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On Property (Saw) and others spectral properties type Weyl-Browder theorems

Sobre la propiedad (Saw) y otras propiedades espectrales tipo teoremas de Weyl-Browder

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Abstract. An operator T acting on a Banach space X satisfies the property (aw) if $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$, where $\sigma_W(T)$ is the Weyl spectrum of T and $E_a^0(T)$ is the set of all eigenvalues of T of finite multiplicity that are isolated in the approximate point spectrum of T. In this paper we introduce and study two new spectral properties, namely (Saw) and (Sab), in connection with Weyl-Browder type theorems. Among other results, we prove that T satisfies property (Saw) if and only if T satisfies property (aw) and $\sigma_{SBF_+^-}(T) = \sigma_W(T)$, where $\sigma_{SBF_+^-}(T)$ is the upper semi B-Weyl spectrum of T.

Key words and phrases. Semi B-Fredholm operator, a-Weyl's theorem, property (Saw), property (Sab).

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Resumen. Un operador T actuando sobre un espacio de Banach X satisface la propiedad (aw) si $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$, donde $\sigma_W(T)$ es el espectro de Weyl de T y $E_a^0(T)$ es el conjunto de todos los autovalores de T de multiplicidad finita que son aislados en el espectro aproximado puntual de T. En este artículo introducimos y estudiamos dos nuevas propiedades espectrales, llamadas (Saw) y (Sab), en conexión con teoremas tipo Weyl-Browder. Entre

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otros resultados, mostramos que T satisface la propiedad (Saw) si y sólo si T satisface la propiedad (aw) y $\sigma_{SBF_+^-}(T) = \sigma_W(T)$, donde $\sigma_{SBF_+^-}(T)$ es el espectro superiormente semi B-Weyl de T.

Palabras y frases clave. Operador semi-B-Fredholm, teorema de a-Weyl, propiedad (Saw), propiedad (Sab).

1. Introduction and preliminaries

In this paper, we introduce two new spectral properties type Weyl-Browder theorems, namely the properties (Saw) and (Sab), respectively. In addition, we establish the precise relationships between these properties and others variants of Weyl's and Browder's theorem that have been recently introduced in [10, 14, 17, 18, 22, 25] and [27].

Throughout this paper, $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. We refer to [24] for details about notations and terminologies. However, we give the following notations that will be useful in the sequel:

- Browder spectrum: $\sigma_b(T)$
- Weyl spectrum $\sigma_W(T)$
- Upper semi-Browder spectrum $\sigma_{ub}(T)$
- Upper semi-Weyl spectrum $\sigma_{SF_+^-}(T)$
- Left Drazin invertible spectrum $\sigma_{LD}(T)$
- Drazin invertible spectrum $\sigma_D(T)$
- B-Weyl spectrum $\sigma_{BW}(T)$
- Upper semi-B-Weyl spectrum $\sigma_{SBF_+^-}(T)$
- Lower semi-B-Weyl spectrum $\sigma_{SBF_-^+}(T)$
- approximate point spectrum $\sigma_a(T)$
- surjectivity spectrum $\sigma_s(T)$.

Recall that an operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0}$$

is $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$ (see [16]). The operator T is said to have SVEP if it has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, every $T \in L(X)$ has SVEP at each point of the resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{2}$$

It is easily seen from definition of localized SVEP that

$$\lambda \notin \operatorname{acc} \sigma_a(T) \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

where $\operatorname{acc} K$ means the set of all accumulation points of $K \subseteq \mathbb{C}$, and

$$\lambda \notin \operatorname{acc} \sigma_s(T) \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{4}$$

***Remark* 1.1.** The implications (1)-(4) are actually equivalences whenever $\lambda I - T$ is a quasi-Fredholm operator, in particular when $\lambda I - T$ is semi B-Fredholm (see [2]).

Lemma 1.2. ([3, Lemma 2.4]) *Let $T \in L(X)$. Then*

(i) *T is upper semi B-Fredholm and $\alpha(T) < \infty$ if and only if $T \in \Phi_+(X)$.*

(ii) *T is lower semi B-Fredholm and $\beta(T) < \infty$ if and only if $T \in \Phi_-(X)$.*

The following lemma is a particular case of [9, Theorem 3.6].

Lemma 1.3. *For $T \in L(X)$, the following statements are equivalent:*

(i) *$\lambda_0 I - T$ is left Drazin invertible $\Leftrightarrow$ $\lambda_0 I - T$ is quasi-Fredholm with finite ascent,*

(ii) *$\lambda_0 I - T$ is right Drazin invertible $\Leftrightarrow$ $\lambda_0 I - T$ is quasi-Fredholm with finite descent,*

(iii) *$\lambda_0 I - T$ is Drazin invertible $\Leftrightarrow$ $\lambda_0 I - T$ is quasi-Fredholm with finite ascent and descent.*

Denote by iso K, the set of all isolated points of $K \subseteq \mathbb{C}$. If $T \in L(X)$ define

$$\begin{aligned}
E^0(T) &= \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\
E_a^0(T) &= \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\
E(T) &= \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\}, \\
E_a(T) &= \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(\lambda I - T)\}.
\end{aligned}$$

Also, define

$$\Pi^0(T) = \sigma(T) \setminus \sigma_b(T), \qquad \Pi^0_a(T) = \sigma_a(T) \setminus \sigma_{ub}(T),$$

$$\Pi(T) = \sigma(T) \setminus \sigma_D(T), \qquad \Pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T).$$

Let $T \in L(X)$. Following Coburn [15], T is said to satisfy *Weyl's theorem*, in symbols $(\mathcal{W})$, if $\sigma(T) \setminus \sigma_W(T) = E^0(T)$. Following Rakočević ([20], [19]), T is said to satisfy *a-Weyl's theorem*, in symbols $(a\mathcal{W})$, if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0_a(T)$, and T is said to have *property* (w) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. According to Berkani and Koliha [11], T is said to satisfy *generalized Weyl's theorem*, in symbols $(g\mathcal{W})$, if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. Similarly, T is said to satisfy *generalized a-Weyl's theorem*, in symbol $(ga\mathcal{W})$, if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$.

Now, we describe several spectral properties introduced recently in [7, 10, 13, 14, 17, 18, 22, 23, 25, 26] and [27].

Definition 1.4. An operator $T \in L(X)$ is said to have:

(i) property (gw) [7] if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$.

(ii) property (b) [13] if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$.

(iii) property (gb) [13] if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$.

(iv) property (aw) [14] if $\sigma(T) \setminus \sigma_W(T) = E^0_a(T)$.

(v) property (gaw) [14] if $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$.

(vi) property (ab) [14] if $\sigma(T) \setminus \sigma_W(T) = \Pi^0_a(T)$.

(vii) property (gab) [14] if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$.

(viii) property (z) [25] if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0_a(T)$.

(ix) property (gz) [25] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$.

(x) property (az) [25] if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0_a(T)$.

(xi) property (gaz) [25] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$.

(xii) property (ah) [26] if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$.

(xiii) property (gah) [26] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$.

(xiv) property (Bw) [17] if $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$.

(xv) property (Baw) [27] if $\sigma(T) \setminus \sigma_{BW}(T) = E^0_a(T)$.

(xvi) property (Bgw) [22] if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$.

(xvii) property (Bab) [27] if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$.

(xviii) property (Bb) [22] if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$.

(xix) property (Bgb) [22] if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^0(T)$.

(xx) property $(SBaw)$ [10] if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$.

(xxi) property $(SBab)$ [10] if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$.

(xxii) property (Sw) [23] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$.

(xxiii) property (Sb) [23] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^0(T)$.

The properties (Bgw) and (Bgb) are also called (SBw) and (SBb), respectively (see [10]). Property (Bb) was also introduced in [18].

2. Properties (Saw) and (Sab).

According to [24], $T \in L(X)$ has *property* (gv) (resp. (v)) if $\sigma(T)\setminus\sigma_{SBF_+^-}(T) = E(T)$ (resp. $\sigma(T)\setminus\sigma_{SF_+^-}(T) = E^0(T)$) and it was proved in [24, Corollary 2.12], that property (gv) (resp. (v)) is equivalent to property (gz) (resp. (z)). Property (gv) (resp. (v)) also is called property (gt) (resp. (t)) in [21] and property (gh) (resp. (h)) in [26]. In this section we introduce and study two new spectral properties that are independent of property (gv) (and hence of property (gz)).

Definition 2.1. An operator $T \in L(X)$ is said to have *property* (Saw) if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$, and *property* (Sab) if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$.

The next result gives the precise relationship between the properties (Saw) and (z).

Theorem 2.2. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (Saw),*

(ii) *T has property (z) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$,*

(iii) *T has property (v) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

Proof. (i)⇒(ii). Assume that T satisfies property (Saw) and let $\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T)$. Since $\sigma(T) \setminus \sigma_{SF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, we have $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. As T satisfies property (Saw), it follows that $\lambda \in E_a^0(T)$. This shows $\sigma(T) \setminus \sigma_{SF_+^-}(T) \subseteq E_a^0(T)$.

To show the opposite inclusion $E_a^0(T) \subseteq \sigma(T) \setminus \sigma_{SF_+^-}(T)$, let $\lambda \in E_a^0(T)$. Then $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(\mathrm{T})$ and $0 < \alpha(\lambda I - T) < \infty$. Since T satisfies property (Saw), we have $\lambda \in \sigma(T)$ and $\lambda I - T$ is upper semi-B-Weyl. Thus, $\lambda I - T$ is an upper semi B-Fredholm operator and $\alpha(\lambda I - T) < \infty$. By Lemma 1.2, we have that $\lambda I - T$ is an upper semi-Fredholm operator, so $\lambda I - T$ is an upper semi-Weyl operator. Hence, $\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T)$ and so $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$. This shows that T satisfies property (z). Consequently, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$ and $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$. Therefore, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T)$ and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.

(ii)⇒(i). Suppose that T satisfies property (z) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$. Then, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, and hence T satisfies property (Saw).

(ii)⇔(iii). The equivalence between properties (z) and (v) has been proved in [24, Corollary 2.12]. ☑

From Theorem 2.2, we observe that property (Saw) implies property (z). But the converse of this implication does not hold in general, as we can see in the following example.

Example 2.3. Let $U \in L(\ell^2(\mathbb{N}))$ be defined by

$$U(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots).$$

Then, $\sigma(U) = \{0, 1\}$, $\sigma_{SF_+^-}(U) = \{1\}$, $\sigma_{SBF_+^-}(U) = \varnothing$ and $E_a^0(U) = \{0\}$. In consequence

$$\sigma(U) \setminus \sigma_{SF_+^-}(U) = E_a^0(U), \qquad \sigma(U) \setminus \sigma_{SBF_+^-}(U) \neq E_a^0(U).$$

Hence, T satisfies property (z), but not property (Saw).

Similarly to Theorem 2.2, we have the following result.

Theorem 2.4. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (Sab),*

(ii) *T has property (az) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$,*

(iii) *T has property (gaz) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

Proof. The proof (i)⇔(ii), is similar to the proof (i)⇔(ii) of Theorem 2.2. The proof (ii)⇔(iii), follows of the equivalence between properties (az) and (gaz) proved in [25, Corollary 3.5]. ☑

From Theorem 2.4, we note that property (Sab) implies property (az). But the converse is not true in general as shown by the following example.

Example 2.5. Consider the operator $T = 0$ defined on the Hilbert space $\ell^2(\mathbb{N})$. Then, $\sigma(T) = \sigma_{SF_+^-}(T) = \{0\}$, $\sigma_{SBF_+^-}(T) = \varnothing$ and $\Pi_a^0(T) = \varnothing$. Therefore, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \{0\} \setminus \{0\} = \varnothing = \Pi_a^0(T)$ and $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \{0\} \setminus \varnothing = \{0\} \neq \varnothing = \Pi_a^0(T)$. Thus, T satisfies property (az), but not property (Sab).

The next result gives the relationship between the properties (Saw) and $(SBaw)$.

Theorem 2.6. *Let $T \in L(X)$. Then T has property (Saw) if and only if T has property $(SBaw)$ and $\sigma(T) = \sigma_a(T)$.*

***Proof*.** Sufficiency: Assume that T satisfies property (Saw) and let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Since $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$, we have $\lambda \in E_a^0(T)$ and so, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E_a^0(T)$.

To show the opposite inclusion $E_a^0(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, let $\lambda \in E_a^0(T)$. Then, $\lambda \in \sigma_a(T)$ and since T satisfies property (Saw), it follows that $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$ and so, $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Hence, $E_a^0(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ and T satisfies property $(SBaw)$. Consequently, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$ and $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$. Therefore, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ and $\sigma(T) = \sigma_a(T)$.

Necessity: Assume that T satisfies property $(SBaw)$ and $\sigma(T) = \sigma_a(T)$. Then, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$. Thus, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$ and T satisfies property (Saw). ☑

Similarly to Theorem 2.6, we have the following result.

Theorem 2.7. *Let $T \in L(X)$. Then T has property (Sab) if and only if T has property $(SBab)$ and $\sigma(T) = \sigma_a(T)$.*

***Proof*.** The proof is similar to the proof of Theorem 2.6. ☑

The next example shows that, in general, property $(SBaw)$ (resp. $(SBab)$) does not imply property (Saw) (resp. (Sab)).

Example 2.8. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$ and U be the operator defined in Example 2.3. Define an operator T on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus U$. Then, $\sigma(T) = \mathbf{D}(0, 1)$, the closed unit disc on $\mathbb{C}$, $\sigma_a(T) = \Gamma \cup \{0\}$, where Γ denotes the unit circle of $\mathbb{C}$ and $\sigma_{SBF_+^-}(T) = \Gamma$. Moreover, $E_a^0(T) = \{0\}$ and $\Pi_a^0(T) = \{0\}$. Therefore, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$ and

$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$. Thus, T satisfies both properties $(SBaw)$ and $(SBab)$. On the other hand, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) \neq E_a^0(T)$ and $\sigma(T) \setminus \sigma_{SBF_+^-}(T) \neq \Pi_a^0(T)$. Therefore, T does not satisfy property (Saw) or property (Sab).

Corollary 2.9. *Let $T \in L(X)$. Then:*

(i) *T has property (Saw) if and only if T has property (Sw).*

(ii) *T has property (Sab) if and only if T has property (Sb).*

Proof. (i) Suppose that T satisfies property (Saw). By Theorem 2.6, $\sigma(T) = \sigma_a(T)$ and so, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T) = E^0(T)$. Therefore, T satisfies property (Sw).

Conversely, assume that T satisfies property (Sw). By [23, Theorem 2.25], $\sigma(T) = \sigma_a(T)$ and so, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T) = E_a^0(T)$. Therefore, T satisfies property (Saw).

(ii) The proof is similar to the proof of (i). Just use both Theorem 2.7 and [23, Theorem 2.28]. ☑

The next result gives the relationship between the properties (Saw) and (Baw).

Theorem 2.10. *Let $T \in L(X)$. Then T has property (Saw) if and only if T has property (Baw) and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.*

Proof. Sufficiency: Suppose that T satisfies property (Saw), that is $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then $\lambda I - T$ is a B-Weyl operator, and hence it is upper semi-B-Weyl. Thus $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$, and so $\sigma(T) \setminus \sigma_{BW}(T) \subseteq E_a^0(T)$.

To show the opposite inclusion $E_a^0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$, let $\lambda \in E_a^0(T)$. Since T satisfies property (Saw), $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(\mathrm{T})$ and $\lambda I - T$ is an upper semi-B-Fredholm operator, hence quasi-Fredholm. By Theorem 2.6, $\lambda \in \operatorname{iso} \sigma(\mathrm{T})$, it follows that both T and T^* have SVEP in λ, and by Remark 1.1, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. From Lemma 1.3, we have $\lambda I - T$ is a Drazin invertible operator, and hence it is B-Weyl. Therefore, $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, and so $E_a^0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$. This shows that T satisfies property (Baw). Consequently, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$ and $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$. Therefore, $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.

Necessity: Suppose that T satisfies property (Baw) and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Then, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$, and so T satisfies property (Saw). ☑

Similarly to Theorem 2.10, we have the following result.

Theorem 2.11. *Let $T \in L(X)$. Then T has property (Sab) if and only if T has property (Bab) and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.*

Proof. The proof is similar to the proof of Theorem 2.10. ☑

The next example shows that, in general, property (Baw) (resp. (Bab)) does not imply property (Saw) (resp. (Sab)).

Example 2.12. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$. Define an operator T on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = 0 \oplus R$. Then, $\sigma(T) = \mathbf{D}(0,1)$, $\sigma_{SBF_+^-}(T) = \Gamma$. Moreover, $\sigma_{BW}(T) = \mathbf{D}(0,1)$, $E_a^0(T) = \varnothing$ and $\Pi_a^0(T) = \varnothing$. Therefore, $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) \neq E_a^0(T) = \Pi_a^0(T)$. Thus, T satisfies property (Baw) and hence property (Bab), but not property (Saw) or property (Sab).

The next result gives the relationship between the properties (Saw) and (aw).

Theorem 2.13. *Let $T \in L(X)$. Then T has property (Saw) if and only if T has property (aw) and $\sigma_{SBF_+^-}(T) = \sigma_W(T)$.*

Proof. Sufficiency: Assume that T satisfies property (Saw); then it also satisfies property (Baw) by Theorem 2.10. Property (Baw) implies by [27, Theorem 3.3] that T satisfies property (aw). Consequently, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$ and $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$. Therefore, $\sigma_{SBF_+^-}(T) = \sigma_W(T)$.

Necessity: Suppose that T satisfies property (aw) and $\sigma_{SBF_+^-}(T) = \sigma_W(T)$. Then, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_W(T) = E_a^0(T)$, and hence T satisfies property (Saw). ☑

Similarly to Theorem 2.13, we have the following result.

Theorem 2.14. *Let $T \in L(X)$. Then T has property (Sab) if and only if T has property (ab) and $\sigma_{SBF_+^-}(T) = \sigma_W(T)$.*

Proof. The proof is similar to the proof of Theorem 2.13. Just use both Theorem 2.11 and [27, Theorem 3.6]. ☑

The next example shows that, in general, property (aw) (resp. (ab)) does not imply property (Saw) (resp. (Sab)).

Example 2.15. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$. Then, $\sigma(R) = \mathbf{D}(0,1)$, $\sigma_{SBF_+^-}(R) = \Gamma$. Moreover, $\sigma_W(R) = \mathbf{D}(0,1)$, $E_a^0(R) = \varnothing$ and $\Pi_a^0(R) = \varnothing$. Therefore, $\sigma(R) \setminus \sigma_W(R) = E_a^0(R)$ and $\sigma(R) \setminus \sigma_{SBF_+^-}(R) \neq E_a^0(R) = \Pi_a^0(R)$. Thus, R satisfies property (aw) and hence property (ab), but not property (Saw) or property (Sab).

As a consequences of the results above, we have the following two corollaries, where we give new characterizations of the properties (Sw) and (Sb), respectively.

Corollary 2.16. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (Sw),*

(ii) *T has property (z) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

(iii) *T has property (v) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

(iv) *T has property $(SBaw)$ and $\sigma(T) = \sigma_a(T)$.*

(v) *T has property (Baw) and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.*

(vi) *T has property (aw) and $\sigma_{SBF_+^-}(T) = \sigma_W(T)$.*

Corollary 2.17. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (Sb),*

(ii) *T has property (az) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

(iii) *T has property (gaz) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

(iv) *T has property $(SBab)$ and $\sigma(T) = \sigma_a(T)$.*

(v) *T has property (Bab) and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.*

(vi) *T has property (ab) and $\sigma_{SBF_+^-}(T) = \sigma_W(T)$.*

Recall that $T \in L(X)$ is said to satisfy *a-Browder's theorem* (resp. *generalized a-Browder's theorem*) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$ (resp. $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$). From [8, Theorem 2.2] (see also [4, Theorem 3.2(ii)]), a-Browder's theorem and generalized a-Browder's theorem are equivalent. It is well known that a-Browder's theorem for T implies Browder's theorem for T, i.e. $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$. Also by [8, Theorem 2.1], Browder's theorem for T is equivalent to generalized Browder's theorem for T, i.e. $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$.

The next result gives the relationship between the properties (Saw) and (Sab).

Theorem 2.18. *Let $T \in L(X)$. Then T has property (Saw) if and only if T has property (Sab) and $E_a^0(T) = \Pi_a^0(T)$.*

***Proof*.** Sufficiency: Assume that T satisfies property (Saw); then it satisfies property $(SBaw)$, by Theorem 2.6. Property $(SBaw)$ implies by [10, Theorem 2.12] that T satisfies a-Weyl's theorem, o equivalently T satisfies a-Browder's theorem and $E_a^0(T) = \Pi_a^0(T)$ [5, Theorem 2.14]. Since that T satisfies property (Saw), it follows that $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$, and hence $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$.

Necessity: Suppose that T satisfies property (Sab) and $E_a^0(T) = \Pi_a^0(T)$. Then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T) = E_a^0(T)$ and so T satisfies property (Saw). ☑

The next example shows that property (Sab) does not imply property (Saw).

Example 2.19. Consider $Q \in L(\ell^2(\mathbb{N}))$ defined as

$$Q(x_1, x_2, x_3, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Then, $\sigma(Q) = \sigma_{SBF_+^-}(Q) = \{0\}$, $E_a^0(Q) = \{0\}$ and $\Pi_a^0(Q) = \varnothing$. We thus have

$$\sigma(Q) \setminus \sigma_{SBF_+^-}(Q) = \Pi_a^0(Q), \qquad \sigma(Q) \setminus \sigma_{SBF_+^-}(Q) \neq E_a^0(Q).$$

Hence, T satisfies property (Sab), but not property (Saw).

Corollary 2.20. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (Saw),*

(ii) *T has properties (Sab) and $(SBaw)$.*

(iii) *T has property (Sab) and satisfies a-Weyl's theorem.*

***Proof*.** Follows immediately from the proof of Theorem 2.18. ☑

The following two examples show that the properties (Saw) (resp. (Sab)) and (gv) (or (gz)) are independient.

Example 2.21. Consider the operator T defined in Example 2.5. Then, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \{0\} \setminus \varnothing = \{0\} \neq \varnothing = E_a^0(T) = \Pi_a^0(T)$, and hence T does not satisfy property (Saw) or property (Sab). Moreover, we have $E(T) = \{0\}$. Consequently, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \{0\} \setminus \varnothing = \{0\} = E(T)$ and so, T satisfies property (gv).

Example 2.22. Let Q be defined for each $x = (\xi_i) \in \ell^1$ by

$$Q(\xi_1, \xi_2, \xi_3, \cdots, \xi_k, \cdots) = (0, \alpha_1\xi_1, \alpha_2\xi_2, \cdots, \alpha_{k-1}\xi_{k-1}, \cdots),$$

where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. It follows from [11, Example 3.12], that

$$\overline{R(Q^n)} \neq R(Q^n), \qquad n = 1, 2, \cdots .$$

Define the operator T on $X = \ell^1 \oplus \ell^1$ by $T = Q \oplus 0$. Then, $N(T) = \{0\} \oplus \ell^1$, $\sigma(T) = \{0\}$, $E(T) = \{0\}$, $E_a^0(T) = \varnothing$. Since $R(T^n) = R(Q^n) \oplus \{0\}$, $R(T^n)$ is not closed for any $n \in \mathbb{N}$; so T is not an upper semi B-Weyl operator and $\sigma_{SBF_+^-}(T) = \{0\}$. We then have,

$$\sigma(T) \setminus \sigma_{SBF_+^-}(T) \neq E(T), \qquad \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T).$$

Hence, T satisfies property (Saw) and hence property (Sab), but does not satisfy property (gv).

In the next theorem we give a new characterization of operators having property (Sb).

Theorem 2.23. Let $T \in L(X)$. Then T has property (Sb) if and only if T has (Bb) and $\operatorname{ind}(\lambda I - T) = 0$ for all $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$.

Proof. Sufficiency: Assume that T satisfies property (Sb); then it also satisfies property (Bb), by [23, Theorem 2.6]. Let $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, as T satisfies property (Sb), then $\lambda \in \Pi^0(T)$. Thus $\lambda \in \operatorname{iso}\sigma(T)$ and $\operatorname{ind}(\lambda I - T) = 0$.

Necessity: Suppose that T satisfies property (Bb) and $\operatorname{ind}(\lambda I - T) = 0$ for all $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, then $\lambda I - T$ is an upper semi-B-Fredholm operator with $\operatorname{ind}(\lambda I - T) = 0$, it follows that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, and as T satisfied property (Bb), $\lambda \in \Pi^0(T)$. This shows $\sigma(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \Pi^0(T)$. Conversely, let $\lambda \in \Pi^0(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_b(T)$ and hence $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. Thus, $\Pi^0(T) \subseteq \sigma(T) \setminus \sigma_{SBF_+^-}(T)$ and T satisfies property (Sb). ☑

Corollary 2.24. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (Saw),*

(ii) *T has property (Sb) and $E_a^0(T) = \Pi_a^0(T)$,*

(iii) *T has property (Bgw) and $\sigma(T) = \sigma_a(T)$,*

(iv) *T has property (Bw) and $\operatorname{ind}(\lambda I - T) = 0$ for all $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$.*

Proof. The equivalence (i)⇔(ii) follows immediately from Theorem 2.18 and Corollary 2.9. The equivalence (i)⇔(iii) follows immediately from Corollary 2.9 and [23, Theorem 2.25]. The equivalence (i)⇔(iv) follows immediately from Corollary 2.9 and [23, Theorem 2.8]. ☑

Similarly to Corollary 2.24, we have the following result.

Corollary 2.25. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (Sab),*

(ii) *T has property (Bgb) and $\sigma(T) = \sigma_a(T)$,*

(iii) *T has property (Bb) and $\mathrm{ind}(\lambda I - T) = 0$ for all $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$.*

Proof. The equivalence (i)⇔(ii) follows immediately from Corollary 2.9 and [23, Theorem 2.8]. The equivalence (i)⇔(iii) follows immediately from Corollary 2.9 and Theorem 2.23. ☑

Theorem 2.26. *Suppose that $T \in L(X)$ has property (Saw). Then:*

(i) *T satisfies generalized a-Browder's theorem and $\sigma(T) = \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_a(T)$.*

(ii) *T satisfies generalized Browder's theorem and $\sigma(T) = \sigma_{BW}(T) \cup \mathrm{iso}\,\sigma_a(T)$.*

Proof. (i) By [6, Theorem 2.4], it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. Let $\lambda \notin \sigma_{SBF_+^-}(T)$. We have the following two cases.

Case 1. $\lambda \notin \sigma(T)$.

Case 2. $\lambda \in \sigma(T)$.

In Case 1, clearly T has SVEP at λ. In Case 2, we have $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$ and since T satisfies property (Saw), it follows that $\lambda \in E_a^0(T)$. Therefore, $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$, and so T has SVEP at λ again.

To show the equality $\sigma(T) = \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$, observe first that $\sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T}) \subseteq \sigma(\mathrm{T})$ holds for every $T \in L(X)$. To show the opposite inclusion, suppose that $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_{SBF_+^-}(T)$. Then $\lambda \in E_a^0(T)$, since T satisfies property (Saw). Therefore, $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$, and so $\sigma(T) \subseteq \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$. This shows that $\sigma(T) = \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$.

(ii) Follows from (i), by using the fact that generalized a-Browder's theorem implies generalized Browder's theorem, and the equality $\sigma(T) = \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$, implies the inclusion $\sigma(T) \subseteq \sigma_{BW}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$, leading to $\sigma(T) = \sigma_{BW}(T) \cup \mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$. ☑

For $T \in L(X)$, define $\Pi_+(T) = \sigma(T) \setminus \sigma_{LD}(T)$. The precise relationship between generalized a-Browder's theorem and property (Saw) is described by the following theorem.

Theorem 2.27. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (Saw),*

(ii) *T satisfies generalized a-Browder's theorem and $\Pi_+(T) = E_a^0(T)$.*

Proof. (i)⇒(ii) Assume that T satisfies property (Saw). Then, by Theorem 2.26, it is sufficient to prove $\Pi_+(T) = E_a^0(T)$. Indeed, $E_a^0(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$ $= \sigma(T) \setminus \sigma_{LD}(T) = \Pi_+(T)$, since T satisfies property (Saw) and generalized a-Browder's theorem by Theorem 2.27.

(ii)⇒(i) If T satisfies generalized a-Browder's theorem and $\Pi_+(T) = E_a^0(T)$, then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{LD}(T) = \Pi_+(T) = E_a^0(T)$. Therefore, T satisfies property (Saw). ☑

Corollary 2.28. *If $T \in L(X)$ has SVEP at each $\lambda \notin \sigma_{SBF_+^-}(T)$, then T has property (Saw) if and only if $E_a^0(T) = \Pi_+(T)$.*

Proof. The hypothesis T has SVEP at each $\lambda \notin \sigma_{SBF_+^-}(T)$, implies that T satisfies generalized a-Browder's theorem. Therefore, if $E_a^0(T) = \Pi_+(T)$, then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{LD}(T) = \Pi_+(T) = E_a^0(T)$. ☑

***Remark* 2.29.** It was proved in [12, Lemma 2.4], that if T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$ (resp. T has SVEP at every $\lambda \notin \sigma_{SBF_-^+}(T)$), then $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T) = \sigma_D(T)$ and $\sigma_a(T) = \sigma(T)$ (resp. $\sigma_{BW}(T^*) = \sigma_{SBF_+^-}(T^*) = \sigma_D(T^*)$ and $\sigma_a(T^*) = \sigma(T^*)$). Under the above results, clearly we have that if T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$ (resp. T has SVEP at every $\lambda \notin \sigma_{SBF_-^+}(T)$), then the properties (Bw), (Sw), (Saw), (Bgw), (Baw) and $(SBaw)$ are equivalent for T (resp. for T^*). In the same form, we obtain equivalence for the properties (Bb), (Sb), (Sab), (Bgb), (Bab) and $(SBab)$.

Theorem 2.30. *Suppose that $T \in L(X)$ has property property (Saw). Then:*

(i) $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_{SF_+^-}(T) = \sigma_W(T) = \sigma_{LD}(T) = \sigma_D(T) = \sigma_{ub}(T) = \sigma_b(T)$ *and* $\sigma(T) = \sigma_a(T)$.

(ii) $\Pi^0(T) = \Pi_a^0(T) = \Pi(T) = \Pi_a(T) = E^0(T) = E_a^0(T)$ *and* $E(T) = E_a(T)$.

(iii) *The properties $(SBaw)$, $(SBab)$, (Bgw), (Bgb), (w), (b), (gb), (z), (az), (gaz), (v), (ah), (gah), (Baw), (Bab), (gab), (Bw), (Bb), (aw), (ab), $(\mathcal{B})$, $(a\mathcal{B})$, $(g\mathcal{B})$, $(ga\mathcal{B})$, $(\mathcal{W})$ and $(a\mathcal{W})$ for T are equivalent, and T satisfies each of these properties.*

(iv) *The properties* (gw), (gz), (gv), (gaw), $(g\mathcal{W})$ *and* $(ga\mathcal{W})$ *for* T *are equivalent.*

Proof. (i) By Theorem 2.6, the equality $\sigma(T) = \sigma_a(T)$ holds. The equalities $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_{SF_+^-}(T) = \sigma_W(T)$ follows from Theorems 2.2, 2.10 and 2.13. Since the inclusions $\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_{ub}(T) \subseteq \sigma_b(T)$ and $\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_D(T) \subseteq \sigma_b(T)$ hold, it is sufficient to prove $\sigma_{SBF_+^-}(T) = \sigma_b(T)$. Indeed, $\sigma_{SBF_+^-}(T) = \sigma_W(T) = \sigma_b(T)$, since T satisfies property (Saw) and generalized Browder's theorem or equivalently Browder's theorem, by Theorem 2.27.

(ii) The equalities $\Pi^0(T) = \Pi^0_a(T) = \Pi(T) = \Pi_a(T)$, $E^0(T) = E^0_a(T)$ and $E(T) = E_a(T)$ follows from (i). By Theorem 2.18, the equality $\Pi^0_a(T) = E^0_a(T)$ holds.

(iii) By Theorem 2.2, T satisfies property (z), and the equivalence between all properties follows from (i) and (ii).

(iv) Follows from (i) and (ii). ☑

Similarly to Theorem 2.30, we have the following result.

Theorem 2.31. *Suppose that* $T \in L(X)$ *has property* (Sab). *Then:*

(i) $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_{SF_+^-}(T) = \sigma_W(T) = \sigma_{LD}(T) = \sigma_D(T) = \sigma_{ub}(T) = \sigma_b(T)$ *and* $\sigma(T) = \sigma_a(T)$.

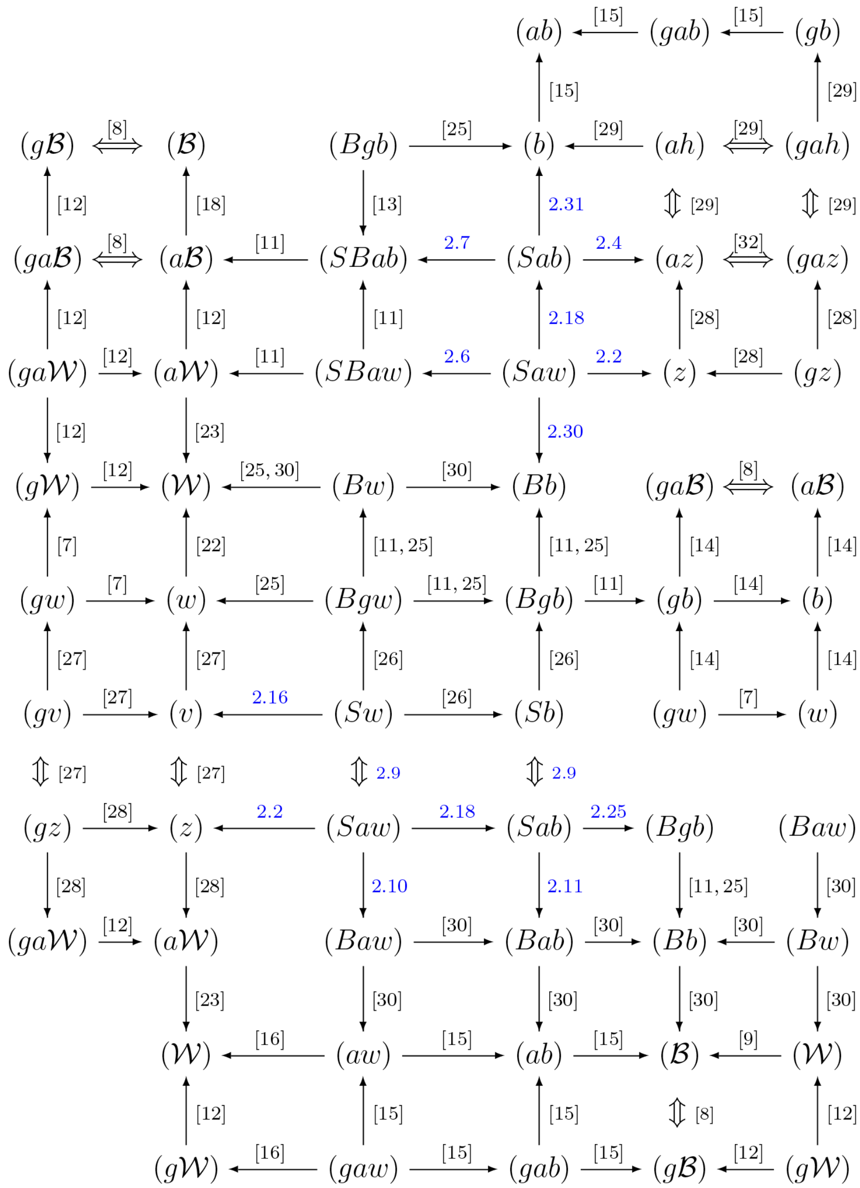
(ii) $\Pi^0(T) = \Pi^0_a(T) = \Pi(T) = \Pi_a(T)$, $E^0(T) = E^0_a(T)$ *and* $E(T) = E_a(T)$.

(iii) *The properties* $(SBab)$, (Bgb), (b), (gb), (az), (gaz), (ah), (gah), (Bab), (gab), (Bb), (ab), $(\mathcal{B})$, $(a\mathcal{B})$, $(g\mathcal{B})$ *and* $(ga\mathcal{B})$ *for* T *are equivalent, and* T *satisfies each of these properties.*

(iv) *The properties* $(SBaw)$, (Bgw), (w), (z), (v), (Baw), (Bw), (aw), $(\mathcal{W})$ *and* $(a\mathcal{W})$ *for* T *are equivalent.*

(v) *The properties* (gw), (gz), (gv), (gaw), $(g\mathcal{W})$ *and* $(ga\mathcal{W})$ *for* T *are equivalent.*

In the following diagram arrows mean implications between the properties defined above. The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (numbers in square brackets).


(ab) [15] (gab) [15] (gb)
[15] [29]
(gℬ) [8] (ℬ) (Bgb) [25] (b) [29] (ah) [29] (gah)
[12] [18] [13] 2.31 [29] [29]
(gaℬ) [8] (aℬ) [11] (SBab) 2.7 (Sab) 2.4 (az) [32] (gaz)
[12] [12] [11] 2.18 [28] [28]
(ga𝒲) [12] (a𝒲) [11] (SBaw) 2.6 (Saw) 2.2 (z) [28] (gz)
[12] [23] 2.30
(g𝒲) [12] (𝒲) [25, 30] (Bw) [30] (Bb) (gaℬ) [8] (aℬ)
[7] [22] [11, 25] [11, 25] [14] [14]
(gw) [7] (w) [25] (Bgw) [11, 25] (Bgb) [11] (gb) [14] (b)
[27] [27] [26] [26] [14] [14]
(gv) [27] (v) 2.16 (Sw) [26] (Sb) (gw) [7] (w)
[27] [27] 2.9 2.9
(gz) [28] (z) 2.2 (Saw) 2.18 (Sab) 2.25 (Bgb) (Baw)
[28] [28] 2.10 2.11 [11, 25] [30]
(ga𝒲) [12] (a𝒲) (Baw) [30] (Bab) [30] (Bb) [30] (Bw)
[23] [30] [30] [30] [30]
(𝒲) [16] (aw) [15] (ab) [15] (ℬ) [9] (𝒲)
[12] [15] [15] [8] [12]
(g𝒲) [16] (gaw) [15] (gab) [15] (gℬ) [12] (g𝒲)

References

[1] P. Aiena, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer, 2004.

[2] ———, *Quasi-Fredholm operators and localized SVEP*, Acta Sci. Math. (Szeged) **73** (2007), 251–263.

[3] P. Aiena, E. Aponte, and E. Balzan, *Weyl type Theorems for left and right polaroid operators*, Integr. Equ. Oper. Theory **66** (2010), 1–20.

[4] P. Aiena, M. T. Biondi, and C. Carpintero, *On drazin invertibility*, Proc. Amer. Math. Soc. **136** (2008), 2839–2848.

[5] P. Aiena, C. Carpintero, and E. Rosas, *Some characterization of operators satisfying a-Browder's theorem*, J. Math. Anal. Appl. **311** (2005), 530–544.

[6] P. Aiena and L. Miller, *On generalized a-Browder's theorem*, Studia Math. **180** (2007), 285–299.

[7] M. Amouch and M. Berkani, *On the property* (*gw*), Mediterr. J. Math. **5** (2008), 371–378.

[8] M. Amouch and H. Zguitti, *On the equivalence of Browder's theorem and generalized Browder's theorem*, Glasgow Math. J. **48** (2006), 179–185.

[9] M. Berkani, *Restriction of an operator to the range of its powers*, Studia Math. **140** (2000), 163–175.

[10] M. Berkani, M. Kachad, H. Zariouh, and H. Zguitti, *Variations on a-Browder-type theorems*, Sarajevo J. Math. **9** (2013), 271–281.

[11] M. Berkani and J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003), 359–376.

[12] M. Berkani, M. Sarih, and H. Zariouh, *Browder-type theorems and SVEP*, Mediterr. J. Math. **8** (2011), 399–409.

[13] M. Berkani and H. Zariouh, *Extended Weyl type theorems*, Math. Bohemica **134** (2009), 369–378.

[14] ———, *New extended Weyl type theorems*, Mat. Vesnik **62** (2010), 145–154.

[15] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.

[16] J. K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.

[17] A. Gupta and N. Kashayap, *Property* (Bw) *and Weyl type theorems*, Bull. Math. Anal. Appl. **3** (2011), 1–7.

[18] ———, *Variations on Weyl type theorems*, Int. J. Contemp. Math. Sciences **8** (2012), 189–198.

[19] V. Rakočević, *On a class of operators*, Mat. Vesnik **37** (1985), 423–426.

[20] ———, *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl. **34** (1989), 915–919.

[21] M. H. M. Rashid, *Properties* (t) *and* (gt) *for bounded linear operators*, Mediterr. J. Math. **11** (2014), 729–744.

[22] M. H. M. Rashid and T. Prasad, *Variations of Weyl type theorems*, Ann Funct. Anal. Math. **4** (2013), 40–52.

[23] ———, *Property* (Sw) *for bounded linear operators*, Asia-European J. Math. **8** (2015), 14 pages, DOI: 10.1142/S1793557115500126.

[24] J. Sanabria, C. Carpintero, E. Rosas, and O. García, *On generalized property* (v) *for bounded linear operators*, Studia Math. **212** (2012), 141–154.

[25] H. Zariouh, *Property* (gz) *for bounded linear operators*, Mat. Vesnik **65** (2013), 94–103.

[26] ———, *New version of property* (az), Mat. Vesnik **66** (2014), 317–322.

[27] H. Zariouh and H. Zguitti, *Variations on Browder's theorem*, Acta Math. Univ. Comenianae **81** (2012), 255–264.

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