REM WORKING PAPER SERIES

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REM Working Paper 069-2019

February 2019

REM – Research in Economics and Mathematics
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ISSN 2184-108X

PERMANENCE IN POLYMATRIX REPLICATORS

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Abstract. Generally a biological system is said to be permanent if under small perturbations none of the species goes to extinction. In 1979 P. Schuster, K. Sigmund, and R. Wolff [14] introduced the concept of permanence as a stability notion for systems that models the self-organization of biological macromolecules. After, in 1987 W. Jansen [8], and J. Hofbauer and K. Sigmund [5] give sufficient conditions for permanence in the usual replicators. In this paper we extend these results for polymatrix replicators.

1. Introduction

In the 1970's J. M. Smith and G. R. Price [10] applied the theory of strategic games developed by J. von Neumann and O. Morgenstern [17] in the 1940's to investigate the dynamical processes of biological populations, giving rise to the feld of the Evolutionary Game Theory (EGT).

Some classes of ordinary differential equations (odes) which plays a central role in EGT are the Lotka-Volterra systems (LV), the replicator equation, the bimatrix replicator and the polymatrix replicator.

In 1979 P. Schuster, K. Sigmund, and R. Wolff introduced in [14] the concept of permanence as a stability notion for systems that models the self-organization of biological macromolecules.

Generally, we say that a biological system is permanent if, for small perturbations, none of the species goes to extinction.

The Lotka-Volterra systems, independently introduced in 1920s by A. J. Lotka [9] and V. Volterra [16], are perhaps the most widely known systems used in scientific areas as diverse as physics, chemistry, biology, and economy.

Another classical model widely used is the replicator equation which in some sense J. Hofbauer [4] proved is equivalent to the LV system.

The replicator equation was introduced by P. Taylor and L. Jonker [15]. It models the time evolution of the probability distribution of strategical behaviors within a biological population. Given a *payoff matrix* $A \in \mathrm{M}_n(\mathbb{R})$, the replicator equation refers to the following ode

$$x_i' = x_i\ \left((Ax)_i - x^t\, A\, x\right), \quad i = 1, \ldots, n,$$

2010 *Mathematics Subject Classification.* 34D05, 34D20, 37B25, 37C75, 37N25, 37N40, 91A22, 92D15.

Key words and phrases. Permanence, Polymatrix Replicator, Stability, Replicator equation, Evolutionary game theory, Lotka-Volterra.

on the simplex $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ : \sum_{j=1}^n x_j = 1\}$.

In the case we want to model the interaction between two populations (or a population divided in two groups, for example, males and females), where each group have a different set of strategies (asymmetric games), and all interactions involve individuals of different groups, the comon used model is the bimatrix replicator, that first appeared in [11] and [13]. Given two payoff matrices $A \in \mathrm{M}_{n\times m}(\mathbb{R})$ and $B \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, for the strategies in each group, the bimatrix replicator refers to the ode

$$\begin{cases} x_i' & = x_i\, ((Ay)_i - x^t\, A\, y) \quad i = 1, \ldots, n \\ y_j' & = y_j\, ((Bx)_j - y^t\, B\, x) \quad j = 1, \ldots, m \end{cases}$$

on the product of simplices $\Delta^{n-1} \times \Delta^{m-1}$. Each state in this case is a pair of frequency vectors, representing respectively the two groups' strategic behavioral frequencies. It describes the time evolution of the strategy usage frequencies in each group.

Suppose now that we want to study a population divided in a finite number of groups, each of them with finitely many behavioral strategies. Bilateral interactions between individuals of any two groups (including the same) are allowed, but competition takes place inside the groups, i.e., the relative success of each strategy is evaluated within the corresponding group.

H. Alishah and P. Duarte [1] introduced the model that they designated as polymatrix games to study this kind of populations. In [2] H. Alishah, P. Duarte and T. Peixe study particular classes of polymatrix games, namely the conservative and dissipative. The system of odes, designated as the polymatrix replicator, that model this game, will be presented later in section 3. The phase space of these systems are prisms, products of simplexes $\Delta^{n_1-1} \times \ldots \Delta^{n_p-1}$, where p is the number of groups and n_j the number of behavioral strategies inside the j-th group, for $j = 1, \ldots, p$. This class of evolutionary systems includes both the replicator (the case of only one group of individuals) and bimatrix replicator models (the case of two groups of individuals).

In 1987 J. Hofbauer and K. Sigmund [5] and W. Jansen [8] give sufficient conditions for permanence in the usual replicators. Besides we introduce the concept of permanece in the polymatrix replicators, in this paper we also extend these results for polymatrix replicators.

This paper is organized as follows. In section 2, we recall some basic properties of the replicator equation, its relation with the LV systems and the concept of permanence. In section 3 we introduce and recall the definition of polymatrix replicator. In section 4 we extend the concept of permanence to polymatrix replicators and the results given by J. Hofbauer and K. Sigmund [5] and W. Jansen [8]. We also present a result that applies to the special class of dissipative polymatrix replicators. Finally, in section 5 we illustrate our main results of permanence in polymatrix replicators with two examples.

2. Replicator equation and Permanence

In this section we present some elementary definitions and properties of the replicator equation. For more details on the subject see [6] for instance.

Consider a population where individuals interact with each other according to a set of n possible strategies. The state of the population concerning this interaction is fully described by a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where x_i represents the frequency of individuals using strategy i, for $i = 1, \ldots, n$. The set of all population states is the simplex $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ : \sum_{j=1}^n x_j = 1\}$.

If an individual using strategy i interacts with an individual using strategy j, the coefficient a_{ij} represents the average payoff for that interaction. Let $A = (a_{ij}) \in M_n(\mathbb{R}^n)$ be the matrix consisting of these a_{ij}'s. Assuming random encounters between individuals of that population, the average payoff for strategy i is given by

$$(Ax)_i = \sum_{k=1}^{n} a_{ik} x_k \,,$$

and the global average payoff of all population strategies is given by

$$x^T A\, x = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i x_k \,.$$

The logarithmic growth rate $\frac{dx_i}{dt}/x_i$ of the frequency of strategy i is equal to the payoff difference $(Ax)_i - x^T A\, x$, which yields the *replicator equation*

$$\frac{dx_i}{dt} = x_i \left((Ax)_i - x^T A\, x \right) , \quad i = 1, \ldots, n \,, \tag{2.1}$$

defined on the simplex Δ^{n-1}, that is invariant under (2.1).

The replicator equation models the frequency evolution of certain strategical behaviours within a biological population. In fact, the equation says that the logarithmic growth of the usage frequency of each behavioural strategy is directly proportional to how well that strategy fares within the population.

This system of odes was introduced in 1978 by P. Taylor and L. Jonker [15] and was designated as *replicator equation* by P. Schuster and K. Sigmund [12] in 1983.

In 1981 J. Hofbauer [4] stated an important relation between the LV systems and the replicator equation. The replicator equation is a cubic equation on the compact set Δ^{n-1} while the LV equation is quadratic on $\mathbb{R}^n_+$. However, Hofbauer proved that the replicator equation in n

variables $x_1, \ldots, x_n$ is equivalent to the LV equation in $n-1$ variables $y_1, \ldots, y_{n-1}$ (see also [6]).

Theorem 2.1. *There exists a differentiable invertible map from $\hat{S}_n = \{x \in \Delta^{n-1} : x_n > 0\}$ onto $\mathbb{R}_+^{n-1}$ mapping the orbits of the replicator equation*

$$\frac{dx_i}{dt} = x_i \left((Ax)_i - x^T A\, x\right), \quad i = 1, \ldots, n\,, \tag{2.2}$$

to time re-parametrization of the orbits of the LV equation

$$\frac{dy_i}{dt} = y_i \left(r_i + \sum_{j=1}^{n-1} a'_{ij} y_j \right), \quad i = 1, \ldots, n-1\,, \tag{2.3}$$

where $r_i = a_{in} - a_{nn}$ and $a'_{ij} = a_{ij} - a_{nj}$.

In LV systems the existence of an equilibrium point in $\mathbb{R}_+^n$ is related with the orbit's behaviour, as we can see by the following proposition (see [6] or [3], for instance).

Proposition 2.2. *System (2.3) admits an interior equilibrium point iff $\mathrm{int}(\mathbb{R}_+^n)$ contains α or ω-limit points.*

The following result shows that the time average of the orbits in LV systems is related to the values that the coordinate functions take on the equilibrium points. In fact, if there exists a unique interior equilibrium point and if the solution does not converge to the boundary neither to infinity, then its time average converges to the equilibrium point.

Proposition 2.3. *Suppose that $x(t)$ is a solution of (2.3) such that $0 < m \leq x_i(t) \leq L$, for all $t \geq 0$ and $i \in \{1, \ldots, n\}$. Then, there exists a sequence $(T_k)_{k\in\mathbb{N}}$ such that $T_k \to +\infty$ and an equilibrium point $q \in \mathbb{R}_+^n$ such that*

$$\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} x(t)\, dt = q.$$

Moreover, if system (2.3) has only one equilibrium point $q \in \mathbb{R}_+^n$, then

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^{T} x(t)\, dt = q.$$

Proof. A proof of this proposition can be seen in [3]. $\square$

By Proposition 2.2, a LV system admits an ω-limit point in $\mathrm{int}(\mathbb{R}_+^n)$ if and only if it has an equilibrium point in $\mathrm{int}(\mathbb{R}_+^n)$. Hence, from Theorem 2.1 we have that

Proposition 2.4. *If the replicator (2.1) has no equilibrium point in $\mathrm{int}(\Delta^{n-1})$, then every solution converges to the boundary of Δ^{n-1}.*

J. Hofbauer in [7] prove also a natural extension of Theorem 2.3 in LV systems to the replicator equation.

Theorem 2.5. *If the replicator* (2.1) *admits a unique equilibrium point* $q \in \operatorname{int}(\Delta^{n-1})$*, and if the* ω*-limit of the orbit of* $x(t)$ *is in* $\operatorname{int}(\Delta^{n-1})$*, then*

$$\lim_{t\to\infty} \frac{1}{T} \int_0^T x(t)\, dt = q\,.$$

We recall now the concept of *permanence* in the replicator equation, that is a stability notion introduced by Schuster *et al.* in [14].

Definition 2.6. A replicator equation (2.1) defined on Δ^{n-1} is said to be *permanent* if there exists $\delta > 0$ such that, for all $x \in \operatorname{int}(\Delta^{n-1})$,

$$\liminf_{t\to\infty} d\left(\varphi^t(x), \partial\Delta^{n-1}\right) > \delta\,,$$

where φ^t denotes the flow determined by system (2.1).

In the context of biology, a system to be permanent means that sufficiently small perturbations cannot lead any species to extinction.

The following theorem due to Jansen [8] is also valid for LV systems.

Theorem 2.7. *Let* X *be the replicator vector field defined by* (2.1)*. If there is a point* $p \in \operatorname{int}(\Delta^{n-1})$ *such that for all boundary equilibria* $x \in \partial\Delta^{n-1}$*,*

$$p^T A\, x > x^T A\, x\,, \tag{2.4}$$

then the vector field X *is permanent.*

This Theorem 2.7 is a corollary of the following theorem which gives sufficient conditions for a system to be permanent. This result is stated and proved by Hofbauer and Sigmund in [5, Theorem 1] or [6, Theorem 12.2.1].

Theorem 2.8. *Let* $P : \Delta^{n-1} \longrightarrow \mathbb{R}$ *be a smooth function such that* $P = 0$ *on* $\partial\Delta^{n-1}$ *and* $P > 0$ *on* $\operatorname{int}(\Delta^{n-1})$*. Assume there is a continuous function* $\Psi : \Delta^{n-1} \longrightarrow \mathbb{R}$ *such that*

(1) *for any orbit* $x(t)$ *in* $\operatorname{int}(\Delta^{n-1})$*,* $\frac{d}{dt} \log P(x(t)) = \Psi(x(t))$ *,*

(2) *for any orbit* $x(t)$ *in* $\partial\Delta^{n-1}$*,* $\exists\, T > 0$ *s. t.* $\int_0^T \Psi(x(t))\, dt > 0$.

Then the vector field X *is permanent.*

3. Polymatrix Replicator

In this section we present the definition of polymatrix replicator. For more details on the subject, namely some of its properties or special classes, see [1] and [2].

Consider a population divided in p groups, labelled by an integer α ranging from 1 to p. Individuals of each group $\alpha \in \{1, \ldots, p\}$ have exactly n_α strategies to interact with other members of the population, including the same group. The strategies of a group α are labelled by positive integers i in the range

$$n_1 + \ldots + n_{\alpha-1} < i \leq n_1 + \ldots + n_\alpha \, .$$

Hence, all strategies of the population are labelled by the integers i from 1 to n, where $n = n_1 + \ldots + n_p$.

We write $i \in \alpha$ to mean that i is a strategy of the group α. If an individual using strategy $i \in \alpha$ interacts with an individual using strategy $j \in \beta$, for some $\alpha, \beta \in \{1, \ldots, p\}$, the entry a_{ij} represents the average payoff for that interaction. Thus, the payoff matrix $A = (a_{ij}) \in M_n(\mathbb{R}^n)$ consisting of these a_{ij}'s can be decomposed into $n_\alpha \times n_\beta$ block matrices $A^{\alpha,\beta}$, with entries a_{ij}, where $i \in \alpha$, $j \in \beta$, and α and β range from 1 to p.

Let $\underline{n} = (n_1, \ldots, n_p)$. The *state* of the population is described by a point $x = (x^\alpha)_\alpha$ in the *prism*

$$\Gamma_{\underline{n}} := \Delta^{n_1-1} \times \ldots \times \Delta^{n_p-1} \subset \mathbb{R}^n \, ,$$

where $\Delta^{n_\alpha-1} = \{x \in \mathbb{R}_+^{n_\alpha} : \sum_{i=1}^{n_\alpha} x_i = 1\}$, $x^\alpha = (x_j)_{j\in\alpha}$ and the entry x_j represents the usage frequency of strategy j within the group α. The prism $\Gamma_{\underline{n}}$ is a $(n-p)$-dimensional simple polytope whose affine support is the $(n-p)$-dimensional subspace of $\mathbb{R}^n$ defined by the p equations

$$\sum_{i\in\alpha} x_i = 1, \quad 1 \leq \alpha \leq p \, .$$

Assuming random encounters between individuals of that population, for each group $\alpha \in \{1, \ldots, p\}$, the average payoff for strategy $i \in \alpha$, is given by

$$(Ax)_i = \sum_{k=1}^{n} a_{ik} x_k \, ,$$

and the average payoff of all strategies in α is given by

$$\sum_{j\in\alpha} x_j \, (Ax)_j \, ,$$

which can also be written as

$$\sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha,\beta} x^\beta \, .$$

The growth rate $\frac{dx_i}{dt}/x_i$ of the frequency of strategy $i \in \alpha$, for each $\alpha \in \{1, \ldots, p\}$, is equal to the payoff difference $(Ax)_i - \sum_{j\in\alpha} x_j \, (Ax)_j$, which yields the following ode on the prism $\Gamma_{\underline{n}}$,

$$\frac{dx_i}{dt} = x_i \left((A\,x)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha,\beta} x^\beta \right) , \quad \forall\, i \in \alpha, \; 1 \leq \alpha \leq p \, , \qquad (3.1)$$

called the *polymatrix replicator*.

Notice that interactions between individuals of any two groups (including the same) are allowed. Notice also that this equation implies that competition takes place inside the groups, i.e., the relative success of each strategy is evaluated within the corresponding group.

The flow $\phi^t_{\underline{n},A}$ of this equation leaves the prism $\Gamma_{\underline{n}}$ invariant. Hence, by compactness of $\Gamma_{\underline{n}}$, this flow is complete. The underlying vector field on $\Gamma_{\underline{n}}$ will be denoted by $X_{A,\underline{n}}$.

In the case $p = 1$, we have $\Gamma_{\underline{n}} = \Delta^{n-1}$ and (3.1) is the usual replicator equation associated to the payoff matrix A.

When $p = 2$, and $A^{11} = A^{22} = 0$, $\Gamma_{\underline{n}} = \Delta^{n_1-1} \times \Delta^{n_2-1}$ and (3.1) becomes the bimatrix replicator equation associated to the pair of payoff matrices (A^{12}, A^{21}).

More generally, it also includes the replicator equation for n-person games (when $A^{\alpha,\alpha} = 0$ for all $\alpha \in \{1, \ldots, p\}$.

4. Permanence in the Polymatrix Replicator

In this section we extend to polymatrix replicators the definition and some properties of *permanence* stated in the context of LV and replicator systems.

If an orbit in the interior of the state space converges to the boundary, this corresponds to extinction. Despite we give a formal definition of permanence in polymatrix replicators (see Definition 4.2), as we saw in the context of the LV systems and the replicator equation, we say that a system is *permanent* if there exists a compact set K in the interior of the state space such that all orbits starting in the interior of the state space end up in K. This means that the boundary of the state space is a repellor.

Consider a polymatrix replicator (3.1) and $X := X_{A,\underline{n}}$ its associated vector field defined on the d-dimensional prism $\Gamma_{\underline{n}}$.

The following result is an extension of the average principle in LV systems (see Theorem 2.3) and replicator equation (see Theorem 2.5) to the framework of the polymatrix replicator systems.

Proposition 4.1 (Average Principle). *Let $x(t) \in \operatorname{int}(\Gamma_{\underline{n}})$ be an interior orbit of the vector field X such that for some $\varepsilon > 0$ and some time sequence $T_k \to +\infty$, as $k \to +\infty$, one has*

(1) $d\,(x(T_k), \partial\Gamma_{\underline{n}}) \geq \varepsilon$ *for all* $k \geq 0$,

(2) $\displaystyle\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} x(t)\,dt = q,$

(3) $\displaystyle\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \pi_\alpha\,(x(t))^T\, A\,x(t)\,dt = a_\alpha$ *for all* $\alpha \in \{1, \ldots, p\}$.

Then q is an equilibrium of X and $a_\alpha = \pi_\alpha(q)^T A\, q$, for all $\alpha \in \{1,\ldots,p\}$. Moreover,

$$\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} x(t)^T A\, x(t)\, dt = q^T A\, q\,.$$

Proof. Let $\alpha \in \{1,\ldots,p\}$ and $i,j \in \alpha$. Observe that from (2) we obtain

$$\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} (Ax)_i\, dt = (Aq)_i\,.$$

By (1) we have for all k, $\varepsilon < x_i^\alpha(T_k) < 1-\varepsilon$. Hence

$$\begin{aligned}
(Aq)_i - (Aq)_j &= e_i^T Aq - e_j^T Aq \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \left(e_i^T Ax - e_j^T Ax\right)\, dt \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \left(\log \frac{x_i^\alpha(T_k)}{x_j^\alpha(T_k)} - \log \frac{x_i^\alpha(0)}{x_j^\alpha(0)} \right) = 0\,.
\end{aligned}$$

It follows that q is an equilibrium of X, and for all $i,j \in \alpha$, $\alpha = 1,\ldots,p$, $(Aq)_i = (Aq)_j = \pi_\alpha(q)^T\, A\, q$.

Finally, using (1)-(3),

$$\begin{aligned}
0 &= \lim_{k\to+\infty} \frac{1}{T_k} \left(\log x_i^\alpha(T_k) - \log x_i^\alpha(0)\right) \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \frac{\frac{dx_i^\alpha}{dt}(t)}{x_i^\alpha(t)}\, dt \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \left((Ax)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha\beta}\, x^\beta \right) dt \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \left((Ax)_i - \pi_\alpha(x)^T A\, x \right)\, dt \\
&= (Aq)_i - \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \pi_\alpha(x)^T A\, x\, dt = (Aq)_i - a_\alpha\,,
\end{aligned}$$

which implies that $a_\alpha = \pi_\alpha(q)^T\, A\, q$, and hence

$$\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} x^T A\, x\, dt = q^T A\, q\,.$$

□

The definition of permanence in the replicator equation (see Definition 2.6) can be naturally extended to the polymatrix replicator, as follows.

Definition 4.2. Given a vector field X defined in $\Gamma_{\underline{n}}$, we say that the associated flow φ_X^t is *permanent* if there exists $\delta > 0$ such that

$x \in \operatorname{int}(\Gamma_{\underline{n}})$ implies

$$\liminf_{t\to+\infty} d\left(\varphi_X^t(x), \partial\Gamma_{\underline{n}}\right) \geq \delta .$$

The following theorem extends Theorem 2.8 for polymatrix replicators.

Theorem 4.3. *Let $\Phi : \Gamma_{\underline{n}} \to \mathbb{R}$ be a smooth function such that $\Phi = 0$ on $\partial\Gamma_{\underline{n}}$ and $\Phi > 0$ on $\operatorname{int}(\Gamma_{\underline{n}})$. Assume there is a continuous function $\Psi : \Gamma_{\underline{n}} \to \mathbb{R}$ such that*

(1) for any orbit $x(t)$ in $\operatorname{int}(\Gamma_{\underline{n}})$, $\frac{d}{dt}\log\Phi(x(t)) = \Psi(x(t))$,

(2) for any orbit $x(t)$ in $\partial\Gamma_{\underline{n}}$, $\exists\, T > 0$ s. t. $\int_0^T \Psi(x(t))\, dt > 0$.

Then the vector field X is permanent.

Proof. In the proof of Theorem 2.8, Sigmund and Hofbauer [6, Theorem 12.2.1] use an argument that is abstract and applicable to a much wider class of systems, including polymatrix replicator systems. □

Remark 4.4. Sigmund and Hofbauer in [6, Theorem 12.2.2] prove that for the conclusion in Theorem 4.3 it is enough to check (2) for all ω-limit orbits in $\partial\Gamma_{\underline{n}}$. Thus, defining

(2') for any ω-limit orbit $x(t)$ in $\partial\Gamma_{\underline{n}}$, $\int_0^T \Psi(x(t))\, dt > 0$ for some $T > 0$,

we have that condition (2') implies (2).

The k-dimensional face skeleton of $\Gamma_{\underline{n}}$, denoted by $\partial_k\Gamma_{\underline{n}}$, is the union of all j-dimensional faces of $\Gamma_{\underline{n}}$ with $j \leq k$. In particular, the edge skeleton of $\Gamma_{\underline{n}}$ is the union $\partial_1\Gamma_{\underline{n}}$ of all vertices and edges of $\Gamma_{\underline{n}}$.

The following theorem extends Theorem 2.7 to polymatrix replicators.

Theorem 4.5. *If there is a point $q \in \operatorname{int}(\Gamma_{\underline{n}})$ such that for all boundary equilibria $x \in \partial\Gamma_{\underline{n}}$,*

$$q^T A\, x > x^T A\, x \,, \tag{4.1}$$

then X is permanent.

Proof. The proof we present here is essentially an adaptation of the argument used in the proof of Theorem 13.6.1 in [6].

Take the given point $q \in \operatorname{int}(\Gamma_{\underline{n}})$ and consider $\Phi : \Gamma_{\underline{n}} \to \mathbb{R}$,

$$\Phi(x) := \prod_{i=1}^{n} (x_i)^{q_i} \ .$$

We can easily see that $\Phi = 0$ on $\partial\Gamma_{\underline{n}}$ and $\Phi > 0$ on $\operatorname{int}(\Gamma_{\underline{n}})$.

Consider now the continuous function $\Psi : \Gamma_{\underline{n}} \to \mathbb{R}$,

$$\Psi(x) := q^T A\, x - x^T A\, x \ .$$

We have that
$$\frac{d}{dt}\log\Phi(x(t)) = \Psi(x(t))\,.$$
It remains to show that for any orbit $x(t)$ in $\partial\Gamma_{\underline{n}}$, there exists a $T>0$ such that
$$\int_0^T \Psi\left(x(t)\right)\, dt > 0\,. \tag{4.2}$$
We will prove by induction in $k\in\mathbb{N}$ that if $x(t)\in\partial_k\Gamma_{\underline{n}}$ then (4.2) holds for some $T>0$.

If $x(t)\in\partial_0\Gamma_{\underline{n}}$ then $x(t)\equiv q'$ for some vertex q' of $\Gamma_{\underline{n}}$. Since by (4.1) $\Psi(q')>0$, (4.2) follows. Hence the induction step is true for $k=0$.

Assume now that conclusion (4.2) holds for every orbit $x(t)\in\partial_{m-1}\Gamma_{\underline{n}}$, and consider an orbit $x(t)\in\partial_m\Gamma_{\underline{n}}$. Then there is an m-dimensional face $\sigma\in K^m(\Gamma_{\underline{n}})$ that contains $x(t)$. We consider two cases:

(i) If $x(t)$ converges to $\partial\sigma$, i.e., $\lim_{t\to+\infty} d\left(x(t),\partial\sigma\right)=0$ then the ω-limit of $x(t)$, $\omega(x)$, is contained in $\partial\sigma$. By induction hypothesis, (4.2) holds for all orbits inside $\omega(x)$, and consequently, by Remark 4.4 the same is true about $x(t)$.

(ii) If $x(t)$ does not converge to $\partial\sigma$, there exists $\varepsilon>0$ and a sequence $T_k\to+\infty$ such that $d\left(x(T_k),\partial\sigma\right)\geq\varepsilon$ for all $k\geq 0$. Let us write
$$\bar{x}(T)=\frac{1}{T}\int_0^T x\, dt \quad\text{and}\quad a_\alpha(T)=\frac{1}{T}\int_0^T \pi_\alpha(x)^T A\, x\, dt$$
for all $\alpha=1,\ldots,p$. Since the sequences $\bar{x}(T_k)$ and $a_\alpha(T_k)$ are bounded, there is a subsequence of T_k, that we will keep denoting by T_k, such that $\bar{x}(T_k)$ and $a_\alpha(T_k)$ converge, say to q' and a_α, respectively, for all $\alpha=1,\ldots,p$. By Proposition 4.1, q' is an equilibrium point in σ and $a_\alpha=\pi_\alpha(q')^T A\, q'$. Therefore
$$\frac{1}{T_k}\int_0^{T_k}\Psi\left(x(t)\right)\, dt$$
converges to $q^T A\, q' - q'^T A\, q'$, which by (4.1) is positive. This implies (4.2) and hence proves the permanence of X. ☐

A particular class of interest in the setting of the polymatrix replicators is the *dissipative polymatrix replicator*. For formal definitions and properties on conservative and dissipative polymatrix replicators see [2]. For this class of systems we have the following remark.

Remark 4.6. If a dissipative polymatrix replicator only has one interior equilibrium, then it is permanent. In fact, if a dissipative system only has one interior equilibrium, it follows that the ω-limit of any interior point is that single equilibrium point (see [2, Definition 5.1 and Proposition 17]). Then permanence follows by definition (see Definition 4.2).

5. Example

We present here two examples of polymatrix replicators that are permanent. We prove that the first example is permanent because it satisfies the condition (4.1) of Theorem 4.5. The second example, we prove that it is permanent by Remark 4.6, because it is dissipative and have a unique interior equilibrium. However, it does not satisfy the condition (4.1) of Theorem 4.5.

There is much more to analyse in the structure/dynamics of these two examples, but this will be done in future work to appear.

All computations and pictures presented in this section were done with *Wolfram Mathematica* and *Geogebra* software.

5.1. **Example 1.** Consider a population divided in 4 groups where individuals of each group have exactly 2 strategies to interact with other members of the population, whose the associated payoff matrix is

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 & -100 & 100 & -100 & 100 \\ -1 & 1 & 1 & -1 & 100 & -100 & 100 & -100 \\ 101 & -101 & -10 & 10 & -1 & 1 & -100 & 100 \\ -101 & 101 & 10 & -10 & 1 & -1 & 100 & -100 \\ 1 & -1 & 100 & -100 & -190 & 190 & -101 & 101 \\ -1 & 1 & -100 & 100 & 190 & -190 & 101 & -101 \\ 1 & -1 & 5 & -5 & 100 & -100 & -100 & 100 \\ -1 & 1 & -5 & 5 & -100 & 100 & 100 & -100 \end{pmatrix}.$$

The phase space of the associated polymatrix replicator defined by the payoff matrix A is the *prism*

$$\Gamma_{(2,2,2,2)} := \Delta^1 \times \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^4 .$$

Besides the 16 vertices of $\Gamma_{(2,2,2,2)}$ (see Table 1), this system has 2 equilibria on $3d$-faces of $\Gamma_{(2,2,2,2)}$ (see Table 2), 6 equilibria on $2d$-faces of $\Gamma_{(2,2,2,2)}$ (see Table 3), and 12 equilibria on $1d$-faces (the edges) of $\Gamma_{(2,2,2,2)}$ (see Table 4).

All these equilibria belong to $\partial\Gamma_{(2,2,2,2)}$ and satisfy

(1) $(A\mathbf{x})_1 = (A\mathbf{x})_2\,, (A\mathbf{x})_3 = (A\mathbf{x})_4\,, (A\mathbf{x})_5 = (A\mathbf{x})_6\,, (A\mathbf{x})_7 = (A\mathbf{x})_8$,

(2) $x_1 + x_2 = 1\,, x_3 + x_4 = 1\,, x_5 + x_6 = 1\,, x_7 + x_8 = 1\,,$

where x_i is the i^{th}-component of the corresponding equilibrium $\mathbf{x}$.

We have that all equilibria on $\partial\Gamma_{(2,2,2,2)}$ satisfy (4.1). In fact, we have that for all equilibria $\mathbf{x} \in \partial\Gamma_{(2,2,2,2)}$,

$$(\mathbf{x} - \mathbf{q})^T A\,\mathbf{x} < 0,$$

whith

$$\mathbf{q} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in \mathrm{int}\Gamma_{(2,2,2,2)},$$

Vertices of $\Gamma_{(2,2,2,2)}$	$f(\mathbf{v_i})$
$\mathbf{v_1} = (1,0,1,0,1,0,1,0)$	-394
$\mathbf{v_2} = (1,0,1,0,1,0,0,1)$	-4
$\mathbf{v_3} = (1,0,1,0,0,1,1,0)$	-392
$\mathbf{v_4} = (1,0,1,0,0,1,0,1)$	-6
$\mathbf{v_5} = (1,0,0,1,1,0,1,0)$	-602
$\mathbf{v_6} = (1,0,0,1,1,0,0,1)$	-592
$\mathbf{v_7} = (1,0,0,1,0,1,1,0)$	-204
$\mathbf{v_8} = (1,0,0,1,0,1,0,1)$	-198
$\mathbf{v_9} = (0,1,1,0,1,0,1,0)$	-198
$\mathbf{v_{10}} = (0,1,1,0,1,0,0,1)$	-204
$\mathbf{v_{11}} = (0,1,1,0,0,1,1,0)$	-592
$\mathbf{v_{12}} = (0,1,1,0,0,1,0,1)$	-602
$\mathbf{v_{13}} = (0,1,0,1,1,0,1,0)$	-6
$\mathbf{v_{14}} = (0,1,0,1,1,0,0,1)$	-392
$\mathbf{v_{15}} = (0,1,0,1,0,1,1,0)$	-4
$\mathbf{v_{16}} = (0,1,0,1,0,1,0,1)$	-394

TABLE 1. The vertices of $\Gamma_{(2,2,2,2)}$ and the value of $f(\mathbf{v_i})$, where $f(\mathbf{x}) = (\mathbf{x}-\mathbf{q})^T A\mathbf{x}$ and $\mathbf{q} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \in \mathrm{int}\Gamma_{(2,2,2,2)}$.

Equilibria on $3d$-faces of $\Gamma_{(2,2,2,2)}$	$f(\mathbf{q_i})$
$\mathbf{q_1} = \left(0.05266, 0.9473, 0.93275, 0.0672483, 0.991199, \frac{9049}{1028189}, 0, 1\right)$	-201.7
$\mathbf{q_2} = \left(0.9473, 0.05266, 0.0672483, 0.93275, \frac{9049}{1028189}, 0.991199, 1, 0\right)$	-201.7

TABLE 2. The equilibria on $3d$-faces of $\Gamma_{(2,2,2,2)}$ and the value of $f(\mathbf{q_i})$, where $f(\mathbf{q_i}) = (\mathbf{x}-\mathbf{q})^T A\mathbf{x}$ and $\mathbf{q} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \in \mathrm{int}\Gamma_{(2,2,2,2)}$.

as we can see in Table 1, Table 2, Table 3, and Table 4.

Hence, by Theorem 4.5, we can conclude that the system defined by the payoff matrix A is permanent.

5.2. **Example 2.** Consider a population divided in 3 groups where individuals of each group have exactly 2 strategies to interact with other members of the population, whose the associated payoff matrix is

$$A = \begin{bmatrix} 0 & -102 & 0 & 79 & 0 & 18 \\ 102 & 0 & 0 & -79 & -18 & 9 \\ 0 & 0 & 0 & 0 & 9 & -18 \\ -51 & 51 & 0 & 0 & 0 & 0 \\ 0 & 102 & -79 & 0 & -18 & -9 \\ -102 & -51 & 158 & 0 & 9 & 0 \end{bmatrix}.$$

Equilibria on $2d$-faces of $\Gamma_{(2,2,2,2)}$	$f(\mathbf{q_i})$
$\mathbf{q_3} = \left(0,1,0,1,\frac{9803}{29100},\frac{19297}{29100},\frac{893}{2910},\frac{2017}{2910}\right)$	-19.2
$\mathbf{q_4} = \left(0,1,1,0,\frac{9649}{14550},\frac{4901}{14550},\frac{994}{1455},\frac{461}{1455}\right)$	-76.7
$\mathbf{q_5} = \left(0,1,\frac{171}{400},\frac{229}{400},\frac{29}{40},\frac{11}{40},0,1\right)$	-197.4
$\mathbf{q_6} = \left(1,0,0,1,\frac{4901}{14550},\frac{9649}{14550},\frac{461}{1455},\frac{994}{1455}\right)$	-76.7
$\mathbf{q_7} = \left(1,0,1,0,\frac{19297}{29100},\frac{9803}{29100},\frac{2017}{2910},\frac{893}{2910}\right)$	-19.2
$\mathbf{q_8} = \left(1,0,\frac{229}{400},\frac{171}{400},\frac{11}{40},\frac{29}{40},1,0\right)$	-197.4

TABLE 3. The equilibria on $2d$-faces of $\Gamma_{(2,2,2,2)}$ and the value of $f(\mathbf{q_i})$, where $f(\mathbf{q_i}) = (\mathbf{x}-\mathbf{q})^T A\mathbf{x}$ and $\mathbf{q} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \in \mathrm{int}\Gamma_{(2,2,2,2)}$.

Equilibria on $1d$-faces of $\Gamma_{(2,2,2,2)}$	$f(\mathbf{q_i})$
$\mathbf{q_9} = \left(0,1,0,1,1,0,\frac{97}{100},\frac{3}{100}\right)$	-5.94
$\mathbf{q_{10}} = \left(1,0,1,0,0,1,\frac{3}{100},\frac{97}{100}\right)$	-5.94
$\mathbf{q_{11}} = \left(0,1,1,0,0,1,\frac{1}{50},\frac{49}{50}\right)$	-593.96
$\mathbf{q_{12}} = \left(1,0,0,1,1,0,\frac{49}{50},\frac{1}{50}\right)$	-593.96
$\mathbf{q_{13}} = \left(0,1,1,0,\frac{47}{95},\frac{48}{95},1,0\right)$	-207.1
$\mathbf{q_{14}} = \left(1,0,0,1,\frac{48}{95},\frac{47}{95},0,1\right)$	-207.1
$\mathbf{q_{15}} = \left(0,1,\frac{2}{5},\frac{3}{5},1,0,0,1\right)$	-307.2
$\mathbf{q_{16}} = \left(1,0,\frac{3}{5},\frac{2}{5},0,1,1,0\right)$	-307.2
$\mathbf{q_{17}} = \left(0,1,0,1,\frac{1}{2},\frac{1}{2},0,1\right)$	-203
$\mathbf{q_{18}} = \left(1,0,1,0,\frac{1}{2},\frac{1}{2},1,0\right)$	-203
$\mathbf{q_{19}} = \left(0,1,\frac{1}{2},\frac{1}{2},0,1,0,1\right)$	-488
$\mathbf{q_{20}} = \left(1,0,\frac{1}{2},\frac{1}{2},1,0,1,0\right)$	-488

TABLE 4. The equilibria on $1d$-faces of $\Gamma_{(2,2,2,2)}$ and the value of $f(\mathbf{q_i})$, where $f(\mathbf{q_i}) = (\mathbf{x}-\mathbf{q})^T A\mathbf{x}$ and $\mathbf{q} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \in \mathrm{int}\Gamma_{(2,2,2,2)}$.

The phase space of the associated polymatrix replicator defined by the payoff matrix A is the *prism*

$$\Gamma_{(2,2,2)} := \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^3 .$$

This system only has one interior equilibrium,

$$\mathbf{q} = \left(\frac{1}{2}, \frac{1}{2}, \frac{71}{158}, \frac{87}{158}, \frac{2}{3}, \frac{1}{3}\right) \in \mathrm{int}(\Gamma_{(2,2,2)})\,.$$

Moreover, besides the 8 vertices of $\Gamma_{(2,2,2)}$,

$$\begin{aligned} \mathbf{v_1} &= (1,0,1,0,1,0)\ , & \mathbf{v_2} &= (1,0,1,0,0,1)\ , \\ \mathbf{v_3} &= (1,0,0,1,1,0)\ , & \mathbf{v_4} &= (1,0,0,1,0,1)\ , \\ \mathbf{v_5} &= (0,1,1,0,1,0)\ , & \mathbf{v_6} &= (0,1,1,0,0,1)\ , \\ \mathbf{v_7} &= (0,1,0,1,1,0)\ , & \mathbf{v_8} &= (0,1,0,1,0,1)\ , \end{aligned}$$

it has 2 equilibria on two opposite $2d$-faces of $\Gamma_{(2,2,2)}$,

$$\mathbf{q_1} = \left(\frac{7}{17}, \frac{10}{17}, \frac{37}{79}, \frac{42}{79}, 1, 0\right), \quad \text{and} \quad \mathbf{q_2} = \left(\frac{23}{34}, \frac{11}{34}, \frac{65}{158}, \frac{93}{158}, 0, 1\right),$$

as represented in Figure 1. In fact, all these equilibria satisfy

(1) $(A\mathbf{x})_1 = (A\mathbf{x})_2\,, (A\mathbf{x})_3 = (A\mathbf{x})_4\,, (A\mathbf{x})_5 = (A\mathbf{x})_6$,
(2) $x_1 + x_2 = 1\,, x_3 + x_4 = 1\,, x_5 + x_6 = 1\,,$

where x_i is the i^{th}-component of the corresponding equilibrium $\mathbf{x}$.

Consider the positive diagonal matrix

$$D = \begin{bmatrix} \frac{1}{51} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{51} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{79} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{79} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \end{bmatrix},$$

and the affine subspace (as defined in [2])

$$H_{(2,2,2)} = \{\mathbf{x} \in \mathbb{R}^6 \,:\, x_1 = -x_2,\ x_3 = -x_4,\ x_5 = -x_6\}\,.$$

The quadratic form $Q_{AD} : H_{(2,2,2)} \longrightarrow \mathbb{R}$ defined by $Q_{AD}(\mathbf{x}) = \mathbf{x}^T AD\mathbf{x}$, is $Q_{AD}(\mathbf{x}) = -2\,x_3^2 \le 0$. Hence, by [2, Definition 5.1], this system is dissipative.

Since the interior equilibrium is unique, by Remark 4.6, it follows that the system is permanent. However, this second example does not satisfies the condition (4.1) of Theorem 4.5. In fact we have that do not exists any $\mathbf{q} \in \mathrm{int}\left(\Gamma_{(2,2,2)}\right)$ such that

$$\mathbf{x}^T A\,\mathbf{x} - \mathbf{q}^T A\,\mathbf{x} < 0\,,$$

for all equilibria $\mathbf{x} \in \partial\Gamma_{(2,2,2)}$.

We give now a brief description of the dynamics of this example, as illustrated by the plot of three orbits in Figure 1. This system has a strict global Lyapunov function $h : \mathrm{int}(\Gamma_{(2,2,2)}) \to \mathbb{R}$ for X_A, defined by

$$h(\mathbf{x}) = -\sum_{i=1}^{6} \frac{q_i}{d_i} \log x_i\,,$$

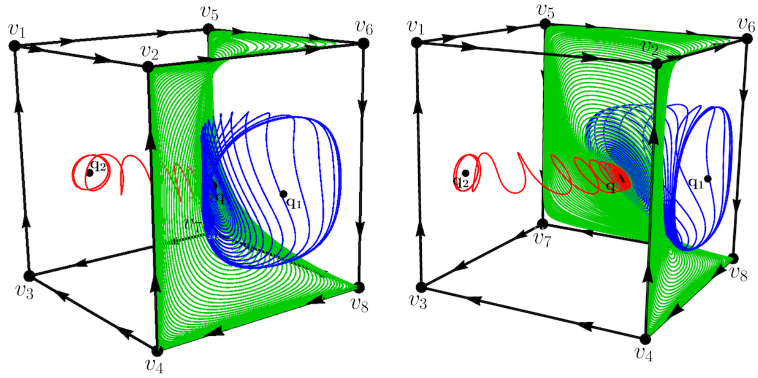

FIGURE 1. Two different perspectives of the polytope $\Gamma_{(2,2,2)}$ where the polymatrix replicator given by the payoff matrix A is defined. Namelly, the plot of its equilibria and three interior orbits.

where X_A is the associated vector field, q_i is the i^{th}-component of the equilibrium $\mathbf{q}$, and d_i is the entry (i,i) of matrix D. In fact this function h has an absolute minimum at $\mathbf{q}$ and satisfy $\dot{h} = Dh_{\mathbf{x}}(X_A) < 0$ for all $\mathbf{x} \in \text{int}(\Gamma_{(2,2,2)})$ with $\mathbf{x} \neq \mathbf{q}$. Hence, by Proposition 13 and Proposition 17 in [2], the ω-limit of any interior point $\mathbf{x} \in \text{int}(\Gamma_{(2,2,2)})$ is the equilibrium $\mathbf{q}$. The equilibria $\mathbf{q_1}$ and $\mathbf{q_2}$ in faces of $\Gamma_{(2,2,2)}$ are centres in each corresponding face, *i.e.*, for any initial condition in one of these faces the corresponding orbit will be periodic around the equilibrium point in that same face.

Acknowledgements

The author was supported by FCT - Fundação para a Ciência e a Tecnologia, under the project CEMAPRE - UID/MULTI/00491/2013 through national funds.

The author also wishes to express his gratitude to Pedro Duarte for stimulating conversations, continuous encouragement and suggestions.

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