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# On geodesic mappings of Riemannian spaces with cyclic Ricci tensor

Sándor Bácsó<sup>a</sup>, Robert Tornai<sup>a</sup>, Zoltán Horváth<sup>b</sup>

<sup>a</sup>University of Debrecen, Faculty of Informatics  
 bacsos@unideb.hu, tornai.robert@inf.unideb.hu

<sup>b</sup>Ferenc Rákóczi II. Transcarpathian Hungarian Institute  
 zolee27@kmf.uz.ua

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## Abstract

An  $n$ -dimensional Riemannian space  $V^n$  is called a Riemannian space with cyclic Ricci tensor [2, 3], if the Ricci tensor  $R_{ij}$  satisfies the following condition

$$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0,$$

where  $R_{ij}$  the Ricci tensor of  $V^n$ , and the symbol “,” denotes the covariant derivation with respect to Levi-Civita connection of  $V^n$ .

In this paper we would like to treat some results in the subject of geodesic mappings of Riemannian space with cyclic Ricci tensor.

Let  $V^n = (M^n, g_{ij})$  and  $\bar{V}^n = (M^n, \bar{g}_{ij})$  be two Riemannian spaces on the underlying manifold  $M^n$ . A mapping  $V^n \rightarrow \bar{V}^n$  is called geodesic, if it maps an arbitrary geodesic curve of  $V^n$  to a geodesic curve of  $\bar{V}^n$ . [4]

At first we investigate the geodesic mappings of a Riemannian space with cyclic Ricci tensor into another Riemannian space with cyclic Ricci tensor.

Finally we show that, the Riemannian - Einstein space with cyclic Ricci tensor admit only trivial geodesic mapping.

*Keywords:* Riemannian spaces, geodesic mapping.

*MSC:* 53B40.

## 1. Introduction

Let an  $n$ -dimensional  $V^n$  Riemannian space be given with the fundamental tensor  $g_{ij}(x)$ .  $V^n$  has the Riemannian curvature tensor  $R_{jkl}^i$  in the following form:

$$R_{ijk}^h(x) = \partial_j \Gamma_{ik}^h(x) + \Gamma_{ik}^\alpha(x) \Gamma_{j\alpha}^h(x) - \partial_k \Gamma_{ij}^h(x) - \Gamma_{ij}^\alpha(x) \Gamma_{k\alpha}^h(x), \quad (1.1)$$

where  $\Gamma_{jk}^i(x)$  are the coefficients of Levi-Civita connection of  $V^n$ .

The Ricci curvature tensor we obtain from the Riemannian curvature tensor using of the following transvection:  $R_{jk\alpha}^\alpha(x) = R_{jk}(x)$ <sup>1</sup>.

**Definition 1.1.** [2, 3] A Riemannian space  $V^n$  is called a Riemannian space with cyclic Ricci tensor, if the Ricci tensor of  $V^n$  satisfies the following equation:

$$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0, \quad (1.2)$$

where the symbol "," means the covariant derivation with respect to Levi-Civita connection of  $V^n$ .

**Definition 1.2.** [4] Let two Riemannian spaces  $V^n$  and  $\bar{V}^n$  be given on the underlying manifold  $M_n$ . The maps:  $\gamma : V^n \rightarrow \bar{V}^n$  is called geodesic (projective) mappings, if any geodesic curve of  $V^n$  coincides with a geodesic curve of  $\bar{V}^n$ .

It is wellknown, that the the geodesic curve  $x^i(t)$  of  $V^n$  is a result of the second order ordinary differential equations in a canonical parameter  $t$ :

$$\frac{d^2 x^i}{dt^2} + \Gamma_{\alpha\beta}^i(x) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0. \quad (1.3)$$

We need in the investigations the next:

**Theorem 1.3.** [4] *The maps:  $V^n \rightarrow \bar{V}^n$  is geodesic if and only if exists a gradient vector field  $\psi_i(x)$ , which satisfies the following condition:*

$$\bar{\Gamma}_{jk}^i(x) = \Gamma_{jk}^i(x) + \delta_j^i \psi_k(x) + \delta_k^i \psi_j(x), \quad (1.4)$$

and

**Definition 1.4.** [1] A Riemannian space  $V^n$  is called Einstein space, if exists a  $\rho(x)$  scalar function, which satisfies the equation:

$$R_{ij} = \rho(x) g_{ij}(x). \quad (1.5)$$

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<sup>1</sup>The Roman and Greek indices run over the range  $1, \dots, n$ ; the Roman indices are free but the Greek indices denote summation.

## 2. Geodesic mappings of Riemannian spaces with cyclic Ricci tensors

It is easy to get the next equations [4]:

$$\bar{R}_{ij} = R_{ij} + (n-1)\psi_{ij}, \quad (2.1)$$

where  $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j$  and

$$\bar{R}_{ij,k} = \frac{\partial \bar{R}_{ij}}{\partial x^k} - \bar{\Gamma}_{ik}^\alpha(x)\bar{R}_{\alpha j} - \bar{\Gamma}_{jk}^\alpha(x)\bar{R}_{\alpha i}, \quad (2.2)$$

where  $\bar{\Gamma}_{ik}^\alpha(x)$  are components of Levi-Civita connection if  $\bar{V}^n$ .

At now we suppose, that  $\bar{V}^n$  in a Riemannian space with cyclic Ricci tensor, that is

$$\bar{R}_{ij,k} + \bar{R}_{jk,i} + \bar{R}_{ki,j} = 0. \quad (2.3)$$

Using (2.2) we can rewrite (2.3) in the following form:

$$\begin{aligned} & \frac{\partial \bar{R}_{ij}}{\partial x^k} - \bar{\Gamma}_{ik}^\alpha(x)\bar{R}_{\alpha j} - \bar{\Gamma}_{jk}^\alpha(x)\bar{R}_{\alpha i} + \\ & \frac{\partial \bar{R}_{jk}}{\partial x^i} - \bar{\Gamma}_{ji}^\alpha(x)\bar{R}_{\alpha k} - \bar{\Gamma}_{ki}^\alpha(x)\bar{R}_{\alpha j} + \\ & \frac{\partial \bar{R}_{ki}}{\partial x^j} - \bar{\Gamma}_{kj}^\alpha(x)\bar{R}_{\alpha i} - \bar{\Gamma}_{ij}^\alpha(x)\bar{R}_{\alpha k} = 0. \end{aligned} \quad (2.4)$$

From (1.4) and (2.1) we can compute:

$$\begin{aligned} & \frac{\partial(R_{ij} + (n-1)\psi_{ij})}{\partial x^k} - (\Gamma_{ik}^\alpha(x) + \psi_i(x)\delta_k^\alpha + \psi_k(x)\delta_i^\alpha)(R_{\alpha j} + (n-1)\psi_{\alpha j}) - \\ & \quad - (\Gamma_{jk}^\alpha(x) + \psi_j(x)\delta_k^\alpha + \psi_k(x)\delta_j^\alpha)(R_{\alpha i} + (n-1)\psi_{\alpha i}) + \\ & \frac{\partial(R_{jk} + (n-1)\psi_{jk})}{\partial x^i} - (\Gamma_{ji}^\alpha(x) + \psi_j(x)\delta_i^\alpha + \psi_i(x)\delta_j^\alpha)(R_{\alpha k} + (n-1)\psi_{\alpha k}) - \\ & \quad - (\Gamma_{ki}^\alpha(x) + \psi_k(x)\delta_i^\alpha + \psi_i(x)\delta_k^\alpha)(R_{\alpha j} + (n-1)\psi_{\alpha j}) + \\ & \frac{\partial(R_{ki} + (n-1)\psi_{ki})}{\partial x^j} - (\Gamma_{kj}^\alpha(x) + \psi_k(x)\delta_j^\alpha + \psi_j(x)\delta_k^\alpha)(R_{\alpha i} + (n-1)\psi_{\alpha i}) - \\ & \quad - (\Gamma_{ij}^\alpha(x) + \psi_i(x)\delta_j^\alpha + \psi_j(x)\delta_i^\alpha)(R_{\alpha k} + (n-1)\psi_{\alpha k}) = 0. \end{aligned}$$

That is

$$\left. \begin{aligned} & \frac{\partial R_{ij}}{\partial x^k} - \Gamma_{ik}^\alpha(x)R_{\alpha j} - \Gamma_{jk}^\alpha(x)R_{\alpha i} + \\ & + \frac{\partial R_{jk}}{\partial x^i} - \Gamma_{ji}^\alpha(x)R_{\alpha k} - \Gamma_{ki}^\alpha(x)R_{\alpha j} + \\ & + \frac{\partial R_{ki}}{\partial x^j} - \Gamma_{kj}^\alpha(x)R_{\alpha i} - \Gamma_{ij}^\alpha(x)R_{\alpha k} + \end{aligned} \right\} R_{ij,k} + R_{jk,i} + R_{ki,j}$$

$$\begin{aligned}
& + (n-1) \frac{\partial \psi_{ij}}{\partial x^k} - (n-1) \Gamma_{ik}^\alpha(x) \psi_{\alpha j} - \psi_i(x) R_{kj} - (n-1) \psi_i(x) \psi_{kj} - \\
& \quad - \psi_k(x) R_{ij} - (n-1) \psi_k(x) \psi_{ij} - (n-1) \Gamma_{jk}^\alpha(x) \psi_{\alpha i} - \psi_j(x) R_{ki} - \\
& \quad - (n-1) \psi_j(x) \psi_{ki} - \psi_k(x) R_{ji} - (n-1) \psi_k(x) \psi_{ji} + \\
& + (n-1) \frac{\partial \psi_{jk}}{\partial x^i} - (n-1) \Gamma_{ji}^\alpha(x) \psi_{\alpha k} - \psi_j(x) R_{ik} - (n-1) \psi_j(x) \psi_{ik} - \\
& \quad - \psi_i(x) R_{jk} - (n-1) \psi_i(x) \psi_{jk} - (n-1) \Gamma_{ki}^\alpha(x) \psi_{\alpha j} - \psi_k(x) R_{ij} - \\
& \quad - (n-1) \psi_k(x) \psi_{ij} - \psi_i(x) R_{kj} - (n-1) \psi_i(x) \psi_{kj} + \\
& + (n-1) \frac{\partial \psi_{ki}}{\partial x^j} - (n-1) \Gamma_{kj}^\alpha(x) \psi_{\alpha i} - \psi_k(x) R_{ji} - (n-1) \psi_k(x) \psi_{ji} - \\
& \quad - \psi_j(x) R_{ki} - (n-1) \psi_j(x) \psi_{ki} - (n-1) \Gamma_{ij}^\alpha(x) \psi_{\alpha k} - \psi_i(x) R_{jk} - \\
& \quad - (n-1) \psi_i(x) \psi_{jk} - \psi_j(x) R_{ik} - (n-1) \psi_j(x) \psi_{ik} = 0.
\end{aligned}$$

If we suppose, that  $V^n$  has cyclic Ricci tensor we have:

$$\begin{aligned}
& (n-1) \left( \frac{\partial \psi_{ij}}{\partial x^k} - \Gamma_{ik}^\alpha(x) \psi_{\alpha j} - \Gamma_{jk}^\alpha(x) \psi_{\alpha i} \right) + \\
& + (n-1) \left( \frac{\partial \psi_{jk}}{\partial x^i} - \Gamma_{ji}^\alpha(x) \psi_{\alpha k} - \Gamma_{ki}^\alpha(x) \psi_{\alpha j} \right) + \\
& + (n-1) \left( \frac{\partial \psi_{ki}}{\partial x^j} - \Gamma_{kj}^\alpha(x) \psi_{\alpha i} - \Gamma_{ij}^\alpha(x) \psi_{\alpha k} \right) + \\
& \quad - 4\psi_i(x) R_{jk} - 4\psi_j(x) R_{ki} - 4\psi_k(x) R_{ij} - \\
& - (n-1)(4\psi_i(x) \psi_{jk} + 4\psi_j(x) \psi_{ki} + 4\psi_k(x) \psi_{ij}) = 0.
\end{aligned}$$

That is

$$\begin{aligned}
& (n-1)(\psi_{ij,k} + \psi_{jk,i} + \psi_{ki,j}) - \\
& \quad - 4(\psi_i(x) R_{jk} + \psi_j(x) R_{ki} + \psi_k(x) R_{ij}) - \\
& - 4(n-1)(\psi_i(x) \psi_{jk} + \psi_j(x) \psi_{ki} + \psi_k(x) \psi_{ij}) = 0.
\end{aligned} \tag{2.5}$$

**Theorem 2.1.**  $V^n$  and  $\bar{V}^n$  Riemannian spaces with cyclic Ricci tensors have common geodesics, that is  $V^n$  and  $\bar{V}^n$  have a geodesic mapping if and only if exists a  $\psi_i(x)$  gradient vector, which satisfies the condition:

$$\begin{aligned}
& (n-1)(\psi_{ij,k} + \psi_{jk,i} + \psi_{ki,j}) - \\
& \quad - 4(\psi_i(x) R_{jk} + \psi_j(x) R_{ki} + \psi_k(x) R_{ij}) - \\
& - 4(n-1)(\psi_i(x) \psi_{jk} + \psi_j(x) \psi_{ki} + \psi_k(x) \psi_{ij}) = 0.
\end{aligned}$$

### 3. Consequences

A) If  $\psi_{ij} = 0$ , then  $\bar{R}_{ij} = R_{ij}$ , and  $\psi_{i,j} = \psi_i\psi_j$ , so we obtain:

$$\psi_i(x)R_{jk} + \psi_j(x)R_{ki} + \psi_k(x)R_{ij} = 0. \quad (3.1)$$

B) If the  $V^n$  is a Riemannian space with cyclic Ricci tensor and at the same time is a Einstein space, then we get

$$\rho\psi_i(x)g_{jk} + \rho\psi_j(x)g_{ki} + \rho\psi_k(x)g_{ij} = 0$$

that is

$$n\psi_i(x) + 2\psi_i(x) = 0, \quad (3.2)$$

so

$$(n + 2)\psi_i(x) = 0. \quad (3.3)$$

It means

**Theorem 3.1.** *A Riemannian-Einstein space  $V^n$  with cyclic Ricci tensor admits into  $\bar{V}^n$  with cyclic Ricci tensor only trivial (affin) geodesic mapping.*

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