Annales Mathematicae et Informaticae
43 (2014) pp. 55–65
http://ami.ektf.hu

On the existence of the generalized Gauss composition of means*

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Submitted April 1, 2014 — Accepted July 16, 2014

Abstract

The paper deals with the generalized Gauss composition of arbitrary means. We give sufficient conditions for the existence of this generalized Gauss composition. Finally, we show that these conditions cannot be improved or changed.

Keywords: means, power means, Gauss composition of means, Archimedean composition of means

MSC: 26E60

1. Introduction

In this part we recall some basic definitions. Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively.

Let $I \subset \mathbb{R}$ be a non-empty open interval. A function $M : I^2 \to I$ is called a *mean* on I if for all $x, y \in I$

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\}.$$

*Supported by VEGA Grant no. 1/1022/12.

It is obvious that $M(x,x) = x$ for all $x \in I$.

The mean $M : I^2 \to I$ is called *symmetric* if

$$M(x,y) = M(y,x)$$

for all $x, y \in I$.

The mean $M : I^2 \to I$ is called a *strict* mean on I if it is continuous on I^2 and for all $x, y \in I$ with $x \neq y$

$$\min\{x,y\} < M(x,y) < \max\{x,y\}.$$

The mean $M : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is called *homogeneous* if

$$M(zx, zy) = zM(x,y),$$

for all $x, y, z \in \mathbb{R}^+$.

Classical examples for two-variable strict means on $\mathbb{R}^+$ are:

– The arithmetic, the geometric and the harmonic mean

$$A(x,y) := \frac{x+y}{2}, \quad G(x,y) := \sqrt{xy}, \quad H(x,y) := \frac{2xy}{x+y}.$$

– The power means, also called Hölder means, of exponent p

$$M_p(x,y) := \begin{cases} \left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}} & \text{if} \quad p \neq 0 \\ \lim\limits_{p\to 0} \left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}} & \text{if} \quad p = 0\,. \end{cases}$$

The case $p = 1$ corresponds to the arithmetic mean, $p = 0$ to the geometric mean, and $p = -1$ to the harmonic mean. It is well known that

$$\lim_{p\to-\infty} M_p(x,y) = \lim_{p\to-\infty} \left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}} = \min\{x,y\}$$

and

$$\lim_{p\to\infty} M_p(x,y) = \lim_{p\to\infty} \left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}} = \max\{x,y\}\,.$$

These means are called the minimum and maximum mean, respectively.

– The logarithmic mean

$$L(x,y) := \begin{cases} \frac{y-x}{\ln y - \ln x} & \text{if} \quad x \neq y \\ x & \text{if} \quad x = y\,. \end{cases}$$

This area has been studied by many mathematicians. For this paper we were inspired by [2, 3, 4, 5, 7].

Let $M, N : I^2 \to I$ be means on I and $a, b \in I$. Consider the sequences (a_n) and (b_n) defined by the Gauss iteration in the following way:

$$\begin{aligned} a_1 &:= a, & b_1 &:= b, \\ a_{n+1} &:= M(a_n, b_n), & b_{n+1} &:= N(a_n, b_n) \quad (n \in \mathbb{N}). \end{aligned} \tag{1.1}$$

If the limits $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ exist and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

then this common limit is called the *Gauss composition* of the means M and N for the numbers a and b, and is denoted by $M \otimes N(a, b)$. We say that the means M and N are composable in the sense of Gauss (or G-composable). For some applications of Gauss composition see for example [7] or [8].

We can similarly define the Archimedean composition mean of the means M and N (see [11, pp. 77–78]): consider the sequences (a_n) and (b_n) defined by

$$\begin{aligned} a_1 &:= a, & b_1 &:= b, \\ a_{n+1} &:= M(a_n, b_n), & b_{n+1} &:= N(a_{n+1}, b_n) \quad (n \in \mathbb{N}). \end{aligned} \tag{1.2}$$

If the limits $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ exist and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

then this common limit is called the *Archimedean composition* mean of the means M and N for the numbers a and b, and is denoted by $M \boxtimes N(a, b)$. We say that the means M and N are composable in the sense of Archimedes (or A-composable).

There is a known relation between Gauss composition of means and Archimedean composition mean of means (see in [11], p. 79):

$$M \boxtimes N(a, b) = M \otimes N(M, \Pi_2)(a, b), \tag{1.3}$$

where $\Pi_2(a, b) = b$ and $N(M, \Pi_2)(a, b) = N(M(a, b), \Pi_2(a, b))$.

It is known (see [1], [6]) that if M, N are strict means on I, then $M \otimes N(a, b)$ exists for every $a, b \in I$.

In this paper we generalise this result. We will show the following: if the means M_1, M_2 (not necessarily continuous) may be bounded "from one direction" by strict means then their Gauss composition exists. Finally, a counter-example will show that the continuity of the bounding mean cannot be omitted.

2. Results

Theorem 2.1. *Let M, N be means on I and let L_1, L_2 be continuous means on I such that for each $x, y \in I$ with $x \neq y$*

$$L_1(x, y) > \min\{x, y\} \tag{2.1}$$

and

$$L_2(x,y) < \max\{x,y\}\,. \tag{2.2}$$

If any of the following conditions is fulfilled,

a) for each pair of real numbers $x, y \in I$*:* $L_1(x,y) \le M(x,y)$ *and* $L_1(x,y) \le N(x,y)$*,*

b) for each pair of real numbers $x, y \in I$*:* $L_2(x,y) \ge M(x,y)$ *and* $L_2(x,y) \ge N(x,y)$*,*

c) for each pair of real numbers $x, y \in I$*:* $L_1(x,y) \le M(x,y) \le L_2(x,y)$*,*

then the means M *and* N *are G-composable, i.e. the mean* $M \otimes N(a,b)$ *exists.*

Proof. Let us define the sequences (a_n) and (b_n) by (1.1) and the sequences (c_n) and (d_n) by

$$c_n = \min\{a_n, b_n\} \quad \text{and} \quad d_n = \max\{a_n, b_n\}\,.$$

Then, evidently, the limits

$$\lim_{n\to\infty} c_n = c \quad \text{and} \quad \lim_{n\to\infty} d_n = d$$

exist, and $c \le d$. It is sufficient to prove that $c = d$.

All three cases will be proved by contradiction. Hence assume

$$c < d\,. \tag{2.3}$$

a) From the definitions of (c_n) and (d_n) it follows that at least one of the following two statements is true.

I. The sequence (c_n) has a subsequence $(c_{n_k})_{k=1}^{\infty}$ such that $c_{n_k} = a_{n_k}$ for each $k \in \mathbb{N}$.

II. The sequence (c_n) has a subsequence $(c_{n_k})_{k=1}^{\infty}$ such that $c_{n_k} = b_{n_k}$ for each $k \in \mathbb{N}$.

In case I., from (2.1), (2.3) and from the continuity of L_1, we get the inequality

$$c < L_1(c,d) = \lim_{n\to\infty} L_1(c_n, d_n) = \lim_{k\to\infty} L_1(c_{n_k}, d_{n_k}) = \lim_{k\to\infty} L_1(a_{n_k}, b_{n_k}). \tag{2.4}$$

On the other hand, from condition a) and the definition of the sequence (c_n), we get the inequality

$$L_1(a_{n_k}, b_{n_k}) \le \min\{M(a_{n_k}, b_{n_k}), N(a_{n_k}, b_{n_k})\} = \min\{a_{n_k+1}, b_{n_k+1}\} = c_{n_k+1}.$$

Substituting this back to (2.4) we get the inequality

$$c < \lim_{k\to\infty} c_{n_k+1} = c$$

and this is a contradiction.

In case II., we similarly get the inequality

$$c < L_1(d,c) = \lim_{n\to\infty} L_1(d_n, c_n) = \lim_{k\to\infty} L_1(d_{n_k}, c_{n_k}) = \lim_{k\to\infty} L_1(a_{n_k}, b_{n_k}) \tag{2.5}$$

and

$$L_1(a_{n_k}, b_{n_k}) \le \min\{M(a_{n_k}, b_{n_k}), N(a_{n_k}, b_{n_k})\} = \min\{a_{n_k+1}, b_{n_k+1}\} = c_{n_k+1}\,.$$

Substituting this back to the (2.5) we get the contradiction

$$c < \lim_{k\to\infty} c_{n_k+1} = c\,.$$

b) The proof is analogous to the proof of case a).

c) From the definitions of (c_n) and (d_n) it follows that at least one of the following three statements is true: the sequence (c_n) has a subsequence $(c_{n_k})_{k=1}^{\infty}$, where for each $k \in \mathbb{N}$,

I.
$$c_{n_k} = a_{n_k} \quad \text{and} \quad c_{n_k+1} = a_{n_k+1}$$

II.
$$c_{n_k} = a_{n_k} \quad \text{and} \quad c_{n_k+1} = b_{n_k+1}$$

III.
$$c_{n_k} = b_{n_k} \quad \text{and} \quad c_{n_k+1} = b_{n_k+1}.$$

In case I., from (2.1), (2.3), continuity of L_1 and the condition c), we obtain

$$\begin{aligned} c < L_1(c,d) &= \lim_{n\to\infty} L_1(c_n, d_n) = \lim_{k\to\infty} L_1(c_{n_k}, d_{n_k}) = \\ &= \lim_{k\to\infty} L_1(a_{n_k}, b_{n_k}) \le \lim_{k\to\infty} M(a_{n_k}, b_{n_k}) = \\ &= \lim_{k\to\infty} a_{n_k+1} = \lim_{k\to\infty} c_{n_k+1} = c\,, \end{aligned}$$

which is a contradiction.

In case II., from (2.2), (2.3), continuity of L_2 and the condition c), we obtain

$$\begin{aligned} d > L_2(c,d) &= \lim_{n\to\infty} L_2(c_n, d_n) = \lim_{k\to\infty} L_2(c_{n_k}, d_{n_k}) = \\ &= \lim_{k\to\infty} L_2(a_{n_k}, b_{n_k}) \ge \lim_{k\to\infty} M(a_{n_k}, b_{n_k}) = \\ &= \lim_{k\to\infty} a_{n_k+1} = \lim_{k\to\infty} d_{n_k+1} = d\,, \end{aligned}$$

which is a contradiction, too.

Finally, in case III., from (2.2), (2.3) and the continuity of L_2 and the condition c), we have

$$\begin{aligned} d > L_2(d,c) &= \lim_{n\to\infty} L_2(d_n,c_n) = \lim_{k\to\infty} L_2(d_{n_k},c_{n_k}) = \\ &= \lim_{k\to\infty} L_2(a_{n_k},b_{n_k}) \geq \lim_{k\to\infty} M(a_{n_k},b_{n_k}) = \\ &= \lim_{k\to\infty} a_{n_k+1} = \lim_{k\to\infty} d_{n_k+1} = d\,, \end{aligned}$$

a contradiction. □

From the relation (1.3) we obtain a similar result for the Archimedean composition.

Corollary 2.2. *If the conditions of Theorem 2.1 hold, then the means M and N are A-composable.*

As a consequence of Theorem 2.1 we immediately get the following result (see also [6] and [10]).

Corollary 2.3. *Let M be a strict mean and N an arbitrary mean defined on the interval I. Then the means M and N are G-composable.*

Corollary 2.4. *Let M be an arbitrary power mean or the logarithmic mean, and N an arbitrary mean defined on the interval $\mathbb{R}^+$. Then the means M and N are G-composable.*

Proof. The power means and the logarithmic mean are strict means, hence our statement immediately follows from the previous corollary. □

Remark, that the composition of means defined by non-continuous means may exist if one of them can be bounded by a strict mean.

Corollary 2.5. *Let f be a bounded function on $(\mathbb{R}^+)^2$. Let*

$$M(x,y) = M_{f(x,y)}(x,y) = \begin{cases} \left(\frac{x^{f(x,y)}+y^{f(x,y)}}{2}\right)^{\frac{1}{f(x,y)}} & \text{if} \quad f(x,y)\neq 0 \\ \sqrt{xy} & \text{if} \quad f(x,y)=0 \end{cases},$$

and N be an arbitrary mean defined on $\mathbb{R}^+$. Then there $M\otimes N(a,b)$ exists for each pair of real numbers $a,b\in\mathbb{R}^+$.

If one of the means is bounded by a strict mean, and the other is the maximum-mean (minimum-mean), then from the fact of convergence we can obtain the limit value as well:

Corollary 2.6. *Let L be a continuous mean defined on I, such that for each pair of numbers $x,y\in I$, where $x\neq y$,*

$$L(x,y) > \min\{x,y\}\,,$$

moreover, let M be an arbitrary mean on I, such that for each pair of numbers $x, y \in I$: $L(x,y) \leq M(x,y)$. For every $a, b \in I$, where $a < b$, define the sequence $(a_n^*)_{n=1}^{\infty}$ as follows: $a_1^* = a$ and for each $n \in \mathbb{N}$, $a_{n+1}^* = M(a_n^*, b)$. Then

$$\lim_{n\to\infty} a_n^* = b\,.$$

Proof. The assertion immediately follows from case a) of Theorem 2.1 for the means M and N, where $N(a,b) = \max\{a,b\}$. □

We will show that the continuity condition in Theorem 2.1 cannot be omitted. Apart from trivial means (minimum- and maximum means) there exist other means that are not G-composable.

It is not difficult to construct non-continuous means M and N which are not G-composable.

For $a \in (0,1)$ and $b \in (2,3)$ define

$$M(a,b) = \frac{a+1}{2} \quad \text{and} \quad N(a,b) = \frac{2+b}{2}.$$

Then, for the sequences (a_n) and (b_n) defined by Gauss' iteration, we have $a_n \in (0,1)$ and $b_n \in (2,3)$ for $n = 1,2,3,\ldots$. So, $M \otimes N(a,b)$ does not exist.

The means M, N constructed above are not homogeneous.

On the other hand, we have:

Theorem 2.7. *There exist symmetric, homogeneous means H, K defined on $\mathbb{R}^+$ such that for each pair of real numbers $x, y \in \mathbb{R}^+$ with $x \neq y$*

$$\min\{x,y\} < H(x,y) \leq K(x,y) < \max\{x,y\},$$

however, $H \otimes K(a_1, b_1)$ does not exist for any pair of real numbers $a_1, b_1 \in \mathbb{R}^+$, where $a_1 \neq b_1$.

Proof. Define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$f(x) = \begin{cases} \frac{\ln 4}{\ln \frac{x}{2}} & \text{if} \quad x \in (2,\infty) \\ \frac{\ln 4}{\ln \frac{(k+1)x}{k+2}} & \text{if} \quad x \in (\frac{k+2}{k+1}, \frac{k+1}{k}] \quad \text{for all } k \in \mathbb{N} \\ 1 & \text{if} \quad x = 1 \\ f\left(\frac{1}{x}\right) & \text{if} \quad x \in (0,1) \end{cases}$$

For each pair of positive real numbers x, y put

$$K(x,y) = M_{f(\frac{x}{y})}(x,y) = \begin{cases} \left(\frac{x^{f(\frac{x}{y})} + y^{f(\frac{x}{y})}}{2}\right)^{\frac{1}{f(\frac{x}{y})}} & \text{if} \quad f(\frac{x}{y}) \neq 0 \\ \sqrt{xy} & \text{if} \quad f(\frac{x}{y}) = 0 \end{cases},$$

and

$$H(x,y) = M_{-f(\frac{x}{y})}(x,y) = \begin{cases} \left(\frac{x^{-f(\frac{x}{y})}+y^{-f(\frac{x}{y})}}{2}\right)^{-\frac{1}{f(\frac{x}{y})}} & \text{if} \quad f(\frac{x}{y}) \neq 0 \\ \sqrt{xy} & \text{if} \quad f(\frac{x}{y}) = 0 \end{cases} .$$

Using the fact that the power mean is symmetric and homogeneous along with $f(\frac{x}{y}) = f(\frac{y}{x})$ we get that the means H and K are symmetric and homogeneous, too.

Now, let a_1, b_1 be arbitrary positive real numbers. Without loss of generality we may assume

$$a_1 < b_1 \quad \text{and} \quad a_1 b_1 = 1 \,. \tag{2.6}$$

Contruct the sequences (a_n), (b_n) by

$$a_{n+1} = H(a_n, b_n) \quad \text{and} \quad b_{n+1} = K(a_n, b_n) \,.$$

Evidently the sequence(a_n) is strictly increasing and bounded and the sequence (b_n) is strictly decreasing and bounded. Due to these facts the limits

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b$$

exist and

$$a_1 < a \leq b < b_1 \,. \tag{2.7}$$

Denote $\alpha_n = f\left(\frac{a_n}{b_n}\right)$. Then

$$\begin{aligned} a_{n+1}b_{n+1} &= \left(\frac{a_n^{-\alpha_n} + b_n^{-\alpha_n}}{2}\right)^{\frac{1}{-\alpha_n}} \left(\frac{a_n^{\alpha_n} + b_n^{\alpha_n}}{2}\right)^{\frac{1}{\alpha_n}} \\ &= \left(\frac{a_n^{\alpha_n} + b_n^{\alpha_n}}{\frac{1}{a_n^{\alpha_n}} + \frac{1}{b_n^{\alpha_n}}}\right)^{\frac{1}{\alpha_n}} = a_n b_n \,. \end{aligned}$$

We immediately obtain that for each positive integer n

$$a_n b_n = a_1 b_1 = 1 \tag{2.8}$$

and hence

$$ab = a_1 b_1 = 1 \,. \tag{2.9}$$

It follows that $H \otimes K(a_1, b_1)$ exists if and only if $a = b = 1$.

Consider the function

$$g(x) = \left(\frac{\left(\frac{1}{x}\right)^{f(x^2)} + x^{f(x^2)}}{2}\right)^{\frac{1}{f(x^2)}} .$$

From (2.8) and (2.9) it follows that

$$\begin{aligned} g(b_n) &= \left(\frac{\left(\frac{1}{b_n}\right)^{f(b_n^2)} + b_n^{f(b_n^2)}}{2} \right)^{\frac{1}{f(b_n^2)}} \\ &= \left(\frac{a_n^{f\left(\frac{b_n}{a_n}\right)} + b_n^{f\left(\frac{b_n}{a_n}\right)}}{2} \right)^{\frac{1}{f\left(\frac{b_n}{a_n}\right)}} = b_{n+1}\,. \end{aligned} \tag{2.10}$$

Let $I_1 = (\sqrt{2}, \infty)$. For each positive integer $k \geq 2$, define $I_k = (\sqrt{\frac{k+1}{k}}, \sqrt{\frac{k}{k-1}}]$. Then evidently

$$\bigcup_{k=1}^{\infty} I_k = (1, \infty) \tag{2.11}$$

and

$$I_k \cap I_l = \emptyset \quad \text{if} \quad k \neq l\,. \tag{2.12}$$

Now we will prove the following implication:

$$\text{if} \quad x \in I_k \quad \text{then} \quad g(x) \in I_k\,. \tag{2.13}$$

Let k be an arbitrary positive integer, and x a real number such that $x \in I_k$. So

$$2 = \frac{k+1}{k} < x^2 \quad \text{if} \quad k = 1\,,$$

or

$$\frac{k+1}{k} < x^2 \leq \frac{k}{k-1} \quad \text{if} \quad k \geq 2\,.$$

From the definition of the function f we obtain in both cases that

$$f\left(x^2\right) = \frac{\ln 4}{\ln \frac{x^2 k}{k+1}}\,.$$

Consequently,

$$\begin{aligned} f\left(x^2\right) \ln \frac{x^2 k}{(k+1)} &= \ln 4 \\ \left(\frac{x^2 k}{k+1}\right)^{f\left(x^2\right)} &= 4 \\ x^{2f\left(x^2\right)} &= 4\left(\frac{k+1}{k}\right)^{f\left(x^2\right)} \\ x^{f\left(x^2\right)} &= 2\left(\sqrt{\frac{k+1}{k}}\right)^{f\left(x^2\right)} \end{aligned}$$

Thus,

$$g(x) = \left(\frac{\left(\frac{1}{x}\right)^{f(x^2)} + x^{f(x^2)}}{2} \right)^{\frac{1}{f(x^2)}} > \left(\frac{x^{f(x^2)}}{2} \right)^{\frac{1}{f(x^2)}}$$

$$= \left(\frac{2\left(\sqrt{\frac{k+1}{k}}\right)^{f(x^2)}}{2} \right)^{\frac{1}{f(x^2)}} = \sqrt{\frac{k+1}{k}}\,.$$

On the other hand, from $\frac{1}{x} < x$ and the fact that $g(x)$ is the power mean of the numbers x and $\frac{1}{x}$, we obtain $g(x) < x$. Thus, $g(x) \in I_k$.

Finally, we will show that $H \otimes K(a_1, b_1)$ does not exist. According to (2.9) it is sufficient to show that $b > 1$.

In view of (2.6) and (2.11), there exists a well defined positive integer k such that $b_1 \in I_k$. However, by (2.10) and (2.13),

$$b_n \in I_k$$

for each positive integer n; thus,

$$b_n > \sqrt{\frac{k+1}{k}}\,.$$

It follows that

$$b = \lim_{n\to\infty} b_n \geq \sqrt{\frac{k+1}{k}} > 1\,,$$

which concludes the proof. □

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