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On the monex solutions of backwater problems for uniform open channels (*)

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RESUMO

Começa-se por recordar o estabelecimento da forma mais geral até agora conhecida da equação de Bernoulli para o tratamento unidimensional das correntes líquidas permanentes, a maneira como dela se deduz a correspondente equação diferencial das curvas de regolfo em canais abertos uniformes e ainda o conceito de valores críticos do escoamento.

Estabelecem-se em seguida fórmulas que, de um modo geral, permitem calcular curvas e volumes de regolfo nesses canais.

Introduzido o conceito de solução monex dum problema de regolfo, mostra-se que fornecem soluções monex as secções transversais rectangular e parabólica larguíssimas, bem como a triangular.

Examinada depois a questão da influência da fórmula de movimento uniforme escolhida, indica-se como pode ser unificada a formulação dos problemas de regolfo, seja para canais inclinados, seja para canais horizontais, propondo-se para estes a adopção de um função à qual se atribui o nome de Gagliardi.

Após uma notícia sobre as tábuas das funções de Dupuit e de Gagliardi de interesse para a obtenção de soluções monex usando a fórmula de Chézy, a de Manning e a de Forchheimer, dão-se alguns exemplos numéricos de aplicação.

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RÉSUMÉ

D'abord on rappelle l'établissement de la forme la plus générale jusqu'ici connue de l'équation de Bernoulli pour le traitement unidimensionnel des courants liquides permanents, la manière d'en déduire la correspondante équation différentielle des courbes de remous en canaux découverts uniformes et aussi le concept de valeurs critiques de l'écoulement.

On établit ensuite des formules qui, d'une façon générale, permettent de calculer courbes et volumes de remous dans ces canaux.

Après l'introduction du concept de solution monex d'un problème de remous, on montre que fournissent des solutions monex les sections transversales rectangulaire et parabolique très larges, ainsi que celle triangulaire.

Une fois la question de l'influence de la formule du mouvement uniforme choisie étant examinée, on indique comment peut être unifiée la formulation des problèmes sur des courbes et des volumes de remous, soit en canaux inclinés, soit en canaux horizontaux, pour ceux-ci l'adoption d'une fonction, à laquelle on attribue le nom de Gagliardi, étant proposée.

Après une notice sur les tables des fonctions de Dupuit et de Gagliardi d'intérêt pour l'obtention de solutions monex en employant la formule de Chézy, celle de Manning et celle de Forchheimer, on donne quelques exemples numériques d'application.

SYNOPSIS

To begin with, it is recalled the establishment of the most general form, so far known, of Bernoulli's equation for dealing one-dimensionally with hydraulic steady streams, the way of deducing from it the corresponding differential equation of backwater curves in uniform open channels and also the concept of critical flow values.

Thereafter formulae that, in a general manner, allow the computation of backwater curves and volumes in such channels are established.

The concept of monex solution of a backwter problem being introduced, it is shown that provide monex solutions the triangular and also the very wide rectangular and parabolic cross-sections.

Next, the question of the influence of the selected uniform flow formula is examined and it is pointed out how the formulation of backwater curves and volumes problems can be unified, either for sloping or for horizontal channels; for the last ones it is proposed the adoption of a function to which the name of Gagliardi is given.

After a notice about the tables of Dupuit and Gagliardi functions relevant for obtaining monex solutions using Chézy's, Manning's and Forchheimer's formulae, a few numerical examples of application are offered.

1. Bernoulli's equation for one-dimensional steady flow.

Consider an isothermal steady stream of a homogenous liquid in a uniform gravitational field. Call Q de discharge, γ the weight per unit volume of the liquid and g the acceleration of gravity.

Let us name cross-section of such a stream any section made in it by a surface that cuts at right angles all its streamlines.

Imagine now that in some region a cross-section Ω, of height y, can be traced. A horizontal datum plane having been selected, denote by z_o the elevation of the Ω lowest points and by p_a the pressure at one of its highest points, with elevation z_a.

The mean velocity of flow across Ω is defined by

1) $$V = \frac{Q}{A},$$

where A represents the sectional area. As usual, call α the Coriolis coefficient.

Suppose further that in the neighbourhood of Ω all the streamlines are situated on vertical planes Π, being parallel those which lie on the same Π. More strickly formulated, this hypothesis means to assume that, at all points of Π, the osculating planes of the streamlines are vertical and that all the streamlines with a common osculating plane have coincident principal normals. Under these conditions, Ω is a ruled surface, having for generating lines the afore-said normals and upon which the pressure distribution is a function solely of point elevation.

Calling ψ the least angle to the vertical of the intersection of Ω with any of the Π planes, put

2) $$\xi = \cos\, \psi\,.$$

Assume that ψ, and therefore ξ, has a constant value over Ω, i. e. that are equally inclined the generating lines of the ruled surface Ω, the stream cross-section under consideration. Then we shall name ξ the Ω Boudin coefficient.

Consider the intersection of Ω with the Π plane which contains one of the Ω lowest points O. Take that straight line as a reference n-axis, with O as origin and pointing upwards. It will be

3) $$z_a = z_o + \xi y\,.$$

Let p be the pressure at any point of Ω with elevation

4) $$z = z_o + \xi n\,.$$

As it is well known, the piezometric head at this point will be higher or lower than that at a point of elevation z_a according as the streamlines allow their concavity to be seen from above or from below:

$$z + \frac{p}{\gamma} \gtrless z_a + \frac{p_a}{\gamma} \quad .$$

Some authors, as for example Chow (1959, p. 30), call these situations respectively of concave and of convex flow.

By means of 3) and 4), the above relation becomes

$$\xi n \gtrless \xi y - \frac{p - p_a}{\gamma}$$

and so we can write

$$\text{5)} \qquad \xi n = \varepsilon \, \xi y - \frac{p - p_a}{\gamma}$$

with $\varepsilon \gtrless 1$ according to the afore-said condition.

The Jaeger-Manzanares coefficient β may now be defined thus [Mendonça 1972, eq. 19)]:

$$\text{6)} \qquad \beta = \frac{1}{Q} \int_{\Omega} \varepsilon V \mathrm{d}A \quad .$$

Consequently, β, equal to one in the case of straight streamlines, has a value higher or lower than one according as the flow is concave or convex.

When all the above-mentioned hypotheses are satisfied, the total head of the stream at section Ω may be written

$$z_0 + \lambda y + \frac{p_a}{\gamma} + \frac{\alpha V^2}{2g} \quad ,$$

with

$$\text{7)} \qquad \lambda = \beta \xi \quad .$$

Suppose now that two such cross-sections, Ω_1 and Ω_2, the first upstream from the second one, can be found. Labelling with the subscripts 1 and 2 the values pertaining respectively to Ω_1 and Ω_2, we may write

$$8) \qquad z_{o1} + \lambda_1 y_1 + \frac{p_{a1}}{\gamma} + \frac{\alpha_1 V_1^2}{2g} = z_{o2} + \lambda_2 y_2 + \frac{p_{a2}}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + H_L \ ,$$

where H_L is the loss of head from Ω_1 to Ω_2.

That one is the most general form of Bernoulli's equation, so far known, for dealing one-dimensionally with hydraulic steady flow problems. At first established only for open channels and plane cross-sections (Mendonça 1964a), it was shortly extended to confined flow (Mendonça 1964-1965) and later on to the case in which, as stated, at all points of both cross-sections the streamlines have vertical osculating planes and each one of these sections, though not plane, is a ruled surface of which the generating lines are equally inclined (Mendonça 1972).

2. *Differential equation of the surface profiles for gradually varied steady flow in uniform open channels.*

An open channel is named uniform when it has a cylindrical bed all along which the inner roughness is the same.

In such a channel the so-called gradually varied steady flow refers to a liquid stream for which the following hypotheses nearly hold:

a) the stream free surface is isobaric,

b) the streamlines are everywhere almost parallel.

In equation 8) condition a) implies $p_{a1} = p_{a2}$ and so the terms p_{a1}/γ and p_{a2}/γ may be dropped. Then that equation reduces to

$$9) \qquad z_{o1} + \lambda_1 y_1 + \frac{\alpha_1 V_1^2}{2g} = z_{o2} + \lambda_2 y_2 + \frac{\alpha_2 V_2^2}{2g} + H_L \ .$$

Assumption b) becomes more and more accurate the lower the streamlines are situated. For this reason, in order that plane sections of the stream may be taken as cross-sections — in the sense explained before — they ought to be made normally to the generating lines of the cylindrical bed or, what is the same thing, perpendicularly to the so-called bottom line.

Take this line, directed in the sense of the flow, as the abscissas x-axis. Let x_1 and x_2 be the abscissas of sections Ω_1 and Ω_2, respectively, with reference to an arbitrary origin.

Owing to the way the cross-sections are traced, the bottom line angle to the horizon is equal to ψ. The channel slope, i. e. its bottom longitudinal slope S_0, is defined by

10) $$S_0 = \pm \sin \psi ,$$

where the upper sign is conventionnally ascribed when the bottom descends in the flow direction and the lower one when it ascends. Positive slopes are also called sustaining and negative ones adverse. Of course, horizontal channels are those for which ψ and S_0 are zero.

Now we may write

11) $$z_{01} - z_{02} = \pm (x_2 - x_1) \sin \psi = S_0 \Delta x$$

and

12) $$H_L = (x_2 - x_1) \bar{S} = \bar{S} \Delta x ,$$

where $\bar{S}$ is the mean energy slope in the reach, of length $X = x_2 - x_1$, limited by the sections Ω_1 and Ω_2.

Introducing 11) and 12) into 9), we get

$$(\lambda_2 y_2 + \frac{\alpha_2 V_2}{2g}) - (\lambda_1 y_1 + \frac{\alpha_1 V_1}{2g}) = (S_0 - \bar{S}) \Delta x ,$$

thas is

$$\Delta \left(\lambda y + \frac{\alpha V^2}{2g} \right) = (S_0 - \bar{S}) \Delta x ,$$

or else

13) $$\frac{\Delta E}{\Delta x} = S_0 - \bar{S}$$

if we put

14) $$E = \lambda y + \frac{\alpha V^2}{2g} = \lambda y + \frac{\alpha Q^2}{2gA^2} \ ,$$

a quantity, although improperly, known by the name of specific energy.

When Ω_1 and Ω_2 are infinitely close, 13) becomes

15) $$\frac{dE}{dx} = S_0 - S \ ,$$

in which S obviously means the energy slope or unit loss of head at the section of abscissa x.

Neglecting the variation of α and λ, differentiation of 14) with respect to y yields

16) $$\frac{dE}{dy} = \lambda - \frac{\alpha Q^2}{gA^3} . \frac{dA}{dy} = \lambda - \frac{\alpha Q^2 B}{gA^3} \ ,$$

where

17) $$B = \frac{dA}{dy}$$

denotes the top width, i. e. the breadth of the cross-section at the free surface.

If we put

18) $$Fr = \frac{\alpha Q^2 B}{\lambda g A^3}$$

equation 16) becomes

19) $$\frac{dE}{dy} = \lambda \ (1 - Fr) \ .$$

The division of 15) by 19) supplies the most compact form of the differential equation of stream surface longitudinal profiles, commonly called backwater curves [Mendonça 1964a; for $\lambda = 1$, Mendonça 1945, eq. (211,3)]:

20)
$$\frac{dy}{dx} = \frac{S_0 - S}{\lambda(1 - Fr)} = \frac{S - S_0}{\lambda(Fr - 1)} .$$

3. Critical values.

The subscript k is used hereafter to label the so-called critical values.

Equation $dE/dy = 0$, that is

21)
$$\lambda - \frac{\alpha Q^2 B}{g A^3} = 0 ,$$

may be looked from two different standpoints.

Consider first a set of steady streams having Ω as a common cross-section. Then, A and B are constants, whereas Q behaves as a parameter the values of which make distinguishable the various streams within the set. The value of Q that satisfies to 21) is the critical discharge

22)
$$Q_k = A \sqrt{\frac{\lambda g A}{\alpha B}}$$

and

23)
$$V_k = \frac{Q_k}{A} = \sqrt{\frac{\lambda g A}{\alpha B}}$$

is the critical (mean) velocity.

The uniform flow of mean velocity V_k is also named critical.

We shall call Froude number the ratio of V^2 to V_k^2 :

$$\frac{V^2}{V_k^2} = \frac{Q^2}{A^2} \cdot \frac{\alpha B}{\lambda g A} = \frac{\alpha Q^2 B}{\lambda g A^3} ;$$

this gives meaning to the symbol Fr adopted in 18) and shows that the critical condition corresponds to

24)
$$Fr = 1 .$$

From the second viewpoint we consider the set of steady streams having a common discharge Q. Within this new set the parameter is the critical height y_k, since all the other elements of the critical section Ω_k are functions of y_k. Equation 21) should now be written

25) $$\frac{A_k^3}{B_k} = \frac{\alpha Q^2}{\lambda g} ,$$

the determination of y_k remaining dependent on the shape of Ω_k.

By combining 18) and 25), we get

26) $$Fr = \left(\frac{A_k}{A} \right)^3 \frac{B}{B_k} .$$

Following Bakhmeteff (1932, p. 47), let us define critical slope I with respect to a depth y as that value the channel slope ought to have in order to make critical the uniform flow.

Assuming that the mean velocity V is related to the energy slope and to the hydraulic radius $R = A/U$, where U is the wetted perimeter, by a monomial formula, which we may write (Mendonça 1945)

27) $$V = \sqrt{\chi R^{\varphi} S^{\theta}} ,$$

the critical velocity, being the mean velocity of the uniform critical flow, must be given by

$$V_k = \sqrt{\chi R^{\varphi} I^{\theta}}$$

and it will get the value

28) $$V_{kk} = \sqrt{\chi R_k^{\varphi} I_k^{\theta}}$$

when the section height is the critical one, y_k .

From 23) another expression for V_{kk} may be obtained:

$$V_{kk} = \sqrt{\frac{\lambda g A_k}{\alpha B_k}} .$$

Then, writing 25) in the form

$$Q^2 = \frac{\lambda g A_k^3}{\alpha B_k} = A_k^2 \cdot \frac{\lambda g A_k}{\alpha B_k} ,$$

we see at once that the following identity holds:

29) $$Q = AV = A_k \; V_{kk} \; .$$

which, by 27) and 28), yields

$$A^2 \, R^{\varphi} \, S^{\theta} = A_k^2 \, R_k^{\varphi} \, I_k^{\theta}$$

or

30) $$\frac{1}{S} = \frac{1}{I_k} \left(\frac{A}{A_k} \right)^{\frac{\varphi+2}{\theta}} \left(\frac{U_k}{U} \right)^{\frac{\varphi}{\theta}} .$$

Further on, from 28) and 29), we get

$$Q^2 = A_k^2 \, \chi \, R_k^{\varphi} \, I_k^{\theta}$$

and

31) $$I_k = \sqrt[\theta]{\frac{Q^2 U_k^{\varphi}}{\chi A_k^{\varphi+2}}} \; .$$

4. Computing backwater curves and volumes in general.

4.1. Sloping channels.

Let us label with the subscript zero the normal values relative to Q, i. e. those assumed at any cross-section when the flow is uniform.

Put (unit height)

32) $$u = \frac{y}{y_0} \; .$$

Write

33) $$\left(\frac{A}{A_0} \right)^{\frac{\varphi+2}{\theta} - 3} \left(\frac{U_0}{U} \right)^{\frac{\varphi}{\theta}} \frac{B}{B_0} = u^q$$

and

34) $$\left(\frac{A}{A_0}\right)^{\frac{\varphi+2}{\theta}}\left(\frac{U_0}{U}\right)^{\frac{\varphi}{\theta}} = u^r .$$

The analytical method for solving backwater problems in uniform open channels called principal by the writer (Mendonça 1964a) is based on the hypothesis that q and r can be supposed to keep constant values all along the reach under consideration.

Equation 27) may be written

35) $$S = \sqrt[\theta]{\frac{Q^2U^{\varphi}}{\chi A^{\varphi+2}}}$$

and so we have

36) $$S_0 = \pm\sqrt[\theta]{\frac{Q^2U_0^{\varphi}}{\chi A_0^{\varphi+2}}}$$

where, as in 10), the upper sign refers to sustaining and the lower one to adverse bottom slopes.

From 34), 35) and 36), we get

$$\frac{S}{S_0} = \pm\left(\frac{A_0}{A}\right)^{\frac{\varphi+2}{\theta}}\left(\frac{U}{U_0}\right)^{\frac{\varphi}{\theta}} = \pm u^{-r} ,$$

that implies

37) $$S - S_0 = S_0\left(\frac{S}{S_0} - 1\right) = S_0(\pm u^{-r} - 1) = |S_0|\ (u^{-r} \mp 1) \cdot$$

From 18), 33) and 34), it follows

$$\lambda Fr = \frac{\alpha Q^2 B_0}{g A_0^3} \cdot \left(\frac{A_0}{A}\right)^3 \frac{B}{B_0} = \omega u^{q-r}$$

and

38) $$\lambda(Fr - 1) = \omega u^{q-r} - \lambda$$

with

39) $$\omega = \frac{\alpha Q^2 B_0}{g A_0^3} .$$

Observing that from 32) we get

40) $$dy = y_0 \; du \; ,$$

the introduction of 37) and 38) into 20) yields

41) $$dx = \frac{y_0}{|S_0|} \cdot \frac{\omega u^q - \lambda u^r}{1 \mp u^r} du$$

and, to obtain the distance X from Ω_1 to Ω_2, it suffices to integrate:

42) $$X = x_2 - x_1 = \frac{y_0}{|S_0|} \left(\lambda \int_{u_2}^{u_1} \frac{u^r \; du}{1 \mp u^r} - \omega \int_{u_2}^{u_1} \frac{u^q \; du}{1 \mp u^r} \right) .$$

It is called backater volume, with respect to a given pair of cross-sections, the volume of liquid included in between these sections (Mendonça 1964c).

The infinitesimal backwater volume with respect to the sections of abscissas x and $x + dx$ is

43) $$d\tau = A \; dx$$

and consequently it depends, not only on u, but also on the section shape.

Putting

44) $$A = \sum_i P_i \; y^{s_i} = \sum_i P_i (\frac{y}{y_0})^{s_i} \; y_0^{s_i} = \sum_i y_0^{s_i} \; P_i \; u^{s_i}$$

with P_i and s_i constants ($i = 1, 2, \ldots k$), and paying attention to 41), equation 43) may be written

$$45)\qquad d\tau = \frac{1}{|S_0|} \sum_i y_0^{1+s_i} \; P_i \; \frac{\omega u^{q+s_i} - \lambda u^{r+s_i}}{1 \mp u^r} \, du \quad .$$

Hence, the backwater volume with respect to the sections Ω_1 and Ω_2 is

$$46)\qquad \tau = \frac{1}{|S_0|} \sum_i y_0^{1+s_i} \; P_i \left(\lambda \int_{u_2}^{u_1} \frac{u^{r+s_i}\, du}{1 \mp u^r} - \omega \int_{u_2}^{u_1} \frac{u^{q+s_i}\, du}{1 \mp u^r} \right) .$$

Since 1964 that the writer has been showing how a certain two-valued diparametric function of the non-negative real variable introduces itself quite naturally into many problems of Fluid Mechanics (Mendonça 1964a, 1964c, 1977, 1978). Denoted by $D_N^M(u)$ and defined as follows it has been named Dupuit function in honour to Arsène-Jules-Émile-Juvénal Dupuit:

$$47)\qquad D_N^M(u) \begin{cases} \text{minus branch} \begin{cases} u < 1 \;\ldots\ldots\ldots\; \overset{-}{D}{}_N^M(u) = \displaystyle\int_u^{0{,}999} \frac{u^M\, du}{1 - u^N} \\[2ex] u > 1 \;\ldots\ldots\ldots\; \overset{-}{D}{}_N^M(u) = \displaystyle\int_u^{1{,}001} \frac{u^M\, du}{1 - u^N} \end{cases} \\[4ex] \text{plus branch} \;\ldots\ldots\ldots\ldots\ldots\ldots\; \overset{+}{D}{}_N^M(u) = \displaystyle\int_u^{50} \frac{u^M\, du}{1 + u^N} . \end{cases}$$

For backwater problems, the minus branch pertains to sustaining slopes ($S_0 > 0$) and the plus branch to adverse slopes ($S_0 < 0$).

And so formulae 42) and 46) respectively may be written too:

$$48)\qquad X = \frac{y_0}{|S_0|} \left\{ \lambda \left[D_r^r(u_2) - D_r^r(u_1) \right] - \omega \left[D_r^q(u_2) - D_r^q(u_1) \right] \right\} .$$

49) $$\tau = \frac{1}{|S_0|} \left\{ \sum_i y_0^{1+s_i} P_i \left\{ \lambda \left[D_r^{r+s_i}(u_2) - D_r^{r+s_i}(u_1) \right] - \omega \left[D_r^{q+s_i}(u_2) - D_r^{q+s_i}(u_1) \right] \right\} \right. .$$

4.2. Horizontal channels.

When the bed is horizontal, we have

50) $$\begin{cases} S_0 = 0 \\ \lambda = \beta \end{cases}$$

and then equation 20) becomes

51) $$dx = \frac{\beta}{S} (Fr - 1)\, dy \; .$$

We shall call reduced height of the cross-section the ratio

52) $$\upsilon = \frac{y}{y_k} \; ,$$

from which we get

53) $$dy = y_k \, d\upsilon \; .$$

Introducing 26), 30), 52) and 53) into 51), the following form is obtained for the differential equation of backwater curves in horizontal uniform open channels:

54) $$dx = \frac{\beta y_k}{I_k} \left[\left(\frac{A}{A_k} \right)^{\frac{\varphi+2}{\theta} - 3} \left(\frac{U_k}{U} \right)^{\frac{\varphi}{\theta}} \frac{B}{B_k} - \left(\frac{A}{A_k} \right)^{\frac{\varphi+2}{\theta}} \left(\frac{U_k}{U} \right)^{\frac{\varphi}{\theta}} \right] d\upsilon \; .$$

The assumed constancy of q and r make 33) and 34) to imply

55) $$\left(\frac{A}{A_k} \right)^{\frac{\varphi+2}{\theta} - 3} \left(\frac{U_k}{U} \right)^{\frac{\varphi}{\theta}} \frac{B}{B_k} = \left(\frac{y}{y_k} \right)^{q} = \upsilon^{q}$$

and

$$56) \qquad \left(\frac{A}{A_k}\right)^{\frac{\varphi+2}{\theta}} \left(\frac{U_k}{U}\right)^{\frac{\varphi}{\theta}} = \left(\frac{y}{y_k}\right)^{r} = \upsilon^{r} .$$

This allows 54) to get the more simple look

$$57) \qquad dx = \frac{\beta y_k}{I_k} (\upsilon^q - \upsilon^r) \; d\upsilon ,$$

of immediate integration:

$$58) \qquad X = \frac{\beta y_k}{I_k} \left[\left(\frac{\upsilon_2^{q+1}}{q+1} - \frac{\upsilon_2^{r+1}}{r+1}\right) - \left(\frac{\upsilon_1^{q+1}}{q+1} - \frac{\upsilon_1^{r+1}}{r+1}\right) \right] .$$

In honour to Gagliardi, who suggested its usefulness by studying two particular cases (Gagliardi 1974), we shall call

$$59) \qquad G_{E_2}^{E_1}(\upsilon) = \frac{\upsilon^{E_1}}{E_1} - \frac{\upsilon^{E_2}}{E_2}$$

the Gagliardi function. Introduced into 58), it yields

$$60) \qquad X = \frac{\beta y_k}{I_k} \left[G_{r+1}^{q+1}(\upsilon_2) - G_{r+1}^{q+1}(\upsilon_1) \right] .$$

Equation 44) can also be written

$$61) \qquad A = \sum_i P_i \, y^{s_i} = \sum_i P_i \left(\frac{y}{y_k}\right)^{s_i} y_k^{s_i} = \sum_i y_k^{s_i} \, P_i \, \upsilon^{s_i}$$

and so, combining 43) with 57), we find

$$d\tau = \frac{\beta}{I_k} \sum_i y_k^{1+s_i} \, P_i (\upsilon^{q+s_i} - \upsilon^{r+s_i}) \; d\upsilon ,$$

that gives by integration

$$\tau = \frac{\beta}{I_k} \sum_i y_k^{1+s_i} P_i \left[\left(\frac{\upsilon_2^{q+s_i+1}}{q+s_i+1} - \frac{\upsilon_2^{r+s_i+1}}{r+s_i+1} \right) - \left(\frac{\upsilon_1^{q+s_i+1}}{q+s_i+1} - \frac{\upsilon_1^{r+s_i+1}}{r+s_i+1} \right) \right], \tag{62}$$

or rather, using Gagliardi function,

$$\tau = \frac{\beta}{I_k} \sum_i y_k^{1+s_i} P_i \left[G_{r+s_i+1}^{q+s_i+1}(\upsilon_2) - G_{r+s_i+1}^{q+s_i+1}(\upsilon_1) \right] \ (*). \tag{63}$$

5. What is meant by «monex solutions».

Formulae 48), 49), 60) and 63) are of a quite general use. But it happens that, in most cases, q and r vary along the reach from Ω_1 to Ω_2. And so, we must proceed either with estimated mean values of the parameters or by dividing the reach into shorter ones wherein such variation might be neglected. Both these ways can give good results in practice, even though theoretically inaccurate.

However, there are some particular cases in which, not only q, r, P_i and s_i remain constant all along the reach, but also this constancy is independent of the uniform flow formula that has been chosen, provided it be a monomial one, i. e. whatever be the values taken by χ, φ and θ in 27). These are the cases that supply the solutions we call *monex*, i. e. mon(omially) ex(act).

Three have been identified: that of the triangular, and those of the very wide rectangular and parabolic cross-sections. Since the «very wide» assumption can be satisfied only approximately, from a strict viewpoint none but the first of these three cases can effectively be realized.

As it will be seen later, contrariwise to the rectangular and the parabolic sections, the triangular one does not need to be symmetric with respect to an axis perpendicular to its horizontal top line.

(*) Formulae 62) and 63) are other forms of that established in 1964 [Mendonça 1964d, eq. 6)].

6. *About the cross-sectional shapes known to provide monex solutions.*

6.1. *Very wide rectangular section.*

The rectangular shape implies

64) $$\begin{cases} B = B_0 = B_k = \text{const.} \\ A = By \end{cases}$$

and the very wide (theoretically infinitely wide) condition means that the

65) $$U = B$$

assumption holds (Mendonça 1945, p. 28).

From 64) and 65), we get

66) $$\begin{cases} \dfrac{A}{A_0} = \dfrac{By}{By_0} = \dfrac{y}{y_0} = u \\ \dfrac{U}{U_0} = \dfrac{B}{B_0} = 1 = u^0 \end{cases}$$

or else

67) $$\begin{cases} \dfrac{A}{A_k} = \dfrac{By}{By_k} = \dfrac{y}{y_k} = \upsilon \\ \dfrac{U}{U_k} = \dfrac{B}{B_k} = 1 = \upsilon^0 \; . \end{cases}$$

Then it follows

68) $$\begin{cases} \left(\dfrac{A}{A_0}\right)^{\frac{\varphi+2}{\theta}-3} \left(\dfrac{U_0}{U}\right)^{\frac{\varphi}{\theta}} \dfrac{B}{B_0} = u^{\frac{\varphi+2}{\theta}-3} \\ \left(\dfrac{A}{A_0}\right)^{\frac{\varphi+2}{\theta}} \left(\dfrac{U_0}{U}\right)^{\frac{\varphi}{\theta}} = u^{\frac{\varphi+2}{\theta}} \end{cases}$$

and

69) $$\begin{cases} \left(\dfrac{A}{A_k}\right)^{\frac{\varphi+2}{\theta}-3} \left(\dfrac{U_k}{U}\right)^{\frac{\varphi}{\theta}} \dfrac{B}{B_k} = \upsilon^{\frac{\varphi+2}{\theta}-3} \\ \left(\dfrac{A}{A_k}\right)^{\frac{\varphi+2}{\theta}} \left(\dfrac{U_k}{U}\right)^{\frac{\varphi}{\theta}} = \upsilon^{\frac{\varphi+2}{\theta}} , \end{cases}$$

that, by comparison with 33), 34), 55) and 56), yield

70) $$q = \frac{\varphi+2}{\theta} - 3$$

and

71) $$r = \frac{\varphi+2}{\theta} .$$

Furthermore, equations 64) give too

$$A = By_0(\frac{y}{y_0}) = By_0u = B\ y_k(\frac{y}{y_k}) = By_k\,\upsilon ,$$

which, compared to 44) and 61) shows we have in this case

72) $$\begin{cases} i = 1 \\ s_i = s_1 = 1 \\ P_i = P_1 = B\ . \end{cases}$$

6.2. Very wide parabolic section.

The parabola is assumed to have its axis perpendicular to the section top line, This implies (Mendonça 1945, p. 79)

73) $$\begin{cases} A = \dfrac{2}{3} By \\ \dfrac{B}{y} = \dfrac{B_0}{y_0} = \dfrac{B_k}{y_k} = \text{const.} \end{cases}$$

Moreover, 65) holds under the very wide hypothesis. Then we get

74) $$\begin{cases} \dfrac{B}{B_0} = \dfrac{U}{U_0} = \left(\dfrac{y}{y_0}\right)^{\frac{1}{2}} = u^{\frac{1}{2}} \\ \dfrac{A}{A_0} = \dfrac{By}{B_0 y_0} = u^{\frac{3}{2}} \end{cases}$$

or else

75) $$\begin{cases} \dfrac{B}{B_k} = \dfrac{U}{U_k} = \left(\dfrac{y}{y_k}\right)^{\frac{1}{2}} = \upsilon^{\frac{1}{2}} \\ \dfrac{A}{A_k} = \dfrac{By}{B_k y_k} = \upsilon^{\frac{3}{2}}\ . \end{cases}$$

It follows

76) $$\begin{cases} \left(\dfrac{A}{A_0}\right)^{\frac{\varphi+2}{\theta}-3} \left(\dfrac{U_0}{U}\right)^{\frac{\varphi}{\theta}} \dfrac{B}{B_0} = u^{\frac{\varphi+3}{\theta}-4} \\ \left(\dfrac{A}{A_0}\right)^{\frac{\varphi+2}{\theta}} \left(\dfrac{U_k}{U}\right)^{\frac{\varphi}{\theta}} = u^{\frac{\varphi+3}{\theta}} \end{cases}$$

and

77) $$\begin{cases} \left(\dfrac{A}{A_k}\right)^{\frac{\varphi+2}{\theta}-3} \left(\dfrac{U_k}{U}\right)^{\frac{\varphi}{\theta}} \dfrac{B}{B_k} = \upsilon^{\frac{\varphi+3}{\theta}-4} \\ \left(\dfrac{A}{A_k}\right)^{\frac{\varphi+2}{\theta}} \left(\dfrac{U_k}{U}\right)^{\frac{\varphi}{\theta}} = \upsilon^{\frac{\varphi+3}{\theta}}\ . \end{cases}$$

So, both 76) and 77) furnish

78) $$q = \frac{\varphi+3}{\theta} - 4$$

and

79) $$r = \frac{\varphi+3}{\theta}\ .$$

On the other hand, from equations 73) we also get

$$A = \frac{2}{3} By = \frac{2}{3} B_0 y_0 u^{\frac{3}{2}} = \frac{2}{3} B_k y_k \upsilon^{\frac{3}{2}}$$

and, by comparison with 44) and 61),

80)
$$\begin{cases} i = 1 \\ s_i = s_1 = \dfrac{3}{2} \\ P_i = P_1 = \dfrac{2}{3} B_0 y_0^{-\frac{1}{2}} = \dfrac{2}{3} B_k \; y_k^{-\frac{1}{2}} . \end{cases}$$

6.3. Triangular section.

As it has already been said, the triangle does not need to be isosceles.

Let δ_1 and δ_2 be the angles of the sides to the horizon. Then $c_1 = \cot \delta_1$ and $c_2 = \cot \delta_2$ are their slopes. And we have

81)
$$\begin{cases} A = \dfrac{1}{2} (c_1 + c_2) \, y^2 \\ B = (c_1 + c_2) \, y \\ U = \left(\sqrt{1 + c_1^2} + \sqrt{1 + c_2^2} \right) y . \end{cases}$$

From these equations we get

82)
$$\begin{cases} \dfrac{A}{A_0} = u^2 \\ \dfrac{B}{B_0} = \dfrac{U}{U_0} = u \end{cases}$$

or else

83)
$$\begin{cases} \dfrac{A}{A_k} = \upsilon^2 \\ \dfrac{B}{B_k} = \dfrac{U}{U_k} = \upsilon \ . \end{cases}$$

Then it follows

84)
$$\begin{cases} \left(\dfrac{A}{A_0}\right)^{\frac{\varphi+2}{\theta}-3} \left(\dfrac{U_0}{U}\right)^{\frac{\varphi}{\theta}} \dfrac{B}{B_0} = u^{\frac{\varphi+4}{\theta}-5} \\ \left(\dfrac{A}{A_0}\right)^{\frac{\varphi+2}{\theta}} \left(\dfrac{U_0}{U}\right)^{\frac{\varphi}{\theta}} = u^{\frac{\varphi+2}{\theta}} \ . \end{cases}$$

and

85)
$$\begin{cases} \left(\dfrac{A}{A_k}\right)^{\frac{\varphi+4}{\theta}-3} \left(\dfrac{U_k}{U}\right)^{\frac{\varphi}{\theta}} \dfrac{B}{B_k} = \upsilon^{\frac{\varphi+2}{\theta}-5} \\ \left(\dfrac{A}{A_k}\right)^{\frac{\varphi+4}{\theta}} \left(\dfrac{U_k}{U}\right)^{\frac{\varphi}{\theta}} = \upsilon^{\frac{\varphi+4}{\theta}} \ . \end{cases}$$

Equations 84) and 85) imply

86)
$$q = \frac{\varphi + 4}{\theta} - 5$$

and

87)
$$r = \frac{\varphi + 4}{\theta} \ .$$

Finally, the first of equations 81) shows that, in the case of the triangular section, we have

88)
$$\begin{cases} i = 1 \\ s_1 = s_1 = 2 \\ P_1 = P_1 = \dfrac{1}{2}\,(c_1 + c_2) \ . \end{cases}$$

7. *Influence of the uniform flow formula.*

The results so far obtained confirm the statement that the three named sectional shapes provide exact solutions whatever be the monomial uniform flow formula selected.

Such a choice has however a very important influence on the final results, since q, r, y_0 and I_k depend on χ, φ and θ.

This will be stressed by considering three of the more widely used formulae, namely those of Chézy ($V = C\sqrt{RS}$), Manning ($V = R^{2/3} S^{1/2}/n$) and Forchheimer ($V = \lambda_F R^{0.7} S^{0.5}$) .

In all of them is $\theta = 1$, and so q and r depend only on φ. Formulae 70), 78) and 86) give for the three sectional shapes $q = \varphi - 1$. Formulae 71), 79) and 87) are reduced to $r=\varphi+2=q+3$, $r=\varphi+3=q+4$ and $r=\varphi+4=q+5$, respectively.

Table 1 may then be constructed.

TABLE 1

Values of q and r for Chézy, Manning and Forchheimer formulae

Formula	φ	Cross-sectional shape					
		Very wide rectangular		Very wide parabolic		Triangular	
		q	r	q	r	q	r
Chézy	1	0	3	0	4	0	5
Manning	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{1}{3}$	$\frac{13}{3}$	$\frac{1}{3}$	$\frac{16}{3}$
Forchheimer	$\frac{7}{5}$	$\frac{2}{5}$	$\frac{17}{5}$	$\frac{2}{5}$	$\frac{22}{5}$	$\frac{2}{5}$	$\frac{27}{5}$

Let us now investigate into the dependence of I_k, or rather λ/I_k, on both shape and uniform flow formula.

For $\theta = 1$, 31) may be written

89) $$\frac{\lambda}{I_k} = \frac{\lambda \gamma A_k^{\varphi+2}}{Q^2 \; U_k^{\varphi}} \quad .$$

From 25), we get

$$\frac{1}{Q^2} = \frac{\alpha \; B_k}{\lambda g \; A_k^3} \quad ,$$

that, when introduced into 89), yields

90) $$\frac{\lambda}{I_k} = \frac{\alpha \chi B_k A_k^{\varphi-1}}{g \; U_k^{\varphi}} \quad .$$

For the very wide rectangular section, 64) and 65) transform 90) into

91) $$\frac{\lambda}{I_k} = \frac{\alpha \chi}{g} \, y_k^{\varphi-1} \quad .$$

For the very wide parabolic section, the introduction of 65) and 73) into 90) gives

92) $$\frac{\lambda}{I_k} = \frac{\alpha \chi}{g} \left(\frac{2}{3} \right)^{\varphi-1} y_k^{\varphi-1}$$

Finally, for the triangular section, by means of 81), 90) becomes

93) $$\frac{\lambda}{I_k} = \frac{\alpha \chi}{g} \left(\frac{1}{2} \right)^{\varphi-1} \left(\frac{c_1 + c_2}{\sqrt{1 + c_1^2} + \sqrt{1 + c_2^2}} \right)^{\varphi} y_k^{\varphi-1} .$$

Each one of equations 91), 92) and 93) generates different expressions according to the value of φ and the signification of χ in the selected uniform flow formula. In the three of these formulae chosen for exemplifying, the values of φ are those on Table 1 and χ is always solely dependent on roughness, being usually denoted, as it was mentioned above, by C^2 in the Chézy's formula, by $1/n^2$ in the Manning's and by λ_F^2 in the Forchheimer's.

8. *Unified formulation of backwater curves and volumes problems.*

8.1. *Sloping channels.*

Equations 72), 80) and 88) show that for the three sectional shapes under consideration we have $i = 1$, a fact that allows the unification of 48) and 49) into the single formula

94) $$J = \frac{K}{|S_0|}\left\{\lambda\left[\overset{M_1}{D_N}(u_2) - \overset{M_1}{D_N}(u_1)\right] - \omega\left[\overset{M_2}{D_N}(u_2) - \overset{M_2}{D_N}(u_1)\right]\right\} .$$

Combining the afore-said equations with Table 1, Table 2, which gives full meaning to 94), can be constructed. In it

95) $$\tau_1 = \frac{\tau}{B_0} ,$$

i. e. the backater volume per unit normal breadth, refers to the very wide sections (rectangular and parabolic).

Another feature common to the cases under study, shown by Table 2, is that M is always non-negative, a condition which simplifies the analytic expressions of Dupuit function because, according to the rules given in previous papers (Mendonça 1948 and 1978), the first two items (in the 1978 paper named G_1, G_2 for $S_0 > 0$ and H_1, H_2 for $S_0 < 0$) are both zero.

This means that all those expressions to be used for obtaining the monex solutions may be written as follows:

96) $$\bar{D}^{-a/p}_{b/p}(u) = \left\{ -Y + \frac{p}{b}\left[-\log_e Z + \sum_{j=1}^{m} (L_j + T_j) \right] \right|_{u=u}^{u=c} ,$$

with

97) $$\begin{cases} L_j = -\cos \dfrac{2(a+p)j\pi}{b} \log_e \left(w^2 - 2\,w \cos \dfrac{2j\pi}{b} + 1\right) \\ T_j = 2 \sin \dfrac{2(a+p)j\pi}{b} \arctan \dfrac{w - \cos \dfrac{2j\pi}{b}}{\sin \dfrac{2j\pi}{b}} \\ u < 1 \rightarrow c = 0.999 \\ u > 1 \rightarrow c = 1.001 \end{cases}$$

and

98) $$\overset{+a/p}{D}_{b/p}(u) = \left\{ \left[Y + \frac{p}{b} - \log_e Z + \right.\right.$$

$$\left.\left. + \frac{b-a-\frac{1}{2}}{\left|b-a-\frac{1}{2}\right|} \sum_{j=1}^{m} (L'_j + T'_j) \right] \right|_{u=u}^{u=50}$$

with

99) $$\begin{cases} L'_j = -\cos \dfrac{(a+p)(2j-1)\pi}{b} \log_e \left(w^2 - 2\,w \cos \dfrac{(2j-1)\pi}{b} + 1\right) \\ T'_j = 2 \sin \dfrac{(a+p)(2j-1)\pi}{b} \arctan \dfrac{w - \cos \dfrac{(2j-1)\pi}{b}}{\sin \dfrac{(2j-1)\pi}{b}} \end{cases} .$$

TABLE 2

Values of K, N, M_1 and M_2 in formula 94)

Cross-section	Uniform flow formula	N	Distance ($J = X$)			Volume ($J = \tau$ or τ_1)		
			K	M_1	M_2	K	M_1	M_2
Very wide rectangular	Chézy	3	y_0	3	0	y_0^2	4	1
	Manning	$\frac{10}{3}$		$\frac{10}{3}$	$\frac{1}{3}$		$\frac{13}{3}$	$\frac{4}{3}$
	Forchheimer	$\frac{17}{5}$		$\frac{17}{5}$	$\frac{2}{5}$		$\frac{22}{5}$	$\frac{7}{5}$
Very wide parabolic	Chézy	4		4	0	$\frac{2}{3} y_0^2$	$\frac{11}{2}$	$\frac{3}{2}$
	Manning	$\frac{13}{3}$		$\frac{13}{3}$	$\frac{1}{3}$		$\frac{35}{6}$	$\frac{11}{6}$
	Forchheimer	$\frac{22}{5}$		$\frac{22}{5}$	$\frac{2}{5}$		$\frac{59}{10}$	$\frac{19}{10}$
Triangular	Chézy	5		5	0	$\frac{c_1 + c_2}{2} y_0^3$	7	2
	Manning	$\frac{16}{3}$		$\frac{16}{3}$	$\frac{1}{3}$		$\frac{22}{3}$	$\frac{7}{3}$
	Forchheimer	$\frac{27}{5}$		$\frac{27}{5}$	$\frac{2}{5}$		$\frac{37}{5}$	$\frac{12}{5}$

In 96) to 99) we set

100) $$w = u^{\frac{1}{p}}$$

and

101)
$$\begin{cases} M = \dfrac{a}{p} \\ N = \dfrac{b}{p} \ , \end{cases}$$

p being obviously the least common denominator of the rational numbers M and N.

Examining closely the rules afore-mentioned, the following facts, true for the cases under consideration, are easily disclosed.

The value of Y is determined by

102)
$$\begin{cases} a \geqslant b \rightarrow \ Y = \dfrac{p}{a - b + p} u^{\frac{a-b+p}{p}} \\ a < b \rightarrow \ Y = 0 \ . \end{cases}$$

Somewhat more elaborated is the determination of Z and m.

In 96), i. e. for $S_0 > 0$ (sustaining slopes), m is equal to the greatest whole number less then $b/2$, and Z may be obtained by

103)
$$\begin{cases} b \text{ uneven} & \rightarrow \ Z = |w - 1| \\ b \text{ even} \begin{cases} a + p \text{ even} & \rightarrow \ Z = |w^2 - 1| \\ a + p \text{ uneven} & \rightarrow \ Z = \dfrac{|w - 1|}{w + 1} \ . \end{cases} \end{cases}$$

In 98), i. e. for $S_0 < 0$ (adverse slopes), m is the greatest integer less than $(b + 1)/2$, and Z is given by

104)
$$\begin{cases} b \text{ uneven} \begin{cases} a + p \text{ even} & \rightarrow \ Z = w + 1 \\ a + p \text{ uneven} & \rightarrow \ Z = \dfrac{1}{w + 1} \end{cases} \\ b \text{ even} & \rightarrow \ Z = 1 \ . \end{cases}$$

These facts, combined with Table 2, made it quite easy to build the Tables 3, 4 and 5.

TABLE 3

Values to be inserted into 96) and 98) when Chézy formula is selected

Section	Backwater	M	a	b	p	Y	Z		m	
							$S_0 > 0$	$S_0 < 0$	$S_0 > 0$	$S_0 < 0$
Very wide rectangular	Curve $J = X$	M_1	3	3	1	u	$\|u-1\|$	$u+1$	1	1
		M_2	0			0		$\frac{1}{u+1}$		
	Volume $J = \tau_1$	M_1	4			$\frac{1}{2} u^2$				
		M_2	1			0		$u+1$		
Very wide parabolic	Curve $J = X$	M_1	4	4		u	$\frac{\|u-1\|}{u+1}$	1		2
		M_2	0			0				
	Volume $J = \tau_1$	M_1	11	8	2	$\frac{2}{5} u^{\frac{5}{2}}$	$\frac{\|w-1\|}{w+1}$		3	4
		M_2	3			0				
Triangular	Curve $J = X$	M_1	5	5	1	u	$\|u-1\|$	$u+1$	2	
		M_2	0			0		$\frac{1}{u+1}$		
	Volume $J = \tau$	M_1	7			$\frac{1}{3} u^3$		$u+1$		
		M_2	2			0		$\frac{1}{u+1}$		

TABLE 4

Values to be inserted into 96) and 98) when
Manning formula is selected

<table>
<tr><th rowspan="2">Section</th><th rowspan="2">Backwater</th><th rowspan="2">M</th><th rowspan="2">a</th><th rowspan="2">b</th><th rowspan="2">p</th><th rowspan="2">Y</th><th colspan="2">Z</th><th colspan="2">m</th></tr>
<tr><th>$S_o > 0$</th><th>$S_o < 0$</th><th>$S_o < 0$</th><th>$S_o > 0$</th></tr>
<tr><td rowspan="4">Very wide rectangular</td><td rowspan="2">Curve $J=X$</td><td>M_1</td><td>10</td><td rowspan="4">10</td><td rowspan="6">3</td><td>u</td><td>$\frac{\lvert w-1\rvert}{w+1}$</td><td rowspan="4">1</td><td rowspan="4">4</td><td rowspan="4">5</td></tr>
<tr><td>M_2</td><td>1</td><td>0</td><td rowspan="2">$\lvert w^2-1\rvert$</td></tr>
<tr><td rowspan="2">Volume $J=\tau_1$</td><td>M_1</td><td>13</td><td>$\frac{1}{2}u^2$</td></tr>
<tr><td>M_2</td><td>4</td><td>0</td><td>$\frac{\lvert w-1\rvert}{w+1}$</td></tr>
<tr><td rowspan="4">Very wide parabolic</td><td rowspan="2">Curve $J=X$</td><td>M_1</td><td>13</td><td rowspan="2">13</td><td>u</td><td rowspan="2">$\lvert w-1\rvert$</td><td rowspan="2">$w+1$</td><td rowspan="2" colspan="2">6</td></tr>
<tr><td>M_2</td><td>1</td><td>0</td></tr>
<tr><td rowspan="2">Volume $J=\tau_1$</td><td>M_1</td><td>35</td><td rowspan="2">26</td><td rowspan="2">6</td><td>$\frac{2}{5}u^{\frac{5}{2}}$</td><td rowspan="3">$\frac{\lvert w-1\rvert}{w+1}$</td><td rowspan="6">1</td><td rowspan="2">12</td><td rowspan="2">13</td></tr>
<tr><td>M_2</td><td>11</td><td>0</td></tr>
<tr><td rowspan="4">Triangular</td><td rowspan="2">Curve $J=X$</td><td>M_1</td><td>16</td><td rowspan="4">16</td><td rowspan="4">3</td><td>u</td><td rowspan="4">7</td><td rowspan="4">8</td></tr>
<tr><td>M_2</td><td>1</td><td>0</td><td>$\lvert w^2-1\rvert$</td></tr>
<tr><td rowspan="2">Volume $J=\tau$</td><td>M_1</td><td>22</td><td>$\frac{1}{3}u^3$</td><td>$\frac{\lvert w-1\rvert}{w+1}$</td></tr>
<tr><td>M_2</td><td>7</td><td>0</td><td>$\lvert w^2-1\rvert$</td></tr>
</table>

TABLE 5

Forchheimer formula is selected
Values to be inserted into 96) and 98) when

<table>
<tr><th rowspan="2">Section</th><th rowspan="2">Backwater</th><th rowspan="2">M</th><th rowspan="2">a</th><th rowspan="2">b</th><th rowspan="2">p</th><th rowspan="2">Y</th><th colspan="2">Z</th><th colspan="2">m</th></tr>
<tr><th>$S_o > 0$</th><th>$S_o < 0$</th><th>$S_o > 0$</th><th>$S_o < 0$</th></tr>
<tr><td rowspan="4">Very wide rectangular</td><td rowspan="2">Curve $J = X$</td><td>M_1</td><td>17</td><td rowspan="4">17</td><td rowspan="6">5</td><td>u</td><td rowspan="4">$|w-1|$</td><td>$w+1$</td><td colspan="2" rowspan="4">8</td></tr>
<tr><td>M_2</td><td>2</td><td>0</td><td rowspan="2">$\frac{1}{w+1}$</td></tr>
<tr><td rowspan="2">Volume $J = \tau_1$</td><td>M_1</td><td>22</td><td>$\frac{1}{2}u^2$</td></tr>
<tr><td>M_2</td><td>7</td><td>0</td><td>$w+1$</td></tr>
<tr><td rowspan="4">Very wide parabolic</td><td rowspan="2">Curve $J = X$</td><td>M_1</td><td>22</td><td rowspan="2">22</td><td>u</td><td rowspan="4">$\frac{|w-1|}{w+1}$</td><td rowspan="4">1</td><td rowspan="2">10</td><td rowspan="2">11</td></tr>
<tr><td>M_2</td><td>2</td><td>0</td></tr>
<tr><td rowspan="2">Volume $J = \tau_1$</td><td>M_1</td><td>59</td><td rowspan="2">44</td><td rowspan="2">10</td><td>$\frac{2}{5}u^{\frac{5}{2}}$</td><td rowspan="2">21</td><td rowspan="2">22</td></tr>
<tr><td>M_2</td><td>19</td><td>0</td></tr>
<tr><td rowspan="4">Triangular</td><td rowspan="2">Curve $J = X$</td><td>M_1</td><td>27</td><td rowspan="4">27</td><td rowspan="4">5</td><td>u</td><td rowspan="4">$|w-1|$</td><td>$w+1$</td><td colspan="2" rowspan="4">13</td></tr>
<tr><td>M_2</td><td>2</td><td>0</td><td>$\frac{1}{w+1}$</td></tr>
<tr><td rowspan="2">Volume $J = \tau$</td><td>M_1</td><td>37</td><td>$\frac{1}{3}u^3$</td><td>$w+1$</td></tr>
<tr><td>M_2</td><td>12</td><td>0</td><td>$\frac{1}{w+1}$</td></tr>
</table>

8.2. Horizontal channels.

Paying attention to Table 1 and to equations 72), 80), 88), 91), 92) and 93), and remembering 50), we may unify 60) and 63) by writing

105) $$J = \frac{\alpha\chi}{g}\left[G_{E_2}^{E_1}(\upsilon_2) - G_{E_2}^{E_1}(\upsilon_1)\right] K_3^{E_3} K_4^{E_4} K_5^{E_5} K_6^{E_6} y_k^{E_7} ,$$

where K_3, K_4, K_5 and K_6 mean

106) $$K_3 = \frac{2}{3} ,$$

107) $$K_4 = \frac{1}{2} ,$$

108) $$K_5 = \frac{1}{\sqrt{1 + c_1^2} + \sqrt{1 + c_2^2}}$$

and

109) $$K_6 = c_1 + c_2 ,$$

the values of χ, E_1, E_2, E_3, E_4, E_5, E_6 and E_7 being given by Tables 6 and 7.

TABLE 6

**Values to be inserted into 105)
for tracing backwater curves**

$(J = X)$

<table>
<tr><th>Section</th><th>Formula</th><th>χ</th><th>E_1</th><th>E_2</th><th>E_3</th><th>E_4</th><th>E_5</th><th>E_6</th><th>E_7</th></tr>
<tr><td rowspan="3">Very wide rectangular</td><td>Chézy</td><td>C^2</td><td>1</td><td>4</td><td colspan="4" rowspan="3">0</td><td>1</td></tr>
<tr><td>Manning</td><td>$\frac{1}{n^2}$</td><td>$\frac{4}{3}$</td><td>$\frac{13}{3}$</td><td>$\frac{4}{3}$</td></tr>
<tr><td>Forchheimer</td><td>λ_F^2</td><td>$\frac{7}{5}$</td><td>$\frac{22}{3}$</td><td>$\frac{7}{5}$</td></tr>
<tr><td rowspan="3">Very wide parabolic</td><td>Chézy</td><td>C^2</td><td>1</td><td>5</td><td>0</td><td colspan="3" rowspan="3">0</td><td>1</td></tr>
<tr><td>Manning</td><td>$\frac{1}{n^2}$</td><td>$\frac{4}{3}$</td><td>$\frac{16}{3}$</td><td>$\frac{1}{3}$</td><td>$\frac{4}{3}$</td></tr>
<tr><td>Forchheimer</td><td>λ_F^2</td><td>$\frac{7}{5}$</td><td>$\frac{27}{5}$</td><td>$\frac{2}{5}$</td><td>$\frac{7}{5}$</td></tr>
<tr><td rowspan="3">Triangular</td><td>Chézy</td><td>C^2</td><td>1</td><td>6</td><td rowspan="3">0</td><td>0</td><td colspan="3">1</td></tr>
<tr><td>Manning</td><td>$\frac{1}{n^2}$</td><td>$\frac{4}{3}$</td><td>$\frac{19}{5}$</td><td>$\frac{1}{3}$</td><td colspan="3">$\frac{4}{3}$</td></tr>
<tr><td>Forchheimer</td><td>λ_F^2</td><td>$\frac{7}{5}$</td><td>$\frac{32}{5}$</td><td>$\frac{2}{5}$</td><td colspan="3">$\frac{7}{5}$</td></tr>
</table>

TABLE 7

Values to be inserted into 105)
for obtaining backwater volumes

($J = \tau_1 = \tau/B_k$ for the very wide sections; $J = \tau$ for the triangular one)

<table>
<tr><th>Section</th><th>Formula</th><th>χ</th><th>E_1</th><th>E_2</th><th>E_3</th><th>E_4</th><th>E_5</th><th>E_6</th><th>E_7</th></tr>
<tr><td rowspan="3">Very wide rectangular</td><td>Chézy</td><td>C^2</td><td>2</td><td>5</td><td colspan="4" rowspan="3">0</td><td>2</td></tr>
<tr><td>Manning</td><td>$\frac{1}{n^2}$</td><td>$\frac{7}{3}$</td><td>$\frac{16}{3}$</td><td>$\frac{7}{3}$</td></tr>
<tr><td>Forchheimer</td><td>λ_F^2</td><td>$\frac{12}{5}$</td><td>$\frac{27}{5}$</td><td>$\frac{12}{5}$</td></tr>
<tr><td rowspan="3">Very wide parabolic</td><td>Chézy</td><td>C^2</td><td>$\frac{5}{2}$</td><td>$\frac{13}{2}$</td><td>1</td><td colspan="3" rowspan="3">0</td><td>2</td></tr>
<tr><td>Manning</td><td>$\frac{1}{n^2}$</td><td>$\frac{17}{6}$</td><td>$\frac{41}{6}$</td><td>$\frac{4}{3}$</td><td>$\frac{7}{3}$</td></tr>
<tr><td>Forchheimer</td><td>λ_F^2</td><td>$\frac{29}{10}$</td><td>$\frac{69}{10}$</td><td>$\frac{7}{5}$</td><td>$\frac{12}{5}$</td></tr>
<tr><td rowspan="3">Triangular</td><td>Chézy</td><td>C^2</td><td>3</td><td>8</td><td rowspan="3">0</td><td colspan="2">1</td><td>2</td><td>3</td></tr>
<tr><td>Manning</td><td>$\frac{1}{n^2}$</td><td>$\frac{10}{3}$</td><td>$\frac{25}{3}$</td><td colspan="2">$\frac{4}{3}$</td><td>$\frac{7}{3}$</td><td>$\frac{10}{3}$</td></tr>
<tr><td>Forchheimer</td><td>λ_F^2</td><td>$\frac{17}{5}$</td><td>$\frac{42}{5}$</td><td colspan="2">$\frac{7}{5}$</td><td>$\frac{12}{5}$</td><td>$\frac{17}{5}$</td></tr>
</table>

9. *About the numerical tables of Dupuit function for solving backwater problems with monex solutions.*

9.1. *Chézy formula.*

The combination «very wide rectangular section plus Chézy formula» is the classical one. The most detailed table available for computing backwater curves was given in a paper (Mendonça 1964b) that also contains a notice of those published before.

For the same combination, but for computing backwater volumes, the first published tables have been included into a set intended to solve problems referring to rectangular channels of any width (Mendonça 1964e).

It has been recently found that such combination, although not quite satisfactory for turbulent flow (owing to variation of C and α), is absolutely accurate for the laminar flow of Newtonian liquids (because then that variation does not take place); at the same time, it became clear that — mainly in the case of shallow streams located in Zone 3 of Boudin-Bakhmeteff — those first published tables, with only three decimal places, are somewhat inadequate, so that new tables, with 5 and 6 decimal places respectively for sustaining and adverse sloping channels, and more detailed, have been added (Mendonça 1977).

Besides, for the afore-said combination, the analytical expressions of Dupuit function are simple enough to allow the direct use of any of the many models of electronic pocket «scientific» calculators so popular nowadays.

Tables for computing backwater curves in very wide parabolic and in triangular channels with sustaining slopes, Chézy formula being adopted, were first published respectively by Tolkmitt (1898) and by Puppini (1911).

More accurate tables for these last cases, including adverse slopes, and for many others, are contained in a still unpublished work (Mendonça unpubl.), where all of them correspond to the following description:

110)

$$\begin{cases} \overset{-M}{D_N}(u), \ 4 \text{ dec.} \\ u = 0(0.02)0.64(0.01)0.85(0.005)0.95(0.002)0.996(0.001) \\ \quad 0.999;\ 1.001(0.001)1.006(0.002)1.05(0.005)1.11(0.01)1.3 \\ \quad (0.02)1.4(0.05)1.7(0.1)2.2(0.2)4(0.5)6(1)15(5)20(10)30 \\ \quad (20)50;\ \infty\ ; \\ \overset{+M}{D_N}(u), \ 4 \text{ dec.} \\ u = 0(0.2)1.4(0.05)1.7(0.1)2.2(0.2)3(0.5)6(1)10(5)20(10) \\ \quad 30(20)50;\ \infty\ . \end{cases}$$

Among them are those of $D_3^1(u)$, $D_{3/2}^{3/2}(u)$ and $D_5^2(u)$. From these and in the same order, equation

$$111) \quad \int \frac{u^{a/p}du}{1 \mp u^{b/p}} = \mp \frac{pu^{\frac{a-b+p}{p}}}{a-b+p} \pm \int \frac{u^{\frac{a-b}{p}}du}{1 \mp u^{b/p}}$$

allows one to deduce the values of $D_3^4(u)$, $D_{8/2}^{11/2}(u)$ and $D_5^7(u)$, so that it is covered all the field taken into account in Table 3.

9.2. *Manning formula.*

The first author to solve correctly the problem of computing backwater curves in very wide rectangular channels using Manning formula was Supino (1934) (*), whose paper contains short tables of $\bar{D}_{10/3}^{1/3}(u)$ for $u \leqslant 1$ and of $\dot{D}_{10/3}^{1/3}(0) - \dot{D}_{10/3}^{1/3}(u)$. Its values are written with 5 decimal places, but only the first two are reliable.

Organized according to scheme 110) and intended for inclusion in the unpublished book mentioned above, correct tables of $D_{10/3}^{1/3}(u)$ and $D_{10/3}^{10/3}(u)$ were computed by the writer from July to December 1964. They are reproduced in Table 9, placed at the end of this paper.

The problem was recently retaken by Gagliardi (1974) (**), who also gave a correct treatment to the case of the triangular section. His paper contains tables of functions which differ from $D_{10/3}^{10/3}(u)$, $D_{10/3}^{1/3}(u)$, $D_{16/3}^{16/3}(u)$ and $D_{16/3}^{1/3}(u)$ by constants (***) and that may therefore be put on their places into 94). Writing on the left members the symbols used by Gagliardi, those functions are namely

(*) Supino refers to Manning-Strickler formula. A historically better name is however Gauckler-Strickler formula (Mendonça 1945, p. 9-10).

(**) Gagliardi, an Italian engineer, attributed the original integration to Gunder (1942), thus quite surprisingly ignoring the priority of his distinguished compatriote Prof. Giulio Supino.

(***) In the minus branches, these constants are not the same for $u < 1$ and $u > 1$, a fact that, of course, is irrelevant.

$$\bar{\mathrm{b}}(\mathrm{x}) = \bar{D}^{10/3}_{10/3}(u) + \text{const.},$$

$$\bar{\psi}_0(\mathrm{x}) = \bar{D}^{\,1/3}_{10/3}(u) + \text{const.},$$

$$\bar{\mathrm{c}}(\mathrm{x}) = \overset{+}{D}{}^{10/3}_{10/3}(u) - \overset{+}{D}{}^{10/3}_{10/3}(0)$$

$$\bar{\varepsilon}_0(\mathrm{x}) = -\overset{+}{D}{}^{\,1/3}_{10/3}(u) + \overset{+}{D}{}^{\,1/3}_{10/3}(0)$$

$$\mathrm{b}(\mathrm{x}) = \bar{D}^{16/3}_{16/3}(u) + \text{const.},$$

$$\psi_0(\mathrm{x}) = \bar{D}^{\,1/3}_{16/3}(u) + \text{const.},$$

$$\mathrm{c}(\mathrm{x}) = \overset{+}{D}{}^{16/3}_{16/3}(u) - \overset{+}{D}{}^{16/3}_{10/3}(0) \;,$$

$$\varepsilon_0(\mathrm{x}) = -\overset{+}{D}{}^{\,1/3}_{16/3}(u) + \overset{+}{D}{}^{\,1/3}_{10/3}(0) \;.$$

All these tables give the functions values with only three decimal places and for the following intervals and ranges of the argument (*):

0(0.02)0.6(0.01)0.95(0.005)1.02(0.01)1.2(0.02)1.5(0.05)2 (0.1)3(0.5)5(1)10(10)20.

For the other cases listed in Table 3 no published or unpublished tables are known to be available by now (**).

9.3. Forchheimer's and other uniform flow formulae.

By following the rules given above, all the other cases of Dupuit function relative to the monex solutions can easily be tabulated by anyone who has at his disposal a digital computer powerful enough. Such a task the writer leaves to the interested reader.

Nevertheless, for the sake of exemplifying the use of Forchheimer formula (Section 11), tables of $D^{17/5}_{17/5}$, $D^{\,2/5}_{17/5}$, $D^{22/5}_{22/5}$ and $D^{\,2/5}_{22/5}$,

(*) For the 4 last named functions (Gagliardi 1974, Tabella 2), certainly by misprint, the line corresponding to the value 0.38 of the argument has been omitted. It can be easily inferred from Tabella 1 of the same paper: b(0.38) = = c(0.38) = 0.000, $\psi_0(0.38) = -0.207$, $\varepsilon_0(0.38) = 0.206$.

(**) The paper of Rastrelli (1937) about the case of channels with sustaining slope and very wide parabolic section is unfortunately marred by an error committed still at the establishment of the surface profiles differential equation: from $A = 2By/3$ the author deduced $dA/dy = 2B/3$, overlooking the relation, that he correctly wrote two lines below, $B = B_0\sqrt{y/y_0}$.

picked out from Mendonça (unpubl.), are given at the end of this paper (Table 9).

For the case of backwater curves in very wide rectangular channels with sustaining slopes, Forchheimer's and also one of the Gröger formulae were chosen by Kozeny (1924, 1928). His tables (Weyrauch 1930, p. 248; Forchheimer 1935, p. 245) are however quite rudimentary and unreliable (Mendonça 1945, p. 64).

10. About the Gagliardi function.

The tables of this function given by Gagliardi (1974) refer to backwater curves in channels with triangular and very wide rectangular sections, using Manning formula, i. e. contain values of $G_{19/3}^{4/3}(u)$ and $G_{13/3}^{4/3}(u)$. They are written with only three decimal places.

We have built a table covering all the cases of the monex solutions and giving the function values with 5 decimal places, which is contained in a photostatic preliminary version of this paper. It corresponds to the following description:

112) $G_{E_2}^{E_1}(u)$, 5 dec.

$(E_1, E_2) = (17/5, 42/5), (10/3, 25/3), (29/10, 69/10), (17/6, 41/6), (3, 8), (12/5, 27/5), (7/3, 16/3), (5/2, 13/2), (2, 5), (7/5, 22/5), (4/3, 13/3), (7/5, 27/5), (7/5, 32/5), (4/3, 16/3), (4/3, 19/3), (1, 4), (1, 5), (1, 6)$;

$u = 0(0.02)1.4(0.04)1.6(0.05)2(0.1)2.8(0.2)4(0.5)5;\ \infty$.

However, the analytic expression 59) is so simple that we have not found pertinent to include that table in the present printed version.

11. Examples.

11.1. Sustaining slope.

Consider a very wide rectangular channel. Let $S_0 = 0.0004$ be the bottom slope, $y_0 = 1.75$ m the normal section height, $g = 9.81$ m s^{-2} the acceleration of gravity and $\lambda_F = 35$ the Forchheimer roughness coefficient. We wish to determine the length of the reach bounded by the cross-sections of heights $y_1 = 1.61$ m and $y_2 = 0.84$ m (backwater curve of Type $\mathscr{M}_2$).

Since for a very wide rectangular section we have $R_0 = y_0$, from Forchheimer formula, $V_0 = \lambda_F R_0^{0.7} S_0^{0.5}$, we get

$$\frac{Q}{A_0} = \frac{Q}{y_0 B} = V_0 = 35 \times (1.75)^{0.7} (0.0004)^{0.5}$$
$$= 1.035677817.$$

Then, putting $\alpha = 1.1$, 39) gives

$$\omega = \frac{\alpha Q^2 B_0}{g A_0^3} = \frac{\alpha B_0 Q^2}{g y_0 B_0 A_0^2} = \frac{\alpha V_0^2}{g y_0} = \frac{1.1 \ (1.035677817)^2}{9.81 \times 1.75}$$
$$= 0.0687282012 \ .$$

Further, we have $u_1 = y_1 / y_0 = 0.92$ and $u_2 = y_2 / y_0 = 0.48$.
Assumtng $\lambda = 0.999$ and taking Table 2 into account, 94) becomes

$$X = \frac{1.75}{0.0004} \left\{ 0.999 \left[\bar{D}_{3.4}^{3.4}(0.48) - \bar{D}_{3.4}^{3.4}(0.92) \right] - \right.$$
$$\left. - 0.0687282012 \left[\bar{D}_{3.4}^{0.4}(0.48) - \bar{D}_{3.4}^{0.4}(0.92) \right] \right\}$$

or, picking from Table 9 the values of Dupuit function needed,

$$X = 4375 \left[0.999 \ (1.5449 - 1.2386) - 0.0687282012 \ (1.9685 - 1.3077) \right]$$
$$= 1140 \text{ m} \ .$$

11.2. Adverse slope.

Now the channel has a very wide parabolic section and an adverse slope $S_0 = - 0.0004$. Apart from this, the data are identical to those of the example given in Section *11.1.* The backwater curve is obviously of Type $\mathscr{A}_2$.

In this case we have, from 73) and 65),

$$A_0 = \frac{2}{3} B_0 y_0$$

and

$$R_0 = \frac{A_0}{U_0} = \frac{A_0}{B_0} = \frac{2}{3} y_0$$

so that it is

$$V_0 = 35 \left(\frac{2 \times 1.75}{3} \right)^{0.7} (0.0004)^{0.5}$$
$$= 0.7797597122 \ .$$

Thus, 39) gives

$$\omega = \frac{3 \alpha V_0^2}{2g \ y_0} = \frac{3 \times 1.1 \ (0.7797597122)^2}{2 \times 9.81 \times 1.75}$$
$$= 0.0584384211$$

and 94) becomes

$$X = 4375 \left\{ 0.999 \left[\overset{+}{D}{}^{4.4}_{4.4}(0.48) - \overset{+}{D}{}^{4.4}_{4.4}(0.92) \right] - \right.$$
$$\left. - 0.0584384211 \left[\overset{+}{D}{}^{0.4}_{4.4}(0.48) - \overset{+}{D}{}^{0.4}_{4.4}(0.92) \right] \right\}$$
$$= 4375 \left[0.999 \ (48.9063 - 48.8225) - 0.0584384211 \ (0.5955 - \right.$$
$$\left. - 0.2909) \right]$$
$$= 288.4 \text{ m} \ .$$

11.3. Horizontal beds.

11.3.1. First example.

Consider three horizontal channels lying to the north of Rome, Italy, at some place where the acceleration of gravity is $g = 9.80392$ m s^{-2}. Suppose that they have a common value of B_k, but the following shapes:

a) very wide rectangular,
b) very wide parabolic,
c) triangular, with $c_1 = c_2 = 9$,

and that for all of them it is $A_k/B_k = 1$ m .

We wish to calculate the distance from the section of height $y_1 = 2.4$ m to that $y_2 = 2.1$ m high and also the volume of water in between them, using Chézy, Manning and Forchheimer formulae, with $C = \frac{1}{n} = \lambda_F = 50$, and supposing the discharge per unit top breadth at critical water level to be $Q_1 = Q/B_k = 3\ m^3\ s^{-1}/m$.

The flow being turbulent and the backwater curves of Type $\mathscr{H}_2$, we shall assume $\alpha = 1.08$ (*).

For the rectangular cross-section it is

$$A_k = B_k\ y_k$$

and then

$$y_k = \frac{A_k}{B_k} = 1\ \text{m}\ ;$$

for the parabolic we have (73)

$$A_k = \frac{2}{3} B_k\ y_k\ ,$$

from which we get

$$y_k = \frac{3\ A_k}{2\ B_k} = 1.5\ \text{m}\ ;$$

finally for the triangular section, from 81), i. e.

$$A_k = \frac{c_1 + c_2}{2}\ y_k^2$$

and

$$B_k = (c_1 + c_2)\ y_k\ ,$$

it follows

$$y_k = \frac{2\ A_k}{B_k} = 2\ \text{m}\ .$$

(*) Observe that so 25) implies $\beta = \alpha\, Q^2/gA_k^2 = 3^2\alpha/g = 9 \times 1.08/9.80382 = 0.99144$, a value which is compatible with the curve Type.

And so

$$B_k = (9 + 9) \times 2 = 18 \times 2 = 36 \text{ m}$$

is the common value of the critical top widths.

Consequently, with the help of Tables 6 and 7, we obtain the following results from formula 105).

A) Chézy formula.

Aa) Rectangular section.

For $y_k = 1$ m, we have $\upsilon_1 = y_1/y_k = 2.4$ and $\upsilon_2 = y_2/y_k = 2.1$. Thus we get:

$$X = \frac{\alpha C^2}{g}\left[G_4^1(\upsilon_2) - G_4^1(\upsilon_1)\right] y_k$$

$$= \frac{1.08 \times 50^2}{9.80392}\left[G_4^1(2.1) - G_4^1(2.4)\right] y_k$$

$$= 275.4000441 \ (-2.76203 + 5.89440) \times 1$$
$$= 275.4000441 \times 3.13237$$
$$= 862.65 \text{ m} ,$$

$$\tau_1 = \frac{\alpha C^2}{g}\left[G_5^2(2.1) - G_5^2(2.4)\right] y_k^2$$

$$= 275.4000441 \ (-5.96320 + 13.04525)$$
$$= 1950.396882 \text{ m}^3/\text{m} ,$$
$$\tau = 36 \times 1950.396882$$
$$= 70214.288 \text{ m}^3.$$

Ab) Parabolic section.

Now we have $\upsilon_1 = y_1/y_k = 2.4/1.5 = 1.6$ and $\upsilon_2 = y_2/y_k = 2.1/1.5 = 1.4$; and then:

$$X = \frac{\alpha C^2}{g}\left[G_5^1(1.4) - G_5^1(1.6)\right] y_k$$

$$= 275.4000441 \ (0.32435 + 0.49715) \times 1.5$$
$$= 339.36 \text{ m} ,$$

$$\tau_1 = \frac{\alpha C^2}{g}\left[G^{5/2}_{13/2}(1.4) - G^{5/2}_{13/2}(1.6) \right] \times \frac{2}{3} y_k^2$$

$$= 275.4000441 \ (-0.44298 + 1.96961) \times \frac{2}{3} (1.5)^2$$

$$= 630.650954 \ \mathrm{m^3/m} \ ,$$

$$\tau = 22703.434 \ \mathrm{m^3} \ .$$

Ac) Triangular section.

In this case the reduced heights are $\upsilon_1 = 2.4/2 = 1.2$ and $\upsilon_2 = 2.1/2 = 1.05$. Hence:

$$X = \frac{\alpha C^2}{g}\left[G^1_8(1.05) - G^1_8(1.2) \right] . \frac{c_1 + c_2}{\sqrt{1+c_1^2} + \sqrt{1+c_2^2}} \, y_k$$

$$= 275.4000441 \ (0.82665 - 0.70234) \times \frac{18}{2\sqrt{82}} \times 2$$

$$= 68.05 \ \mathrm{m},$$

$$\tau = \frac{\alpha C^2}{g}\left[G^3_8(1.05) - G^3_8(1.2) \right] \times$$

$$\times \ \frac{1}{2}(c_1 + c_2)^2 \frac{1}{\sqrt{1+c_2^2} + \sqrt{1+c_2^2}} \, y_k^3$$

$$= 275.4000441 \ (0.20119 - 0.03852) \times \frac{18^2}{2} \times \frac{2^3}{2\sqrt{82}}$$

$$= 3205.823 \ \mathrm{m^3} \ .$$

B) Manning formula.

Ba) Rectangular section.

$$X = \frac{\alpha}{n^2 g}\left[G^{4/3}_{13/3}(2.1) - G^{4/3}_{13/3}(2.4) \right] y_k^{4/3}$$

$$= 275.4000441 \ (-3.73036 + 7.84089)$$

$$= 1132.04 \ \mathrm{m} \ ,$$

$$\tau_1 = \frac{\alpha}{n^2 g}\left[G^{7/3}_{16/3}(2.1) - G^{7/3}_{16/3}(2.4) \right] y_k^{7/3}$$

$$= 275.4000441 \ (-7.38598 + 16.68407)$$

$$= 2560.694396 \ m^3/m ,$$

$$\tau = 92184.998 \ m^3 .$$

Bb) Parabolic section.

$$X = \frac{\alpha}{n^2 g}\left[G^{4/3}_{16/3}(1.4) - G^{4/3}_{16/3}(1.6) \right] (\frac{2}{3})^{1/3} \ y_k^{4/3}$$

$$= 275.4000441 \ (0.04652 + 0.89601) \ (2/3)^{1/3} \ (1.5)^{4/3}$$

$$= 389.36 \ m ,$$

$$\tau_1 = \frac{\alpha}{n^2 g}\left[G^{17/6}_{41/6}(1.4) - G^{17/6}_{41/6}(1.6) \right] (\frac{2}{3})^{4/3} \ y_k^{7/3}$$

$$= 275.4000441 \ (-0.54285 + 2.29562) \ (\frac{2}{3})^{4/3} \ (1.5)^{7/3}$$

$$= 724.0694029 \ m^3/m ,$$

$$\tau = 26066.499 \ m^3.$$

Bc) Triangular section.

$$X = \frac{\alpha}{n^2 g}\left[G^{4/3}_{19/3}(1.05) - G^{4/3}_{19/3}(1.2) \right] (\frac{1}{2})^{1/3} \left(\frac{18}{2\sqrt{82}} \right)^{4/3} \ y_k^{4/3}$$

$$= 275.4000441 \ (0.58535 - 0.45538) \ (1/2)^{1/3} \times$$

$$\times \ (18/2\sqrt{82})^{4/3} \times 2^{4/3}$$

$$= 71.00 \ m ,$$

$$\tau = \frac{\alpha}{n^2 g}\left[G^{10/3}_{25/3}(1.05) - G^{10/3}_{25/3}(1.2) \right] (\frac{1}{2})^{\frac{4}{3}} \times$$

$$\times \left(\frac{1}{\sqrt{1 + c_1^2} + \sqrt{1 + c_2^2}} \right)^{\frac{4}{3}} (c_1 + c_2)^{\frac{7}{3}} \ y_k^{\frac{10}{3}}$$

$$= 275.4000441 \ (0.17278 - 0.00257) \ (1/2)^{4/3} \times$$

$$\times \ (1/2\sqrt{82})^{4/3} \times 18^{7/3} \times 2^{10/3}$$

$$= 3347.5649 \ m^3 .$$

C) Forchheimer formula.

Ca) Rectangular section.

$$X = \frac{\alpha\lambda_F^2}{g}\left[G_{22/5}^{7/5}(2.1) - G_{22/5}^{7/5}(2.4)\right] y_k^{7/5}$$
$$= 275.4000441\ (-\ 3.92893 + 8.26915)$$
$$= 1195.30\ \text{m}\ ,$$

$$\tau_1 = \frac{\alpha\lambda_F^2}{g}\left[G_{27/5}^{12/5}(2.1) - G_{27/5}^{12/5}(2.4)\right] y_k^{12/5}$$
$$= 275.4000441\ (-\ 7.70393 + 17.52253)$$
$$= 2704.042873\ \text{m}^3/\text{m}\ ,$$
$$\tau = 97345.543\ \text{m}^3.$$

Cb) Parabolic section.

$$X = \frac{\alpha\lambda_F}{g}\left[G_{27/5}^{7/5}(1.4) - G_{27/5}^{7/5}(1.6)\right] (2/3)^{2/5}\ y_k^{7/5}$$
$$= 275.4000411\ (0.00461 + 0.96420)\ (2/3)^{2/5}\ (1.5)^{7/5}$$
$$= 400.22\ \text{m}\ ,$$

$$\tau_1 = \frac{\alpha\lambda_F^2}{g}\left[G_{69/10}^{29/10}(1.4) - G_{69/10}^{29/10}(1.6)\right] (2/3)^{7/5}\ y_k^{12/5}$$
$$= 275.4000441\ (-\ 0.56228 + 2.36418)\ (2/3)^{7/5}\ (1.5)^{12/5}$$
$$= 744.3650092\ \text{m}^3/\text{m}\ ,$$
$$\tau = 26797.140\ \text{m}^3\ .$$

Cc) Triangular section.

$$X = \frac{\alpha\lambda_F^2}{g}\left[G_{32/5}^{7/5}(1.05) - G_{32/5}^{7/5}(1.2)\right] (1/2)^{2/5}\ (c_1 + c_2)^{7/5} \times$$
$$\times \left(\sqrt{1 + c_1^2} + \sqrt{1 + c_2^2}\right)^{-7/5}\ y_k^{7/5}$$
$$= 275.4000411\ (0.55126\ -\ 0.42013)\ (1/2)^{2/5} \times$$
$$\times\ (18/2\sqrt{82})^{7/5} \times 2^{7/5}$$
$$= 71.61\ \text{m}\ ,$$

$$\tau = \frac{\alpha\lambda_F^2}{g}\left[G^{17/5}_{42/5}(1.05) - G^{17/5}_{42/5}(1.2)\right](1/2)^{7/5}\ (c_1 + c_2)^{12/5} \times$$
$$\times \left(\sqrt{1+c_1^2} + \sqrt{1+c_2^2}\right)^{-7/5} y_k^{17/5}$$
$$= 275.4000441\ (0.16783 + 0.00392)\ (1/2)^{7/5}\ (2\sqrt{82})^{-7/5} \times$$
$$\times 18^{12/5} \times 2^{17/5}$$
$$= 3376.471 \text{ m}^3 .$$

D) Summary of results.

The results of this example, summarized below, show the paramount importance of cross-sectional shape:

Distances (m)

Formula	Rectangular	Parabolic	Triangular
Chézy	862.65	339.36	68.05
Manning	1132.04	389.36	71.00
Forchheimer	1195.30	400.22	71.61

Volumes (m^3) *for* $B_k = 36$ m

Formula	Rectangular	Parabolic	Triangular
Chézy	70 214.29	22 703.43	3205.82
Manning	92 185.00	26 066.50	3347.56
Forchheimer	97 345.54	26 797.14	3376.47 .

11.3.2. Second example.

Assuming $g = 9.805$ m s^{-2} and $Q_1 = Q/B_k = 3.052$ $m^3 s^{-1}/m$, one wishes do trace the complete ideal (*) backwater curves of Type $\mathscr{H}_3$ in channels having the three cross-sectional shapes under consideration, with $c_1 = c_2 = 9$ for the triangular one. Manning formula is to be chosen putting $n = 0.02$.

(*) «Ideal» because at the extremities they have no real meaning, since the hypothesis b) of Section 2 is not satisfied. As a rule, the real backwater curve extends only from the contracted section of a stream coming out of a bottom sluice-gate to the beginning of a hydraulic jump.

The following estimates will be adopted: $\alpha = 1.08$, $\beta = 1.026$. We have

$$\frac{\alpha Q_1}{\beta g} = \frac{1.08 \times (3.052)^2}{1.026 \times 9.805} = 0.9999950616 \approx 1 \ .$$

Hence, by means of equations 25), 64), 73) and 81), it can easily be seen that, just like in the first example, the critical depth y_k is 1 m for the very wide rectangular, 1.5 m for the very wide parabolic and 2 m for the triangular section.

Choosing as origin the point where the curve crosses the bottom line, 105) may be simply written

113) $$x = W \ G_{E_2}^{E_1}(\upsilon) = W \left(\frac{\upsilon^{E_1}}{E_1} - \frac{\upsilon^{E_2}}{E_2} \right).$$

The value of α/n^2g being $1.08/(0.02)^2 \times 9.805 = 275.3697093$, we get

$$W = \frac{\alpha}{n^2g} = 275.3697093,$$

$$W = \frac{\alpha}{n^2g} (2/3)^{1/3} (1.5)^{4/3} = 413.054564$$

and

$$W = \frac{\alpha}{n^2g} (1/2)^{1/3} (18/2\sqrt{82})^{4/3} \times 2^{4/3} = 546.252712$$

for the values of W to introduce into 113) according to the cross-sectional shape.

Then, Table 8 can be constructed and the figure «Horizontal beds — Graphic representation of the results of the second example» drawn.

As this figure shows, each one of the backwater curves has a point of inflection. Their coordinates can be found as follows.

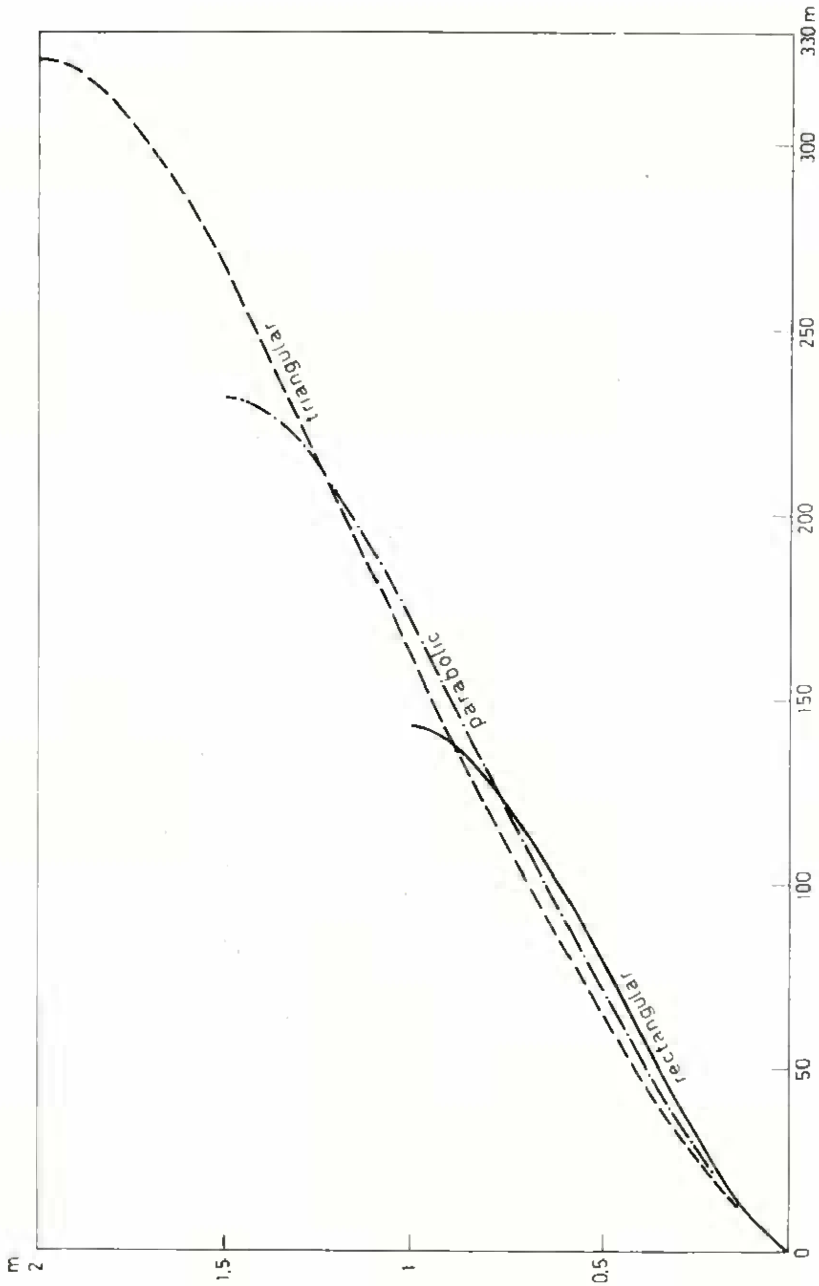
The first and the second derivatives of function 113) are

$$\frac{dx}{d\upsilon} = W (\upsilon^{E_1-1} - \upsilon^{E_2-1})$$

TABLE 8

Computation of backater curves of Type $\mathscr{H}_1$

v	Gagliardi function, $E_1 = 4/3$			Cross-section					
	E_2			Rectangular		Parabolic		Triangular	
	13/3	16/3	19/3	y	x	y	x	y	x
0.02	0.00407	0.00407	0.00407	0.02	1.12	0.03	1.68	0.04	2.22
0.04	0.01026	0.01026	0.01026	0.04	2.83	0.06	4.24	0.08	5.60
0.06	0.01762	0.01762	0.01762	0.06	4.85	0.09	7.28	0.12	9.62
0.08	0.02585	0.02585	0.02585	0.08	7.12	0.12	10.68	0.16	14.12
0.10	0.03480	0.03481	0.03481	0.10	9.58	0.15	14.38	0.20	19.02
0.12	0.04437	0.04439	0.04439	0.12	12.22	0.18	18.34	0.24	24.25
0.44	0.24442	0.24864	0.25012	0.44	67.31	0.66	102.70	0.88	136.63
0.46	0.25834	0.26334	0.26517	0.46	71.14	0.69	108.77	0.92	144.85
0.48	0.27228	0.27813	0.28036	0.48	74.98	0.72	114.88	0.96	153.15
0.50	0.28619	0.29299	0.29568	0.50	78.81	0.75	121.02	1.00	161.52
0.52	0.30005	0.30788	0.31111	0.52	82.62	0.78	127.17	1.04	169.94
0.54	0.31382	0.32279	0.32661	0.54	86.42	0.81	133.33	1.08	178.41
0.56	0.32748	0.33768	0.34217	0.56	90.18	0.84	139.48	1.12	186.91
0.58	0.34099	0.35251	0.35776	0.58	93.90	0.87	145.61	1.16	195.43
0.60	0.35432	0.36725	0.37333	0.60	97.57	0.90	151.69	1.20	203.93
0.90	0.50552	0.54481	0.57069	0.90	139.20	1.35	225.04	1.80	311.74
0.92	0.51030	0.55090	0.57797	0.92	140.52	1.38	227.55	1.84	315.72
0.94	0.51411	0.55581	0.58391	0.94	141.57	1.41	229.58	1.88	318.96
0.96	0.51692	0.55945	0.58835	0.96	152.34	1.44	231.08	1.92	321.39
0.98	0.51864	0.56172	0.59114	0.98	142.82	1.47	232.02	1.96	322.91
1.00	0.51923	0.56250	0.59211	1.00	142.98	1.50	232.34	2.00	323.44



Horizontal beds — Graphic representation of the results of the second example

and

$$\frac{d^2x}{d\upsilon^2} = W \left[(E_1 - 1)\, \upsilon^{E_1-2} - (E_2 - 1)\, \upsilon^{E_2-2} \right].$$

Thereupon the condition for inflection, $d^2x/d\upsilon^2 = 0$, i. e.

$$(E_1 - 1)\, \upsilon^{E_1-2} = (E_2 - 1)\, \upsilon^{E_2-2},$$

yields

$$\upsilon = \left(\frac{E_1 - 1}{E_2 - 1} \right)^{\frac{1}{E_2 - E_1}} .$$

And so, for the very wide rectangular section ($E_1 = 4/3$, $E_2 = 13/3$) we have

$$\upsilon = (0.1)^{1/3} = 0.4641588834,$$

from which we get the coordinates of the point of inflection:

$$\begin{cases} x \approx 71.94 \\ y \approx 0.464 \end{cases};$$

in the same manner, for the very wide parabolic cross-section ($E_1 = 4/3$, $E_2 = 16/3$), one obtains

$$\upsilon = (1/13)^{1/4} = 0.5266403878,$$

$$\begin{cases} x \approx 129.22 \\ y \approx 0.790 \end{cases}$$

and for the triangular ($E_1 = 4/3$, $E_2 = 19/3$)

$$\upsilon = (1/16)^{1/5} = 0.5743491775,$$

$$\begin{cases} x \approx 193.02 \\ y \approx 1.149 \end{cases}.$$

REFERENCES

BAKHMETEFF, B. A. 1932 — *Hydraulics of Open Channels*. New York and London: McGraw-Hill Book Company, Inc.

CHOW, VEN TE 1959 — *Open-Channel Hydraulics*. New York, Toronto, London: McGraw-Hill Book Company.

FORCHHEIMER, PHILIPP 1935 — *Tratado de Hidráulica*. Transl. 3d German ed. Barcelona, Madrid, Buenos Aires, Rio de Janeiro: Editorial Labor, S. A.

GAGLIARDI, LUIGI 1974 — Integrazione dell'equazione differenziale dei profili di rigurgito per gli alvei triangolare e rettangulare molto largo. *Atti Acad. Sci. Lett. Arti Palermo* [4]34(1), 1974-75. = *Publ. Ist. Idr. Univ. Palermo,* 111.

GUNDER, DWIGHT F. 1942 — Profile Curves for Open-Channel Flow. *Proc. Amer. Soc. civ. Engrs* 68 (4): 535-541.

KOŽENY, J. 1924 — Berechnung des Staues in breiten Gerinnen. *Wasserk. u. Wasserw.* 19: 363.

KOŽENY, J. 1928 — Berechnung des Senkungskurve in regelmässigen breiten Gerinnen. *Wasserk. u. Wasserw.* 23: 232.

MENDONÇA, P. DE VARENNES E 1945 — *Curvas de Regolfo*. Lisboa: Livraria Ferin, Lda.

MENDONÇA, P. DE VARENNES E 1948 — Algumas reflexões sobre o traçado das curvas de regolfo em canais com leito cilíndrico de pequeno declive. *An. Inst. Sup. Agron.* 16: 1-22.

MENDONÇA, P. DE VARENNES E 1964a — Nouvelles méthodes analytiques de calcul des courbes de remous en canaux découverts uniformes. *Bol. Ord. Engrs* 9 (1): 57-76.

MENDONÇA, P. DE VARENNES E 1964b — Uma nova tábua da função de Dupuit--Bresse. *Bol. Ord. Engrs* 9 (3): 287-299.

MENDONÇA, P. DE VARENNES E 1964c — Avatars de la fonction de Dupuit. *Bol. Ord. Engrs* 9 (4): 381-389.

MENDONÇA, P. DE VARENNES E 1964d — Volumes de regolfo em canais uniformes horizontais. *Bol. Ord. Engrs* 9 (5): 535-539.

MENDONÇA, P. DE VARENNES E 1964e — Backwater volumes in uniform rectangular channels. *Bol. Ord. Engrs* 9 (6): 627-646.

MENDONÇA, P. DE VARENNES E 1964-1965 — *Excertos das Lições de Hidráulica Geral e Agrícola*, 2d ed. Lisboa: Instituto Superior de Agronomia.

MENDONÇA, P. DE VARENNES E 1972 — Sobre uma forma do teorema de Bernoulli na Hidráulica. *Mem. Acad. Ciên. Lisboa — Classe Ciên.* 16: 203-211.

MENDONÇA, P. DE VARENNES E 1973 — Backwater curves and volumes in uniform rectangular channels. *An. Inst. Sup. Agron.* 34: 9-25.

MENDONÇA, P. DE VARENNES E 1977 — Problèmes de remous en écoulements à deux dimensions d'un liquide visqueux sur une plaque plane, attention spéciale étant prêtée au régime laminaire. I — Liquides newtoniens. *An. Inst. Sup. Agron.* 37: 223-255.

MENDONÇA, P. DE VARENNES E 1978 — Problèmes de remous en écoulements à deux dimensions d'un liquide visqueux sur une plaque plane, attention spéciale étant prêtée au régime laminaire. II — Liquides non-newtoniens à loi monomiale. *An. Inst. Sup. Agron.* 38: 9-37.

MENDONÇA, P. DE VARENNES E unpubl. — *Tables of Dupuit Function*. Unpublished.

PUPPINI, U. 1911 — I profili di rigurgito nei canali ristretti. *Monit. tecn.*

RASTRELLI, A. 1937 — Sul moto permanente in alvei prismatici a sezione parabolica. *Boll. Fac. Agraria Univ. Pisa* 13: 535-545.

SUPINO, GIULIO 1934 — Sui profili di rigurgito nei canali scoperti. *Ric. Ingegn.* 2 (3).

TOLKMITT, G. 1898 — *Grundlagen der Wasserbaukunst*. Berlin: Wilhelm Ernst & Sohn.

WEYRAUCH, R. 1930 — *Hydraulisches Rechnen*. 6. Aufl. v. A. Strobel. Stuttgart: Konrad Wittwer.

TABLE 9

Some values of Dupuit function

u	$\bar{D}_{10/3}^{10/3}$	$\bar{D}_{10/3}^{1/3}$	$\bar{D}_{17/5}^{17/5}$	$\bar{D}_{17/5}^{2/5}$	$\bar{D}_{22/5}^{22/5}$	$\bar{D}_{22/5}^{2/5}$
0.000	1.5894	2.3061	1.5544	2.2306	1.1606	1.8498
0.020	1.5894	2.3021	1.5544	2.2276	1.1606	1.8468
0.040	1.5894	2.2959	1.5544	2.2227	1.1606	1.8419
0.060	1.5894	2.2885	1.5544	2.2167	1.1606	1.8359
0.080	1.5894	2.2803	1.5544	2.2098	1.1606	1.8290
0.100	1.5894	2.2713	1.5544	2.2021	1.1606	1.8214
0.120	1.5894	2.2617	1.5543	2.1938	1.1606	1.8131
0.140	1.5893	2.2516	1.5543	2.1850	1.1606	1.8043
0.160	1.5893	2.2409	1.5543	2.1756	1.1606	1.7949
0.180	1.5892	2.2298	1.5542	2.1658	1.1606	1.7851
0.200	1.5892	2.2183	1.5542	2.1554	1.1606	1.7748
0.220	1.5890	2.2063	1.5541	2.1447	1.1605	1.7640
0.240	1.5889	2.1940	1.5539	2.1335	1.1605	1.7529
0.260	1.5887	2.1813	1.5538	2.1219	1.1605	1.7414
0.280	1.5884	2.1682	1.5535	2.1099	1.1604	1.7295
0.300	1.5881	2.1547	1.5532	2.0975	1.1603	1.7173
0.320	1.5877	2.1409	1.5528	2.0848	1.1602	1.7047
0.340	1.5872	2.1267	1.5524	2.0716	1.1600	1.6918
0.360	1.5866	2.1122	1.5518	2.0581	1.1598	1.6785
0.380	1.5858	2.0973	1.5511	2.0442	1.1596	1.6649
0.400	1.5849	2.0820	1.5502	2.0299	1.1593	1.6509
0.420	1.5838	2.0664	1.5492	2.0152	1.1589	1.6366
0.440	1.5825	2.0503	1.5480	2.0000	1.1584	1.6220
0.460	1.5810	2.0338	1.5466	1.9845	1.1578	1.6070
0.480	1.5793	2.0169	1.5449	1.9685	1.1570	1.5917
0.500	1.5772	1.9995	1.5430	1.9520	1.1561	1.5760
0.520	1.5749	1.9817	1.5407	1.9350	1.1550	1.5599
0.540	1.5721	1.9633	1.5381	1.9174	1.1537	1.5433
0.560	1.5690	1.9443	1.5351	1.8993	1.1521	1.5264
0.580	1.5653	1.9247	1.5316	1.8806	1.1503	1.5089
0.600	1.5612	1.9044	1.5276	1.8611	1.1481	1.4910
0.620	1.5564	1.8834	1.5230	1.8410	1.1456	1.4725
0.640	1.5509	1.8616	1.5178	1.8200	1.1425	1.4533
0.650	1.5479	1.8503	1.5149	1.8091	1.1408	1.4435
0.660	1.5447	1.8388	1.5118	1.7981	1.1390	1.4335
0.670	1.5412	1.8271	1.5084	1.7867	1.1370	1.4233
0.680	1.5375	1.8151	1.5049	1.7752	1.1348	1.4129
0.690	1.5336	1.8028	1.5011	1.7633	1.1325	1.4023
0.700	1.5294	1.7902	1.4970	1.7511	1.1300	1.3915
0.710	1.5248	1.7772	1.4926	1.7386	1.1273	1.3804
0.720	1.5200	1.7640	1.4879	1.7257	1.1243	1.3691
0.730	1.5148	1.7503	1.4828	1.7125	1.1211	1.3575
0.740	1.5092	1.7362	1.4774	1.6989	1.1176	1.3456
0.750	1.5032	1.7217	1.4716	1.6848	1.1138	1.3333
0.760	1.4967	1.7068	1.4654	1.6703	1.1097	1.3207
0.770	1.4898	1.6913	1.4586	1.6553	1.1053	1.3077
0.780	1.4823	1.6752	1.4514	1.6397	1.1005	1.2943
0.790	1.4743	1.6586	1.4436	1.6235	1.0952	1.2805
0.800	1.4655	1.6412	1.4351	1.6067	1.0895	1.2661
0.810	1.4561	1.6231	1.4259	1.5891	1.0822	1.2512
0.820	1.4459	1.6043	1.4160	1.5707	1.0763	1.2357

TABLE 9

(Continued)

u	$\bar{D}_{10/3}^{10/3}$	$\bar{D}_{10/3}^{1/3}$	$\bar{D}_{17/5}^{17/5}$	$\bar{D}_{17/5}^{2/5}$	$\bar{D}_{22/5}^{22/5}$	$\bar{D}_{22/5}^{2/5}$
0.820	1.4459	1.6043	1.4160	1.5707	1.0763	1.2357
0.830	1.4348	1.5844	1.4051	1.5514	1.0688	1.2195
0.840	1.4226	1.5636	1.3933	1.5311	1.0606	1.2025
0.850	1.4093	1.5416	1.3804	1.5096	1.0515	1.1846
0.855	1.4022	1.5301	1.3734	1.4984	1.0465	1.1753
0.860	1.3947	1.5182	1.3661	1.4869	1.0414	1.1658
0.865	1.3869	1.5060	1.3585	1.4750	1.0359	1.1559
0.870	1.3786	1.4934	1.3504	1.4626	1.0302	1.1457
0.875	1.3700	1.4803	1.3420	1.4499	1.0241	1.1352
0.880	1.3608	1.4667	1.3330	1.4366	1.0176	1.1244
0.885	1.3511	1.4526	1.3236	1.4229	1.0108	1.1131
0.890	1.3409	1.4380	1.3136	1.4086	1.0036	1.1015
0.895	1.3300	1.4227	1.3030	1.3937	0.9959	1.0893
0.900	1.3185	1.4068	1.2917	1.3781	0.9877	1.0767
0.905	1.3062	1.3901	1.2798	1.3618	0.9789	1.0635
0.910	1.2931	1.3726	1.2670	1.3447	0.9695	1.0496
0.915	1.2791	1.3542	1.2533	1.3267	0.9594	1.0351
0.920	1.2641	1.3347	1.2386	1.3077	0.9486	1.0198
0.925	1.2479	1.3140	1.2227	1.2875	0.9368	1.0036
0.930	1.2303	1.2920	1.2056	1.2660	0.9241	0.9864
0.935	1.2113	1.2685	1.1869	1.2430	0.9102	0.9680
0.940	1.1904	1.2432	1.1665	1.2182	0.8950	0.9483
0.945	1.1675	1.2159	1.1441	1.1914	0.8782	0.9270
0.950	1.1421	1.1860	1.1192	1.1622	0.8595	0.9038
0.952	1.1311	1.1732	1.1085	1.1497	0.8514	0.8939
0.954	1.1196	1.1600	1.0972	1.1367	0.8429	0.8836
0.956	1.1076	1.1461	1.0854	1.1232	0.8340	0.8729
0.958	1.0949	1.1316	1.0730	1.1090	0.8246	0.8617
0.960	1.0815	1.1165	1.0599	1.0942	0.8147	0.8500
0.962	1.0674	1.1006	1.0461	1.0786	0.8043	0.8378
0.964	1.0525	1.0839	1.0315	1.0623	0.7932	0.8249
0.966	1.0366	1.0662	1.0160	1.0450	0.7814	0.8113
0.968	1.0197	1.0475	0.9994	1.0267	0.7688	0.7969
0.970	1.0016	1.0277	0.9817	1.0072	0.7554	0.7816
0.972	0.9822	1.0065	0.9627	0.9864	0.7409	0.7654
0.974	0.9613	0.9837	0.9422	0.9642	0.7253	0.7479
0.976	0.9385	0.9592	0.9199	0.9402	0.7083	0.7291
0.978	0.9137	0.9326	0.8956	0.9141	0.6897	0.7088
0.980	0.8864	0.9035	0.8688	0.8856	0.6693	0.6865
0.982	0.8561	0.8714	0.8391	0.8541	0.6465	0.6619
0.984	0.8221	0.8355	0.8058	0.8190	0.6210	0.6346
0.986	0.7833	0.7950	0.7678	0.7793	0.5918	0.6036
0.988	0.7383	0.7482	0.7237	0.7334	0.5580	0.5680
0.990	0.6849	0.6930	0.6714	0.6793	0.5178	0.5260
0.992	0.6193	0.6256	0.6071	0.6132	0.4683	0.4747
0.994	0.5343	0.5388	0.5238	0.5282	0.4042	0.4087
0.996	0.4139	0.4166	0.4058	0.4084	0.3132	0.3160
0.997	0.3283	0.3301	0.3218	0.3236	0.2485	0.2503
0.998	0.2073	0.2082	0.2032	0.2041	0.1569	0.1578
0.999	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.000	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$

TABLE 9

(Continued)

u	$\bar{D}^{10/3}_{10/3}$	$\bar{D}^{1/3}_{10/3}$	$\bar{D}^{17/5}_{17/5}$	$\bar{D}^{2/5}_{17/5}$	$\bar{D}^{22/5}_{22/5}$	$\bar{D}^{2/5}_{22/5}$
1.001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.002	0.2086	0.2077	0.2045	0.2036	0.1581	0.1572
1.003	0.3309	0.3291	0.3244	0.3227	0.2509	0.2491
1.004	0.4178	0.4151	0.4097	0.4070	0.3169	0.3142
1.005	0.4854	0.4818	0.4760	0.4724	0.3682	0.3646
1.006	0.5408	0.5363	0.5302	0.5258	0.4103	0.4057
1.008	0.6284	0.6221	0.6161	0.6100	0.4769	0.4705
1.010	0.6966	0.6885	0.6831	0.6751	0.5289	0.5207
1.012	0.7526	0.7427	0.7380	0.7283	0.5715	0.5615
1.014	0.8002	0.7885	0.7846	0.7731	0.6078	0.5960
1.016	0.8416	0.8280	0.8252	0.8119	0.6394	0.6257
1.018	0.8782	0.8629	0.8611	0.8461	0.6674	0.6519
1.020	0.9111	0.8940	0.8934	0.8766	0.6926	0.6753
1.022	0.9410	0.9221	0.9228	0.9042	0.7155	0.6963
1.024	0.9684	0.9477	0.9497	0.9293	0.7365	0.7155
1.026	0.9938	0.9712	0.9745	0.9524	0.7559	0.7331
1.028	1.0173	0.9930	0.9976	0.9737	0.7740	0.7494
1.030	1.0393	1.0132	1.0192	0.9936	0.7909	0.7645
1.032	1.0600	1.0320	1.0395	1.0121	0.8069	0.7786
1.034	1.0795	1.0497	1.0587	1.0294	0.8219	0.7918
1.036	1.0980	1.0664	1.0768	1.0458	0.8361	0.8042
1.038	1.1155	1.0821	1.0940	1.0612	0.8497	0.8159
1.040	1.1322	1.0970	1.1104	1.0759	0.8626	0.8270
1.042	1.1482	1.1111	1.1261	1.0897	0.8749	0.8375
1.044	1.1634	1.1246	1.1411	1.1030	0.8868	0.8475
1.046	1.1781	1.1375	1.1555	1.1156	0.8981	0.8570
1.048	1.1922	1.1497	1.1693	1.1276	0.9091	0.8661
1.050	1.2058	1.1615	1.1826	1.1392	0.9196	0.8748
1.055	1.2377	1.1889	1.2140	1.1660	0.9444	0.8951
1.060	1.2671	1.2137	1.2429	1.1905	0.9673	0.9134
1.065	1.2944	1.2365	1.2697	1.2129	0.9887	0.9302
1.070	1.3200	1.2576	1.2948	1.2335	1.0087	0.9456
1.075	1.3440	1.2770	1.3184	1.2527	1.0276	0.9599
1.080	1.3667	1.2952	1.3408	1.2705	1.0455	0.9731
1.085	1.3883	1.3122	1.3619	1.2878	1.0625	0.9855
1.090	1.4088	1.3281	1.3821	1.3029	1.0787	0.9971
1.095	1.4284	1.3431	1.4013	1.3176	1.0942	1.0080
1.100	1.4471	1.3573	1.4198	1.3316	1.1091	1.0182
1.105	1.4651	1.3708	1.4376	1.3448	1.1234	1.0279
1.110	1.4825	1.3835	1.4545	1.3574	1.1372	1.0371
1.120	1.5153	1.4072	1.4869	1.3807	1.1635	1.0541
1.130	1.5462	1.4289	1.5172	1.4020	1.1882	1.0696
1.140	1.5752	1.4488	1.5458	1.4215	1.2116	1.0837
1.150	1.6027	1.4671	1.5729	1.4396	1.2339	1.0966
1.160	1.6290	1.4841	1.5987	1.4564	1.2552	1.1086
1.170	1.6540	1.5000	1.6234	1.4720	1.2757	1.1197
1.180	1.6781	1.5148	1.6471	1.4866	1.2954	1.1300
1.190	1.7012	1.5287	1.6699	1.5003	1.3144	1.1397
1.200	1.7236	1.5418	1.6919	1.5132	1.3328	1.1487
1.210	1.7452	1.5542	1.7132	1.5254	1.3506	1.1572
1.220	1.7661	1.5658	1.7339	1.5369	1.3680	1.1652

TABLE 9

(Continued)

u	$\bar{D}^{10/3}_{10/3}$	$\bar{D}^{1/3}_{10/3}$	$\bar{D}^{17/5}_{17/5}$	$\bar{D}^{2/5}_{17/5}$	$\bar{D}^{22/5}_{22/5}$	$\bar{D}^{2/5}_{22/5}$
1.220	1.7661	1.5658	1.7339	1.5369	1.3680	1.1652
1.230	1.7865	1.5769	1.7540	1.5478	1.3849	1.1727
1.240	1.8063	1.5874	1.7735	1.5582	1.4015	1.1798
1.250	1.8256	1.5974	1.7925	1.5681	1.4176	1.1865
1.260	1.8444	1.6070	1.8111	1.5775	1.4335	1.1929
1.270	1.8628	1.6160	1.8293	1.5864	1.4490	1.1990
1.280	1.8808	1.6247	1.8471	1.5950	1.4642	1.2047
1.290	1.8985	1.6331	1.8645	1.6032	1.4792	1.2102
1.300	1.9158	1.6410	1.8816	1.6111	1.4939	1.2154
1.320	1.9495	1.6560	1.9149	1.6259	1.5227	1.2252
1.340	1.9821	1.6699	1.9471	1.6396	1.5507	1.2342
1.360	2.0137	1.6828	1.9784	1.6523	1.5779	1.2424
1.380	2.0445	1.6947	2.0088	1.6642	1.6046	1.2500
1.400	2.0745	1.7059	2.0385	1.6752	1.6308	1.2570
1.450	2.1467	1.7309	2.1100	1.7000	1.6941	1.2723
1.500	2.2156	1.7524	2.1782	1.7212	1.7552	1.2853
1.550	2.2819	1.7711	2.2439	1.7398	1.8145	1.2962
1.600	2.3460	1.7875	2.3075	1.7561	1.8723	1.3056
1.650	2.4084	1.8020	2.3693	1.7705	1.9290	1.3138
1.700	2.4693	1.8150	2.4298	1.7834	1.9848	1.3209
1.800	2.5877	1.8371	2.5434	1.8054	2.0941	1.3326
1.900	2.7025	1.8553	2.6615	1.8234	2.2013	1.3417
2.000	2.8146	1.8704	2.7731	1.8385	2.3069	1.3491
2.100	2.9247	1.8832	2.8826	1.8512	2.4114	1.3550
2.200	3.0331	1.8942	2.9906	1.8621	2.5149	1.3599
2.400	3.2465	1.9118	3.2032	1.8796	2.7202	1.3672
2.600	3.4564	1.9253	3.4125	1.8931	2.9239	1.3725
2.800	3.6640	1.9358	3.6196	1.9036	3.1265	1.3763
3.000	3.8700	1.9443	3.8252	1.9121	3.3283	1.3792
3.200	4.0747	1.9512	4.0295	1.9190	3.5297	1.3814
3.400	4.2785	1.9569	4.2331	1.9246	3.7308	1.3831
3.600	4.4816	1.9616	4.4359	1.9294	3.9316	1.3844
3.800	4.6842	1.9656	4.6383	1.9334	4.1322	1.3855
4.000	4.8864	1.9690	4.8403	1.9368	4.3327	1.3863
4.500	5.3905	1.9757	5.3440	1.9434	4.8336	1.3879
5.000	5.8933	1.9804	5.8465	1.9481	5.3341	1.3889
5.500	6.3953	1.9839	6.3483	1.9516	5.8345	1.3896
6.000	6.8968	1.9865	6.8496	1.9542	6.3347	1.3900
7.000	7.8988	1.9902	7.8514	1.9579	7.3350	1.3906
8.000	8.9000	1.9926	8.8525	1.9603	8.3351	1.3909
9.000	9.9008	1.9942	9.8532	1.9620	9.3352	1.3911
10.000	10.9013	1.9954	10.8536	1.9631	10.3353	1.3912
11.000	11.9017	1.9963	11.8540	1.9640	11.3353	1.3913
12.000	12.9020	1.9969	12.8542	1.9647	12.3353	1.3914
13.000	13.9022	1.9974	13.8544	1.9652	13.3353	1.3914
14.000	14.9024	1.9979	14.8546	1.9656	14.3353	1.3914
15.000	15.9026	1.9982	15.8547	1.9659	15.3353	1.3915
20.000	20.9029	1.9992	20.8550	1.9669	20.3354	1.3915
30.000	30.9032	1.9999	30.8552	1.9676	30.3354	1.3915
50.000	50.9033	2.0002	50.8553	1.9679	50.3354	1.3916
∞	∞	2.0004	∞	1.9681	∞	1.3916

TABLE 9

(Continued)

u	$\overset{+10/3}{D}_{10/3}$	$\overset{+1/3}{D}_{10/3}$	$\overset{+17/5}{D}_{17/5}$	$\overset{+2/5}{D}_{17/5}$	$\overset{+22/5}{D}_{22/5}$	$\overset{+2/5}{D}_{22/5}$
0.00	48.8351	0.9908	48.8422	0.9605	48.9097	0.8487
0.02	48.8351	0.9867	48.8422	0.9575	48.9097	0.8457
0.04	48.8351	0.9805	48.8422	0.9526	48.9097	0.8408
0.06	48.8351	0.9732	48.8422	0.9466	48.9097	0.8348
0.08	48.8351	0.9649	48.8422	0.9397	48.9097	0.8279
0.10	48.8351	0.9560	48.8422	0.9320	48.9097	0.8203
0.12	48.8351	0.9464	48.8421	0.9238	48.9097	0.8120
0.14	48.8350	0.9363	48.8421	0.9149	48.9097	0.8032
0.16	48.8350	0.9257	48.8421	0.9056	48.9097	0.7938
0.18	48.8349	0.9146	48.8420	0.8958	48.9097	0.7840
0.20	48.8349	0.9032	48.8420	0.8855	48.9097	0.7737
0.22	48.8348	0.8914	48.8419	0.8749	48.9096	0.7630
0.24	48.8346	0.8792	48.8417	0.8638	48.9096	0.7519
0.26	48.8344	0.8667	48.8416	0.8524	48.9096	0.7404
0.28	48.8342	0.8540	48.8413	0.8407	48.9095	0.7286
0.30	48.8338	0.8409	48.8410	0.8287	48.9094	0.7165
0.32	48.8334	0.8277	48.8407	0.8164	48.9093	0.7041
0.34	48.8330	0.8142	48.8402	0.8039	48.9092	0.6913
0.36	48.8324	0.8005	48.8397	0.7911	48.9090	0.6783
0.38	48.8317	0.7866	48.8390	0.7781	48.9087	0.6650
0.40	48.8308	0.7726	48.8382	0.7649	48.9084	0.6515
0.42	48.8299	0.7585	48.8373	0.7516	48.9080	0.6378
0.44	48.8287	0.7443	48.8362	0.7381	48.9075	0.6239
0.46	48.8274	0.7299	48.8350	0.7244	48.9069	0.6098
0.48	48.8259	0.7156	48.8336	0.7107	48.9063	0.5955
0.50	48.8242	0.7011	48.8319	0.6969	48.9054	0.5811
0.52	48.8223	0.6867	48.8301	0.6830	48.9044	0.5666
0.54	48.8202	0.6722	48.8280	0.6691	48.9033	0.5119
0.56	48.8178	0.6578	48.8257	0.6552	48.9019	0.5372
0.58	48.8151	0.6434	48.8231	0.6413	48.9004	0.5225
0.60	48.8122	0.6291	48.8203	0.6274	48.8986	0.5078
0.62	48.8089	0.6149	48.8171	0.6136	48.8966	0.4930
0.64	48.8054	0.6008	48.8137	0.5998	48.8942	0.4783
0.66	48.8016	0.5868	48.8099	0.5861	48.8916	0.4637
0.68	48.7974	0.5729	48.8059	0.5726	48.8887	0.4492
0.70	48.7929	0.5592	48.8015	0.5591	48.8854	0.4347
0.72	48.7881	0.5457	48.7967	0.5458	48.8818	0.4205
0.74	48.7829	0.5324	48.7916	0.5327	48.8778	0.4064
0.76	48.7773	0.5192	48.7861	0.5198	48.8734	0.3925
0.78	48.7714	0.5063	48.7803	0.5070	48.8686	0.3788
0.80	48.7652	0.4936	48.7741	0.4944	48.8633	0.3652
0.82	48.7585	0.4812	48.7675	0.4821	48.8577	0.3522
0.84	48.7515	0.4689	48.7606	0.4699	48.8516	0.3393
0.86	48.7442	0.4570	48.7533	0.4581	48.8456	0.3267
0.88	48.7365	0.4452	48.7456	0.4464	48.8380	0.3144
0.90	48.7284	0.4338	48.7376	0.4350	48.8305	0.3025
0.92	48.7199	0.4226	48.7292	0.4238	48.8225	0.2909
0.94	48.7111	0.4116	48.7204	0.4129	48.8141	0.2796
0.96	48.7020	0.4010	48.7113	0.4023	48.8052	0.2687
0.98	48.6925	0.3905	48.7018	0.3919	48.7959	0.2582
1.00	48.6827	0.3804	48.6920	0.3818	48.7861	0.2480

TABLE 9

(Continued)

u	$\overset{+10/3}{D}_{10/3}$	$\overset{+1/3}{D}_{10/3}$	$\overset{+17/5}{D}_{17/5}$	$\overset{+2/5}{D}_{17/5}$	$\overset{+22/5}{D}_{22/5}$	$\overset{+2/5}{D}_{22/5}$
1.00	48.6827	0.3804	48.6920	0.3818	48.7861	0.2480
1.02	48.6725	0.3706	48.6818	0.3719	48.7759	0.2382
1.04	48.6620	0.3610	48.6713	0.3623	48.7652	0.2287
1.06	48.6512	0.3516	48.6605	0.3529	48.7542	0.2196
1.08	48.6401	0.3425	48.6493	0.3438	48.7427	0.2109
1.10	48.6287	0.3337	48.6379	0.3350	48.7308	0.2025
1.12	48.6169	0.3251	48.6261	0.3264	48.7186	0.1944
1.14	48.6049	0.3168	48.6141	0.3180	48.7059	0.1866
1.16	48.5926	0.3087	48.6017	0.3099	48.6930	0.1792
1.18	48.5801	0.3009	48.5891	0.3020	48.6796	0.1721
1.20	48.5673	0.2933	48.5762	0.2944	48.6660	0.1653
1.22	48.5542	0.2859	48.5631	0.2870	48.6520	0.1588
1.24	48.5409	0.2788	48.5497	0.2798	48.6378	0.1526
1.26	48.5273	0.2718	48.5361	0.2728	48.6232	0.1466
1.28	48.5135	0.2651	48.5223	0.2661	48.6084	0.1409
1.30	48.4995	0.2586	48.5082	0.2595	48.5933	0.1355
1.32	48.4853	0.2522	48.4939	0.2532	48.5780	0.1303
1.34	48.4709	0.2461	48.4794	0.2470	48.5624	0.1253
1.36	48.4563	0.2402	48.4647	0.2410	48.5466	0.1205
1.38	48.4414	0.2344	48.4498	0.2352	48.5306	0.1160
1.40	48.4264	0.2288	48.4347	0.2296	48.5144	0.1116
1.45	48.3882	0.2156	48.3963	0.2163	48.4731	0.1016
1.50	48.3489	0.2034	48.3568	0.2040	48.4308	0.0927
1.55	48.3088	0.1920	48.3164	0.1926	48.3876	0.0847
1.60	48.2678	0.1815	48.2752	0.1821	48.3435	0.0775
1.65	48.2261	0.1718	48.2333	0.1723	48.2988	0.0711
1.70	48.1837	0.1628	48.1906	0.1632	48.2535	0.0653
1.80	48.0971	0.1466	48.1036	0.1470	48.1614	0.0555
1.90	48.0085	0.1326	48.0146	0.1329	48.0677	0.0475
2.00	47.9183	0.1204	47.9240	0.1207	47.9727	0.0409
2.10	47.8267	0.1098	47.8320	0.1100	47.8768	0.0354
2.20	47.7339	0.1004	47.7389	0.1006	47.7801	0.0309
2.40	47.5457	0.0849	47.5501	0.0850	47.5851	0.0239
2.60	47.3547	0.0726	47.3586	0.0727	47.3887	0.0188
2.80	47.1618	0.0628	47.1653	0.0629	47.1912	0.0151
3.00	46.9674	0.0548	46.9705	0.0549	46.9930	0.0123
3.50	46.4771	0.0404	46.4795	0.0404	46.4959	0.0078
4.00	45.9832	0.0309	45.9851	0.0309	45.9974	0.0052
4.50	45.4873	0.0244	45.4888	0.0244	45.4982	0.0037
5.00	44.9900	0.0198	44.9913	0.0198	44.9988	0.0027
5.50	44.4920	0.0163	44.4931	0.0163	44.4991	0.0020
6.00	43.9935	0.0137	43.9944	0.0137	43.9993	0.0015
7.00	42.9955	0.0100	42.9961	0.0100	42.9996	0.0010
8.00	41.9967	0.0076	41.9972	0.0076	41.9998	0.0006
9.00	40.9975	0.0060	40.9979	0.0060	40.9998	0.0005
10.00	39.9981	0.0048	39.9984	0.0048	39.9999	0.0003
15.00	34.9993	0.0020	34.9994	0.0020	35.0000	0.0001
20.00	29.9997	0.0010	29.9997	0.0010	30.0000	0.0000
30.00	19.9999	0.0004	19.9999	0.0004	20.0000	0.0000
50.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
∞	$-\infty$	-0.0002	$-\infty$	-0.0002	$-\infty$	-0.0000