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Sierpinski-like triangle-patterns in Bi- and Fibo-nomial triangles

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Abstract

In this paper we introduce the notion of generalized (p-order) Sierpinski-like triangle-pattern, and we define the Bi- and Fibo-nomial triangles ($P_\triangle$, $F_\triangle$) and their divisibility patterns ($P_{\triangle(p)}, F_{\triangle(p)}$), respect to p. We proof that if p is an odd prime then these divisibility patterns actually are generalized Sierpinski-like triangle-patterns.

Keywords: Fibonacci sequence, Binomial triangle, Fibonomial triangle, Sierpinski pattern

MSC: 11B39, 11B65

1. Introduction

Several authors investigated the divisibility patterns of Bi- and Fibo-nomial triangles. Long (see [1]) showed that, modulo p (where p denotes a prime), Binomial triangles (also called Pascal's triangle) have self-similar structures (upon scaling by the factor p). Holte proofed similar results for Fibonomial triangles (see [2, 3]). Wells investigated (see [4]) the parallelisms between modulo 2 patterns of Bi- and Fibo-nomial triangles. In this paper we introduce the notion of generalized (p-order) Sierpinski-like triangle-pattern, and we proof that if p is an odd prime then the divisibility patterns, respect to p, of the Bi- and Fibo-nomial triangles are generalized Sierpinski-like triangle-patterns.

2. Sierpinsky like binary triangle patterns

Definition 2.1. We define $S(a, p, k)$ as generalized Sierpinsky-like binary triangle pattern, where: a denotes the side-length of the starting triangle, p denotes the order of the pattern, and k denotes the level of the pattern. The first level pattern is an equilateral number triangle with side-lengths equal to a, and all elements equal to 1 (row i, $1 \leq i \leq a$, contains i elements equal to 1). We construct the p-th order ($p > 1$), $(k+1)$-th level pattern from the p-th order, k-th level pattern ($k \geq 1$) as follows:

- We multiply the k-th level triangle $1 + 2 + \ldots + p$ times and we arrange them in p rows (row i, $1 \leq i \leq p$, will contain i k-th level triangle) in such a way that each triangle touches its neighbour triangles at a corner.
- The remaining free positions are filled by zeros.

Figure 1 shows the 3rd order, 1st, 2nd and 3rd level patterns, if the starting side-length is 3. If we choose as starting side-length 4, then we have the patterns from Figure 2.

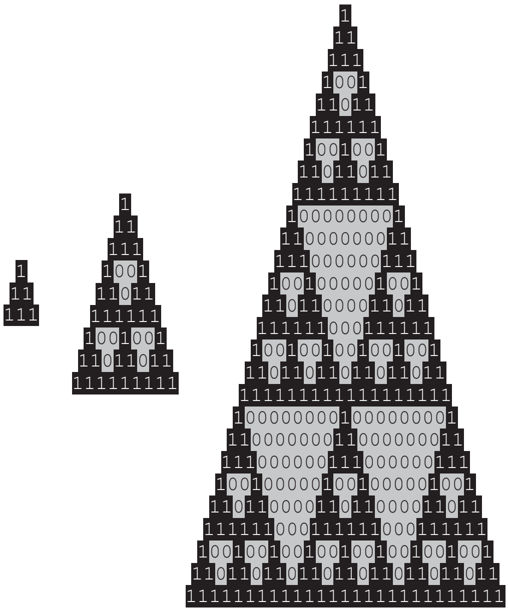

Figure 1: The 3$^{\text{rd}}$ order, 1$^{\text{st}}$ (a), 2$^{\text{nd}}$ (b) and 3$^{\text{rd}}$ (c) level patterns, if the starting side-length is 3.

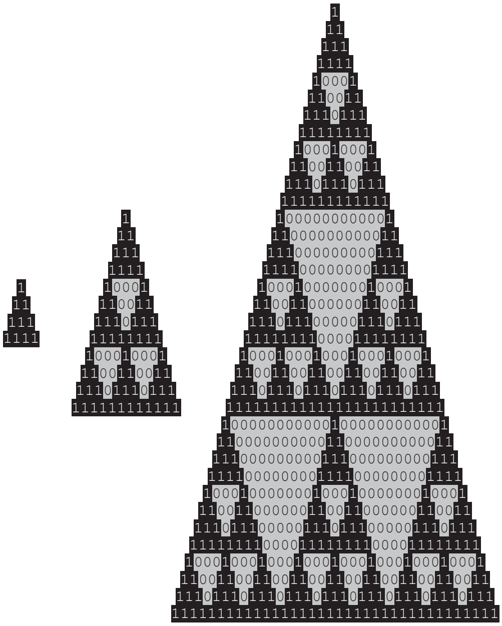

Figure 2: The 3^{rd} order, 1^{st} (a), 2^{nd} (b) and 3^{rd} (c) level patterns, if the starting side-length is 4.

3. Patterns in the prime-factorization of n and F_n

Definition 3.1. For any prime $p \geq 2$, we define sequence $x(r,p)_{r\geq 1}$ as the sequence of the powers of p in the prime-factorization of n.

Let a_p denote the subscript of the first natural number which is divisible by p. Evidently, $a_p = p$.

It is trivial that sequence $x(r,p)_{r\geq 1}$ can be constructed as follows:

- Step 0: We start with $x(r,p)_{r\geq 1} = 0$
- Step 1: All a_p-th elements 0 are increased with 1.
- Step 2: All p-th elements 1 are increased with 1.
- Step k: … All p-th elements equal to $(k-1)$ are increased with 1 …

Let n_k denote the subscript of the first term of sequence $x(r,p)_{r\geq 1}$ that is equal to a given $k \geq 1$. Evidently, $n_k = p^k$.

Definition 3.2. The well-known Fibonacci sequence is defined as follows:

$$F_0 = 0, F_1 = 1$$
$$F_r = F_{r-1} + F_{r-2}, r > 1$$

Definition 3.3. For any prime $p \geq 2$, we define sequence $y(r,p)_{r\geq 1}$ as the sequence of the powers of p in the prime-factorization of F_r.

Let b_p denote the subscript of the first Fibonacci number which is divisible by p (restricted period of $F \pmod p$). Two well-known results (for proofs see [5, 6]):

Lemma 3.4. *For any $i \geq 1$, $b_i \mid r$ if and only if $i \mid F_r$.*

Lemma 3.5. *Let p be an odd prime and suppose p^t divides F_r but p^{t+1} does not divide F_r for some $t \geq 1$. If p does not divide v then p^{t+1} divides $F_{r\cdot v\cdot p}$ but p^{t+2} does not divide $F_{r\cdot v\cdot p}$.*

A well-known conjecture in this subject:

Conjecture. For any prime p, F_{b_p} is divisible by p exactly once.

Assuming the validity of the above conjecture an immediate consequence of lemmas 3.4 and 3.5 is that sequence $y(r,p)_{r\geq 1}$ can be constructed as follows:

- Step 0: We start with $y(r,p)_{r\geq 1} = 0$
- Step 1: All b_p-th elements 0 are increased with 1.
- Step 2: All p-th elements 1 are increased with 1. (for $p = 2$ all p-th elements 1 are increased with 2)
- Step k: … All p-th elements appeared in step $(k-1)$ are increased with 1 …

Let m_k denote the subscript of the first term of sequence $y(r,p)_{r\geq 1}$ that is equal to a given $k \geq 1$. Evidently, $m_1 = b_p$.

Two immediate properties of sequence y are:

Property 1. Sequence y is characterized by several symmetry points: terms from symmetric positions are identical.

$$y_r = y_{j\cdot m_k - r} = y_{j\cdot m_k + r} = y_{p\cdot m_k - r}, \text{ for any } 0 < r < m_k,\ j = 1\ldots(p-1).$$
$$y_r = y_{j\cdot(m_k/p) - r} = y_{j\cdot(m_k/p) + r} = y_{m_k - r}, \text{ for any } 0 < r < \frac{m_k}{p},\ j = 1\ldots(p-1).$$

Proof. Trivially results from Lemmas 3.4 and 3.5. □

Property 2. For a fixed d the sum of the terms of a subsequence of length d is minimal for the leftmost (starting with index 1) subsequence and maximal for the rightmost (ending with index m_k) one. We define

$$v(i,d) = y_{i+1-d} + \ldots + y_i,\ d = 1\ldots m_k,\ i = d\ldots m_k,$$
$$u(i,d) = y_i + \ldots + y_{i+d-1},\ d = 1\ldots m_k,\ i = 1\ldots(m_k + 1 - d).$$

We have for a fixed d

a) $v(i, d) < v(m_k, d)$, for any $i = d \dots m_k - 1$

b) $u(1, d) \le u(i, d)$, for any $i = 2 \dots (m_k + 1 - d)$

Proof. (a): According to the way sequence y was built we have:

- Step 0: All terms are 0 and consequently $v(i, d) = v(m_k, d)$, for any $i = d \dots m_k - 1$.
- Steps $1 \dots (k-1)$: Since the increasing operations take place in equidistant positions, and the term from position m_k is increased in each step, we have $v(i, d) \le v(m_k, d)$, for any $i = d \dots m_k - 1$.
- Step k: Since in this step only the term from position m_k is increased, we have $v(i, d) < v(m_k, d)$, for any $i = d \dots m_k - 1$. □

Proof. (b): According to the way sequence y was built we have:

- Step 0: All terms are 0 and consequently $u(1, d) = u(i, d)$, for any $i = 2 \dots (m_k + 1 - d)$.
- Steps $1 \dots k$: The number of equidistant increases along a fixed length sequence decreases as the position of the first increase increases. Since in each step the position of the first increase (if it exists) of the leftmost subsequence of length d is maximal (relative to subsequences that start in positions $i > 1$), we have $u(1, d) \le u(i, d)$, for any $i = 2 \dots (m_k + 1 - d)$. □

Note that properties 1 and 2 hold even we do not assume the validity of the above conjecture. Since sequences x and y were constructed in a similar way, Lemmas 3.4 and 3.5 hold for sequence x too (m_k has to be replaced by n_k).

4. Bi- and Fibo-nomial triangles

Definition 4.1. We define the r rows height Binomial triangle (also called Pascal triangle) ($P_\Delta(r)$) as an equilateral number triangle with rows indexed by $i = 0 \dots (r-1)$, the elements of rows indexed by $j = 0 \dots i$, and term (i, j) equal to:

$$P_\Delta[i, j] = \frac{\prod\limits_{i+1-j}^{i} t}{\prod\limits_{1}^{j} t}$$

Changing t by F_t in the definition of Binomial triangle we receive the corresponding Fibonomial triangle.

Definition 4.2. We define the r rows height Fibonomial triangle ($F_\Delta(r)$) as an equilateral number triangle with rows indexed by $i = 0 \dots (r-1)$, the elements of rows indexed by $j = 0 \dots i$, and term (i, j) equal to

$$F_\Delta[i,j] = \frac{\prod\limits_{i+1-j}^{i} F_t}{\prod\limits_{1}^{j} F_t}$$

Definition 4.3. We also define the mod p binary Bi- and Fibo-nomial triangles $(P_{\Delta(p)}, F_{\Delta(p)})$ as follows: term (i,j) in the binary triangle is 0, if p divide term (i,j) in the corresponding Bi- or Fibo-nomial triangle, otherwise it is 1.

$$P_{\Delta(p)}[i,j] = \begin{cases} 0 & \text{if } p \mid P_{\Delta(p)}[i,j] \\ 1 & \text{otherwise} \end{cases}$$

$$F_{\Delta(p)}[i,j] = \begin{cases} 0 & \text{if } p \mid F_{\Delta(p)}[i,j] \\ 1 & \text{otherwise} \end{cases}$$

Figures 1 and 2 (triangles c) shows the $n_3 = 27$ and $m_3 = 36$ row height mod 3 binary Bi- and Fibo-nomial triangles, respectively.

5. Main result

Lemma 5.1. *Considering triangle $F_{\Delta(p)}$ (p an odd prime), for any i $(0 \le i < m_k)$ segments $F_{\Delta(p)}[i, 0 \dots i]$, $F_{\Delta(p)}[m_k + i, 0 \dots i]$ and $F_{\Delta(p)}[m_k + i, m_k \dots (m_k + i)]$ are identical.*

Proof. For $i = 0$ the validity of this lemma results trivially from the definition of $F_{\Delta(p)}$. In the case of $0 < i < m_k$ terms $F_{\Delta(p)}[i,j]$ and $F_{\Delta(p)}[m_k + i, j]$ $(j = 1 \dots i)$ are identical since the denominators of terms $F_{\Delta(p)}[i,j]$ and $F_{\Delta(p)}[m_k + i, j]$ are identical, and the exponents of p in the factorizations of the numerators of these terms are also identical. These exponents, $\sum_{m_k+r+1-i}^{m_k+r} x_t$ and $\sum_{r+1-i}^{r} x_t$, are equals since $y_{m_k+j} = y_j$ for any $j = 1 \dots r$. Since both row i and row $m_k + i$ are symmetrical, it results that the segments of the first $i + 1$ and last $i + 1$ elements of row $m_k + i$ are identical. □

Lemma 5.2. *Considering triangle $F_{\Delta(p)}$ (p an odd prime), for any i and j, where $0 \le i < m_k$ and $i + 1 \le j < m_k$, term $F_{\Delta(p)}[m_k + i, j]$ equals zero.*

Proof. With respect to the exponent of p in the factorizations of term $F_\Delta[m_k + i, j]$ we have

$$\sum_{m_k+r+1-i}^{m_k+r} x_t - \sum_{1}^{i} x_t = \sum_{m_k+r+1-i}^{m_k} x_t - \sum_{r+1}^{i} x_t > 0.$$

The equality results from Property 1 and the inequality results from Property 2.b. Consequently, $F_\Delta[m_k + i, j]$ is dividable by p. □

Lemma 5.3. *Considering triangle $F_{\Delta(p)}$ (p an odd prime), segments $F_{\Delta(p)}[m_k + i, 0 \dots m_k + i]$ and $F_{\Delta(p)}[f \cdot m_k + i, g \dots (g + m_k + i)]$, where $0 \le i < m_k, 1 < f < p$ and $0 \le g < f$, are identical.*

Proof. With respect to the exponent of p in the factorizations of term $F_\Delta[f \cdot m_k + i, g + j]$, where $0 \le j \le m_k + i$, we have

$$\begin{aligned}
&\sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r} x_t - \sum_{1}^{g\cdot m_k+i} x_t \\
&= \sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r-g\cdot m_k} x_t + \sum_{f\cdot m_k+r-g\cdot m_k+1}^{f\cdot mk} x_t + \sum_{f\cdot m_k+1}^{f\cdot m_k+r} x_t - \sum_{1}^{g\cdot m_k} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t \\
&= \sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r-g\cdot m_k} x_t + \sum_{f\cdot m_k+r-g\cdot m_k+1}^{f\cdot mk} x_t + \sum_{(f-g)\cdot m_k+1}^{(f-g)\cdot m_k+r} x_t - \sum_{1}^{g\cdot m_k} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t \\
&= \sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r-g\cdot m_k} x_t + \sum_{(f-g)\cdot m_k+1}^{f\cdot mk} x_t - \sum_{1}^{g\cdot m_k} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t \\
&= \sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r-g\cdot m_k} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t \\
&= \sum_{(f-g)\cdot m_k+r+1-i}^{(f-g)\cdot m_k+r} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t = \sum_{m_k+r+1-i}^{m_k+r} x_t - \sum_{1}^{i} x_t.
\end{aligned}$$

Which equals to the exponent of p in the factorizations of term $F_\Delta[m_k + i, j]$. □

Theorem 5.4. *For odd prime $p, P_{\Delta(p)}(n_k)$ is identical with $S(n_1, p, k)$.*

The proof of this theorem follows the same train of thought as the next one.

Theorem 5.5. *For odd prime $p, F_{\Delta(p)}(m_k)$ is identical with $S(m_1, p, k)$.*

Proof. We use mathematical induction. For $k = 1$ it is trivial that $F_{\Delta(p)}(1)$ is identical with $S(m_1, p, 1)$. Assuming that $F_{\Delta(p)}(k)$ is identical with $S(m_1, p, k)$, we prove that $F_{\Delta(p)}\ (k + 1)$ is identical with a $S(m_1, p, k + 1)$. Lemmas 5.1 and 5.2 show that rows $[m_k \dots 2 \cdot m_k)$ follow the Sierpinski pattern. Lemma 5.3 shows: since segments $[j \cdot m_k \dots (j+1)m_k)$, $(j = 2 \dots (p-1))$ can be viewed as translations of segment $[m_k \dots 2 \cdot m_k)$, these also follow the Sierpinski pattern. □

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