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Commutator identities on group algebras*

Tibor Juhász

Institute of Mathematics and Informatics
Eszterházy Károly College
juhaszti@ektf.hu

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Abstract

Let K be a field of characteristic $p > 2$, and G a nilpotent group with commutator subgroup of order p^n. Denote by $(KG)_*$ the set of symmetric elements of the group algebra KG with respect to an oriented classical involution. Then KG satisfies all Lie commutator identities of degree $p^n + 1$ or more. We will show that $(KG)_*$ satisfies a Lie commutator identity of degree less than $p^n + 1$ if and only if G' is not cyclic. Consequently, if G' is cyclic, then the Lie nilpotency index and the Lie derived length of $(KG)_*$ are just the same as of KG, namely $p^n + 1$ and $\lceil \log_2(p^n + 1) \rceil$, respectively. The corresponding results on the set of symmetric units of KG are also obtained.

Keywords: Group ring, involution, polynomial identity, group identity, derived length, Lie nilpotency index, nilpotency class

MSC: 16W10, 16S34, 16U60, 16N40

1. Introduction

The Lie derived length and the Lie nilpotency index of group algebras and their certain subsets have been studied separately for many decades. Both of these properties can be characterized by specific polynomial identities, where the polynomials are multilinear Lie monomials. In this paper we investigate group algebras satisfying general multilinear Lie monomial (Lie commutator) identities, and from that draw conclusions about the above properties.

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Let KG denote the group algebra of a group G over a field K. Then KG, with the Lie commutator $[x, y] = xy - yx$ serving as the Lie bracket, can be considered as a Lie algebra. Let S be a nonempty subset of KG. We will consider the elements of S as *Lie commutators of weight* 1 on S, and inductively, an element $[x, y]$ of KG, where x and y are Lie commutators of weight u and v on S with $u+v = r$, will be called a *Lie commutator of weight* r on S.

Denote by $K\langle x_1, \dots, x_m\rangle$ the polynomial ring in the non-commuting indeterminates $x_1, \dots, x_m$ over K. The set S is said to *satisfy a polynomial identity* if there exists a nonzero polynomial in $K\langle x_1, \dots, x_m\rangle$ such that $f(s_1, \dots, s_m) = 0$ for all $s_1, \dots, s_m \in S$. Let now X be the set of the indeterminates in $K\langle x_1, \dots, x_m\rangle$. A Lie commutator of weight r on X is called a *multilinear Lie monomial of degree* r, if it is linear in each of its indeterminates. We will say that the subset S of KG *satisfies a Lie commutator identity of degree* r, if there exists a nonzero multilinear Lie monomial f of degree r in $K\langle x_1, \dots, x_m\rangle$ with $f(s_1, \dots, s_m) = 0$ for all $s_1, \dots, s_m \in S$. Then we also say: S satisfies the Lie commutator identity $f(x_1, \dots, x_m) = 0$. We will denote by $f(S)$ the image of the set S under the polynomial function f.

For subsets $V, W \subseteq KG$, by the symbol $[V, W]$ we mean the subspace of KG generated by all Lie commutators $[v, w]$ with $v \in V, w \in W$. Set $\gamma_1(S) = \delta^{[0]}(S) = S$, and by induction, let $\gamma_{n+1}(S) = [\gamma_n(S), S]$ and $\delta^{[n+1]}(S) = [\delta^{[n]}(S), \delta^{[n]}(S)]$. S is said to be *Lie nilpotent*, if $\gamma_n(S) = 0$, and *Lie solvable*, if $\delta^{[n]}(S) = 0$ for some integer n. The first such n is called the *Lie nilpotency index* or the *Lie derived length* of S and denoted by $t_L(S)$ and $\mathrm{dl}_L(S)$, respectively. It is obvious that S is Lie nilpotent of index n, or Lie solvable of derived length n, if and only if it satisfies the polynomial identity

$$[x_1, \dots, x_n] = 0, \tag{1.1}$$

or

$$[x_1, \dots, x_{2^n}]^\circ = 0, \tag{1.2}$$

respectively, where the Lie commutators on the left-hand sides are defined inductively to be

$$[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$$

and

$$[x_1, \dots, x_{2^n}]^\circ = [[x_1, \dots, x_{2^{n-1}}]^\circ, [x_{2^{n-1}+1}, \dots, x_{2^n}]^\circ]$$

with $[x_1, x_2]^\circ = [x_1, x_2]$, and n is the least such integer.
Besides Lie nilpotence and Lie solvability, many other properties can be originated from Lie commutator identities. For example, KG is said to be *Lie centre-by-metabelian* (or *Lie centrally metabelian*), if $\delta^{[2]}(KG)$ is central in KG, or, equivalently, KG satisfies the Lie commutator identity

$$[[[x_1, x_2], [x_3, x_4]], x_5] = 0 \tag{1.3}$$

of degree 5. However, as we will see, the identities (1.1) and (1.2) play special roles.

For a prime p we say that G is p-abelian, if its commutator subgroup G' is a finite p-group. By definition, the 0-abelian groups are the abelian groups. In what follows, p will always denote the characteristic of the field K. According to [7], KG satisfies a polynomial identity if and only if G has a p-abelian subgroup of finite index. Now, assume that the Lie ideal L of KG satisfies the Lie commutator identity $f(x_1,\dots,x_m)=0$. If f is of degree 1, then $f(x_1,\dots,x_m)=x_i$ for some $i\in\{1,\dots,m\}$, so $L=\delta^{[0]}(L)=f(L)$. Suppose that there exists k such that $\delta^{[k]}(L)\subseteq f(L)$ whenever f is of degree less than r. Let now f be of degree r. Then f can be expressed as the Lie commutator of the multilinear Lie monomials f_1 and f_2 of degrees less than r. By the inductive hypothesis, there exist k_1,k_2 such that $\delta^{[k_1]}(L)\subseteq f_1(L)$ and $\delta^{[k_2]}(L)\subseteq f_2(L)$. Assume that $k_1\le k_2$, and let $k=k_2+1$. Then

$$\begin{aligned}\delta^{[k]}(L)=[\delta^{[k_2]}(L),\delta^{[k_2]}(L)]&\subseteq[\delta^{[k_1]}(L),\delta^{[k_2]}(L)]\\ &\subseteq[f_1(L),f_2(L)]=f(L).\end{aligned}$$

We have just proved that if L satisfies a Lie commutator identity, then L is Lie solvable. The converse is trivial.

The Lie solvable group algebras are described in [6]: KG is Lie solvable if and only if one of the following conditions holds: (i) $p\ne 2$, and G is p-abelian; (ii) $p=2$, and G has a 2-abelian subgroup of index at most 2. Consequently, for $p=0$, KG satisfies a Lie commutator identity precisely if G is abelian, and then, of course, KG satisfies all Lie commutator identities of degree at least 2. Therefore, in the sequel we can restrict ourselves to the case only when $p>0$ and G is nonabelian. In [6], a necessary and sufficient condition can also be found for the Lie nilpotence of the group algebra KG: KG is Lie nilpotent if and only if G is nilpotent and p-abelian. It is easy to check that if $S\subseteq KG$ is Lie nilpotent of class n (in other words, S satisfies (1.1)), then S satisfies all Lie commutator identities of degree at least n.

Applying Theorems 3 and 6 of [5], it is not so hard to derive that on group algebras, all Lie commutator identities of degree r are equivalent while $r\le 4$. Nevertheless, according to [9], the group algebra $\mathbb{F}_3D_6$, where $\mathbb{F}_3$ denotes the field of three elements and D_6 the dihedral group of order 6, satisfies the identity (1.3), but, by [6], it does not satisfy (1.1) for $n=5$. It is worth mentioning here that the question of the equivalence of Lie commutator identities of the same degree is raised in the "Dniester Notebook: Unsolved Problems in the Theory of Rings and Modules" (see Problem 2.6 in [8, p. 482]).

Let now $*$ be an involution of the group algebra KG, and let $(KG)_*=\{x\in KG : x^*=x\}$ the set of symmetric elements with respect to $*$. Evidently, $(KG)_*$ is a subspace of KG, but not always closed under Lie commutator. Although the classification of all involutions of group algebras is still open, the exploration of the algebraic properties of symmetric elements is an extensively studied area of group algebras. Most of the results are known with respect to the so-called *classical involution*, which sends every element of G into its inverse. By $*$ we will understand a more general involution introduced by S. P. Novikov. Let $\sigma\colon G\to$

$\{\pm 1\}$ a homomorphism and let $*\colon KG \to KG$ be given by

$$\left(\sum_{g\in G} \alpha_g g\right)^* = \sum_{g\in G} \alpha_g \sigma(g) g^{-1}.$$

This involution is called *oriented classical involution* of KG. According to [3], it can happen that $(KG)_*$ satisfies a Lie commutator identity, but the whole KG does not satisfy the same identity.

Now, we will assign group commutators to Lie monomials. Let τ be the mapping from the set of all Lie commutators on the subset $X = \{x_1, \dots, x_m\}$ of $K\langle x_1, \dots, x_m\rangle$ into the free group F with generators $u_1, \dots, u_n$, given by $\tau(x_i) = u_i$, and for the Lie commutator $[x, y]$ of weight $r > 1$ on X, let $\tau([x, y])$ be the group commutator of $\tau(x)$ and $\tau(y)$. The word w in F will be called a *multilinear group commutator of degree* r, if it is the image of a multilinear Lie monomial of degree r under τ. Denote by $U(S)$ the set of units of the set $S \subseteq KG$. We will say that $U(S) \neq \emptyset$ satisfies a *group commutator identity of degree* r, if there exists a nontrivial multilinear group commutator $w(u_1, \dots, u_n)$ of degree r in the free group with generators $v_1, \dots, v_n$ such that $w(h_1, \dots, h_n) = 1$ for all $h_1, \dots, h_n \in U(S)$.

We will say that $U(S)$ is *nilpotent of class* $n-1$, or *solvable of length* n, if $U(S)$ satisfies the group commutator identity $(v_1, \dots, v_n) = 1$, or $(v_1, \dots, v_{2^n})^\circ = 1$, respectively, where the group commutators $(v_1, \dots, v_n)$ and $(v_1, \dots, v_{2^n})^\circ$ are defined by induction, analogously to (1.1) and (1.2), and n is the first such integer. The nilpotency class and the derived length of $U(S)$ will be denoted by $\mathrm{cl}(U(S))$ and $\mathrm{dl}(U(S))$, respectively.

Our main theorem is the following.

Theorem 1.1. *Let K be a field of characteristic $p > 2$, and let G be a nilpotent p-abelian group with cyclic commutator subgroup. Then:*

(i) $(KG)_*$ satisfies no Lie commutator identity of degree less than $|G'| + 1$;

(ii) provided that G is torsion, $U_*(KG)$ satisfies no group commutator identity of degree less than $|G'| + 1$.

By Theorem 1 of [2], if G' is not cyclic, then $t_L(KG) \leq |G'|$, or in other words, KG satisfies all Lie commutator identities of degree at least $|G'|$. Combining this result with Theorem 1.1, we can state the next corollary.

Corollary 1.2. *Let K be a field of characteristic $p > 2$, and let G be a nilpotent p-abelian group. Then the group algebra KG satisfies all Lie commutator identities of degree $|G'| + 1$ or more, and $U(KG)$ satisfies all group commutator identities of degree $|G'|+1$ or more. Furthermore, $(KG)_*$ satisfies a Lie commutator identity of degree less than $|G'| + 1$ if and only if G' is not cyclic. Provided that G is torsion, $U_*(KG)$ satisfies a group commutator identity of degree less than $|G'| + 1$ if and only if G' is not cyclic.*

Finally, we draw conclusions about the Lie nilpotency index and the Lie derived length of $(KG)_*$, such as the nilpotency class and derived length of $U_*(KG)$.

Corollary 1.3. *Let K be a field of characteristic $p > 2$, and let G be a nilpotent p-abelian group. Then $t_L((KG)_*) \leq |G'| + 1$, with equality if and only if G' is cyclic. Provided that G is torsion, $\mathrm{cl}(U_*(KG)) \leq |G'|$, with equality if and only if G' is cyclic.*

Corollary 1.4. *Let K be a field of characteristic $p > 2$, and let G be a nilpotent p-abelian group with cyclic commutator subgroup. Then*

$$\mathrm{dl}_L((KG)_*) = \mathrm{dl}_L(KG) = \lceil \log_2(|G'| + 1) \rceil.$$

In addition, if G is torsion, then $\mathrm{dl}(U_*(KG)) = \mathrm{dl}_L(KG)$.

2. Proof of Theorem 1.1

Let G be a finite p-group with cyclic commutator subgroup of order p^n, where p is an odd prime, and let K be a field of characteristic p. We will denote by $\omega(KG)$ and $\omega(KG')$ the augmentation ideals of KG and KG', respectively. The assumption guarantees that they are nilpotent ideals, and by Lemma 3 of [1], the relations

$$\begin{aligned}
&[\omega(KG')^m, \omega(KG)^l] \subseteq \omega(KG)^{l-1}\omega(KG')^{m+1};\\
&[\omega(KG)^k, \omega(KG)^l] \subseteq \omega(KG)^{k+l-2}\omega(KG');\\
&[\omega(KG)^k\omega(KG')^m, \omega(KG)^l\omega(KG')^n] \subseteq \omega(KG)^{k+l-2}\omega(KG')^{n+m+1}
\end{aligned} \tag{2.1}$$

hold for all $k, l, m, n \geq 1$. By definition, $\omega(KG)^0 = KG$.

We will also use the following well-known identity: for any $g \in G$ and integer k

$$g^k - 1 \equiv k(g-1) \pmod{\omega(KG)^2}. \tag{2.2}$$

Let I_r denote the ideal $\omega(KG)^3\omega(KG')^{r-1} + KG\omega(KG')^r$ of KG, and let S be the subspace of KG spanned by the elements

$$(a-1)(a^{-1}-1),\ (b-1)(b^{-1}-1),\ (ab-1)((ab)^{-1}-1),$$

with $a, b \in G$ such that the commutator $x = (a, b)$ is of order p^n. For the multilinear Lie monomial f we will denote by w_f the multilinear group commutator $\tau(f)$.

Lemma 2.1. *S satisfies no Lie commutator identity of degree less than $p^n + 1$, and $1 + S$ satisfies no group commutator identity of degree less than $p^n + 1$.*

Proof. We show that for arbitrary multilinear Lie commutator $f(x_1, \ldots, x_m)$ of degree r, and for any element v of the set $V = \{(a-1)^2, (b-1)^2, (a-1)(b-1)\}$ there exist $s_1, \ldots, s_m \in S$ such that

$$f(s_1, \ldots, s_m) \equiv w_f(1 + s_1, \ldots, 1 + s_m) - 1 \equiv v(x-1)^{r-1} \pmod{I_r}.$$

This goes by induction on r. If $r=1$, then $f(S)=S$, and using (2.2) we have

$$-(a-1)(a^{-1}-1)\equiv(a-1)^2 \pmod{\omega(KG)^3},$$
$$-(b-1)(b^{-1}-1)\equiv(b-1)^2 \pmod{\omega(KG)^3}$$

and

$$\begin{aligned}-(ab-1)((ab)^{-1}-1)&\equiv(ab-1)^2=((a-1)(b-1)+(a-1)+(b-1))^2\\ &\equiv(a-1)^2+(b-1)^2+2(a-1)(b-1) \pmod{\omega(KG)^3}.\end{aligned}$$

Hence,

$$\begin{aligned}2^{-1}((a-1)(a^{-1}-1)+(b-1)(b^{-1}-1)-(ab-1)((ab)^{-1}-1))\\ \equiv(a-1)(b-1) \pmod{\omega(KG)^3}.\end{aligned}$$

As $\omega(KG)^3\subseteq I_1$, the claim is true for $r=1$. Assume the claim for all Lie commutator identity of degree less than r, and let f be a multilinear Lie commutator of degree r. Then f can be expressed as a Lie commutator of the multilinear Lie commutators f_1 and f_2 of degree d and $r-d$, respectively. By the inductive hypothesis, for all $v_1,v_2\in V$ there exist $s_1,\dots,s_m\in S$ such that

$$f_1(s_1,\dots,s_m)\equiv w_{f_1}(1+s_1,\dots,1+s_m)-1\equiv v_1(x-1)^{d-1} \pmod{I_d},$$
$$f_2(s_1,\dots,s_m)\equiv w_{f_2}(1+s_1,\dots,1+s_m)-1\equiv v_2(x-1)^{r-d-1} \pmod{I_{r-d}}.$$

Now we can apply (2.1) and the equality

$$KG\omega(KG')^k=\omega(KG')^k+\omega(KG)\omega(KG')^k$$

which holds for any $k\geq 1$ to get that both $[I_s,I_t]$ and $[\omega^2(KG)\omega(KG')^{s-1},I_t]$ are subsets of I_{s+t} for any $s,t\geq 1$. Then

$$f_(s_1,\dots,s_m)\equiv[v_1(x-1)^{d-1},v_2(x-1)^{r-d-1}] \pmod{I_r},$$

furthermore,

$$\begin{aligned}&[v_1(x-1)^{d-1},v_2(x-1)^{r-d-1}]\\ &\quad=v_1[(x-1)^{d-1},v_2(x-1)^{r-d-1}]+[v_1,v_2(x-1)^{r-d-1}](x-1)^{d-1}\\ &\quad=v_1[(x-1)^{d-1},v_2](x-1)^{r-d-1}+[v_1,v_2](x-1)^{r-2},\end{aligned}$$

and by using the first relation of (2.1) we have

$$f(s_1,\dots,s_m)\equiv[v_1,v_2](x-1)^{r-2} \pmod{I_r}. \tag{2.3}$$

It remains to compute the Lie commutators $[v_1,v_2]$ for all possible v_1 and v_2. According to [1] (see p. 4911),

$$\begin{aligned}[(a-1)^2,(b-1)^2]&\equiv4(a-1)(b-1)(x-1) \pmod{I_2},\\ [(a-1)^2,(a-1)(b-1)]&\equiv2(a-1)^2(x-1) \pmod{I_2},\\ [(b-1)^2,(a-1)(b-1)]&\equiv2(b-1)^2(x-1) \pmod{I_2}.\end{aligned} \tag{2.4}$$

For the sake of completeness, we confirm here the first congruence, the other two can be obtained similarly. Clearly,

$$\begin{aligned}[(a-1)^2,(b-1)^2] &= (a-1)[a,(b-1)^2]+[a,(b-1)^2](a-1)\\ &= (a-1)(b-1)[a,b]+(a-1)[a,b](b-1)\\ &\quad +(b-1)[a,b](a-1)+[a,b](b-1)(a-1).\end{aligned}$$

Furthermore, $[a,b]=ba(x-1)=(ba-1)(x-1)+(x-1)$ and $(g-1)(h-1)=(h-1)(g-1)+hg((g,h)-1)$ for any $g,h\in G$, so every summand on the right hand side is congruent to $(a-1)(b-1)(x-1)$ modulo I_2. This implies the required congruence.

So, by (2.4), for any $v\in V$ we can choose v_1 and v_2 such that

$$f(s_1,\dots,s_m)\equiv \alpha v(x-1)^{r-1} \pmod{I_r},$$

for some $\alpha\in K\setminus\{0\}$.

For the sake of brevity, we write $1+\underline{s}$ instead of $(1+s_1,\dots,1+s_m)$. Then

$$\begin{aligned} w_f(1+\underline{s}) &= (w_{f_1}(1+\underline{s}),w_{f_2}(1+\underline{s}))\\ &= 1+w_{f_1}(1+\underline{s})^{-1}w_{f_2}(1+\underline{s})^{-1}[w_{f_1}(1+\underline{s}),w_{f_2}(1+\underline{s})]\\ &\equiv 1+w_{f_1}(1+\underline{s})^{-1}w_{f_2}(1+\underline{s})^{-1}f(s_1,\dots,s_m)\\ &\equiv 1+\alpha v(x-1)^{r-1} \pmod{I_r}.\end{aligned}$$

Let k be an integer for which x_k divides the polynomial $f(x_1,\dots,x_m)$; let $s'_k=\alpha^{-1}s_k$, and $s'_i=s_i$ for all $i\neq k$. Then

$$f(s'_1,\dots,s'_m)\equiv w_f(1+s'_1,\dots,1+s'_m)-1\equiv v(x-1)^{r-1} \pmod{I_r},$$

and the induction is done.

Now, applying the results of [4] we show that $w=v(x-1)^{r-1}\notin I_r$ for $r=p^n$. Denote by t the weight of the element $x-1$. Then $t\geq 2$, and $w\in\omega(KG)^{2+t(r-1)}\setminus\omega(KG)^{3+t(r-1)}$. Since $\omega(KG)^i$ has a basis over K consisting of regular elements of weight not less than i, we have that $I_r=\omega(KG)^3\omega(KG')^{r-1}\subseteq\omega(KG)^{3+t(r-1)}$. Consequently, $w\notin I_r$. This means that $f(S)$ contains a nonzero element for any Lie commutator identity f of degree p^n or less. □

As every element of G has odd order, the orientation σ has to be trivial, so all elements of S belong to $(KG)_*$, further $1+S\subseteq U_*(KG)$. This implies the following statement.

Lemma 2.2. *Let K be a field of characteristic $p>2$, and let G be a finite p-group with cyclic commutator subgroup. Then*

(i) $(KG)_*$ satisfies no Lie commutator identity of degree less than $|G'|+1$;

(ii) $U_*(KG)$ satisfies no group commutator identity of degree less than $|G'|+1$.

Now, we are ready to prove our main theorem. We will use that the subspace $(KG)_*$ of KG is spanned by the set $\{g + \sigma(g)g^{-1} : g \in G\}$.

Proof of Theorem 1.1. Let $f(x_1, \dots, x_m)$ be a multilinear Lie commutator of degree less than $|G'| + 1$.

According to Theorem 1.7 of [10], the FC-group G is isomorphic to a subgroup of the direct product of the torsion FC-group G/A and the torsion-free abelian group G/T, where A is a maximal torsion free central subgroup, and T is the torsion part of G. Hence, $G' \cong (G/A)'$. Assume that $A \subseteq \ker\sigma$. Then the involution $*$ induces the involution

$$\left(\sum_{\overline{g}\in G/A} \alpha_g \overline{g}\right)^{\star} = \sum_{\overline{g}\in G/A} \alpha_g \sigma(g)\overline{g}^{-1},$$

on $K[G/A]$, which is also an oriented classical involution, and the elements of $(K[G/A])_\star$ are exactly the homomorphic images of the elements of $(KG)_*$ under the natural homomorphism $\varphi\colon KG \to K[G/A]$. Choose the elements $\overline{g}, \overline{h} \in G/A$ such that $(G/A)' = \langle(\overline{g}, \overline{h})\rangle$. As a finitely generated torsion nilpotent group, $H = \langle \overline{g}, \overline{h}\rangle$ is finite, and it is the direct product of its Sylow subgroups. Denote by P the Sylow p-subgroup of H. Since G' is a p-group, we have that $P' = H' \cong G'$. By applying (i) of Lemma 2.2 for the finite p-group P, we obtain that there exist elements $s_1, \dots, s_m \in (KG)_*$ such that $\varphi(s_1), \dots, \varphi(s_m) \in (KP)_\star$ and

$$f(\varphi(s_1), \dots, \varphi(s_m)) \neq 0.$$

Then $\varphi(f(s_1, \dots, s_m)) \neq 0$, and $f(s_1, \dots, s_m) \neq 0$, as desired.

In the remaining case when $A \not\subseteq \ker\sigma$, let us take an element a from $A \setminus \ker\sigma$. Then $G = \ker\sigma \cup a\ker\sigma$, and as a is central in G, it follows that $(\ker\sigma)' = G'$. Now we may repeat the proof to have that $(K\ker\sigma)_*$ does not satisfy f. Since $(K\ker\sigma)_* \subseteq (KG)_*$, the first part of the theorem is proved.

Assume that G is torsion, and denote by P the Sylow p-subgroup of the finite nilpotent group $H = \langle g, h\rangle$, where $g, h \in G$ such that $\langle(g, h)\rangle = G'$. Then $P' = G'$, and by (ii) of Lemma 2.2, $U_*(KP)$ satisfies no Lie commutator identity of degree less than $|G'| + 1$, so is $U_*(KG)$. $\square$

References

[1] Bagiński, C., A note on the derived length of the unit group of a modular group algebra, *Comm. Algebra*, Vol. 30 (2002), no. 10, 4905–4913.

[2] Bovdi, V., Spinelli, E., Modular group algebras with maximal Lie nilpotency indices, *Publ. Math. (Debrecen)*, Vol. 65 (2004), no. 1-2, 243–252.

[3] Castillo, J.H., Polcino, C.M., Lie properties of symmetric elements under oriented involutions, *Comm. Algebra*, Vol. 40 (2012), no. 12, 4404–4419.

[4] Jennings, S.A., The structure of the group ring of a p-group over a modular field, *Trans. Amer. Math. Soc.*, Vol. 50 (1941), 175–185.

[5] Levin, F., Rosenberger, G., Lie metabelian group rings, *North-Holland Math. Stud.*, Vol. 126 (1986), 153–161 (to appear in Group and semigroup rings (Johannesburg, 1985)).

[6] Passi, I.B.S., Passman, D.S., Sehgal, S.K., Lie solvable group rings, *Canad. J. Math.*, Vol. 25 (1973), 748–757.

[7] Passman, D.S. Group rings satisfying a polynomial identity, *J. Algebra*, Vol. 20 (1972), 221–225.

[8] Sabinin, L., Sbitneva, L., Shestakov, I., Non-associative algebras and its applications, CRC Press, (2006)

[9] Sharma, R.K., Srivastava, J.B., Lie centrally metabelian group rings, *J. Algebra*, Vol. 151 (1992), 476–486.

[10] Tomkinson, M.J., FC-groups, Research Notes in Mathematics, Vol. 96, Pitman (Advanced Publishing Program), Boston, (1984).