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On *h*-perfect numbers

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Abstract

Let $\sigma(x)$ denote the sum of the divisors of x. The diophantine equation $\sigma(x) + \sigma(y) = 2(x + y)$ equalizes the abundance and deficiency of x and y. For $x = n$ and $y = hn$ the solutions n are called h-perfect since the classical perfect numbers occur as solutions for $h = 1$. Some results on h-perfect numbers are determined.

Keywords: perfect numbers, amicable numbers

MSC: 11A25

1. Introduction

Let $\sigma(n)$ denote the sum of the divisors of n, that is,

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \quad \text{for} \quad n = \prod_{i=1}^{r} p_i^{\alpha_i}.$$

Since the classical antiquity there exist two famous problems for $\sigma(n)$.

At first it is asked for perfect numbers n fulfilling

$$\sigma(n) = 2n.$$

All even perfect numbers are of the form $n = (2^p - 1)2^{p-1}$ where p is a prime number and where $2^p - 1$ is a so-called Mersenne prime number, too. Nearly 50 such prime numbers are known. The existence of odd perfect numbers is still unknown.

Secondly, it is asked for amicable number pairs x, y such that

$$\sigma(x) - x = y \quad \text{and} \quad \sigma(y) - y = x.$$

Several thousand pairs are known. It remains unknown whether there are infinitely many pairs.

Nonperfect numbers n are called abundant if $\sigma(n) > 2n$ and called deficient if $\sigma(n) < 2n$. Then it may be asked for perfect number pairs x, y fulfilling the diophantine equation

$$\sigma(x) + \sigma(y) = 2(x + y), \tag{1.1}$$

that is, x and y equalize abundance and deficiency.

There exist many solutions x, y of (1.1). For fixed d let X and Y be the sets of solutions x and y of $\sigma(x) = 2x + d$ and $\sigma(y) = 2y - d$, respectively. The sets X and Y are finite (see [1], p. 169). Then all pairs x, y with $x \in X$ and $y \in Y$ are solutions of (1.1).

It may be remarked that perfect and amicable numbers are special cases of (1.1): Perfect numbers for $x = y$ and amicable numbers for $\sigma(x) = \sigma(y)$.

Here it is proposed to consider the special class of solutions of (1.1) when y is a multiple of x, that is,

$$\sigma(n) + \sigma(hn) = 2(n + hn) = 2n(h + 1). \tag{1.2}$$

If $h = 1$ then n is a perfect number. Therefore solutions n of (1.2) may be called h-perfect numbers. Some results on h-perfect numbers are determined in the following.

2. Powers of two

For $h = 2^t$ all h-perfect numbers are dependent on a sequence of certain prime numbers being similar to Mersenne prime numbers.

Theorem 2.1. *A number n is 2^t-perfect, $t \geq 1$, if and only if it holds $n = 2^\alpha((2^t + 1)2^\alpha - 1)$ where $(2^t + 1)2^\alpha - 1$ is a prime number.*

Proof. Suppose that n is 2^t-perfect, $t \geq 1$.

If $(n, 2) = 1$ then equation (1.2) implies

$$\sigma(n) + \sigma(n2^t) = \sigma(n)(1 + 2^{t+1} - 1) = \sigma(n)2^{t+1} = 2n(1 + 2^t).$$

Since the left term of (1.2) is divisible by 2^{t+1} whereas the right term of (1.2) is divisible by 2 only, odd 2^t-perfect numbers do not exist.

If $n = s2^\alpha$, $\alpha \geq 1$, $(s, 2) = 1$ then equation (1.2) yields

$$\sigma(s2^\alpha) + \sigma(s2^{t+\alpha}) = 2(s2^\alpha + s2^{t+\alpha}).$$

This is equivalent to

$$\sigma(s)((2^t + 1)2^\alpha - 1) = (2^t + 1)2^\alpha s \quad \text{with} \quad s = v((2^t + 1)2^\alpha - 1),\ v \geq 1, \tag{2.1}$$

since $((2^t + 1)2^\alpha - 1, (2^t + 1)2^\alpha) = 1$.

If $v > 1$ then equation (2.1) determines

$$v((2^t+1)2^\alpha - 1) + v + 1 \le \sigma(v((2^t+1)2^\alpha - 1)) = v(2^t+1)2^\alpha,$$

a contradiction.

If $v = 1$ and if $s = (2^t+1)2^\alpha - 1$ is a composite number then equation (2.1) yields

$$(2^t+1)2^\alpha < \sigma((2^t+1)2^\alpha - 1) = (2^t+1)2^\alpha,$$

again a contradiction.

If $v = 1$ and if $s = (2^t+1)2^\alpha - 1$ is a prime number then equations (2.1) and (1.2) are fulfilled and $n = s2^\alpha$ is 2^t-perfect. □

In [2] the first 16 and 12 prime numbers $p = (2^t+1)2^\alpha - 1$ are listed for $t = 1$ and $t = 2$, respectively. Thus 10, 44, 184, 752, 12224, 49024,... are the first 2-perfect numbers. The question for odd 2^t-perfect numbers, $t \ge 1$, is completely answered by nonexistence whereas it is still open in the classical case of perfect numbers.

3. Nonexistence

For some classes of values of h it can be proved that h-perfect numbers do not exist.

Theorem 3.1. *For $h = c2^t$, $(c, 2) = 1$, $c \ge 3$, there are no even h-perfect numbers if $c + 2 < 2^{t+2}$ and there are no h-perfect numbers if $c + 2 < 2^{t+1}$.*

Proof. For even n let $n = r2^\alpha$, $\alpha \ge 1$, $(r, 2) = 1$. Now suppose that n is $c2^t$-perfect for $c + 2 < 2^{t+2}$. Equation (1.2) implies

$$(2^{\alpha+1} - 1)\sigma(r) + (2^{\alpha+t+1} - 1)\sigma(cr) = r2^{\alpha+1}(c2^t + 1).$$

Using $\sigma(cr) \ge cr + \sigma(r)$ it follows

$$\sigma(r)(2^{\alpha+1} - 1 + 2^{\alpha+t+1} - 1) \le (2^{\alpha+1} + c)r.$$

Then $\sigma(r) \ge r$ together with $\alpha \ge 1$ determines

$$2^{t+1} \le 2^{\alpha+t+1} \le c + 2,$$

a contradiction.

For odd n suppose that n is $c2^t$-perfect for $c + 2 < 2^{t+1}$. Equation (1.2) implies

$$\sigma(n) + (2^{t+1} - 1)\sigma(cn) = 2n(1 + c2^t).$$

With $\sigma(cn) \ge cn + \sigma(n)$ it follows

$$2^{t+1}\sigma(n) \le (c + 2)n$$

and with $\sigma(n) \ge n$ the contradiction

$$2^{t+1} \le c + 2$$

is obtained. □

For $h < 100$ by Theorem 3.1 no h-perfect numbers occur if $h = 12$, 20, 24, 40, 48, 56, 72, 80, 88, or 92.

The following theorem presents another example of partial nonexistence.

Theorem 3.2. *There is no even 3^t-perfect number, $t \geq 1$.*

Proof. Suppose that $n = r2^\alpha$ is an h-perfect number for $h = 3^t$, $t \geq 1$, $\alpha \geq 1$, $(r, 2) = 1$. Equation (1.2) yields

$$\sigma(r)(2^{\alpha+1} - 1) + \sigma(r3^t)(2^{\alpha+1} - 1) = r2^{\alpha+1}(1 + 3^t). \tag{3.1}$$

Case I: $(r, 3) = 1$. It follows

$$\sigma(r)(2^{\alpha+1} - 1)(1 + (3^{t+1} - 1)/2) = r2^{\alpha+1}(1 + 3^t)$$

and equivalently

$$\sigma(r)(2^{\alpha+1} - 1)(1 + 3^{t+1}) = r2^{\alpha+2}(1 + 3^t).$$

With $\sigma(r) \geq r$ the inequality

$$(2^{\alpha+1} - 1)(1 + 3^{t+1}) \leq 2^{\alpha+2}(1 + 3^t)$$

is obtained being equivalent to

$$(3^t - 1)2^{\alpha+1} \leq 1 + 3^{t+1}.$$

This is a contradiction for $\alpha, t \geq 1$ excluded $\alpha = t = 1$. Then, however, the left term of (3.1) is divisible by 3 and, in the contrary, 3 does not divide the right term of (3.1) due to $(r, 3) = 1$.

Case II: $r = s3^\beta$, $\beta \geq 1$, $(s, 3) = 1$, and $(s, 2) = 1$ since $(r, 2) = 1$. By equation (3.1) it follows

$$\sigma(s)(2^{\alpha+1} - 1)(3^{\beta+1} + 3^{\beta+t+1} - 2) = s2^{\alpha+2}3^\beta(1 + 3^t)$$

and with $\sigma(s) \geq s$

$$2^{\alpha+1}3^{\beta+1} + 2^{\alpha+1}3^{t+\beta+1} - 2^{\alpha+2} - 3^{\beta+1} - 3^{t+\beta+1} + 2 \leq 2^{\alpha+2}3^{t+\beta} + 2^{\alpha+2}3^\beta.$$

This inequality is equivalent to

$$(3^\beta(1 + 3^t) - 2)(2^{\alpha+1} - 3) \leq 4$$

yielding a contradiction for $\alpha, \beta, t \geq 1$. □

4. Even perfect-perfect numbers

For some values of h there exist only a small number of h-perfect numbers.

Theorem 4.1. *For $h = 6$ only 13 is h-perfect and for any other even perfect number h there are no h-perfect numbers.*

Proof. Let $h = (2^p - 1)2^{p-1}$ be an even perfect number, that is, p and $2^p - 1$ both are prime numbers. Suppose that n is an h-perfect number.

For even n, that is, $n = r2^\alpha$, $\alpha \geq 1$, $(r, 2) = 1$, Theorem 3.1 implies the condition $2^p + 1 \geq 2^{p+1}$ being impossible.

For odd n two cases are distinguished.

Case I: $n = r(2^p - 1)^\alpha = rq^\alpha$, $\alpha \geq 1$, $(r, 2^p - 1) = (r, q) = 1$. By equation (1.2),

$$\sigma(rq^\alpha) + \sigma(r2^{p-1}q^{\alpha+1}) = 2rq^\alpha(1 + q2^{p-1})$$

and hence

$$\sigma(r)(q^{\alpha+1} - 1 + (2^p - 1)(q^{\alpha+2} - 1)) = r(q - 1)(2q^\alpha + 2^p q^{\alpha+1}).$$

With $\sigma(r) \geq r$ and $2^p - 1 = q$ this yields

$$q^{\alpha+1} - 1 + q^{\alpha+3} - q \leq 2q^{\alpha+1} + q^{\alpha+3} + q^{\alpha+2} - 2q^\alpha - q^{\alpha+2} - q^{\alpha+1}$$

and thus the contradiction

$$2q^\alpha \leq q + 1.$$

Case II: $(n, 2^p - 1) = (n, q) = 1$. Equation (1.2) yields

$$\sigma(n) + \sigma(nq2^{p-1}) = 2n(1 + q2^{p-1}),$$

$$\sigma(n) + \sigma(n)(2^p - 1)(q + 1) = n(2 + q2^p),$$

and thus

$$\sigma(n)(1 + q(q + 1)) = n(2 + q(q + 1)).$$

Since $(1 + q(q + 1), 2 + q(q + 1)) = 1$ it is necessary that

$$\sigma(n) = v(2 + q(q + 1)) \quad \text{with} \quad n = v(1 + q(q + 1)),\ v \geq 1. \tag{4.1}$$

If $v > 1$ in equation (4.1) then

$$v(1 + q(q + 1)) + v + 1 \leq \sigma(n) = v(2 + q(q + 1))$$

is a contradiction.

If $v = 1$ in equation (4.1) and if $1 + q(q + 1)$ is a composite number then

$$2 + q(q + 1) < \sigma(n) = 2 + q(q + 1)$$

is a contradiction.

It remains that $v = 1$ in equation (4.1) and $1 + q(q + 1)$ is a prime number. This, however, is impossible for odd prime numbers p since 3 divides $1 + q(q + 1) = 1 + (2^p - 1)2^p$ due to $2^p \equiv -1 \pmod 3$. Thus $p = 2$ determines $1 + q(q + 1) = 13$ as the unique solution of equations (4.1) and (1.2) for $h = (2^2 - 1)2^{2-1} = 6$. □

5. Small values of h

For $h \leq 16$ the discussion is completed for $h = 2$, 4, 6, 8, 12, and 16. For $h = 3$, 9, and 10 even h-perfect numbers do not exist. So far no h-perfect numbers are known for $h = 3$, 9, 10, and 13. The numbers $n = 14$ and $n = 7030$ are 5-perfect, $n = 135$ and $n = 1365$ are 7-perfect, $n = 182$ is 11-perfect, $n = 5$ and $n = 118$ are 14-perfect, and $n = 455$ is 15-perfect.

Finally, there are two corollaries for the Fibonacci number $F_7 = 13$ as consequences of Theorems 3.1 and 4.1.

Corollary 5.1. *Only 13 is an h-perfect number for any even perfect number h.*

Corollary 5.2. *Only 13 is a* $3 \cdot 2^t$*-perfect number for any* $t \geq 1$.

References

[1] Sierpinski, W., Elementary Theory of Numbers. Warszawa 1964.

[2] Online Ecyclopedia of Integer Sequences (OEIS), A007505 and A050522.