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On second order non-homogeneous recurrence relation

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Abstract

We consider here the sequence g_n defined by the non-homogeneous recurrence relation $g_{n+2} = g_{n+1} + g_n + At^n$, $n \geq 0$, $A \neq 0$ and $t \neq 0$, α, β where α and β are the roots of $x^2 - x - 1 = 0$ and $g_0 = 0$, $g_1 = 1$.

We give some basic properties of g_n.Then using Elmore's technique and exponential generating function of g_n we generalize g_n by defining a new sequence G_n. We prove that G_n satisfies the recurrence relation $G_{n+2} = G_{n+1} + G_n + At^n e^{xt}$.

Using Generalized circular functions we extend the sequence G_n further by defining a new sequence $Q_n(x)$. We then state and prove its recurrence relation. Finally we make a note that sequences $G_n(x)$ and $Q_n(x)$ reduce to the standard Fibonacci Sequence for particular values.

1. Introduction

The Fibonacci Sequence $\{F_n\}$ is defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, n \geq 0 \tag{1.1}$$

with

$$F_0 = 0, \quad \text{and} \quad F_1 = 1.$$

We consider here a slightly more general non-homogeneous recurrence relation which gives rise to a generalized Fibonacci Sequence which we call *The Pseudo Fibonacci Sequence*. But before defining this sequence let us state some identities for the Fibonacci Sequence.

2. Some Identities for $\{F_n\}$

Let α and β be the distinct roots of $x^2 - x - 1$=0, with

$$\alpha = \frac{(1+\sqrt{5})}{2} \quad \text{and} \quad \beta = \frac{(1-\sqrt{5})}{2}. \tag{2.1}$$

Note that

$$\alpha + \beta = 1, \quad \alpha\beta = -1 \quad \text{and} \quad \alpha - \beta = \sqrt{5}. \tag{2.2}$$

Binets formula for $\{F_n\}$ is given by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}. \tag{2.3}$$

Generating function for $\{F_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{(1 - x - x^2)}. \tag{2.4}$$

Exponential Generating Function for $\{F_n\}$ is given by

$$E(x) = \sum_{n=0}^{\infty} \frac{F_n x^n}{n!} = \frac{e^{\alpha x} - e^{\beta x}}{\sqrt{5}}. \tag{2.5}$$

3. Elmores Generalisation of $\{F_n\}$

Elmore [1] generalized the Fibonacci Sequence $\{F_n\}$ as follows. He takes $E_0(x) = E(x)$ as in (2.5) and then defines $E_n(x)$ of the generalized sequence $\{E_n(x)\}$ as the n^{th} derivatives with respect to x of $E_0(x)$. Thus we see from (2.5) that

$$E_n(x) = \frac{\alpha^n e^{\alpha x} - \beta^n e^{\beta x}}{\sqrt{5}}.$$

Note that

$$E_n(0) = \frac{\alpha^n - \beta^n}{\sqrt{5}} = F_n.$$

The Recurrence relation for $\{E_n\}$ is given by

$$E_{n+2}(x) = E_{n+1}(x) + E_n(x).$$

4. Definiton of Pseudo Fibonacci Sequence

Let $t \neq \alpha, \beta$ where α, β are as in (2.1). We define the Pseudo Fibonacci Sequence $\{g_n\}$ as the sequence satisfying the following non-homogeneous recurrence relation.

$$g_{n+2} = g_{n+1} + g_n + At^n, n \geq 0, A \neq 0 \quad \text{and} \quad t \neq 0, \alpha, \beta \tag{4.1}$$

with $g_0 = 0$ and $g_1 = 1$. The few initial terms of $\{g_n\}$ are

$$g_2 = 1 + A,$$
$$g_3 = 2 + A + At.$$

Note that for A $=0$ the above terms reduce to those for $\{F_n\}$.

5. Some Identities for $\{g_n\}$

Binet's formula: Let

$$p = p(t) = \frac{A}{t^2 - t - 1}. \tag{5.1}$$

Then g_n is given by

$$g_n = c_1\alpha^n + c_2\beta^n + \frac{At^n}{t^2 - t - 1} \tag{5.2}$$
$$= c_1\alpha^n + c_2\beta^n + pt^n, \tag{5.3}$$

where

$$c_1 = \frac{1 - p(t)(t - \beta)}{\alpha - \beta}, \tag{5.4}$$

$$c_2 = \frac{p(t)(t - \alpha) - 1}{\alpha - \beta}. \tag{5.5}$$

The Generating Function $G(x) = \sum_{n=0}^{\infty} g_n x^n$ is given by

$$G(x) = \frac{x + x^2(A - t)}{(1 - xt)(1 - x - x^2)}, \quad 1 - xt \neq 0. \tag{5.6}$$

Note from (5.6) that if A $=0$

$$G(x) = \frac{x}{1 - x - x^2},$$

which, as in section (2.4), is the generating function for $\{F_n\}$.

The Exponential Generating Function $E^*(x) = \sum_{n=0}^{\infty} \frac{g_n x^n}{n!}$ is given by

$$E^*(x) = c_1 e^{\alpha x} + c_2 e^{\beta x} + pe^{xt}, \tag{5.7}$$

where c_1 and c_2 are as in (5.4) and (5.5) respectively. Note that if A$=0$ we see from (5.3), (5.4) and (5.5) that

$$p = 0, \quad c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = \frac{-1}{\sqrt{5}},$$

so that $E^*(x)$ reduces to $\frac{e^{\alpha x} - e^{\beta x}}{\sqrt{5}}$ which, as in (2.5), is the Exponential generating function for $\{F_n\}$.

6. Generalization of $\{g_n\}$ by applying Elmore's Method

Let

$$E_0^*(x) = E^*(x) = c_1 e^{\alpha x} + c_2 e^{\beta x} + p e^{xt}$$

be the Exponential Generating Function of $\{g_n\}$ as in (5.7). Further, let $G_n(x)$ of the sequence $\{G_n(x)\}$ be defined as the n^{th} derivative with respect to x of $E_0^*(x)$, then

$$G_n(x) = c_1\alpha^n e^{\alpha x} + c_2\beta^n e^{\beta x} + pt^n e^{xt}. \tag{6.1}$$

Note that

$$G_n(0) = c_1\alpha^n + c_2\beta^n + pt^n = g_n, \tag{6.2}$$

which, in turn, reduces to F_n if $A = 0$.

Theorem 6.1. *The sequence $\{G_n(x)\}$ satisfies the non-homogeneous recurrence relation*

$$G_{n+2}(x) = G_{n+1}(x) + G_n(x) + At^n e^{xt}. \tag{6.3}$$

Proof.

$$\begin{aligned} \text{R.H.S.} &= c_1\alpha^{n+1}e^{\alpha x} + c_2\beta^{n+1}e^{\beta x} + pt^{n+1}e^{xt} \\ &\quad + c_1\alpha^n e^{\alpha x} + c_2\beta^n e^{\beta x} + pt^n e^{xt} + At^n e^{xt} \\ &= c_1\alpha^n e^{\alpha x}(\alpha+1) + c_2\beta^n e^{\beta x}(\beta+1) \\ &\quad + pt^n e^{xt}(t+1) + p(t^2-t-1)t^n e^{xt}. \end{aligned} \tag{6.4}$$

Since α and β are the roots of $x^2 - x - 1 = 0$, $\alpha+1 = \alpha^2$ and $\beta+1 = \beta^2$ so that (6.4) reduces to

$$\text{R.H.S} = c_1\alpha^{n+2}e^{\alpha x} + c_2\beta^{n+2}e^{\beta x} + pt^{n+2}e^{xt} = G_{n+2}(x). \qquad \square$$

7. Generalization of Circular Functions

The Generalized Circular Functions are defined by Mikusinsky [2] as follows: Let

$$N_{r,j} = \sum_{n=0}^{\infty} \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, \dots, r-1; \quad r \geq 1, \tag{7.1}$$

$$M_{r,j} = \sum_{n=0}^{\infty} (-1)^r \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, \dots, r-1; \quad r \geq 1. \tag{7.2}$$

Observe that

$$\begin{aligned} &N_{1,0}(t) = e^t, && N_{2,0}(t) = \cosh t, && N_{2,1}(t) = \sinh t, \\ &M_{1,0}(t) = e^{-t}, && M_{2,0}(t) = \cos t, && M_{2,1}(t) = \sin t. \end{aligned}$$

Differentiating (7.1) term by term it is easily established that

$$N_{r,0}^{(p)}(t) = \begin{cases} N_{r,j-p}(t), & 0 \le p \le j \\ N_{r,r+j-p}(t), & 0 \le j < j < p \le r \end{cases} \tag{7.3}$$

In particular, note from (7.3) that

$$N_{r,0}^{(r)}(t) = N_{r,0}(t),$$

so that in general

$$N_{r,0}^{(nr)}(t) = N_{r,0}(t), r \ge 1. \tag{7.4}$$

Further note that

$$N_{r,0}(0) = N_{r,0}^{(nr)}(0) = 1.$$

8. Application of Circular functions to generalize $\{g_n\}$

Using Generalized Circular Functions and Pethe-Phadte technique [3] we define the sequence $Q_n(x)$ as follows. Let

$$Q_0(x) = c_1 N_{r,0}(\alpha^* x) + c_2 N_{r,0}(\beta^* x) + p N_{r,0}(t^* x), \tag{8.1}$$

where $\alpha^* = \alpha^{1/r}$, $\beta^* = \beta^{1/r}$ and $t^* = t^{1/r}$, r being the positive integer. Now define the sequence $\{Q_n(x)\}$ successively as follows:

$$Q_1(x) = Q_0^{(r)}(x), \quad Q_2(x) = Q_0^{(2r)}(x),$$

and in general

$$Q_n(x) = Q_0^{(nr)}(x),$$

where derivatives are with respect to x. Then from (8.1) and using (7.4) we get

$$\begin{aligned} Q_1(x) &= c_1 \alpha N_{r,0}(\alpha^* x) + c_2 \beta N_{r,0}(\beta^* x) + pt N_{r,0}(t^* x), \\ Q_2(x) &= c_1 \alpha^2 N_{r,0}(\alpha^* x) + c_2 \beta^2 N_{r,0}(\beta^* x) + pt^2 N_{r,0}(t^* x), \\ Q_n(x) &= c_1 \alpha^n N_{r,0}(\alpha^* x) + c_2 \beta^n N_{r,0}(\beta^* x) + pt^n N_{r,0}(t^* x). \end{aligned} \tag{8.2}$$

Observe that if $r = 1$, $x = 0$, $A = 0$, $\{Q_n(x)\}$ reduces to $\{F_n\}$.

Theorem 8.1. *The sequence $\{G_n(x)\}$ satisfies the non-homogeneous recurrence relation*

$$Q_{n+2}(x) = Q_{n+1}(x) + Q_n(x) + A t^n N_{r,0}(t^* x). \tag{8.3}$$

Proof.

$$\begin{aligned}\text{R.H.S.} &= c_1\alpha^{n+1}N_{r,0}(\alpha^*x) + c_2\beta^{n+1}N_{r,0}(\beta^*x) + pt^{n+1}N_{r,0}(t^*x)\\ &\quad + c_1\alpha^n N_{r,0}(\alpha^*x) + c_2\beta^n N_{r,0}(\beta^*x) + pt^n N_{r,0}(t^*x) + At^n N_{r,0}(t^*x)\\ &= c_1\alpha^n N_{r,0}(\alpha^*x)(\alpha+1) + c_2\beta^n N_{r,0}(\beta^*x)(\beta+1) + t^n N_{r,0}(t^*x)(pt+p+A). \quad (8.4)\end{aligned}$$

Using the fact that α and β are the roots of $x^2 - x - 1 = 0$ and (5.1) in (8.4) we get

$$\text{R.H.S.} = c_1\alpha^{n+2}N_{r,0}(\alpha^*x) + c_2\beta^{n+2}N_{r,0}(\beta^*x) + pt^{n+2}N_{r,o}(t^*x) = Q_{n+2}(x). \quad \square$$

It would be an interesting exercise to prove 7 identities for $Q_n(x)$ similar to those proved in Pethe-Phadte with respect to $P_n(x)$ [3].

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