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On intersections of the exponential and logarithmic curves

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Abstract

We consider the curves $y = a^x$ and $y = \log_a x$ and their intersecting points for various bases a. Although this problem belongs to the elementary calculus, it turns out that the problem of determining number of these points, for $a \in \langle 0, 1\rangle$, is overlooked, so far. We prove that this number can be $0, 1, 2$ or, even, 3, depending on the base a.

Keywords: exponential function, inflection, stationary point, homeomorphism

MSC: 26A06, 26A09.

1. Introduction

We consider the problem of determining the number of intersecting points of the graphs of the functions $f(x) = a^x$ and $g(x) = \log_a x$ depending on the base a. This problem is reduced to the study of solutions of the system

$$\begin{cases} y = a^x \\ y = \log_a x \end{cases} \tag{1.1}$$

which is equivalent to the equation

$$a^x = \log_a x \tag{1.2}$$

depending on a, $a \in \mathbb{R}^+ \setminus \{1\}$.

Although this problem belongs to the elementary calculus, usually, it was not considered in sufficient detail in the calculus courses on universities worldwide. Moreover, students of mathematics and many professional mathematicians are likely to think that these curves do not intersect, for $a > 1$, and meet at only one point, for $a \in \langle 0, 1\rangle$. This impression is caused by many calculus books, math teachers or professors who usually take nice bases $a = 2, e, 10..$ as standard examples for the exponential and logarithmic curves. However, in [1] and [2] can be found a solution of this problem for $a > 1$. However, for $a \in \langle 0, 1\rangle$, in [1] can be found an incorrect claim (Proposition 1) that the graphs $y = a^x$ and $y = \log_a x$ always meet at only one point. The author's conclusion seems correct at the first glance. Indeed, if we considered these curves for some standard bases $\frac{1}{2}, e^{-1} \ldots$ or if we try to make a sketch of the graphs of the functions $f(x) = a^x$ and $g(x) = \log_a x$, $a \in \langle 0, 1\rangle$, the inference, suggested by the picture, would be the same. Unexpectedly, this is not the case. Counterexample which was a motivation for this work is the base $a = \frac{1}{16}$. Namely, it holds

$$\log_{\frac{1}{16}} \frac{1}{4} = \frac{1}{2}, \quad \left(\frac{1}{16}\right)^{\frac{1}{4}} = \frac{1}{2},$$

$$\log_{\frac{1}{16}} \frac{1}{2} = \frac{1}{4}, \quad \left(\frac{1}{16}\right)^{\frac{1}{2}} = \frac{1}{4}.$$

This means that $\left(\frac{1}{4}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$ are common points of the graphs of the functions $g(x) = \log_{\frac{1}{16}} x$ and $f(x) = \left(\frac{1}{16}\right)^x$. Since the both curves must meet the line $y = x$ at the same point we infer that there are (at least) 3 intersecting points.

The main goal of this paper is to prove:

Theorem 1.1. *The equation* (1.2)*:*
has no solutions, provided $a \in \langle \sqrt[e]{e}, +\infty\rangle$,
has exactly one solution, provided $a \in \left[\frac{1}{e^e}, 1\right\rangle \cup \{\sqrt[e]{e}\}$,
has exactly two solutions, provided $a \in \langle 1, \sqrt[e]{e}\rangle$,
has exactly three solutions, provided $a \in \left\langle 0, \frac{1}{e^e}\right\rangle$.

In order to eliminate any intuitive concluding and to avoid any possible ambiguity and incorrect inferences, which a shallow considering of the graphs might cause, we will conduct the proof of this theorem very strictly (in the mathematical sense). A necessary mathematical tool needed for the proof belongs to elementary calculus and to topology. We will split the proof of the theorem into two separate cases: $a > 1$ and $a < 1$. In the both cases we need the following corollary which is an immediate consequence of the Intermediate value theorem and some elementary facts of mathematical analysis (see e.g. [3]).

Corollary 1.2. *Let* $u\colon [c, d] \to \mathbb{R}$ *be a continuous function such that* $u(c)\, u(d) \leq 0$.

(i) *If* $u(c)\, u(d) < 0$, *then* u *has at least one zero* $x_0 \in \langle c, d\rangle$.

(ii) *If* u *is a strictly monotonic function, then* u *has exactly one zero* $x_0 \in [c, d]$.

Hereinafter, for a real function which is given by a formula we understand that the function domain is the (maximal) natural domain of that formula.

We will consider two (in)equations to be equivalent provided their solution sets coincide.

2. The case $a > 1$

The proof of this case can be given as an assignment to students of mathematics in some elementary courses. It is based on the following, several, auxiliary lemmata whose proofes we leave to the reader. Acctually, proving of these claims could be a good exercise for students in higher classes of a secondary school, providing they have sufficently ambitious math teacher.

Lemma 2.1. *If (x_0, y_0) is a solution of the system (1.1), for $a > 1$, then $x_0 = y_0$.*

Lemma 2.2. *If $a > 1$, the equation (1.2) is equivalent to the equation*

$$a^x = x, \tag{2.1}$$

and thus, the solution sets of (1.2) and (2.1) coincide with the set of zeros of the function $\chi_a(x) = a^x - x$.

Lemma 2.3. *If $a > 1$, the function χ_a is continuously differentiable. It is strictly decreasing on the interval $\langle -\infty, \frac{1}{\ln a}\ln(\frac{1}{\ln a})\rangle$, while it is strictly increasing on the interval $\langle \frac{1}{\ln a}\ln(\frac{1}{\ln a}), +\infty\rangle$. It reaches the global minimum at the point $x_a^* = \frac{1}{\ln a}\ln(\frac{1}{\ln a})$.*

Lemma 2.4. *Let $a > 1$. Then the equation (1.2) has: no zeros if and only if $\chi_a(x_a^*) > 0$; a unique zero if and only if $\chi_a(x_a^*) = 0$; exactly two zeros if and only if $\chi_a(x_a^*) < 0$.*

Let us interpret the previous result in term of the base a, i.e., how does a value $\chi_a(x_a^*)$ depend on a. Since the procedure is the same for all cases, it is sufficient to consider the case $\chi_a(x_a^*) < 0$. This is equivalent to $a^{x_a^*} < x_a^*$, which means

$$a^{\frac{1}{\ln a}\ln\left(\frac{1}{\ln a}\right)} < \frac{1}{\ln a}\ln\left(\frac{1}{\ln a}\right).$$

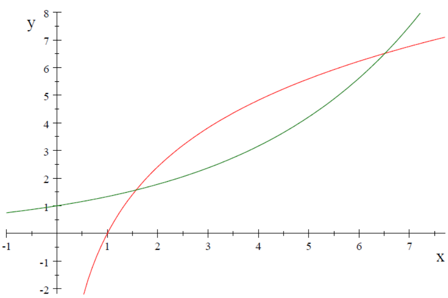
Now, one obtains, in several steps, the following mutually equivalent inequalities

$$\frac{1}{\ln a}\ln\left(\frac{1}{\ln a}\right)\ln a < \ln\left(\frac{1}{\ln a}\ln\left(\frac{1}{\ln a}\right)\right) \Leftrightarrow \ln\left(\frac{1}{\ln a}\right) < \ln\left(\frac{1}{\ln a}\ln\left(\frac{1}{\ln a}\right)\right)$$

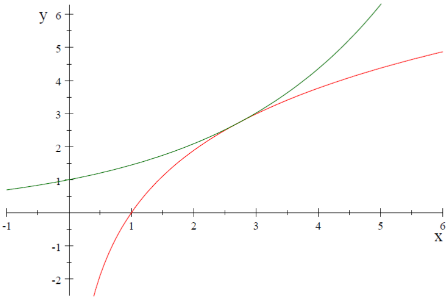
$$\frac{1}{\ln a} < \frac{1}{\ln a}\ln\left(\frac{1}{\ln a}\right) \Leftrightarrow 1 < \ln\left(\frac{1}{\ln a}\right) \Leftrightarrow \ln a < e^{-1} \Leftrightarrow a < e^{e^{-1}}.$$

Thus, the equation (1.2) has: exactly two solutions whenever $a \in \langle 1, \sqrt[e]{e}\rangle$, exactly one solution whenever $a = \sqrt[e]{e}$ (the solution is $x_0 = e$), no solutions whenever $a \in \langle \sqrt[e]{e}, +\infty\rangle$.

Example 2.5.



(a) $a = \frac{4}{3}$ $y = a^x$, $y = \log_a x$



(b) $a = \sqrt[e]{e}$ $y = a^x$, $y = \log_a x$

3. The case $0 < a < 1$

Unlike the previous case, the proof of this case is rather nontrivial. In order to make this proof easier to follow, we will split it into nine simpler claims.

Lemma 3.1. *Let* $0 < a < 1$. *then the curve* $y = a^x$ *(*$y = \log_a x$*) and the line* $y = x$ *meet at a single point* (ξ_a, ξ_a), $\xi_a \in \langle 0, 1\rangle$. *The point* ξ_a *is the solution of the equation* (1.2). *The function* $\zeta : \langle 0, 1\rangle \to \langle 0, 1\rangle$, $\zeta(a) = \xi_a$, *which assigns the point* ξ_a *to each base a, is an increasing homeomorphism whose inverse is given by the rule* $a_\xi = \zeta^{-1}(\xi) = \xi^{\frac{1}{\xi}}$.

Proof. Let λ be the real function given by $\lambda(x) = a^x - x$, for every $0 < a < 1$. Since $\lambda(0) = 1$ and $\lambda(1) < 0$, we may apply Corollary 1.2 on the function λ to infer that the curve $y = a^x$ intersects the line $y = x$. It remains to prove that they meet at exactly one point. Suppose that $\left(\xi, \xi = a^\xi\right)$ and $\left(\xi', \xi' = a^{\xi'}\right)$ are two different intersection points. There is no loss of generality in assuming $\xi < \xi'$. Since the function a^x is strictly decreasing ($a < 1$) it follows that $a^\xi = \zeta > \zeta' = a^{\xi'}$, which is an obvious contradiction.

Given an $x \in \langle 0, 1\rangle$, it is clear that, for $a = x^{\frac{1}{x}}$, it holds $a^x = x$. By examining limits $\lim_{x\to 0^+} x^{\frac{1}{x}} = 0$, $\lim_{x\to 1} x^{\frac{1}{x}} = 1$,and the first derivative $y' = x^{\frac{1}{x}}\frac{1-\ln x}{x^2}$of the function

$$y(x) = \begin{cases} 0, & x = 0, \\ x^{\frac{1}{x}}, & 0 < a < 1, \end{cases}$$

one infers that it is a strictly increasing mapping on the interval $[0, 1]$ and it maps the interval $[0, 1]$ onto itself. Therefore, it is a homeomorphism and its inverse restricted to the interval $\langle 0, 1\rangle$ is the function $\zeta : \langle 0, 1\rangle \to \langle 0, 1\rangle$, $\zeta(a) = \xi_a$, exactly as asserted. □

Lemma 3.2. *If* $0 < a < 1$, *the solution set of the equation* (1.2) *is a nonempty subset of the interval* $\langle 0, 1\rangle$. *If that set is finite, then its cardinality is odd.*

Proof. Let $0 < a < 1$. Then, obviously, since the equation (1.2) is defined only for $x > 0$, it has no solution on the interval $\langle -\infty, 0]$. Further, it holds that $a^x > 0$, $\log_a x \leq 0$, for every $x \in [1, +\infty\rangle$. Therefore, the equation (1.2) has no solution on the interval $[1, +\infty\rangle$. Consequently, by Lemma 3.1, the solution set of (1.2) is a nonempty subset of the interval $\langle 0, 1\rangle$. Let us assume that (x_0, y_0), $x_0 \neq y_0$, is an intersection point of the curves $y = a^x$ and $y = \log_a x$. Then, since these curves are mutually symmetric regarding the line $y = x$, they also meet at the point (y_0, x_0). Therefore, if there are only finitely many intersecting points of these curves, the number of those points which do not belong to the line $y = x$ is even. Now the statement follows by Lemma 3.1. □

Lemma 3.3. *If* $0 < a < 1$, *the solution set of the equation* (1.2) *coincides with the solution set of the equation*

$$a^{a^x} = x, \tag{3.1}$$

i.e., it coincides with the set of zeros of the real function $H_a(x) = a^{a^x} - x$.

Proof. Notice that there are no solution of (3.1) outside of the domain $\langle 0, \infty\rangle$ of the equation (1.2). Because of the injectivity of the exponential function, it is clear that (1.2) is equivalent to (3.1). □

Let us examine the functions $H_a(x) = a^{a^x} - x$ and $\varphi_a(x) = a^{a^x}$ which are, obviously, both continuously differentiable.

Lemma 3.4. *Let* $0 < a < 1$. *The function* φ_a *is strictly increasing, and the lines* $y = 1$ *and* $y = 0$ *are its horizontal asymptotes (from the right side and left side, respectively). The functions* φ_a *and* H_a *are convex on the interval* $\langle -\infty, \overline{x}_a\rangle$, *and the both are concave on the interval* $\langle \overline{x}_a, \infty\rangle$, *where*

$$\overline{x}_a = \log_a \log_a e^{-1}$$

is the common inflection point satisfying $\varphi_a(\overline{x}_a) = e^{-1}$.

Proof. Since $0 < a < 1$, it holds that $\lim\limits_{x\to+\infty} a^{a^x} = a^0 = 1$ and $\lim\limits_{x\to-\infty} a^{a^x} = a^\infty = 0$. Hence, the lines $y = 1$ and $y = 0$ are horizontal asymptotes of the function φ_a indeed.

Since, $\varphi_a'(x) = a^{a^x} a^x \ln^2 a > 0$, for every $x \in \mathbb{R}$,it follows that φ_a is strictly increasing. Further,

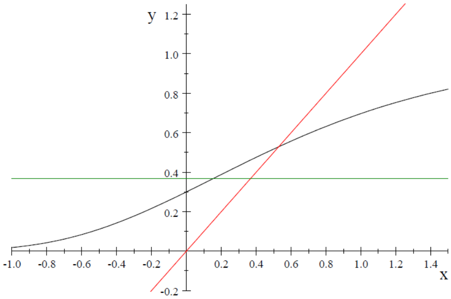
$$\begin{aligned} H_a''(x) = \varphi_a''(x) &= a^{a^x} a^x (\ln^2 a) a^x \ln^2 a + a^{a^x} a^x \ln^3 a = \\ = \varphi_a''(x) &= \underbrace{a^{a^x+x}}_{>0} \underbrace{\ln^3 a}_{<0} (a^x \ln a + 1)\,. \end{aligned}$$

Therefore, $H_a''(x) = 0$ ($\varphi_a''(x) = 0$) if and only if $a^x \ln a + 1 = 0$. Consequently,

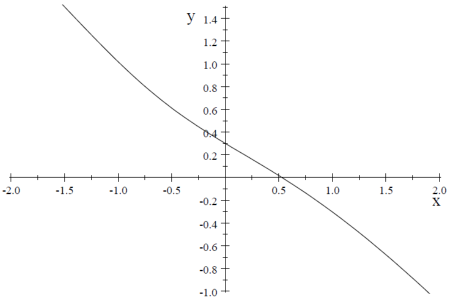
$$\overline{x}_a = \frac{1}{\ln a} \ln(\frac{-1}{\ln a}) = \log_a \log_a e^{-1} \text{ and } \varphi_a(\overline{x}_a) = a^{a^{\log_a \log_a e^{-1}}} = e^{-1}.$$

Now, it is trivial to check that $H_a''(x) = \varphi_a''(x) > 0$, for every $x \in \langle -\infty, \overline{x}_a\rangle$ and $H_a''(x) = \varphi_a''(x) < 0$, for every $x \in \langle \overline{x}_a, \infty\rangle$, which completes the proof. □

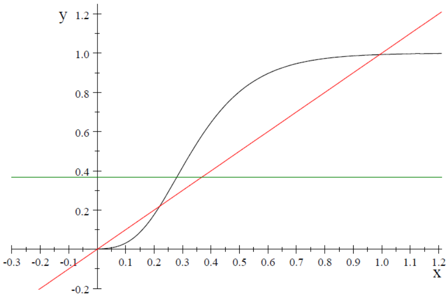
In the figures below, the graphs of the functions φ_a and H_a, for several bases a, $0 < a < 1$, are shown. In order to emphasize the inflection $\left(\overline{x}_a, e^{-1}\right)$ and solutions of the equation (3.1), the graph of the function φ_a is presented along with the lines $y = x$ and $y = e^{-1}$.



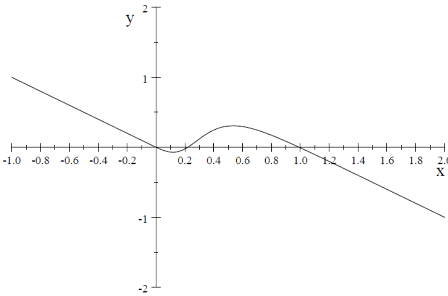
(a) $\varphi_{a=0.3}(x) = 0.3^{0.3^x}$



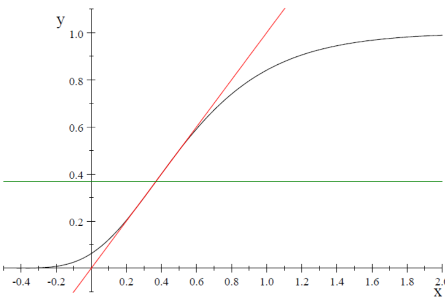
(b) $H_{a=0.3}(x) = 0.3^{0.3^x} - x$



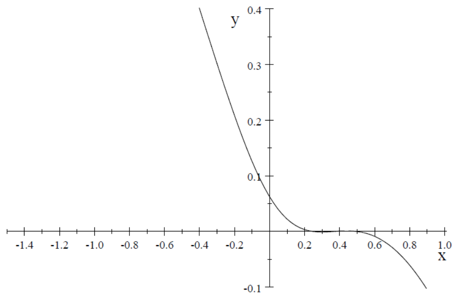
(a) $\varphi_{a=0.001}(x) = 0.001^{0.001^x}$



(b) $H_{a=0.001}(x) = 0.001^{0.001^x} - x$



(a) $\varphi_{a=\frac{1}{16}}(x) = \left(\frac{1}{16}\right)^{\left(\frac{1}{16}\right)^x}$



(b) $H_{a=\frac{1}{16}}(x) = \left(\frac{1}{16}\right)^{\left(\frac{1}{16}\right)^x} - x$

Lemma 3.5. *If $0 < a < 1$ the function H_a has at most two stationary points in* *the interval* $\langle 0,1\rangle$, *i.e., the equation*

$$H_a'(x) = 0$$

has 0, 1 *or* 2 *solutions in the interval* $\langle 0,1\rangle$. *If* $a < e^{-1}$, H_a *has at most two stationary points, while if* $a \geq e^{-1}$, H_a *has at most one stationary point.*

Proof. First,

$$H_a'(x) = 0 \text{ if and only if } a^{a^x} a^x \ln^2 a - 1 = 0.$$

Thus,

$$a^{a^x + x} = \frac{1}{\ln^2 a} \text{ if and only if } a^x + x = \frac{1}{\ln a} \ln(\frac{1}{\ln^2 a}).$$

We need to determine the number of solutions of the equation

$$a^x + x = \frac{1}{\ln a} \ln(\frac{1}{\ln^2 a}) \tag{3.2}$$

on the interval $\langle 0,1\rangle$. Given an $0 < a < 1$, let us define the real function u_a by $u_a(x) = a^x + x$. It holds $u_a'(x) = a^x \ln a + 1$. Now, one can easily verify that $u_a'(\overline{x}_a) = 0$, and conclude that the function u_a is strictly increasing on the interval $\langle \overline{x}_a, \infty\rangle$ and that it is strictly decreasing on the interval $\langle -\infty, \overline{x}_a\rangle$. Notice that

$$\overline{x}_a > 0 \ (\overline{x}_a < 0) \text{ if and only if } a < e^{-1} \ (a > e^{-1})$$

and that $\overline{x}_a = 0$ for $a = e^{-1}$. We infer that the function u_a reaches its global minimum at $\overline{x}_a$, and that

$$u_a(\overline{x}_a) = a^{\overline{x}_a} + \overline{x}_a = a^{\log_a \log_a e^{-1}} + \overline{x}_a = \frac{-1}{\ln a} + \overline{x}_a.$$

Hence, for $a = e^{-1}(\overline{x}_a = 0)$, we have $u(\overline{x}_a) = 1$.
Now, we infer that the number of intersection points of the curve $y = u_a(x)$, for $x \in \langle 0,1\rangle$, and the line $y = \frac{1}{\ln a} \ln(\frac{1}{\ln^2 a})$ coincide with the number of solution of the equation (3.2) in the interval $\langle 0,1\rangle$. Thus, by assuming $a \geq e^{-1}$, we obtain the strict monotonicity of the restriction of function u_a to the interval $\langle 0,1\rangle$, which implies that there are only 0 or 1 intersection points. Suppose that $a < e^{-1}$. Then, since the function u_a is strictly decreasing on the interval $\langle 0, \overline{x}_a]$ and strictly increasing on the interval $[\overline{x}_a, 1\rangle$, there are 0, 1 or 2 intersection points. □

Lemma 3.6. *If* $0 < a < 1$, *the equation* (1.2) *has either one or three solutions.*

Proof. If we assume that (1.2) has infinitely many solutions, then, by Lemmata 3.2 and 3.3, the function H_a has infinitely many zeros in the interval $\langle 0,1\rangle$. Now, by applying Rolle's theorem, one infers that H_a has infinitely many stationary points in $\langle 0,1\rangle$ which contradicts Lemma 3.5. Therefore, by Lemma 3.2, the number of solutions of the equation (1.2) is finite and odd. That number cannot exceed 3 because, by Rolle's theorem, in such a case the function H_a would have at least four stationary points in $\langle 0,1\rangle$ which is, according to Lemma 3.5, impossible. □

Lemma 3.7. *Let $0 < a < 1$. If the equation* (1.2) *has three solutions, then it holds $a < e^{-e}$.*

Proof. If the equation (1.2) has 3 solutions then, by Lemmata 3.2, 3.3 and Rolle's theorem, the function H_a has at least two stationary points in $\langle 0,1\rangle$. Now, by Lemma 3.5, it follows that there are exactly two stationary points of the function H_a in $\langle 0,1\rangle$. It implies that the equation (3.2) has two solutions in $\langle 0,1\rangle$ and $a < e^{-1}$. Consequently, for $x \in \langle 0,1\rangle$, the line $y = \frac{1}{\ln a}\ln(\frac{1}{\ln^2 a})$ meets the curve $y = u_a(x)$ at exactly two points, which is equivalent to

$$u_a(\overline{x}_a) < \frac{1}{\ln a}\ln\left(\frac{1}{\ln^2 a}\right) < 1, \quad a \in \left\langle 0, e^{-1}\right\rangle. \tag{3.3}$$

We propose to find solutions of this system of inequalities, i.e., to solve the system (3.3) in the terms of a. Let us put

$$t = -\frac{1}{\ln a}. \tag{3.4}$$

Notice that this substitution defines a bijective correspondence between $a \in \langle 0, e^{-1}\rangle$ and $t \in \langle 0,1\rangle$. The replacement with t in (3.3) yields the system

$$t - t\ln t < -t\ln t^2 < 1, \quad t \in \langle 0,1\rangle, \tag{3.5}$$

which we need to solve in terms of t. Now, from the first inequality $t - t\ln t < -t\ln t^2$, one obtains the following, mutually equivalent, inequalities

$$t < -t\ln t \Leftrightarrow t(1+\ln t) < 0 \Leftrightarrow 1 + \ln t < 0 \Leftrightarrow t < e^{-1}.$$

Now, by (3.4), one infers that $-\frac{1}{\ln a} < e^{-1}$ which is equivalent to $\ln a < -e$. It follows that $a < e^{-e}$, which means that the solutions of the first inequality of the system (3.3) are all $a \in \langle 0, e^{-e}\rangle$.

Further, the second inequality $-t\ln t^2 < 1$ of the system (3.5) is equivalent to

$$-t\ln t < \frac{1}{2}, \tag{3.6}$$

which is fulfilled for every $t \in \langle 0,1\rangle$. Indeed, by examining the function $w(t) = -t\ln t$ and its derivative $w'(t) = -\ln t - 1$, one can straightforwardly verify that w reaches the global maximum at the point $t_0 = e^{-1}$. Therefore, $w(e^{-1}) = e^{-1} < \frac{1}{2}$ implies (3.6), for every $t \in \langle 0,1\rangle$. Consequently, the solutions of the second inequality of the system (3.3) are all $a \in \left\langle 0, e^{-1}\right\rangle$. Finally, the solution of the system (3.3) is the interval

$$\left\langle 0, e^{-e}\right\rangle = \left\langle 0, e^{-e}\right\rangle \cap \left\langle 0, e^{-1}\right\rangle. \qquad \square$$

Lemma 3.8. *For every $a \in [e^{-e}, 1\rangle$, the equation* (1.2) *has a unique solution. Especially, for $a = e^{-e}$ the solution is e^{-1}.*

Proof. By Lemma 3.6 and 3.7, it follows that, for $a \geq e^{-e}$, (1.2) has only one solution. According to Lemma 3.1, that solution is the point ξ_a such that $a^{\xi_a} = \xi_a = \log_a \xi_a$. Especially, for $a = e^{-e}$, it holds $\xi_a = e^{-1}$. Indeed, $(e^{-e})^{e^{-1}} = (e^{-e})^{\frac{1}{e}} = e^{-1}$. □

Lemma 3.9. *If $a \in \langle 0, e^{-e} \rangle$, then the equation* (1.2) *has exactly three solutions.*

Proof. According to Lemma 3.6, for every $a \in \langle 0, e^{-e} \rangle$ the equation (1.2) has 1 or 3 solutions. Let us prove that the value of an inflection point $\overline{x}_a$ of the function H_a and φ_a ranges from 0 to e^{-1}, for every $a \in \langle 0, e^{-e} \rangle$. By using L'Hospital's rule, one easily evaluates the following limits:

$$\lim_{a\to 0^+} \overline{x}_a = \lim_{a\to 0^+} \frac{\ln(\frac{-1}{\ln a})}{\ln a} = \left[\frac{\infty}{-\infty}\right] = \lim_{a\to 0^+} \frac{-\ln(a)(\frac{1}{\ln^2 a})\frac{1}{a}}{\frac{1}{a}} = \lim_{a\to 0^+} \frac{-1}{\ln a} = 0,$$

$$\lim_{a\to e^{-e}} \overline{x}_a = \frac{1}{-e} \ln \frac{-1}{-e} = \frac{1}{e}.$$

We are claming that the function $\nu : \langle 0, e^{-e} \rangle \to \mathbb{R}$,

$$\nu(a) = \overline{x}_a = \frac{1}{\ln a} \ln\left(\frac{-1}{\ln a}\right),$$

is an increasing mapping. Indeed, from its first derivative

$$\nu'(a) = \frac{-1 - \ln(-\frac{a}{\ln a})}{a \ln^2 a},$$

one infers that

$$\nu'(a) > 0 \quad \text{if and only if} \quad -1 - \ln\left(-\frac{1}{\ln a}\right) > 0,$$

which is equivalent to

$$e^{-1} > -\frac{1}{\ln a} \Leftrightarrow \ln a < -e \Leftrightarrow a < e^{-e}.$$

Hence, $\nu'(a) > 0$, for every $a \in \langle 0, e^{-e} \rangle$. It follows that the function ν is an increasing and bijective mapping onto its image $\nu(\langle 0, e^{-e} \rangle) = \langle 0, e^{-1} \rangle$. Consequently, $\overline{x}_a < e^{-1}$, for every $a \in \langle 0, e^{-e} \rangle$. Now, by Lemma 3.4, it follows that $\overline{x}_a < \varphi_a(\overline{x}_a) = e^{-1}$, which implies that $H_a(\overline{x}_a) > 0$, for every $a \in \langle 0, e^{-e} \rangle$. On the other hand, it holds

$$H_a(1) = \varphi_a(1) - 1 = a^a - 1 < 0.$$

Therefore, by Corollary 1.2 and Lemma 3.3, there exists a solution x_1 of the equation (1.2), $a \in \langle 0, e^{-e} \rangle$, such that $x_1 \in \langle \overline{x}_a, 1 \rangle$. We propose to show that, beside x_1, there exists another solution x_0 of (1.2), $a \in \langle 0, e^{-e} \rangle$, such that $x_0 < \overline{x}_a$. It is sufficient to show that $\xi_a < \overline{x}_a$. First notice that $\xi_a < e^{-1}$. Indeed, since the function $\zeta : \langle 0,1\rangle \to \langle 0,1\rangle$ is an increasing bijection, and

$$\zeta^{-1}\left(e^{-1}\right) = \left(e^{-1}\right)^{\frac{1}{e^{-1}}} = e^{-e},$$

by Lemma 3.1, it follows that $\zeta\left(\langle 0, e^{-e}\rangle\right) = \left\langle 0, e^{-1}\right\rangle$. Now, from $a^{\xi_a} = \xi_a < \frac{1}{e}$, it follows that

$$\log_a a^{\xi_a} = \xi_a = a^{\xi_a} > \log_a e^{-1},$$

which implies that

$$\log_a a^{\xi_a} = \xi_a < \log_a \log_a e^{-1} = \overline{x}_a.$$

Hence, if $a \in \langle 0, e^{-e}\rangle$, the equation (1.2) has two different solutions x_1 and ξ_a. Therefore, by Lemma 3.6, (1.2) has exactly three solutions. $\square$

Remark 3.10. Notice that the point (ξ_a, ξ_a) is the common intersection point of the curves $y = \varphi_a(x)$, $y = a^x$ and $y = \log_a x$. It is interesting to consider what is happening with the inflection point $\left(\overline{x}_a, \frac{1}{e}\right)$ of φ_a and with the intersection point (ξ_a, ξ_a), and how $\overline{x}_a$ is related to the ξ_a and other solutions of (1.2), depending on a base $a \in \langle 0,1\rangle$. By the proof of Lemma 3.9, it is clear that, for $a \in \langle 0, e^{-e}\rangle$, there exist three different solutions x_2, ξ_a and x_1 of (1.2), such that

$$x_2 < \xi_a < \overline{x}_a < x_1.$$

By Lemma 3.1 and by the proof of Lemma 3.9, it follows that, while a ranges from 0 to e^{-e}, $\overline{x}_a$ and ξ_a tents from 0 to $\frac{1}{e}$. For $a = e^{-e}$, all the solutions and inflection merge into one point. Namely, $\xi_a = \overline{x}_a = \frac{1}{e}$ is the unique solution of (1.2), while the inflection point and intersection point coincide with the point $\left(\frac{1}{e}, \frac{1}{e}\right)$. "After that", for $a > e^{-e}$, they separate and $\overline{x}_a$ moves to the left and ξ_a moves to the right.

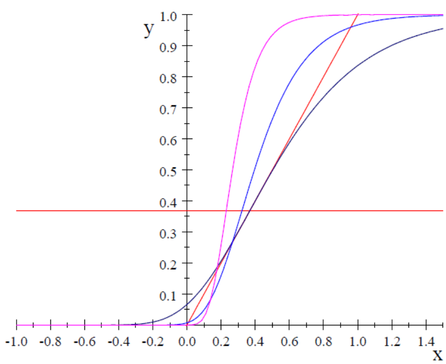


Figure 5: $y = x$, $y = \frac{1}{e}$, $y = \varphi_a(x)$, $a = e^{-10}, e^{-5}, e^{-e}$

If a ranges from e^{-e} to 1, since $\omega : \langle e^{-e}, 1\rangle \to \left\langle -\infty, e^{-1}\right\rangle$ $\omega(a) = \overline{x}_a = \frac{1}{\ln a}\ln\left(\frac{-1}{\ln a}\right)$, is a decreasing bijective mapping, it follows that $\omega(a) = \overline{x}_a$ tends from e^{-1} to $-\infty$ and the unique solution ξ_a of (1.2), by Lemma 3.1, tends from e^{-1} to 1.

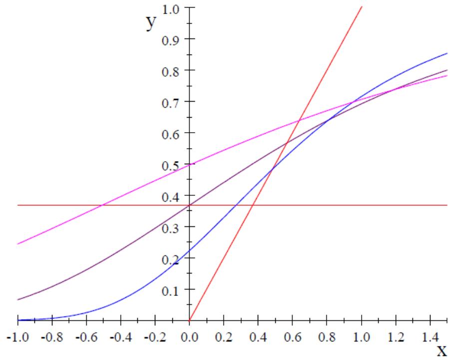


Figure 6: $y = x, y = \frac{1}{e}, y = \varphi_a(x), a = e^{-1.5}, e^{-1}, e^{-0.7}$

In the figures below an initial problem (1.1) is visualized for the bases $a = 0.3, \frac{1}{16}, 0.001$.

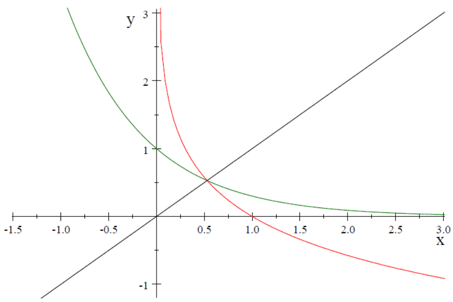


Figure 7: $a = 0.3$ $y = a^x, y = \log_a x$

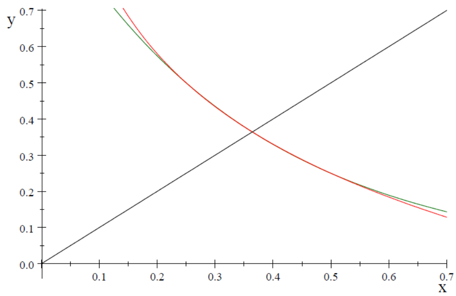


Figure 8: $a = \frac{1}{16}$ $y = a^x, y = \log_a x$

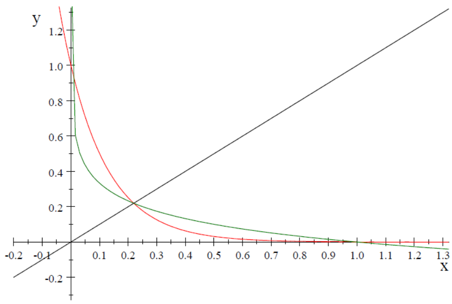


Figure 9: $a = 0.001$ $y = a^x, y = \log_a x$

The problem considered in this paper motivate us to study the equation

$$a^x = \log_b x,$$

for $a, b \in \langle 0, \infty\rangle \setminus \{1\}$ and to state the following problem:

Problem. *Determine the number of all intersecting points of the curves* $y = \log_b x$ *and* $y = a^x$ *depending on bases* a *and* b.

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