Counting permutations
by cyclic peaks and valleys

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Abstract

In this paper, we study the generating functions for the number of permutations having a prescribed number of cyclic peaks or valleys. We derive closed form expressions for these functions by use of various algebraic methods. When restricted to the set of derangements (i.e., fixed point free permutations), the evaluation at -1 of the generating function for the number of cyclic valleys gives the Pell number. We provide a bijective proof of this result, which can be extended to the entire symmetric group.

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1. Introduction

Let $\mathfrak{S}_n$ denote the set of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A *peak* in π is defined to be an index $i \in [n]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = \pi(n+1) = 0$. Let $\mathrm{pk}(\pi)$ denote the number of peaks in π. A *left peak* (resp. an *interior peak*) in π is an index $i \in [n-1]$ (resp. $i \in [2, n-1] = \{2, 3, \ldots, n-1\}$) such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\mathrm{lpk}(\pi)$ (resp. $\mathrm{ipk}(\pi)$) denote the number of left peaks (resp. interior peaks) in π.

Combinatorial statistics on cycle notation have been extensively studied in recent years from several points of view (see, e.g., [7, 8, 9, 10, 14]). We say that π is a *circular permutation* if it has only one cycle. The notions of cyclic peak and cyclic valley are defined on circular permutations as follows. Let $A = \{x_1, x_2, \ldots, x_k\}$ be a finite set of positive integers with $k \geq 1$, and let $\mathcal{C}_A$ be the set of all circular permutations of A. We will write a circular permutation $w \in \mathcal{C}_A$ by using the *canonical presentation* $w = y_1 y_2 \cdots y_k$, where $y_1 = \min A$, $y_i = w^{i-1}(y_1)$ for $2 \leq i \leq k$ and $y_1 = w^k(y_1)$. The number $\mathrm{cpk}(w)$ of *cyclic peaks* (resp. $\mathrm{cval}(w)$ of *cyclic valleys*) of w is defined to be the number of indices $i \in [2, k-1]$ such that $y_{i-1} < y_i > y_{i+1}$ (resp. $y_{i-1} > y_i < y_{i+1}$).

The organization of this paper is as follows. In Section 2, we study the generating functions for the number of cyclic peaks or valleys, providing explicit expressions in both cases. In Section 3, a new combinatorial interpretation for the Pell numbers is obtained by considering the sign-balance of the cyclic valley statistic on the set of derangements. Our argument may be extended to yield a simple sign-balance formula for the entire symmetric group.

We now recall some prior results which we will need in our derivation of the generating function formulas for cyclic peaks and valleys. Define

$$P(\mathfrak{S}_n; q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{pk}(\pi)},$$

$$IP(\mathfrak{S}_n; q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{ipk}(\pi)},$$

$$LP(\mathfrak{S}_n; q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{lpk}(\pi)}.$$

For convenience, set $P(\mathfrak{S}_0; q) = IP(\mathfrak{S}_0; q) = LP(\mathfrak{S}_0; q) = 1$. It is well known (see [4, ex. 3.3.46] and [11, A008303, A008971]) that

$$IP(\mathcal{S}; q, z) = \sum_{n \geq 1} IP(\mathfrak{S}_n; q)\frac{z^n}{n!} = \frac{\sin(z\sqrt{q-1})}{\sqrt{q-1}\cos(z\sqrt{q-1}) - \sin(z\sqrt{q-1})},$$

$$LP(\mathcal{S}; q, z) = \sum_{n \geq 0} LP(\mathfrak{S}_n; q)\frac{z^n}{n!} = \frac{\sqrt{q-1}}{\sqrt{q-1}\cos(z\sqrt{q-1}) - \sin(z\sqrt{q-1})}.$$

See also the related generating function formula found earlier by Entringer [3].

The complement map $\pi \mapsto \pi^c$, defined by $\pi^c(i) = n + 1 - \pi(i)$, shows that $\mathrm{pk}\,(\pi^c) = 1 + \mathrm{ipk}\,(\pi)$, upon considering cases as to whether there is one more, one less, or the same number of interior peaks as valleys in the permutation π. Thus

$$P(\mathfrak{S}_n; q) = qIP(\mathfrak{S}_n; q)$$

for $n \geq 1$. From the preceding, we conclude that

$$P(\mathcal{S}, q, z) = \sum_{n \geq 0} P(\mathfrak{S}_n; q) \frac{z^n}{n!} = 1 + \frac{q \sin(z\sqrt{q-1})}{\sqrt{q-1}\cos(z\sqrt{q-1}) - \sin(z\sqrt{q-1})}.$$

2. Generating functions

Let $\mathcal{C}_n = \mathcal{C}_{[n]}$. For each $w \in \mathcal{C}_n$, we define w' by the mapping

$$\Phi : w = 1y_2 \cdots y_n \mapsto w' = (y_2 - 1)(y_3 - 1) \cdots (y_n - 1).$$

One can verify the following result.

Lemma 2.1. *For $n \geq 2$, the mapping Φ is a bijection of $\mathcal{C}_n$ onto $\mathfrak{S}_{n-1}$ having the properties:*

$$\mathrm{cpk}\,(w) = \mathrm{lpk}\,(w'), \;\; \mathrm{cval}\,(w) = \mathrm{pk}\,(w') - 1.$$

Define

$$CP(\mathcal{C}_n; q) = \sum_{w \in \mathcal{C}_n} q^{\mathrm{cpk}\,(w)}; \;\; CP(\mathcal{C}; q, z) = \sum_{n \geq 1} CP(\mathcal{C}_n; q) \frac{z^n}{n!};$$

$$CV(\mathcal{C}_n; q) = \sum_{w \in \mathcal{C}_n} q^{\mathrm{cval}\,(w)}; \;\; CV(\mathcal{C}; q, z) = \sum_{n \geq 1} CV(\mathcal{C}_n; q) \frac{z^n}{n!}.$$

By Lemma 2.1, we have

$$CP(\mathcal{C}; q, z) = z + \sum_{n \geq 2} CP(\mathcal{C}_n; q) \frac{z^n}{n!}$$

$$= z + \sum_{n \geq 2} LP(\mathfrak{S}_{n-1}; q) \frac{z^n}{n!}$$

$$= \sum_{n \geq 0} LP(\mathfrak{S}_n; q) \frac{z^{n+1}}{(n+1)!}$$

$$= \int_0^z LP(\mathcal{S}; q, z) dz.$$

Along the same lines, we obtain

$$qCV(\mathcal{C}; q, z) = qz + q \sum_{n \geq 2} CV(\mathcal{C}_n; q) \frac{z^n}{n!}$$

$$= qz + \sum_{n \geq 2} P(\mathfrak{S}_{n-1}; q) \frac{z^n}{n!}$$

$$= \int_0^z P(\mathcal{S}; q, z) dz + z(q-1).$$

With the aid of Maple, we can find expressions for the following two antiderivatives:

$$\int \frac{a}{a \cos(az) - \sin(az)} dz = \frac{1}{2\sqrt{1+a^2}} \ln \frac{\sqrt{1+a^2} + \cos(az) + a \sin(az)}{\sqrt{1+a^2} - \cos(az) - a \sin(az)} + C,$$

$$\int \frac{\sin(az)}{a \cos(az) - \sin(az)} dz = \frac{1}{1+a^2} \left(-z + \ln \frac{a}{a \cos(az) - \sin(az)} \right) + C.$$

Therefore, we get

$$CP(\mathcal{C}; q, z) = \frac{1}{2\sqrt{q}} \ln \left[\left(\frac{\sqrt{q} - 1}{\sqrt{q} + 1} \right) \left(\frac{\sqrt{q} + \cos(z\sqrt{q-1}) + \sqrt{q-1} \sin(z\sqrt{q-1})}{\sqrt{q} - \cos(z\sqrt{q-1}) - \sqrt{q-1} \sin(z\sqrt{q-1})} \right) \right],$$

$$CV(\mathcal{C}; q, z) = \frac{1}{q} \ln \frac{\sqrt{q-1}}{\sqrt{q-1} \cos(z\sqrt{q-1}) - \sin(z\sqrt{q-1})} + z \left(1 - \frac{1}{q} \right).$$

We write $\pi \in \mathfrak{S}_n$ as the product of disjoint cycles: $\pi = w_1 w_2 \cdots w_k$. When each of these cycles is expressed in canonical form, we define

$$\mathrm{cpk}(\pi) := \mathrm{cpk}(w_1) + \mathrm{cpk}(w_2) + \cdots + \mathrm{cpk}(w_k),$$
$$\mathrm{cval}(\pi) := \mathrm{cval}(w_1) + \mathrm{cpk}(w_2) + \cdots + \mathrm{cval}(w_k).$$

Both $\mu_1 \colon \pi \mapsto q^{\mathrm{cpk}(\pi)}$ and $\mu_2 \colon \pi \mapsto q^{\mathrm{cval}(\pi)}$ are multiplicative, in the sense that

$$\mu_i(\pi) = \mu_i(w_1) \mu_i(w_2) \cdots \mu_i(w_k), \ i = 1, 2.$$

Using the *exponential formula* [12, Corollary 5.5.5], we have

$$\sum_{n \geq 0} \frac{z^n}{n!} \sum_{\pi \in \mathfrak{S}_n} \mu_i(\pi) = \exp \left(\sum_{n \geq 1} \frac{z^n}{n!} \sum_{w \in \mathcal{C}_n} \mu_i(w) \right).$$

Define

$$CP(\mathfrak{S}_n; q) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{cpk}(w)};$$

$$CV(\mathfrak{S}_n; q) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{cval}(w)}.$$

Accordingly,

$$CP(\mathcal{S}; q, z) = \sum_{n \geq 0} CP(\mathfrak{S}_n; q) \frac{z^n}{n!} = \exp(CP(\mathcal{C}; q, z)),$$

$$CV(\mathcal{S}; q, z) = \sum_{n \geq 0} CV(\mathfrak{S}_n; q) \frac{z^n}{n!} = \exp(CV(\mathcal{C}; q, z)).$$

Combining the prior observations yields the following result.

Theorem 2.2. *We have*

$$CP(\mathcal{S}; q, z)$$
$$= \left[\left(\frac{\sqrt{q} - 1}{\sqrt{q} + 1} \right) \left(\frac{\sqrt{q} + \cos(z\sqrt{q-1}) + \sqrt{q-1}\sin(z\sqrt{q-1})}{\sqrt{q} - \cos(z\sqrt{q-1}) - \sqrt{q-1}\sin(z\sqrt{q-1})} \right) \right]^{\frac{1}{2\sqrt{q}}}, \quad (2.1)$$

$$CV(\mathcal{S}; q, z) = e^{z(1-1/q)} \left(\frac{\sqrt{q-1}}{\sqrt{q-1}\cos(z\sqrt{q-1}) - \sin(z\sqrt{q-1})} \right)^{\frac{1}{q}}. \quad (2.2)$$

We say that $\pi \in \mathfrak{S}_n$ changes direction at position i if either $\pi(i-1) < \pi(i) > \pi(i+1)$ or $\pi(i-1) > \pi(i) < \pi(i+1)$, where $i \in [2, n-1]$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions. Let $R(n, k)$ denote the number of permutations in $\mathfrak{S}_n$ with k alternating runs. It is well known that the numbers $R(n, k)$ satisfy the recurrence relation

$$R(n, k) = kR(n-1, k) + 2R(n-1, k-1) + (n-k)R(n-1, k-2)$$

for $n, k \geqslant 1$, where $R(1, 0) = 1$ and $R(1, k) = 0$ for $k \geqslant 1$ (see [11, A059427]). Let $R_n(q) = \sum_{k \geqslant 1} R(n, k)q^k$. There is an extensive literature devoted to the polynomials $R_n(q)$. The reader is referred to [5, 6, 13] for recent progress on this subject.

In [1], Carlitz proved that

$$H(q, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} R(n+1, k)q^{n-k} = \left(\frac{1-q}{1+q} \right) \left(\frac{\sqrt{1-q^2} + \sin(z\sqrt{1-q^2})}{q - \cos(z\sqrt{1-q^2})} \right)^2.$$

Consider

$$R(q, z) = \sum_{n \geq 0} R_{n+1}(q) \frac{z^n}{n!}.$$

It is clear that $R(q, z) = H(\frac{1}{q}, qz)$. Hence

$$R(q, z) = \left(\frac{q-1}{q+1} \right) \left(\frac{\sqrt{q^2-1} + q\sin(z\sqrt{q^2-1})}{1 - q\cos(z\sqrt{q^2-1})} \right)^2.$$

There is an equivalent expression for $R(q, z)$ (see [2, eq. (20)]):

$$R(q, z) = \left(\frac{q-1}{q+1} \right) \left(\frac{q + \cos(z\sqrt{q^2-1}) + \sqrt{q^2-1}\sin(z\sqrt{q^2-1})}{q - \cos(z\sqrt{q^2-1}) - \sqrt{q^2-1}\sin(z\sqrt{q^2-1})} \right). \quad (2.3)$$

By Theorem 2.2, we can now conclude the following result.

Corollary 2.3. *We have*

$$CP(\mathcal{S}; q, z) = R(\sqrt{q}, z)^{\frac{1}{2\sqrt{q}}},$$
$$CV(\mathcal{S}; q, z) = e^{z(1-1/q)} LP(\mathcal{S}; q, z)^{\frac{1}{q}}.$$

Corollary 2.4. *For $n \geqslant 2$, the total number of cyclic peaks in all the members of $\mathfrak{S}_n$ is given by*

$$\frac{n!(4n+1-6H_n)}{12},$$

and the total number of cyclic valleys in all the members of $\mathfrak{S}_n$ is given by

$$\frac{n!(2n+5-6H_n)}{6},$$

where $H_n = \sum_{j=1}^n \frac{1}{j}$ is the n-th harmonic number.

Proof. It follows from Theorem 2.2 that

$$
\begin{aligned}
\frac{d}{dq} CP(\mathcal{S}; q, z) \mid_{q=1} &= \frac{(z^3 - 3z^2 + 6z + 6(1-z)\ln(1-z))}{12(1-z)^2} \\
&= \frac{z^3 - 3z^2 + 6z}{12(1-z)^2} + \frac{\ln(1-z)}{2(1-z)} \\
&= \sum_{n \geqslant 2} (4n + 1 - 6H_n) \frac{z^n}{12}, \\
\frac{d}{dq} CV(\mathcal{S}; q, z) \mid_{q=1} &= \frac{(-z^3 - 3z^2 + 6z + 6(1-z)\ln(1-z))}{6(1-z)^2} \\
&= \frac{-z^3 - 3z^2 + 6z}{6(1-z)^2} + \frac{\ln(1-z)}{1-z} \\
&= \sum_{n \geqslant 2} (2n + 5 - 6H_n) \frac{z^n}{6},
\end{aligned}
$$

as required. $\square$

Note that
$$CP(\mathfrak{S}_n; 0) = \#\{\pi \in \mathfrak{S}_n : \mathrm{cpk}\,(\pi) = 0\}.$$

Consider a permutation

$$\pi = (\pi(i_1), \dots)(\pi(i_2), \dots) \cdots (\pi(i_j), \dots)$$

counted by $CP(\mathfrak{S}_n; 0)$. Replacing the parentheses enclosing cycles with brackets, we get a partition of $[n]$ with j blocks. Therefore, we obtain

$$CP(\mathfrak{S}_n; 0) = B_n, \tag{2.4}$$

where B_n is the nth *Bell number* [11, A000110], i.e., the number of partitions of $[n]$ into non-empty blocks.

We present now a generating function proof of (2.4). Note that

$$\sum_{n \geqslant 0} CP(\mathfrak{S}_n; 0) \frac{z^n}{n!}$$

$$= \lim_{q \to 0} \left[\left(\frac{\sqrt{q} - 1}{\sqrt{q} + 1} \right) \left(\frac{\sqrt{q} + \cos(z\sqrt{q-1}) + \sqrt{q-1} \sin(z\sqrt{q-1})}{\sqrt{q} - \cos(z\sqrt{q-1}) - \sqrt{q-1} \sin(z\sqrt{q-1})} \right) \right]^{\frac{1}{2\sqrt{q}}}.$$

Denote the limit on the right by L. It is easy to see that L is of the indeterminate form 1^∞. So, by l'Hôpital's rule, we have

$$\ln L = \lim_{q \to 0} \frac{1}{2\sqrt{q}} \ln \left[\left(\frac{\sqrt{q} - 1}{\sqrt{q} + 1} \right) \left(\frac{\sqrt{q} + \cos(z\sqrt{q-1}) + \sqrt{q-1} \sin(z\sqrt{q-1})}{\sqrt{q} - \cos(z\sqrt{q-1}) - \sqrt{q-1} \sin(z\sqrt{q-1})} \right) \right]$$

$$= \cosh z + \sinh z - 1 = e^z - 1.$$

Consequently,

$$\sum_{n \geqslant 0} CP(\mathfrak{S}_n; 0) \frac{z^n}{n!} = e^{e^z - 1},$$

the right-hand side being the exponential generating function of B_n, thus proving (2.4).

3. A new combinatorial interpretation for the Pell numbers

Recall that the *Pell numbers* P_n are defined by the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ for $n \geqslant 2$, with initial values $P_0 = 0$ and $P_1 = 1$ (see [11, A000129]). The Pell numbers are also given, equivalently, by the Binet formula

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}, \quad n \geqslant 0,$$

which implies

$$P_n = \sum_{0 \leqslant r \leqslant \lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} 2^r, \quad n \geqslant 1.$$

By a *fixed point* of a permutation π, we mean an $i \in [n]$ such that $\pi(i) = i$. A fixed point free permutation is called a *derangement*. Let $\mathcal{D}_n$ denote the set of derangements of $[n]$.

Our next result reveals a somewhat unexpected connection between derangements and Pell numbers.

Theorem 3.1. *For $n \geqslant 1$, we have*

$$\sum_{\sigma \in \mathcal{D}_n} (-1)^{\mathrm{cval}(\sigma)} = P_{n-1}. \tag{3.1}$$

Combinatorial proof of Theorem 3.1

Let us assume $n \geqslant 3$ and let $\mathcal{D}_n^+$ and $\mathcal{D}_n^-$ denote the subsets of $\mathcal{D}_n$ whose members contain an even or odd number of cyclic valleys, respectively. To show (3.1), we will define a cval-parity changing involution of $\mathcal{D}_n$ whose survivors belong to $\mathcal{D}_n^+$ and have cardinality P_{n-1}. We will say that a permutation is in *standard form* if the smallest element is first within each cycle, with cycles arranged in increasing order of smallest elements. Let $\mathcal{D}_n^*$ consist of those members $\pi = C_1 C_2 \cdots C_r$ of $\mathcal{D}_n$ in standard form whose cycles C_i satisfy the following two conditions for $1 \leqslant i \leqslant r$:

(a) C_i consists of a set of consecutive integers, and

(b) C_i is either increasing or contains exactly one cyclic peak but no cyclic valleys.

Note that $\mathcal{D}_n^* \subseteq \mathcal{D}_n^+$ and in the lemma that follows, it is shown that $|\mathcal{D}_n^*| = P_{n-1}$.

We now proceed to define an involution of $\mathcal{D}_n - \mathcal{D}_n^*$. Given $\pi = C_1 C_2 \cdots C_r \in \mathcal{D}_n - \mathcal{D}_n^*$ in standard form, let j_0 denote the smallest index j such that cycle C_j violates condition (a) or (b) (possibly both). Let us assume for now that $j_0 = 1$. Then let i_0 be the smallest index i such that either

(I) i is the middle letter of some cyclic valley of C_1, or

(II) i fails to belong to C_1 with at least one member of $[i+1, n]$ belonging to C_1.

Observe that if (I) occurs, then C_1 may be decomposed as

$$C_1 = 1\alpha\gamma\delta\beta,$$

where α is a subset of $[2, i_0 - 1]$ and is increasing, β is a subset of $[2, i_0 - 1]$ and is decreasing, the union of α and β is $[2, i_0 - 1]$ with α or β possibly empty, γ consists of letters in $[i_0 + 1, n]$, and δ starts with the letter i_0. Note in this case that i_0 being the middle letter of some cyclic valley implies γ is non-empty and δ has length at least two. Next observe that if (II) occurs, then $C_1 = 1\alpha\rho\beta$, where α and β are as before and ρ is non-empty. Note that the second cycle C_2 must start with i_0 in this case.

We define an involution by splitting the cycle C_1 into two cycles $L_1 = 1\alpha\gamma\beta$, $L_2 = \delta$ if (I) occurs, and by merging cycles C_1 and C_2 such that the letters of C_2 go between ρ and β if (II) occurs. Note that the former operation removes exactly one cyclic valley (namely, the one involving i_0) since all of the letters of γ are greater than those of β with β decreasing, while the latter operation is seen to add exactly one cyclic valley. Furthermore, the standard ordering of the cycles is preserved by the former operation, by the minimality of i_0.

For $j_0 \geqslant 1$ in general, perform the operations defined above using the cycle C_{j_0} and its successor, treating the letters contained therein as those in $[\ell]$ for some ℓ and leaving the cycles $C_1, C_2, \ldots, C_{j_0-1}$ undisturbed. Let π' denote the resulting derangement. Then it may be verified that the mapping $\pi \mapsto \pi'$ is an involution of $\mathcal{D}_n - \mathcal{D}_n^*$ such that π and π' have opposite cval parity for all π.

For example, if $n = 20$ and $\pi = \mathcal{D}_{20} - \mathcal{D}_{20}^*$ is given by

$$\pi = (1, 3, 5, 4, 2), \ (6, 7), \ (8, 10, 11, 18, 13, 15, 9), \ (12, 17, 14, 20), \ (16, 19),$$

then $j_0 = 3$ and

$$\pi' = (1, 3, 5, 4, 2), \ (6, 7), \ (8, 10, 11, 18, 13, 15, 12, 17, 14, 20, 9), \ (16, 19).$$

Lemma 3.2. *If $n \geqslant 1$, then $|\mathcal{D}_n^*| = P_{n-1}$.*

Proof. Recall that P_m counts the tilings of length $m - 1$ consisting of squares and dominos such that squares may be colored black or white (called *Pell tilings*). To complete the proof, we define a bijection f between $\mathcal{D}_n^*$ and the set of Pell tilings of length $n - 2$, where $n \geqslant 3$. Suppose $\pi = C_1 C_2 \cdots C_r \in \mathcal{D}_n^*$. If $1 \leqslant i < r$, then we convert the cycle C_i into a Pell subtiling as follows. First assume $i = 1$ and let t denote the largest letter of cycle C_1. If $j \in [2, t - 1]$ and occurs to the left (resp. right) of t in C_1, then let the $(j-1)$-st piece of $f(\pi)$ be a white (resp. black) square. To the resulting sequence of $t - 2$ squares, we append a domino. Thus C_1 has been converted to a Pell subtiling of the same length ending in a domino. Repeat for the cycles $C_2, C_3, \ldots, C_{r-1}$, at each step appending the subtiling that results to the current tiling. For cycle C_r, we perform the same procedure, but this time no domino is added at the end. Let $f(\pi)$ denote the resulting Pell tiling of length $n - 2$. It may be verified that the mapping f is a bijection. Note that $f(\pi)$ ends in a domino if and only if cycle C_r has length two and that the number of dominos of $f(\pi)$ is one less than the number of cycles of π. $\qquad\square$

In the remainder of this section, we present a comparable sign-balance result for $\mathfrak{S}_n$. Let $i = \sqrt{-1}$. Note that $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i \sin(ix)$. Setting $q = -1$ in (2.2), we obtain

$$\sum_{n \geqslant 0} \frac{z^n}{n!} \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{cval}\,(\pi)} = \sum_{n \geqslant 0} CV(\mathfrak{S}_n; -1) \frac{z^n}{n!}$$

$$= \frac{1}{\sqrt{2}} e^{2z} (\sqrt{2} \cosh(\sqrt{2}z) - \sinh(\sqrt{2}z)).$$

Equating coefficients yields the following result.

Theorem 3.3. *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{cval}\,(\pi)} = \frac{1}{2}((2 + \sqrt{2})^{n-1} + (2 - \sqrt{2})^{n-1})$$

$$= \sum_{0 \leqslant r \leqslant \lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2r} 2^{n-1-r}. \tag{3.2}$$

Combinatorial proof of Theorem 3.3

Let $\mathfrak{S}_n^+$ and $\mathfrak{S}_n^-$ denote the subsets of $\mathfrak{S}_n$ whose members contain an even or odd number of cyclic valleys. To show (3.2), we seek an involution of $\mathfrak{S}_n$ which changes the parity of $cval$. Indeed, we define a certain extension of the mapping used in the proof of (3.1). Let $\mathfrak{S}_n^*$ consist of those members $\pi = C_1 C_2 \cdots C_r$ of $\mathfrak{S}_n$ in standard form all of whose cycles satisfy the following two properties:

(a) C_i is either a singleton or if it is not a singleton, it comprises a set of consecutive integers when taken together with all singleton cycles between it and the next non-singleton cycle (if there is one), and

(b) C_i is either increasing or contains exactly one cyclic peak but no cyclic valleys.

Note that $\mathfrak{S}_n^* \subseteq \mathfrak{S}_n^+$ and below it is shown that $|\mathfrak{S}_n^*| = \sum_r \binom{n-1}{2r} 2^{n-1-r}$.

We now define an involution of $\mathfrak{S}_n - \mathfrak{S}_n^*$. Given $\pi = C_1 C_2 \cdots C_r \in \mathfrak{S}_n - \mathfrak{S}_n^*$ in standard form, let j_0 denote the smallest index j such that cycle C_j violates condition (a) or (b) (possibly both). Let us assume for now that $j_0 = 1$, the general case being done in a similar manner as will be apparent. Let i_0 denote the smallest index i satisfying conditions (I) or (II) in the proof above for (3.1), where in (II) we must now add the assumption that i belongs to a non-singleton cycle. The involution $\pi \mapsto \pi'$ is then defined in an analogous manner as it was in the proof of (3.1) above except now, in the merging operation, a non-singleton cycle is moved to the first non-singleton cycle which precedes it (with possibly some singletons separating the two).

For example, if $n = 20$ and $\pi = \mathfrak{S}_{20} - \mathfrak{S}_{20}^*$ is given by

$$\pi = (1,3,5,4),\ (2),\ (6,7),\ (8,18,13,15,11),\ (9),\ (10),\ (12,17,14),\ (16,20),(19),$$

then $j_0 = 4$ and

$$\pi' = (1,3,5,4),\ (2),\ (6,7),\ (8,18,13,15,12,17,14,11),\ (9),\ (10),(16,20),\ (19).$$

We now seek the cardinality of $\mathfrak{S}_n^*$. To do so, we will first define a bijection between $\mathfrak{S}_n^*$ and the set $\mathcal{A}_{n-1}$ consisting of sequences $s_1 s_2 \cdots s_{n-1}$ in [4] such that $s_1 = 1$ or 2, with the strings 13 and 24 forbidden. To define it, first observe that members of $\mathfrak{S}_n^*$, $n \geqslant 1$, may be formed recursively from members of $\mathfrak{S}_{n-1}^*$ (on the alphabet $[2, n]$) by performing one of the following operations:

(i) adding 1 as (1),

(ii) either replacing the 1-cycle (2), if it occurs, with $(1, 2)$ or replacing the cycle $(2 c_1 c_2 \cdots)$ with the two cycles $(1 c_1 c_2 \cdots),\ (2)$,

(iii) replacing the cycle $(2 c_1 c_2 \cdots c_s)$, if it occurs where $s \geqslant 1$, with $(12 c_1 c_2 \cdots c_s)$, or

(iv) replacing the cycle $(2 c_1 c_2 \cdots c_s)$, if it occurs where $s \geqslant 1$, with $(1 c_1 c_2 \cdots c_s 2)$.

Note that (iii) or (iv) cannot be performed on a member of $\mathfrak{S}_{n-1}^*$ if 2 occurs as a 1-cycle, that is, if (i) has been performed in the previous step. Let $\mathcal{B}_{n-1}$ denote the set of sequences in [4] of length $n-1$ having first letter 1 or 2, with the strings 13 and 14 forbidden. Thus, adding 1 to a member of $\mathfrak{S}_{n-1}^*$ as described to obtain a member of $\mathfrak{S}_n^*$ may be viewed as writing the final letter of some member of $\mathcal{B}_{n-1}$. From this, we see that members of $\mathcal{B}_{n-1}$ serve as encodings for creating members of $\mathfrak{S}_n^*$, starting with the letter n and working downward. For example, the sequence $w = 21123412243 \in \mathcal{B}_{11}$ would correspond to $\pi = (1,2,5,3),(4),(6,8,9,7),(10),(11,12) \in \mathfrak{S}_{12}^*$. Note that replacing any occurrence of the string 24 within a member of $\mathcal{B}_{n-1}$ with the string 14 is seen to define a bijection with the set $\mathcal{A}_{n-1}$.

Taking the composition of the maps described from $\mathfrak{S}_n^*$ to $\mathcal{B}_{n-1}$ and from $\mathcal{B}_{n-1}$ to $\mathcal{A}_{n-1}$ yields the desired bijection from $\mathfrak{S}_n^*$ to $\mathcal{A}_{n-1}$.

The following lemma will imply $|\mathfrak{S}_n^*|$ is given by the right-hand side of (3.2) and complete the proof.

Lemma 3.4. *If $m \geqslant 1$, then $|\mathcal{A}_m| = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2r} 2^{m-r}$.*

Proof. The $r = 0$ term of the sum clearly counts all of the binary members of $\mathcal{A}_m$, so we need to show that the cardinality of all $\pi \in \mathcal{A}_m$ containing at least one 3 or 4 is given by $\sum_{r=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2r} 2^{m-r}$.

Note that π may be decomposed as

$$\pi = S_1 S_2 \cdots S_\ell, \qquad \ell \geqslant 2, \tag{3.3}$$

where the odd-indexed S_i are maximal substrings containing only letters in $\{1,2\}$ and the even-indexed S_i are maximal substrings containing only letters in $\{3,4\}$. If $\ell = 2r$ is even in (3.3), then choose a sequence of length $2r - 1$ in $[2,m]$, which we will denote by $i_2 < i_3 < \cdots < i_{2r}$ for convenience with $i_1 = 1$. We wish to create members $\pi = \pi_1 \pi_2 \cdots \pi_m \in \mathcal{A}_m$ such that the initial letter of the block S_j is in position i_j for $1 \leqslant j \leqslant 2r$. To do so, we first fill in the positions of π whose indices correspond to elements of $[i_{2j-1}, i_{2j} - 1]$ with letters from $\{1,2\}$ for each $j \in [r]$. Next, we fill the positions of π in $[i_{2j}, i_{2j+1} - 1]$ for $j \in [r-1]$, along with the positions in $[i_{2r}, n]$, with letters from $\{3,4\}$. Note that the letters in positions i_{2j}, $j \in [r]$, are determined by the choice of last letter for the block S_{2j-1}, since the 13 and 24 strings are forbidden. Thus, there are $\binom{m-1}{2r-1} 2^{m-r}$ members of $\mathcal{A}_m$ such that $\ell = 2r$ in (3.3). By similar reasoning, there are $\binom{m-1}{2r} 2^{m-r}$ members of $\mathcal{A}_m$ such that $\ell = 2r + 1$ in (3.3). Combining these two cases, it follows that there are $\left(\binom{m-1}{2r-1} + \binom{m-1}{2r} \right) 2^{m-r} = \binom{m}{2r} 2^{m-r}$ members of $\mathcal{A}_m$ for which $\ell = 2r$ or $\ell = 2r + 1$ in (3.3). Summing over $r \geqslant 1$ gives the cardinality of all members of $\mathcal{A}_m$ containing at least one 3 or 4 and completes the proof. $\qquad\square$

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