i))$ approximates real value of $(\underline{Y}^{\alpha}(t_i),\overline{Y}^{\alpha}(t_i))$ to the original differential equations $(i=4,5,\ldots,N)$.

4.2. Improved Adams Predictor-Corrector Method. In the above section, the predicted values $\underline{y}^0_{i+1},\overline{y}^0_{i+1}$ and the corrected values $\underline{y}_{i+1},\overline{y}_{i+1}$ have the local truncation errors as follows:

$$\begin{aligned}\underline{y}(t_{i+1})-\underline{y}^0_{i+1}&\approx\frac{251}{720}h^5\underline{y}^{(5)}(\underline{x}_i),\\\underline{y}(t_{i+1})-\underline{y}^0_{i+1}&\approx\frac{251}{720}h^5\underline{y}^{(5)}(\underline{x}_i),\\\overline{y}(t_{i+1})-\overline{y}_{i+1}&\approx-\frac{19}{720}h^5\overline{y}^{(5)}(x_i),\\\overline{y}(t_{i+1})-\overline{y}_{i+1}&\approx-\frac{19}{720}h^5\overline{y}^{(5)}(\overline{x}_i).\end{aligned} \tag{35}$$

Thus there exists the error estimations

$$\begin{aligned}\underline{y}(t_{i+1})-\underline{y}^0_{i+1}&\approx-\frac{251}{270}\left(\underline{y}^0_{i+1}-\underline{y}_{i+1}\right),\\\overline{y}(t_{i+1})-\overline{y}^0_{i+1}&\approx-\frac{251}{270}\left(\overline{y}^0_{i+1}-\overline{y}_{i+1}\right),\\\overline{y}(t_{i+1})-\overline{y}_{i+1}&\approx-\frac{19}{720}\left(\underline{y}^0_{i+1}-\underline{y}_{i+1}\right),\\\overline{y}(t_{i+1})-\overline{y}_{i+1}&\approx-\frac{19}{720}\left(\overline{y}^0_{i+1}-\overline{y}_{i+1}\right).\end{aligned} \tag{36}$$

Based on above results, we improve Adams predictor-corrector four-step method into the following iterative computation algorithm.

Algorithm 12 (improved predictor-corrector systems). To approximate the solution of the following fuzzy initial value problem:

$$y'(t)=f(t,y),\quad t_0\le t\le T,$$
$$\underline{y}^{\alpha}(t_0)=\alpha_0,\quad \underline{y}^{\alpha}(t_1)=\alpha_1,$$

$$\underline{y}^{\alpha}(t_2)=\alpha_2,\qquad \underline{y}^{\alpha}(t_3)=\alpha_3,$$
$$\overline{y}^{\alpha}(t_0)=\alpha_4,\qquad \overline{y}^{\alpha}(t_1)=\alpha_5,$$
$$\overline{y}^{\alpha}(t_2)=\alpha_6,\qquad \overline{y}^{\alpha}(t_3)=\alpha_7, \tag{37}$$

positive integer N is chosen.

Step 1. Let $h=(T-t_0)/N$,

$$\begin{aligned}
\underline{w}^{\alpha}(t_0)&=\alpha_0, & \underline{w}^{\alpha}(t_1)&=\alpha_1,\\
\underline{w}^{\alpha}(t_2)&=\alpha_2, & \underline{w}^{\alpha}(t_3)&=\alpha_3,\\
\overline{w}^{\alpha}(t_0)&=\alpha_4, & \overline{w}^{\alpha}(t_1)&=\alpha_5,\\
\overline{w}^{\alpha}(t_2)&=\alpha_6, & \overline{w}^{\alpha}(t_3)&=\alpha_7.
\end{aligned} \tag{38}$$

Step 2. Let $i=3$.

Step 3. Let

$$\begin{aligned}
\underline{p}_{i+1}^{(0)\alpha}&=\underline{w}^{\alpha}(t_i)\\
&\quad+\frac{h}{24}\left[-9\overline{f}^{\alpha}(t_{i-3},w(t_{i-3}))\right.\\
&\qquad+37\underline{f}^{\alpha}(t_{i-2},w(t_{i-2}))\\
&\qquad-59\overline{f}^{\alpha}(t_{i-1},w(t_{i-1}))\\
&\qquad\left.+55\underline{f}^{\alpha}(t_i,w(t_i))\right],\\
\overline{p}_{i+1}^{(0)\alpha}&=\overline{w}^{\alpha}(t_i)\\
&\quad+\frac{h}{24}\left[-9\underline{f}^{\alpha}(t_{i-3},w(t_{i-3}))\right.\\
&\qquad+37\overline{f}^{\alpha}(t_{i-2},w(t_{i-2}))\\
&\qquad-59\underline{f}^{\alpha}(t_{i-1},w(t_{i-1}))\\
&\qquad\left.+55\overline{f}^{\alpha}(t_i,w(t_i))\right].
\end{aligned} \tag{39}$$

Step 4. Let $\underline{m}_{i+1}=\underline{f}(t_{i+1},\underline{p}_{i+1})+(251/270)(\underline{c}_i-\underline{p}_i)$, $\overline{m}_{i+1}=\overline{f}(t_{i+1},\overline{p}_{i+1})+(251/270)(\overline{c}_i-\overline{p}_i)$.

Step 5. Let

$$\begin{aligned}
\underline{c}_{i+1}^{\alpha}&=\underline{y}^{\alpha}(t_i)\\
&\quad+\frac{h}{24}\left[\underline{f}^{\alpha}(t_{i-2},w(t_{i-2}))\right.\\
&\qquad-5\underline{f}^{\alpha}(t_{i-1},w(t_{i-1}))\\
&\qquad+19\underline{f}^{\alpha}(t_i,w(t_i))\\
&\qquad\left.+9\underline{f}^{\alpha}\left(t_{i+1},\underline{w}^{(0)}(t_{i+1})\right)\right],\\
\overline{c}_{i+1}^{\alpha}&=\overline{y}^{\alpha}(t_{i-1})\\
&\quad+\frac{h}{24}\left[\overline{f}^{\alpha}(t_{i-2},w(t_{i-2}))\right.\\
&\qquad-5\underline{f}^{\alpha}(t_{i-1},w(t_{i-1}))\\
&\qquad+19\overline{f}^{\alpha}(t_i,w(t_i))\\
&\qquad\left.+9\overline{f}^{\alpha}\left(t_{i+1},\overline{w}^{(0)}(t_{i+1})\right)\right].
\end{aligned} \tag{40}$$

Step 6. Let

$$\begin{aligned}
\underline{y}_{i+1}&=\underline{c}_{i+1}-\frac{19}{720}\left(\underline{c}_{i+1}-\underline{p}_{i+1}\right),\\
\overline{y}_{i+1}&=\overline{c}_{i+1}-\frac{19}{720}\left(\overline{c}_{i+1}-\overline{p}_{i+1}\right).
\end{aligned} \tag{41}$$

Step 7. $i=i+1$.

Step 8. If $i\le N-1$ goto Step 3.

Step 9. Algorithm is completed and $(\underline{w}^{\alpha}(t_i),\overline{w}^{\alpha}(t_i))$ approximates real value of $(\underline{Y}^{\alpha}(t_i),\overline{Y}^{\alpha}(t_i))$ to the original differential equations ($i=4,5,\dots,N$).

5. Convergence

To integrate the system given in (11) from t_0 to a prefixed $T>t_0$, the interval $[t_0,T]$ is replaced by a set of discrete equally spaced grid points $t_0<t_1<\cdots<t_n=T$, and the exact solution $(\underline{Y}(t,\alpha),\overline{Y}(t,\alpha))$ is approximated by some $(\underline{y}(t,\alpha),\overline{y}(t,\alpha))$. The exact and approximate solutions at $t_i, 0\le i\le N$, are denoted by $Y_i(\alpha)=(\underline{Y}_i(\alpha),\overline{Y}_i(\alpha))$ and $y_i(\alpha)=(\underline{y}_i(\alpha),\overline{y}_i(\alpha))$, respectively. The grid points at which the solution is calculated are $t_i=t_0+ih$, $h=(T-t_0)/N$, $0\le i\le N$.

From (30), the polygon curves

$$\begin{aligned}
\underline{y}(t,h,\alpha)&=\left\{\left[t_0,\underline{y}_0(\alpha)\right],\left[t_1,\underline{y}_1(\alpha)\right],\dots,\left[t_N,\underline{y}_N(\alpha)\right]\right\},\\
\overline{y}(t,h,\alpha)&=\left\{\left[t_0,\overline{y}_0(\alpha)\right],\left[t_1,\overline{y}_1(\alpha)\right],\dots,\left[t_N,\overline{y}_N(\alpha)\right]\right\}
\end{aligned} \tag{42}$$

are the implicit three-step approximation to $\underline{Y}(t,\alpha)$ and $\overline{Y}(t,\alpha)$, respectively, over the interval $[t_0,T]$. The following lemma will be applied to show the convergence of these approximations; that is,

$$\lim_{h\to 0}\underline{y}(t,h,\alpha)=\underline{Y}(t,\alpha),\qquad \lim_{h\to 0}\overline{y}(t,h,\alpha)=\overline{Y}(t,\alpha). \tag{43}$$

Lemma 13. *Let a sequence of numbers* $\{w_n\}_{n=0}^{N}$ *satisfy*

$$|w_{n+1}| \leq A|w_n| + B|w_{n-1}| + C, \quad 0 \leq n \leq N-1, \tag{44}$$

for some given positive A and B, C. Then

$$\begin{aligned}|w_n| \leq &\left(A^{n-1} + \beta_1 A^{n-3}B + \beta_2 A^{n-5}B^2 + \cdots + \beta_s B^{n/2}\right)|w_1| \\ &+ \left(A^{n-2}B + \gamma_1 A^{n-4}B^2 + \cdots + \gamma_t AB^{n/2}\right)|w_0| \\ &+ \left(A^{n-2} + A^{n-3} + \cdots + 1\right)C \\ &+ \left(\delta_1 A^{n-4} + \delta_2 A^{n-5} + \cdots + \delta_m A + 1\right)BC \\ &+ \left(\zeta_1 A^{n-6} + \zeta_2 A^{n-7} + \cdots + \zeta_l A + 1\right)B^2C \\ &+ \left(\lambda_1 A^{n-8} + \lambda_2 A^{n-9} + \cdots + \lambda_p A + 1\right)B^3C + \cdots,\end{aligned} \tag{45}$$

when n is odd and

$$\begin{aligned}|w_n| \leq &\left(A^{n-1} + \beta_1 A^{n-3}B + \beta_2 A^{n-5}B^2 + \cdots + \beta_s B^{(n/2)-1}\right)|w_1| \\ &+ \left(A^{n-2}B + \gamma_1 A^{n-4}B^2 + \cdots + \gamma_t AB^{n/2}\right)|w_0| \\ &+ \left(A^{n-2} + A^{n-3} + \cdots + 1\right)C \\ &+ \left(\delta_1 A^{n-4} + \delta_2 A^{n-5} + \cdots + \delta_m A + 1\right)BC \\ &+ \left(\zeta_1 A^{n-6} + \zeta_2 A^{n-7} + \cdots + \zeta_l A + 1\right)B^2C \\ &+ \left(\lambda_1 A^{n-8} + \lambda_2 A^{n-9} + \cdots + \lambda_p A + 1\right)B^3C + \cdots,\end{aligned} \tag{46}$$

when n is even, where β_s, γ_t, δ_m, ζ_l, λ_p *are constants for all s, t, l, m, and p.*

Theorem 14. *For any arbitrary fixed* $r : 0 \leq r \leq 1$, *the implicit three-step Simpson approximations of* (30) *converge to the exact solutions* $\underline{Y}(t,\alpha)$, $\overline{Y}(t,\alpha)$ *for* $\underline{Y}, \overline{Y} \in C^3[t_0, T]$.

Proof. It is sufficient to show

$$\lim_{h\to 0} \underline{y}_N(\alpha) = \underline{Y}(T,\alpha), \qquad \lim_{h\to 0} \overline{y}_N(\alpha) = \overline{Y}(T,\alpha). \tag{47}$$

By Taylor's theorem, we have

$$\begin{aligned}&\underline{Y}_{n+1}(\alpha) \\ &= \underline{Y}_{n-1}(\alpha) + \frac{h}{3}\left[f\left(t_{n-1}, \underline{Y}_{n-1}(\alpha)\right) + 4f\left(t_n, \underline{Y}_n(\alpha)\right)\right. \\ &\qquad \left. + f\left(t_{n+1}, \underline{Y}_{n+1}(\alpha)\right)\right] - \frac{h^5}{90}\underline{Y}^{(5)}\left(\underline{\xi}_n\right), \\ &\overline{Y}_{n+1}(\alpha) \\ &= \overline{Y}_{n-1}(\alpha) + \frac{h}{3}\left[f\left(t_{n-1}, \overline{Y}_{n-1}(\alpha)\right) + 4f\left(t_n, \overline{Y}_n(\alpha)\right)\right. \\ &\qquad \left. + f\left(t_{n+1}, \overline{Y}_{n+1}(\alpha)\right)\right] - \frac{h^5}{90}\overline{Y}^{(5)}\left(\overline{\xi}_n\right),\end{aligned} \tag{48}$$

where $t_{n-1} < \underline{\xi}_n, \overline{\xi}_n < t_{n+1}$. Consequently

$$\begin{aligned}&\underline{Y}_{n+1}(\alpha) - \underline{y}_{n+1}(\alpha) \\ &= \underline{Y}_{n-1}(\alpha) - \underline{y}_{n-1}(\alpha) + \frac{h}{3}\left\{f\left(t_{n-1}, \underline{Y}_{n-1}(\alpha)\right) - f\left(t_{n-1}, \underline{y}_{n-1}(\alpha)\right)\right\} \\ &+ \frac{4h}{3}\left\{f\left(t_n, \underline{Y}_n(\alpha)\right) - f\left(t_n, \underline{y}_n(\alpha)\right)\right\} \\ &+ \frac{h}{3}\left\{f\left(t_{n+1}, \underline{Y}_{n+1}(\alpha)\right) - f\left(t_{n+1}, \underline{y}_{n+1}(\alpha)\right)\right\} \\ &- \frac{h^5}{90}\underline{Y}^{(5)}\left(\underline{\xi}_n\right), \\ &\overline{Y}_{n+1}(\alpha) - \overline{y}_{n+1}(\alpha) \\ &= \overline{Y}_{n-1}(\alpha) - \overline{y}_{n-1}(\alpha) + \frac{h}{3}\left\{f\left(t_{n-1}, \overline{Y}_{n-1}(\alpha)\right) - f\left(t_{n-1}, \overline{y}_{n-1}(\alpha)\right)\right\} \\ &+ \frac{4h}{3}\left\{f\left(t_n, \overline{Y}_n(\alpha)\right)\left\{-f\left(t_n, \overline{y}_n(\alpha)\right)\right\}\right. \\ &\times \frac{h}{3}\left\{f\left(t_{n+1}, \overline{Y}_{n+1}(\alpha)\right) - f\left(t_{n+1}, \overline{y}_{n+1}(\alpha)\right)\right\} \\ &- \frac{h^5}{90}\overline{Y}^{(5)}\left(\overline{\xi}_n\right).\end{aligned} \tag{49}$$

Denote $w_n = \underline{Y}_n(\alpha) - \underline{y}_n(\alpha)$, $v_n = \overline{Y}_{n+1}(\alpha) - \overline{y}_{n+1}(\alpha)$. Then

$$\begin{aligned}|w_{n+1}| \leq &\frac{4hL_1}{3}|w_{n+1}| + \left(1 + \frac{hL_2}{3}\right)|w_{n-1}| + \frac{hL_3}{3}|w_{n+1}| + \frac{h^5}{90}\underline{M}, \\ |v_{n+1}| \leq &\frac{4hL_4}{3}|v_{n+1}| + \left(1 + \frac{hL_5}{3}\right)|v_{n-1}| \\ &+ \frac{hL_6}{3}|v_{n+1}| + \frac{h^5}{90}\overline{M},\end{aligned} \tag{50}$$

where $\underline{M} = \max_{t_0 \le t \le T} |\underline{Y}^{(5)}(t,\alpha)|$ and $\overline{M} = \max_{t_0 \le t \le T} |\overline{Y}^{(5)}(t,\alpha)|$.

Set

$$L = \max\{L_1, L_2, L_3, L_4, L_5, L_6\} < \frac{3}{h}, \tag{51}$$

then

$$\begin{aligned} |w_{n+1}| &\le \frac{4hL}{3-hL}|w_n| + \frac{3+hL}{3-hL}|w_{n-1}| + \frac{1}{30(3-hL)}h^5\underline{M}, \\ |v_{n+1}| &\le \frac{4hL}{3-hL}|v_n| + \frac{3+hL}{3-hL}|v_{n-1}| + \frac{1}{30(3-hL)}h^5\overline{M} \end{aligned} \tag{52}$$

are obtained, where $|u_n| = |w_n| + |v_n|$, so by Lemma 13, since $w_0 = v_0 = 0$ and $w_1 = v_1 = 0$ we have

$$\begin{aligned} |u_n| \le& \frac{((4hL)/(3-hL))^{n-1} - 1}{(30(hL-3))/(3-hL)} \\ &\times \frac{1}{30(3-hL)}h^5(\underline{M}+\overline{M}) \\ &+ \left\{\delta_1\left(\frac{4hL}{3-hL}\right)^{n-4} + \delta_2\left(\frac{4hL}{3-hL}\right)^{n-5} \right. \\ &\left. + \cdots + \delta_m\left(\frac{4hL}{3-hL}\right) + 1\right\} \\ &\times \left(\frac{3+hL}{3-hL}\right)\left(\frac{1}{30(3-hL)}h^5(\underline{M}+\overline{M})\right) \\ &+ \left\{\zeta_1\left(\frac{4hL}{3-hL}\right)^{n-6} + \zeta_2\left(\frac{4hL}{3-hL}\right)^{n-7} \right. \\ &\left. + \cdots + \zeta_l\left(\frac{4hL}{3-hL}\right) + 1\right\} \\ &\times \left(\frac{3+hL}{3-hL}\right)^2\left(\frac{1}{30(3-hL)}h^5(\underline{M}+\overline{M})\right) \\ &+ \left\{\lambda_1\left(\frac{4hL}{3-hL}\right)^{n-8} + \lambda_2\left(\frac{4hL}{3-hL}\right)^{n-9} \right. \\ &\left. + \cdots + \lambda_p\left(\frac{4hL}{3-hL}\right) + 1\right\} \\ &\times \left(\frac{3+hL}{3-hL}\right)^3\left(\frac{1}{30(3-hL)}h^5(\underline{M}+\overline{M})\right) + \cdots. \end{aligned} \tag{53}$$

If $h \to 0$ then $w_n \to 0$, $v_n \to 0$, which concludes the proof. □

Theorem 15. *For any arbitrary fixed $r : 0 \le r \le 1$, the explicit four-step Milne approximations of (29) converge to the exact solutions $\underline{Y}(t,\alpha)$, $\overline{Y}(t,\alpha)$ for $\underline{Y}, \overline{Y} \in C^3[t_0, T]$.*

Proof. Like Theorem 14, the conclusion can be obtained easily. □

Table 1: Comparisons between the exact solution and the numerical solution.

t	$\underline{Y}(t,r)$	$\underline{y}(t,r)$	Error
0.1	0.963635583813	0.963082998354	0.000552585459
0.2	0.975914722827	0.975304021448	0.000610701379
0.3	0.995959477182	0.995284547778	0.000674929403
0.4	1.022969627904	1.022223715555	0.000745912348
0.5	1.056214668049	1.055390307414	0.000824360635
0.6	1.095026491542	1.094115432142	0.000911059400
0.7	1.138792706547	1.137785830194	0.001006876353
0.8	1.186950506443	1.185837735978	0.001112770464
0.9	1.238981037065	1.237751235509	0.001229801555
1	1.294404203842	1.293045062928	0.001359140914

Table 2: Comparisons between the exact solution and the numerical solution.

t	$Y^1(t,r)$	$y^1(t,r)$	Error
0.1	1.004837418035	1.006647092872	0.001809674836
0.2	1.018730753077	1.020368214584	0.001637461506
0.3	1.040818220681	1.042299857123	0.001481636441
0.4	1.070320046035	1.071660686127	0.001340640092
0.5	1.106530659712	1.107743721032	0.001213061319
0.6	1.148811636094	1.149909259366	0.001097623272
0.7	1.196585303791	1.197578474398	0.000993170607
0.8	1.249328964117	1.250227622045	0.000898657928
0.9	1.306569659740	1.307382799060	0.000813139319
1	1.367879441171	1.368615200053	0.000735758882

6. Numerical Examples

Example 1 (see [33]). Consider the following fuzzy differential equation:

$$\begin{aligned} \tilde{y}'(t) &= -\tilde{y}(t) + t + 1, \quad t \ge 0, \\ \tilde{y}(0) &= (0.96, 1, 1.01). \end{aligned} \tag{54}$$

The exact solution of equation is

$$\begin{aligned} \underline{Y}(t) &= t - 0.025e^{t} + 0.985e^{-t}, \\ Y^1(t) &= t + 1.0e^{-t}, \\ \overline{Y}(t) &= t + 0.025e^{t} + 0.985e^{-t}. \end{aligned} \tag{55}$$

By using the Adams predictor-corrector four-step method with $N = 10$ for some $t \in [0,1]$, the results shown in Tables 1, 2, and 3 are obtained.

And by using the improved Adams predictor-corrector systems with $N = 10$ for some $t \in [0,1]$, the results shown in Tables 4, 5, and 6 are obtained.

The results of Example 1 are shown by Figures 1, 2, and 3.

7. Conclusion

In this paper two numerical methods with higher order of convergence and not much amounts of computation for

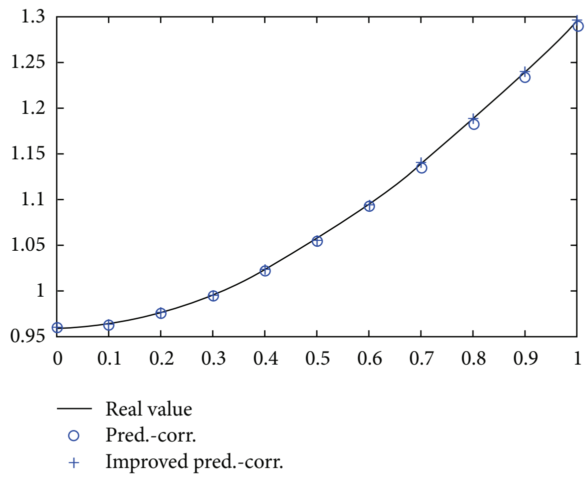

Figure 1: Comparisons between the exact solution $\underline{Y}$ and the numerical solution $\underline{y}$.

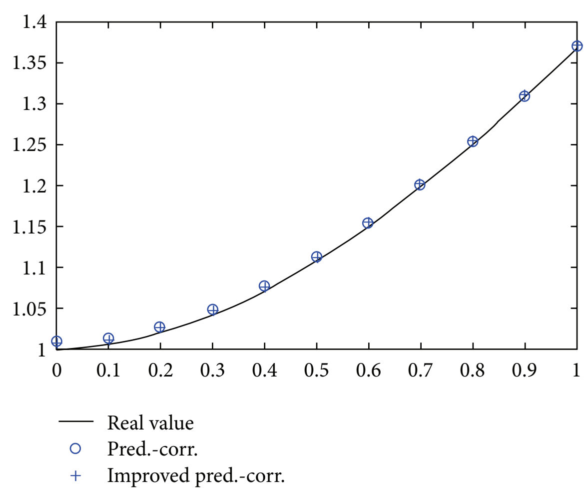

Figure 2: Comparisons between the exact solution Y^1 and the numerical solution y^1.

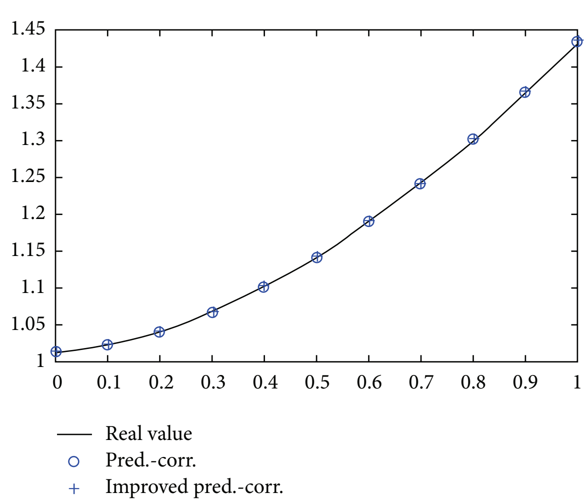

Figure 3: Comparisons between the exact solution $\overline{Y}$ and the numerical solution $\overline{y}$.

Table 3: Comparisons between the exact solution and the numerical solution.

t	$\overline{Y}(t,r)$	$\overline{y}(t,r)$	Error
0.1	1.018894129717	1.019446715176	0.000552585459
0.2	1.036984860735	1.037595562114	0.000610701379
0.3	1.063452417560	1.064127346964	0.000674929403
0.4	1.097560862786	1.098306775134	0.000745912348
0.5	1.138650731584	1.139475092219	0.000824360635
0.6	1.186132431562	1.187043490962	0.000911059400
0.7	1.239480341921	1.240487218275	0.001006876353
0.8	1.298227552867	1.299340323332	0.001112770464
0.9	1.361961192623	1.363190994178	0.001229801555
1	1.430318295265	1.431677436179	0.001359140914

Table 4: Comparisons between the exact solution and the numerical solution.

t	$\underline{Y}(t,r)$	$\underline{y}(t,r)$	Error
0.1	0.963635583813	0.963525066721	$0.110517091807e-003$
0.2	0.975914722827	0.975792582551	$0.122140275815e-003$
0.3	0.995959477182	0.995824491301	$0.134985880757e-003$
0.4	1.022969627904	1.022820445434	$0.149182469764e-003$
0.5	1.056214668049	1.056049795922	$0.164872127069e-003$
0.6	1.095026491542	1.094844279662	$0.182211880038e-003$
0.7	1.138792706547	1.138591331277	$0.201375270747e-003$
0.8	1.186950506443	1.186727952350	$0.222554092849e-003$
0.9	1.238981037065	1.238735076754	$0.245960311115e-003$
1	1.294404203842	1.294132375659	$0.271828182845e-003$

Table 5: Comparisons between the exact solution and the numerical solution.

t	$Y^1(t,r)$	$y^1(t,r)$	Error
0.1	1.004837418035	1.005742255454	$0.904837418035e-003$
0.2	1.018730753077	1.019549483831	$0.818730753077e-003$
0.3	1.040818220681	1.041559038902	$0.740818220682e-003$
0.4	1.070320046035	1.070990366081	$0.670320046035e-003$
0.5	1.106530659712	1.107137190372	$0.606530659712e-003$
0.6	1.148811636094	1.149360447730	$0.548811636093e-003$
0.7	1.196585303791	1.197081889095	$0.496585303791e-003$
0.8	1.249328964117	1.249778293081	$0.449328964117e-003$
0.9	1.306569659740	1.306976229400	$0.406569659740e-003$
1	1.367879441171	1.368247320612	$0.367879441171e-003$

solving fuzzy differential equations were discussed in detail. The proposed algorithms were generated by updating the Adams-Bashforth four-step method and Adams-Moulton three-step method. An example showed that the proposed methods is more efficient and practical than some methods appeared in the literature before.

Table 6: Comparisons between the exact solution and the numerical solution.

t	$\overline{Y}(t,r)$	$\overline{y}(t,r)$	Error
0.1	1.018894129717	1.019004646809	$0.110517091807e-003$
0.2	1.036984860735	1.037107001011	$0.122140275816e-003$
0.3	1.063452417560	1.063587403441	$0.134985880757e-003$
0.4	1.097560862786	1.097710045255	$0.149182469763e-003$
0.5	1.138650731584	1.138815603711	$0.164872127070e-003$
0.6	1.186132431562	1.186314643442	$0.182211880038e-003$
0.7	1.239480341921	1.239681717192	$0.201375270747e-003$
0.8	1.298227552867	1.298450106960	$0.222554092849e-003$
0.9	1.361961192623	1.362207152934	$0.245960311115e-003$
1	1.430318295265	1.430590123448	$0.271828182845e-003$

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