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Manuel Guerra, João Pereira, Miguel St. Aubyn

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Portugal

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An Horizontal Innovation Growth Model with Endogenous Time Allocation and Non-Stable Demography

Manuel Guerra* João Pereira† Miguel St. Aubyn‡

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Abstract

We propose a decentralized endogenous growth model in order to study the transitional dynamics associated with the process of population aging in a small open economy. The model features endogenous time allocation and two growth engines: R&D and human capital accumulation. Per capita output growth is affected negatively by the difference in the rates of growth of labor force and of the total population in the period where the weight of the labor force decreases to a new and lower level. The biggest impact of aging on per capita output growth is during the period where labor force grows at a lower rate than the population unless it is compensated by some other effect. Under some assumptions, a decrease in the corporate tax improves growth.

Keywords: Endogenous Growth; Demographic Changes; Time Allocation; Human Capital; R&D

JEL Classification: O41; O33; J11; J22; J24.

*CEMAPRE, REM – Research In Economics and Mathematics, ISEG – Universidade de Lisboa. Assistant Professor, Dept. of Mathematics, ISEG. Manuel Guerra was partially supported by the Project CEMAPRE - UID/MULTI/00491/2013 financed by FCT/MEC through national funds.

†Ph.D. Student at ISEG – Universidade de Lisboa. Corresponding Author, e-mail: joaopereira9@gmail.com.

‡UECE – Research Unit on Complexity and Economics, REM – Research In Economics and Mathematics, ISEG – Universidade de Lisboa, Rua do Quelhas 6, 1200 - 781 Lisboa, Portugal. UECE – Research Unit on Complexity and Economics is financially supported by FCT (Fundação para a Ciência e a Tecnologia), Portugal. This article is part of the Strategic Project UID/ECO/00436/2013.

1 Introduction

Many advanced economies are going through a process of population aging. We should expect this process to have important effects on the economy. One reason is that there are clustering of some activities at particular phases of the life-cycle, like the specialization in education and the phase of retirement. Another reason is that people make decisions taking into account how old they are and some expectation of remaining lifetime. Hence, changes in the life expectancy should affect all those decisions. Still, another reason is that some of the economy's institutions are particularly sensitive to demographic changes. An obvious example are pension systems, in particular, PAYG (pay as you go) pension systems.

In an economy with a stable age structure ignoring the age dimension may be acceptable. However, if the population is undergoing important demographic changes then we should expect those demographic changes to have a significant effect on the economic behavior of individuals and, as a consequence, on the evolution of the economy.

In this paper we propose a horizontal innovation endogenous growth model to be used in the study of the effects of population aging in the economy. The consumer block is developed in Guerra et al. (2018).[1] In that paper the consumer decides how much to consume and how to allocate time at the intensive margin between leisure, studying and work while facing an age-dependent mortality law. By endogeneizing labor supply and individual human capital investment we are able to capture a behavorial effect of consumers as a reaction to a higher life expectancy and follow those effects to the aggregate.

Our model is based on Romer (1990) and Jones (1995), but is closer to the Jones specification because of the way we eliminate scale effects. Nevertheless, it departs from those models in several directions. One of them is that it features two engines of growth instead of one because there is a human capital externality in the production of individual human capital.

Also, related to the fact that we are modeling a small open economy, we added a catching up term on the production function of ideas. The farthest the economy is from the world technological frontier the stronger will be this effect.[2]

We believe that is of great interest to study the transitional dynamics between two different demographic steady-states, so the model is intended to be applied to a situation where the population is not stable and therefore decisions of consumers cannot be aggregated simply into a per capita variable. People differ by age and also their planning is age dependent. Since the age composition of the population varies, so will all aggregate variables that are simply the cohort aggregation of individual variables, as is the case of total human capital. This will mean, also, that we cannot solve for a balanced growth path as the continuous change in the age composition of the population will prevent variables growing at a constant rate.

With the increased interest in the phenomenon of population aging, many growth models incorporating demographic details have been proposed. Some examples of exogenous growth

[1] Guerra et al. (2018) and the current paper are part of a general equilibrium model developed and simulated numerically in Pereira (2018).

[2] For a review of technological diffusion models, see Benhabib and Spiegel (2005).

models are Heijdra and Reijnders (2016), that use a fully rectangular survival law, and Heijdra and Romp (2009), Heijdra and Reijnders (2018), Ludwig et al. (2012) and Vogel et al. (2017) who use an age-dependent mortality. Except for the latter two, the other studies restrict their attention in the long run.

Regarding endogenous growth models within this line of research, it becomes somewhat natural to consider models in which human capital accumulation is the source of growth. As aggregate human capital is simply a composite of individual human capital and the latter is age-dependent, aggregate human capital becomes a variable directly affected by age composition. This is what Boucekkine et al. (2002) do taking it to the extreme, with the production sector having a linear technology on aggregate human capital.[3] In their study, the survival law is age dependent.

Azomahou et al. (2009) perform an empirical investigation on the relationship between life expectancy and economic growth. Their nonparametric study finds a logistic shape for the relation: convexity at low levels of life expectancy and concavity at high levels. They argue that models relying solely on vintage human capital accumulation, such as in Boucekkine et al. (2002) cannot replicate this relationship, as it tends to be concave. They consider the increasing returns model of Romer (1986) and find that with a constant probability of death this model also does not replicate the empirical relationship uncovered. However, when the same age-dependent survival law as in Boucekkine et al. (2002) is added, the result is inverted. They conclude that life-cycle behavior with a realistic survival law is able to reproduce their finding.[4]

Within the narrower context of endogenous growth with expanding varieties, in which our model lays, there have been also some attempts to integrate demography into the analysis. For example, Prettner (2013), Prettner and Trimborn (2016) and Gehringer and Prettner (2017) are studies with a production sector that follows more closely the Romer - Jones kind of model. They all use a constant mortality rate. We improve on these studies by introducing an age-dependent survival law and by endogenizing labor supply and human capital. With this, we obtain age-structured labor supply and age-structured efficiency, allowing more channels through which changes in the age composition of the population can influence the economy.

This paper is organized as follows. The next section builds the production sector. Section 4 introduces the government sector. Section 5 presents the equilibrium in every market and the general equilibrium. In Section 6 we analyze some analytical results and Section 7 concludes.

[3]Some other examples are de la Croix and Licandro (1999), D'Albis (2007) or Grafenhofer et al. (2007)

[4]Other examples of studies where growth depends on physical capital externalities are Futagami and Nakajima (2001) and Heijdra and Mierau (2011). In the former, a fully rectangular survival law is used. In the latter, consumers face an age-dependent survival law in which uncertainty cannot be fully insured due to imperfect annuity markets.

2 The model

The production block of the economy has three sectors as is common in horizontal innovation models.[5] The research sector uses aggregate human capital, the existing stock of domestic ideas and also foreign ideas to produce new ideas. Each idea gives the blueprint to manufacture a variety of durable good. To each idea is assigned a patent which is auctioned to the highest bidder. After the purchase of the patent, a firm in the durable goods sector may start producing the variety of durable good for which it is the monopolist, by using foregone expenditure. These durable goods are rented by the final goods sector and combined with aggregate human capital produce a homogeneous final good.

We introduce a government sector that manages the taxes, the educational subsidy and the pension system that we introduced in the consumer block, but we add now a corporate tax to the policy tools. As usual, when considering policy options, the government has to respect a solvency condition. We shall also define the functioning of the capital market. The foreign sector closes the model but is not modeled. Foreign sector stocks and flows mirror domestic equilibria. Foreign debt is also subject to a solvency condition.

This is a model of a small open economy and we need to define what we mean by this:

Definition 1 *A small open economy in this model has the following features:*

1. *Financial capital is perfectly mobile. The interest rates the economy faces are the international interest rates which are assumed to be constant.*

2. *There are two interest rates,* r_k *and* r*, with* $r_k > r$ *and where* $r_k - r$ *captures a risk premium.*

3. *There is no migration. Therefore the wage rate is determined domestically.*

4. *Only the output of the final goods sector is tradable internationally. The production of the research sector and of the durable goods sector are sold domestically.*

5. *The research sector may use foreign ideas as an input without paying a cost for it.*[6]

Being a small open economy we let the interest rate determination out of the model, but at the international level the higher interest rate on capital exists because of the risk premium which is assumed to be constant. We do not have any uncertainty at the aggregate level in our economy. Nevertheless, we may think that in the economy there are a large number of risky projects with an average rate of return r_k. On average this rate is known, but each investor does not know if her particular project will succeed or fail. At the economy-wide level the rate will appear like being deterministic. That is why we make this assumption of a higher interest rate associated with a risk premium, although our model has no microfoundations for such risk. Using two different interest rates has also the advantage of letting us simulate

[5]In order to get a comprehensive description of the model, some repetition of the standard horizontal innovation models features is unavoidable.

[6]This assumption of no cost is justified on the grounds that foreign ideas are not being used to produce domestic durable goods but instead they must be adapted to produce domestic innovations.

numerically the model in a more satisfactory way.

2.1 Expectations

In Guerra et al. (2018), where the consumer block was developed we assumed agents have myopic expectations, in the sense that they expect future values of the variables to be the same as current variables. A subscript $_{*t}$ indicates a quantity being forecast or planned at time t for some time τ into the future. Then, for some variable X, myopic expectations mean $X(\tau)_{*t} = X(t)$.

We also assumed in that paper that consumers borrow and lend to a risk-neutral institutional lender, at the rate r plus the mortality risk. This institutional lender will apply those savings in the remaining assets of the economy. Some of this assets will have a return r_k. The per capita extra return of $(r_k - r)$ is distributed evenly to consumers (Section 5.3). We extend our assumption of myopic expectations made in that paper by assuming that each consumer does not know the net balance of other consumers. This means that they optimize taking into consideration the interest rate r and only after their plans are executed, they will receive that extra return.

2.2 Final goods sector

The final goods sector is competitive and composed of a large number of firms, using the same production technology. Firms use durable goods and human capital as inputs to produce a homogenous final good. We use the general convention of normalizing the price of this good to one in order to avoid dragging another parameter around. It is possible to treat the problem of this sector as the one of a representative firm. The constant elasticity of substitution production function is given by:

$$Y(t) = \mu H_Y(t)^{1-\alpha} \left(\int_0^{A(t)} X(i,t)^{\alpha\gamma} di \right)^{1/\gamma}, \qquad \mu, \gamma > 0, 0 < \alpha < 1 \tag{1}$$

In which μ is a scale parameter, γ measures the degree of substitutability between durable goods, $A(t)$ is the number of varieties of durable goods, $X(i,t)$ represents the stock of durable good of variety i. $H_Y(t)$ is embodied human capital used in this sector. Contrary to Romer (1990), for example, we do not define raw labor. Labor is supplied jointly with human capital, hence $H_Y(t)$ is a generational integration of individual labor supply and human capital.

At each moment in time, this sector decides how much human capital and how much quantity of each variety to use. The rental price of a human capital unit employed in this sector is the wage rate $w_Y(t)$ and the cost of the variety is a rental price $p(i,t)$.

The pre-tax instantaneous cash-flow function is given by,

$$\pi_Y(t) = \mu H_Y^{1-\alpha}(t) \left(\int_0^{A(t)} X(i,t)^{\alpha\gamma} di \right)^{1/\gamma} - w_Y(t) H_Y(t) - \int_0^{A(t)} p(i,t) X(i,t)) di \tag{2}$$

The optimization problem is to maximize the present value of its cash-flow net of taxes:

$$\max_{H_Y(t), X(i,t)} \quad V_{Y_{*t}} = \int_t^{\infty} e^{-r_k(\tau-t)} (1 - t_p(\tau)_{*t}) \pi_Y(\tau)_{*t} d\tau \tag{3}$$

With myopic expectations in the corporate tax, $t_p(\tau)_{*t} = t_p(t)$, the tax term can be removed from the integral and ignored. Since there are not any differential equations, the problem can be solved as a static one. Taking the measure of varieties and both rental rates as given, the sector's demand equations result in

$$\frac{\partial \pi_Y(t)}{\partial H_Y(t)} = 0 \Leftrightarrow w_Y(t) = (1-\alpha) \mu H_Y(t)^{-\alpha} \left(\int_0^{A(t)} X(i,t)^{\alpha\gamma} di \right)^{1/\gamma} \tag{4}$$

For human capital, and

$$\begin{aligned} \frac{\partial \pi_Y(t)}{\partial X(i,t)} = 0 \Leftrightarrow & p(i,t) = \alpha \mu H_Y^{1-\alpha}(t) \left(\int_0^{A(t)} X(i,t)^{\alpha\gamma} di \right)^{1/\gamma - 1} X(i,t)^{\alpha\gamma - 1} \\ \Leftrightarrow & X(i,t) = \left(\frac{\alpha \mu H_Y^{1-\alpha}(t) \left(\int_0^{A(t)} X(i,t)^{\alpha\gamma} di \right)^{1/\gamma - 1}}{p(i,t)} \right)^{\frac{1}{1-\alpha\gamma}} \end{aligned} \tag{5}$$

For each durable.

2.3 Durable goods

Each firm in the durable goods sector produces only one variety of durable good. In order to start their business, they have to purchase a patent for the design of that particular good from the research sector. This excludes competition on that particular variety which makes each company a monopolist. Hence, the sector is characterized by monopolistic competition. Since the price of the patent constitutes a fixed sunk cost, the maximization problem of the monopolist is to maximize it's after-tax operational cash-flow by choosing the level of output to be rented to the final goods sector.

In order to produce the durable goods, there are some resources that are diverted at the aggregate level to use in the production of durables. We assume all firms, although producing different durables, share the same technology. We further simplify by assuming that each unit of raw capital can be transformed in one durable good. The user cost of capital is the interest rate r_k plus the depreciation rate of physical capital δ_x. The expected instantaneous pre-tax cash-flow function for each firm is

$$\pi_i(t) = (p(i,t) - (r_k + \delta_x))\, X(i,t) \tag{6}$$

For each firm, the maximization problem is

$$\max_{p(i,t)} \quad V_{i_{*t}} = \int_t^{\infty} e^{-r_k(\tau - t)} (1 - t_p(\tau)_{*t} \pi_i(\tau)_{*t} d\tau$$

Once again, because of myopic expectations, the tax term can be taken out of the integral and the problem is reduced to maximizing the instantaneous pre-tax cash-flow function, taking into account the demand function (5):

$$\begin{aligned}
\frac{\partial \pi_i(t)}{\partial p(i,t)} = 0 &\Leftrightarrow X(i,t) + (p(i,t) - (r_k + \delta_x))\, \frac{\partial X(i,t)}{\partial p(i,t)} = 0 \\
&\Leftrightarrow X(i,t) - (p(i,t) - (r_k + \delta_x))\, \frac{X(i,t)}{(1 - \alpha\gamma) p(i,t)} = 0 \\
&\Leftrightarrow p(i,t) = \bar{p} = \frac{r_k + \delta_x}{\alpha\gamma} \qquad (7)
\end{aligned}$$

If we plug this constant price in (5) we will get the result that all the quantity rented of each variable will be the same for each moment in time,

$$X(i,t) = X(t) \tag{8}$$

This symmetry, which is a usual property of this type of models, happens for two reasons. First, all varieties are being rented, so in the final goods sector there can be made an instantaneous adjustment to the desired level without consideration of the previous level of stock. That problem would appear in the durable goods sector but is washed away by the putty-putty capital hypothesis. Secondly, because the demand equation for rental of durables in the final goods sector, that compares the marginal productivity of each durable with its marginal cost is the same for all durables. What makes this the same is the fact that the interest rate and the depreciation rate are constant, making the price constant and the particular production function of the final goods sector that displays additively separable marginal productivity of durables. The quantity of the durables varies in time because the marginal productivity of the durables depends on aggregate human capital which also varies in time.

The pre tax profit of each firm producing durables is

$$\pi_i(t) = \pi(t) = \frac{(1 - \alpha\gamma)\,(r_k + \delta_x)\, X(t)}{\alpha\gamma} = (1 - \alpha\gamma)\bar{p} X(t) \tag{9}$$

2.4 Research

There is free entry into the research sector, preventing the existence of a monopoly rent. Innovation is produced using human capital, the existing domestic stock of knowledge and adaptation of foreign ideas. Following Romer (1990), the ideas (blueprints, designs) produced in this sector are disembodied knowledge. Each design will have a patent valid for an infinite period of time. This patent will be auctioned to a firm that will produce a variety of durable good based on that particular idea.

We decided to introduce a catching up term towards the technological frontier in the production function of ideas. This is based on the idea put forward by Nelson and Phelps (1966) that links the rate of technology diffusion to the distance to the technological frontier and in which human capital facilitates the process. This catching-up term may also be viewed as capturing international knowledge spillovers. Klenow and Rodríguez-Clare (2005) argue that international knowledge spillovers are necessary to explain some empirical findings on economic growth namely that many countries seem to share a common long-run growth rate despite having rates of investment in all sorts of capital.

Benhabib and Spiegel (2005) put emphasis on the functional specification of TFP growth with technology diffusion. They distinguish between two basic functional forms, exponential and logistic, which are presented, respectively, as:

$$\dot{A}(t) = g(H(t))A(t) + c(H(t))\left(A^*(t) - A(t)\right)$$
$$\dot{A}(t) = g(H(t))A(t) + c(H(t))\frac{A(t)}{A^*(t)}\left(A^*(t) - A(t)\right)$$

In which $g(H(t))$ and $c(H(t))$ are positive functions of human capital and $A(t)$ is the domestic level of TFP and $A^*(t)$ the TFP level of the leading nation. They present a functional form that nests both specifications and test it on data. The results favor the logistic specification. In this specification, there is the possibility of existing a critical human capital threshold below which convergence towards the technology frontier does not occur which may lead to the appearance of the so-called convergence clubs.

Although in our model, A represents the stock of knowledge and not the TFP, we may apply the above reasoning. As we are dealing with a small open economy we assume that the ideas produced in this economy have a negligible impact on the world technological frontier, A^*. This assumption allows us to model A^* as exogenous. We define the world technological frontier as the stock of ideas of the most advanced economy and we assume it grows at some exogenous positive rate. This can be considered a suitable proxy in an institutional environment where ideas can be adapted freely from other countries. We also assume that the domestic production of ideas is increasing on the distance to the world technological frontier (the catching-up effect). The production function of the research sector is

$$\dot{A}(t) = H_A(t)\left[\xi A(t)^{\phi_A} + \zeta\big(A(t)^* - A(t)\big)^+\right] \qquad (10)$$

With parameters $\xi > 0$, $\zeta > 0$ and $0 < \phi_A < 1$. $H_A(t)$ is the quantity of human capital used in this sector. The superscript $'+'$ means that this term is only accounted for when it is positive, i.e., when our economy lags behind the most advanced one.

Note that we are not considering the automatic implementation of foreign knowledge. The foreign knowledge must be transformed in such a way that, together with the domestic knowledge and embodied human capital of researchers, will generate a new, and therefore, patentable idea. The term $\zeta\big(A(t)^* - A(t)\big)^+$ represents the catching up term mentioned earlier - a positive effect of the distance to the world technological frontier on the marginal productivity of researchers.[7] The less developed the economy is, the higher will be the impact of adaptation of foreign ideas. When the economy is the technological leader, the term vanishes.

Real marginal productivity of researchers is divided into two terms, one associated with the 'pure' domestic innovation and another associated with adaptation of foreign knowledge, respectively $\xi A(t)^{\phi_A}$ and $\zeta\big(A(t)^* - A(t)\big)^+$[8]. We consider only productivity measured in quantity of ideas and abstract from their price. For fixed $\xi, \zeta > 0$ equality between these marginal productivities leads to $\xi A(t)^{\phi_A} = \zeta\big(A(t)^* - A(t)\big)^+$. If we start from a position sufficiently far away from the technological leader then the marginal productivity of pure domestic innovation will be lower than the one associated with adaptation of foreign ideas: $\xi A(t)^{\phi_A} < \zeta\big(A(t)^* - A(t)\big)^+$. There is a relative advantage in adapting foreign ideas. As the economy develops and converges to the leading nation the situation will eventually revert.

We have an exponential type specification of the technology diffusion process. We chose this specification, despite the evidence in Benhabib and Spiegel (2005). It is simpler and if the country is not too far away from the leading nation, meaning that it lies somewhere in the middle section of the logistic function, then the exponential specification probably does not differ much in terms of predictive power from the logistic function.

Comparing our functional form with the ones used by Romer (1990) and Jones (1995) and abstracting from the catching up term, our form may be viewed as sitting somewhere in the middle. On one hand, there is a decreasing marginal productivity of current knowledge $\partial^2 \dot{A}(t)/\partial^2 A(t) < 0$ as in Jones, on the other hand, we have constant marginal productivity for the researchers, $\partial^2 \dot{A}(t)/\partial^2 H_A(t) = 0$ which is a feature of the Romer model. This assumption is not far-fetched, as it would be in the case of the final goods sector because researchers use only ideas to produce more ideas. So, it can be reasonably assumed that doubling the number of researchers will double the number of ideas produced. On the overall this results in increasing returns to scale, a property shared with the Romer model and which is not excluded by the Jones model.[9]. Another simplification is that we do not model the use of equipment by the researchers.

The rate of growth of knowledge is:

$$\frac{\dot{A}(t)}{A(t)} = H_A(t)\left[\xi A(t)^{\phi_A - 1} + \zeta\Big(T(t) - 1\Big)^+\right]$$

With $T(t) = A(t)^*/A(t)$. The rate of growth of disembodied knowledge and, therefore, of the number of varieties, is a positive, but a decreasing function of the level of domestic knowledge. There will be a positive upper bound for the growth rate of ideas imposed by the level of human

[7] *Ceteris paribus*, the production of ideas *per* researcher will be higher the farther away the domestic level of knowledge is from the world technological frontier.

[8] This is similar to the discussion of technology diffusion applied to the Romer model present in Benhabib and Spiegel (2005).

[9] In the Jones model, the functional form (in equilibrium) is $\dot{A}(t) = \delta L_A(t)^{\lambda} A(t)^{\phi}$, with $0 < \lambda \leq 1, \phi < 1$. Hence the degree of returns to scale is given by $\lambda + \phi$: ($\dot{A} = f(L_A, A)$, $f(kL_A, kA) = k^{\lambda+\phi}\dot{A}$

capital used in the sector. in the long run, if the economy becomes the most technologically advanced one, we will have $\frac{\dot{A}(t)}{A(t)} = H_A(t)\xi A(t)^{\phi_A - 1}$. Then, avoiding a declining rate of growth in new ideas/blueprints requires an increasing allocation of human capital to research.

The maximization problem of the research sector is

$$\max_{H_A(t)} \quad V_A(t) = \int_t^{\infty} e^{-r_k(\tau - t)}(1 - t_p(\tau)_{*t}\pi_A(\tau)_{*t} d\tau$$

It can also be solved as a static problem. The firms choose how much human capital to hire in order to maximize the instantaneous expected pre-tax cash-flow function:

$$\pi_A(t) = \Big(P_A(t)H_A(t)\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t) - A(t)\big)^+\Big] - w_A(t)H_A(t)\Big) \tag{11}$$

with $P_A(t)$ being the price of the ideas discovered, which is the price of their patents and $w_A(t)$ being the wage rate per unit of human capital employed in this sector. Solving for the optimum amount of labor we get the demand equation for human capital:

$$\frac{\partial \pi_A(t)}{\partial H_A(t)} = 0 \Leftrightarrow w_A(t) = P_A(t)\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t) - A(t)\big)^+\Big] \tag{12}$$

For simplification, patents have an infinite life, granting a monopoly power to their owner for all future periods. Each idea is related to a durable good that can be produced and each idea will be auctioned at a price that, in equilibrium, will be equal to the expected discounted after-tax profit cash flow generated by the production and selling of those durables. Hence, the equilibrium price of a patent in the primary market is

$$P_i(t) = \int_t^{+\infty} e^{-r_k(\tau - t)}(1 - t_p(\tau)_{*t})\pi_i(\tau)_{*t} d\tau \tag{13}$$

An immediate conclusion is that all ideas invented in the same period will have the same price of patents. We will use the notation $P_A(t)$ to refer to the price of the patents of the most recently invented designs. Also, we will use $\pi(t)$ instead of $\pi_i(t)$ because of the symmetry of cash flows. Because of our choice of myopic expectations, the expected future cash-flow generated by the idea is equal to the current cash-flow, which allows simplifying (13) to:

$$P_A(t) = \frac{(1 - t_p(t))\pi(t)}{r_k} = \frac{(1 - t_p(t))(1 - \alpha\gamma)\bar{p}X(t)}{r_k} \tag{14}$$

3 Consumers

The optimization problem of the consumers was solved in Guerra et al. (2018). For economy of space, we will restrain from repeating the relevant expressions for the variables. From the consumer block, consumption ($c_v(t)$), human capital ($h_v(t)$), labor income ($\omega_v(t)$), work effort ($s_v^w(t)$), and assets ($a_v(t)$) will be relevant. The share of time dedicated to leisure will not be used directly. The time dedicated to studying ($s_v^h(t)$) will only be relevant to determine the amount of education subsidies to be paid by the government. [10]Other government expenses, like pensions, and also some government revenues will depend directly on the optimization problem of the consumer block. Individual (age-dependent) variables will be aggregated into economy-wide variables by integrating them with the population per age.

4 Government

In the following, we present the expressions that define the government sector.

The government's primary balance, $B(t)$, is described by:

$$B(t) = Z_0(t) + Z_l(t) + t_p(t)\Pi(t) - (1 - z_p(t))P(t) - E(t) - G(t) \tag{15}$$

Where $Z_0(t)$ are total funds raised by the lump sum tax ($z_0(t)$),

$$Z_0(t) = \int_{-\infty}^{t} L_v(t) z_0(t) dv \tag{16}$$

With $L_v(t)$ being the population born at v, still alive at time t, the population with age $t - v$. $Z_l(t)$ are total funds raised by labor income taxes ($z_l(t)$)[11],

$$Z_l(t) = z_l(t) \int_{-\infty}^{t} L_v(t) h_v(t) w(t) s_v^w(t) dv \tag{17}$$

$\Pi(t)$ are total taxable cash-flow from the firms in the economy. Since the final goods sector and the research sector have zero profits, this represents only profits from the durable goods sector,

$$\Pi(t) = \int_0^{A(t)} \pi_i(t) di - P_A(t)\dot{A}(t) \tag{18}$$

We subtract a term that represents the cost of investment in patents for the firms that start operating in this period. We use the term profit for all firms although, for new firms, the profit in the first period of operation is not $\pi_i(t)$ but ($\pi_i(t) - P_A(t)$) and will be negative if $r_k < (1 - t_p)$, which should be the normal case. We decided to compute revenues from the corporate tax proportional to the entire sector, and not to calculate separately for new firms and already existing firms. This decision was made for simplification reasons. Nevertheless, since in many tax codes there is the possibility to defer the cost of investment through several years, this way of computing the revenues is, in some way, capturing that effect. Notice that for a sufficiently large $\dot{A}$, $\Pi(t)$ may become negative.

[10]The subscript v indexes the time of birth.

[11]Although we use the term taxes loosely, $z_l(t)$ should, in fact, be viewed as including social security contributions, not only paid by the worker but also by the employer since it represents the difference, in percentage, between the costs paid by the firms and net income received by the workers.

$P(t)$ are gross pensions expenditure, so the fourth term on the right-hand side represents net pensions expenditures, as we subtract the funds raised by the tax on pensions ($z_p(t)$),

$$P(t) = \int_{-\infty}^{t} L_v(t) p_v(t) dv$$

with individual pension benefits defined by (19)

$$p_v(t) = \begin{cases} 0, & t \le R_v + v \\ \frac{\theta \pi_R}{R_v - S_v}, & t > R_v + v \end{cases}$$

Where R_v is the mandatory retirement age and S_v the age of completion of mandatory schooling assuming no grade retention. $E(t)$ represents expenditure with educational grants:

$$E(t) = \int_{t-S}^{t} L_v(t) \gamma_e(t) w(t) s_v^h(t) dv \tag{20}$$

with $\gamma_e(t)$ being the educational subsidy and $G(t)$ is the exogenous government consumption.

Government debt (D) evolves according to:

$$\dot{D}(t) = rD(t) - B(t) \tag{21}$$

Moreover, government debt must satisfy an intertemporal solvency condition:

$$\lim_{\tau \to +\infty} D(\tau) e^{-r(\tau - t)} \le 0 \tag{22}$$

5 Equilibrium

In this section, we shall present the conditions for the general equilibrium of the economy which requires equilibrium in each of the production sectors, in the labor market, aggregate goods market and capital market. The foreign sector is not modeled. It closes the model and its dynamics are determined by the dynamics in the other markets.

The first equilibrium property was already mentioned, $X(i,t) = X(t)$. With this symmetry between durable goods, the production of the final goods sector becomes:

$$Y(t) = \mu H_Y(t)^{1-\alpha} A(t)^{1/\gamma} X(i,t)^{\alpha} \tag{23}$$

The demand for durables becomes:

$$X(t) = H_Y(t) \left(\frac{\mu\alpha}{\bar{p}} \right)^{\frac{1}{1-\alpha}} A(t)^{\frac{1-\gamma}{\gamma(1-\alpha)}} \tag{24}$$

and the demand equation for human capital in the final goods sector:

$$w_Y(t) = (1-\alpha)\mu H_Y(t)^{-\alpha} A(t)^{1/\gamma} X(t)^{\alpha} \tag{25}$$

Combining the two previous results we may write the wage rate as a function of A:

$$w_Y(t) = (1-\alpha)\mu \left(\frac{\mu\alpha}{\bar{p}} \right)^{\frac{\alpha}{1-\alpha}} A(t)^{\frac{1-\alpha\gamma}{\gamma(1-\alpha)}} \tag{26}$$

5.1 Labor market

An equilibrium in this market will deliver the optimal distribution of workers between the final goods sector and the research sector and the optimal equilibrium wage rate. Since the only heterogeneity workers have is related to their age, individual human capital and individual labor supply are the same in both sectors. This means that the allocation of human capital between the final goods sector and the research sector reduces itself to an allocation of total workers between those sectors. Total aggregate human capital is defined as

$$H(t) \equiv \int_{-\infty}^{t} L_\nu(t) h_\nu(t) s_\nu^w(t) d\nu \tag{27}$$

The equilibrium in this market will determine

$$H_Y(t) \equiv \int_{-\infty}^{t} L_\nu^Y(t) h_\nu(t) s_\nu^w(t) d\nu, \qquad H_A(t) \equiv \int_{-\infty}^{t} L_\nu^A(t) h_\nu(t) s_\nu^w(t) d\nu$$

In which $L_\nu^Y(t)$ and $L_\nu^A(t)$ are the population born at time ν, with age $t-\nu$ employed in time t in the final goods and research sectors respectively. Notice that we do not restrict the integral limits to the people in labor force. This is not necessary as these people will have $s_\nu^w(t) = 0$ and the integral will only count positively people that are currently working.

Without labor market frictions, in equilibrium the labor income a worker receives in one of these sectors must be identical to what the same worker can receive on the other sector, otherwise the worker would find an incentive to switch jobs. Then we must have,

$$\omega_\nu^Y(t) = w_Y(t) h_\nu(t) s_\nu^w(t) = w_A(t) h_\nu(t) s_\nu^w(t) = \omega_\nu^A(t) \tag{28}$$

We assumed in Guerra et al. (2018) that the wage rate per unit of human capital is independent of the age at which that human capital was acquired. This is consistent with a description of human capital as having a generic nature and not being firm-specific and also consistent with our assumption of no financing by the firms of on-the-job training.

The equilibrium condition (28) imposes that the wage rate must be identical in all sectors employing workers. This will be instrumental to determine how many workers will be allocated to each sector. Due to the linearity of the production of ideas on human capital, people employed in this sector will be determined as the remaining pool of people that are not employed in the final goods sector. Human capital used in the final goods sector is restricted by total human capital. The equilibrium in the labor market is defined by the following set of equations:

$$H(t) = H_Y(t) + H_A(t) \tag{29}$$

$$w_Y(t) = w_A(t) = w(t) \tag{30}$$

With $w_Y(t)$ and $w_A(t)$ coming from (4) and (12), respectively. Expression (12) defines the wage of researchers independently of their employment level. We may think of this as if the wage rate is determined in the research sector and then the manufacturing sector follows this wage rate and employs their optimal level of human capital. Then the research sector employs the remaining free workers. This happens because the marginal productivity of workers in the final goods sector is decreasing but in the research sector it is constant.

The equalization between the two wage rates gives us the equilibrium quantity of human capital employed in the final goods sector. In the following, we use the symmetry property of the durable goods, by using the expressions (25) and (24) and we require also the expression (14) for patents. From equalizing the wage rates we obtain:

$$(1-\alpha)\mu H_Y(t)^{-\alpha}A(t)^{1/\gamma}X(t)^{\alpha} = P_A(t)\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t)-A(t)\big)^+\Big]$$

$$\Leftrightarrow (1-\alpha)\mu H_Y(t)^{-\alpha}A(t)^{1/\gamma}X(t)^{\alpha} = \frac{(1-t_p(t))(1-\alpha\gamma)\bar{p}X(t)}{r_k}\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t)-A(t)\big)^+\Big]$$

$$\Leftrightarrow (1-\alpha)\mu H_Y(t)^{-\alpha}A(t)^{1/\gamma}X(t)^{\alpha-1} = \frac{(1-t_p(t))(1-\alpha\gamma)\bar{p}}{r_k}\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t)-A(t)\big)^+\Big]$$

$$\Leftrightarrow (1-\alpha)\mu H_Y(t)^{-\alpha}A(t)^{1/\gamma}H_Y(t)^{\alpha-1}\left(\frac{\alpha\mu}{\bar{p}}\right)^{\frac{\alpha-1}{1-\alpha}}A(t)^{\frac{(\alpha-1)(1-\gamma)}{\gamma(1-\alpha)}} =$$

$$= \frac{(1-t_p(t))(1-\alpha\gamma)\bar{p}}{r_k}\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t)-A(t)\big)^+\Big]$$

$$\Leftrightarrow \frac{(1-\alpha)\mu H_Y(t)^{-1}A(t)\bar{p}}{\mu\alpha} = \frac{(1-t_p(t))(1-\alpha\gamma)\bar{p}}{r_k}\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t)-A(t)\big)^+\Big]$$

$$\Leftrightarrow H_Y(t) = \frac{A(t)r_k(1-\alpha)}{(1-t_p(t))(1-\alpha\gamma)\alpha\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t)-A(t)\big)^+\Big]} \tag{31}$$

Theoretically, a corner solution with $H_Y(t) = H(t)$ and $H_A(t) = 0$ cannot be excluded, so the correct expression for the equilibrium level of human capital employed in the production of final goods is

$$H_Y(t) = Min\left[\frac{A(t)r_k(1-\alpha)}{(1-t_p(t))(1-\alpha\gamma)\alpha\Big[\xi A(t)^{\phi_A} + \zeta\big(A^*(t)-A(t)\big)^+\Big]}, H(t)\right] \tag{32}$$

Nevertheless, there will be a tendency for a corner solution to be temporary, as the growth in $A(t)$ is interrupted and $(A^*(t) - A(t))$ increases because we assumed $\dot{A}^*(t)/A^*(t) > 0$. This will decrease the demand of $H_Y(t)$. How long this period will last will depend on the behavior of $H(t)$. All the subsequent analysis will be restricted to the case of an interior solution.

The equilibrium level of $H_A(t)$ is given by (29). Both human capital allocations will not be constant, even at the technological frontier. They are not even expected to be constant fractions of $H(t)$ as total human capital fluctuates due to the non-stable age composition of the population.

Clearly, the result that workers of the same age will earn the same regardless of whether they are employed on a factory or employed on a research lab is not realistic but is a straightforward consequence of the nonexistence of any other heterogeneity other than the age of the individuals.

5.2 Aggregate goods market

One of the changes introduced by The System of National Accounts, 2008 to national accounting conventions is related to the treatment of R&D expenses. Before, regardless of whether they resulted from purchases or from internal R&D, they were considered intermediate consumption. With the new System of National Accounts, 2008, R&D expenses are treated as an investment. Since we follow this new convention, we will have another difference in comparison to other models that we have been referring to, like Romer (1990) and Jones (1995). Gross domestic product (GDP) is the sum of the Gross Value Added (GVA) of all sectors. If R&D expenses were treated as intermediate consumption they would need to be subtracted to the production value, but being treated as an investment they are not subtracted.

The GVA of the final goods sector is the output minus the cost of renting the durables. Labor costs do not enter the calculation and rental of physical equipment, contrary to its final acquisition, is considered an intermediate consumption. In the durable goods sector, it is the value of production because the cost of patents is not considered an intermediate consumption, neither is the user cost of capital. Finally, in the research sector, it is the value of production. Then GDP, which we denote by $\bar{Y}$, is:

$$\bar{Y}(t) \equiv Y(t) + P_A(t)\dot{A}(t) \tag{33}$$

Equilibrium in this market is given by the identity

$$NX(t) = \bar{Y}(t) - C(t) - I_K(t) - I_A - G(t) \tag{34}$$

In which $NX(t)$ are net exports, $\bar{Y}(t)$ is GDP, $C(t)$ is aggregate consumption, $I_K(t)$ and $I_A(t)$ are, respectively, investment in physical and incorporeal capital, while G(t) is government consumption. We chose to write the identity for NX, because the trade balance is the variable that will be determined in this market's equilibrium. Aggregate consumption is the cross-generational integration of individual consumption,

$$C(t) = \int_{-\infty}^{t} L_v(t)c_v(t)dv \tag{35}$$

Related to the fact that one unit of raw capital can be transformed into one raw unit of durable goods, there is an aggregate resource constraint that relates the total measure of raw physical capital $K(t)$ with the total measure of the quantity of durables. Like the output of the final goods sector, the price of raw physical capital goods is normalized to one at each point in time. Then, the value of the physical capital stock is,

$$K(t) = \int_0^{A(t)} X(i,t)di = \int_0^{A(t)} X(t)di = A(t)X(t) \tag{36}$$

And the change in the physical capital stock is,

$$\dot{K}(t) = \dot{A}(t)X(t) + A(t)\dot{X}(t) \tag{37}$$

The investment in physical capital is given by,

$$I_K(t) = \dot{K}(t) + \delta_x K(t) \tag{38}$$

Where δ_x is the depreciation rate of physical capital that was used in the maximization problem of the durable goods sector. The investment in incorporeal capital equals the value of the production of the research sector:

$$I_A(t) = P_A(t)\dot{A}(t) \tag{39}$$

Taking (33) and (39) we can rewrite the expression for the trade balance as:

$$NX(t) = Y(t) - C(t) - I_K(t) - G(t) \tag{40}$$

5.3 Capital market

This market equalizes liabilities to assets. There are four assets: claims to the domestic stock of physical and incorporeal capital, government bonds and foreign assets.

The incorporeal capital stock consists of the accumulated quantity of patents, and its value is given by the aggregation of their prices by the total quantity of ideas. We postulate the existence of a secondary market for patents. Since the price of a patent is the expected future net cash flow generated by the production of the durables and since this cash flow at any moment in time is equal for all of them, this means that the price of all patents in the secondary market is the same and will be equal to the price of the patents of the most recent invented varieties. Hence, the value of the incorporeal capital stock is

$$\int_0^{A(t)} P_i(t)di = \int_0^{A(t)} P_A(t)di = P_A(t)A(t) \tag{41}$$

And the investment in this stock is given by $P_A(t)\dot{A}(t)$.

The equilibrium in the capital market is given by:

$$F(t) + \bar{A}(t) = K(t) + P_A(t)A(t) + D(t) \tag{42}$$

In which $F(t)$ is the net foreign asset position, $D(t)$ are the value of the treasury bonds and $\bar{A}(t)$ are the consumer's private savings,

$$\bar{A}(t) = \int_{-\infty}^{t} L_v(t)a_v(t)dv \tag{43}$$

$F(t)$, the net foreign asset position, can be positive or negative. When it is positive, the economy is a net debtor, as it means that private savings are not enough to finance the acquisition of all assets in the economy. A negative sign means that the economy is a net creditor. This is an economy with perfect capital mobility, with the international interest rate determining the return of all forms of capital. As there are two international interest rates, the net foreign asset position has two parts,

$$F(t) = F_1(t) + F_2(t) \tag{44}$$

With two interest rates we need to specify which interest rate applies to each asset:

Assumption 2

The return on $K(t)$ *is* r_k

The return on $P_A(t)A(t)$ *is* r_k

The return on $F_1(t)$ *is* r_k

The return on $\bar{A}(t)$ *is* r

The return on $D(t)$ *is* r

The return on $F_2(t)$ *is* r

The usual result that the mortality rate premium for individual savings disappears at the aggregate level is shown in Appendix A. The model is not completely identified yet, as we need to determine how much of the net foreign position is F_1 and F_2. We need to introduce a further assumption,

Assumption 3

$$F_2(t) = D(t)$$

This means that government bonds will be sold only to non-residents and that the quantity of $F_2(t)$ is only the value of the government bonds issued. Then $F_2(t)$ will be always non-negative, while $F_1(t)$ can have any sign.

The net foreign asset position is related to the trade balance by:

$$\dot{F}_1(t) + \dot{F}_2(t) = r_k F_1(t) + rF_2(t) - NX(t) \tag{45}$$

Plus there is an external position solvency that needs to be met

$$\lim_{\tau \to +\infty} \left(F_1(\tau)e^{-r_k(\tau-t)} + F_2(\tau)e^{-r(\tau-t)} \right) \leq 0 \tag{46}$$

5.3.1 From the capital market to the consumer

There are three effects flowing from the capital market to the consumer. One of them are the dividends from the monopolist sector. The other is the change in the value of the stock of incorporeal capital. Finally, due to the fact that consumers optimize taking in consideration the interest rate r, there is an extra return from applying savings in assets that render a return r_k. All these effects enter the flow $(d(t))$, that each consumer receives from the aggregate economy, stated in the consumer balance sheet equation of Guerra et al. (2018).

The way we determine this flow is through a stock-flow consistency exercise. In Appendix B we show that if we derive (42), taking into consideration (44) and plug it in (45), we will arrive at

$$d(t) = \frac{\dot{P}_A(t)A(t) + tp(t)P_A(t)\dot{A}(t) + (P_A(t)A(t) + K(t) - F_1(t))\,(r_k - r)}{L(t)} \tag{47}$$

5.4 General equilibrium

We are now in a position to give a precise definition of the general equilibrium of the model. Our equilibrium does not have the properties usually associated with balanced growth paths. Constant growth rates will not occur because the population is not in a demographic steady state.

Definition 4 (General equilibrium) *A general equilibrium for this small open economy is achieved when consumers and each of the production sectors solve their optimization problems, equilibrium in the labor, goods and capital markets is obtained and the government and foreign sector solvency conditions are met:*

Demographic block

$L_v(t)$ *and* $m_v(t)$ *(the instantaneous mortality rate) are exogenous and come from the demographic block.*

Consumers optimize (Solved in Guerra et al. (2018))

Individual consumption $c_v(t)$
Individual human capital $h_v(t)$
Labor effort $s_v^w(t)$
Learning effort $s_v^h(t)$
Pension accumulator $\pi_v(R_v)$
Consumer assets $a_v(t)$

Equilibrium of durable goods

$$\begin{aligned} &\bar{p} && \longrightarrow (7) \\ &Y(t) && \longrightarrow (23) \\ &\pi_Y(t) && \longrightarrow 0 \\ &X(t) && \longrightarrow (24) \\ &K(t) && \longrightarrow (36) \\ &\pi(t) && \longrightarrow (9) \end{aligned}$$

Equilibrium of ideas

$$\begin{aligned} &P_A(t) && \longrightarrow (14) \\ &A(t) && \longrightarrow \textit{Integration of } (10) \\ &\pi_A(t) && \longrightarrow 0 \end{aligned}$$

Government

$$
\begin{aligned}
&G(t) &&\longrightarrow \textit{ Exogenous}\\
&Z_0(t) &&\longrightarrow (16)\\
&Z_l(t) &&\longrightarrow (17)\\
&\Pi(t) &&\longrightarrow (18)\\
&P(t) &&\longrightarrow (19)\\
&E(t) &&\longrightarrow (20)\\
&B(t) &&\longrightarrow (15)\\
&D(t) &&\longrightarrow \textit{Integration of } (21)
\end{aligned}
$$

Equilibrium of labor market

$$
\begin{aligned}
&w(t) &&\longrightarrow (26)\\
&H(t) &&\longrightarrow (27)\\
&H_Y(t) &&\longrightarrow (31)\\
&H_A(t) &&\longrightarrow (29)
\end{aligned}
$$

Equilibrium of aggregate goods market

$$
\begin{aligned}
&\bar{Y}(t) &&\longrightarrow (33)\\
&NX(t) &&\longrightarrow (40)\\
&C(t) &&\longrightarrow (35)\\
&I_K(t) &&\longrightarrow (38)\\
&I_A(t) &&\longrightarrow (39)
\end{aligned}
$$

Equilibrium of capital market

$$
\begin{aligned}
&\bar{A}(t) &&\longrightarrow (43)\\
&F(t) &&\longrightarrow (42)\\
&F_1(t) &&\longrightarrow (44)\\
&d(t) &&\longrightarrow (47)
\end{aligned}
$$

Solvency
Consumers are solvent
Government is solvent: (22)
Country is solvent internationally: (46)

6 Results

Most of the model's results are obtained numerically, which is done in Pereira (2018). In the current paper we present the analytical results that we obtained. In the following, we skip the time index, we will use the letter g to refer to growth rates and for notational convenience, we define a variable for the real marginal productivity of researchers:

$$\mathrm{RMP} = \left[\xi A^{\phi_A} + \zeta A\left(T-1\right)^{+}\right] \tag{48}$$

We recall that $T = A^*/A$, measures the distance to frontier.

The growth of ideas is given by

$$g_A = H_A\left[\xi A^{\phi_A - 1} + \zeta\left(T-1\right)^{+}\right] \tag{49}$$

Other relevant growth rates are:

$$g_w = \frac{1-\alpha\gamma}{\gamma-\alpha\gamma} g_A \tag{50}$$

$$g_{H_Y} = g_A - g_{RMP} \tag{51}$$

$$g_{P_A} = g_\pi = g_X = \frac{1-\alpha\gamma}{\gamma(1-\alpha)} g_A - g_{RMP} \tag{52}$$

$$g_K = \frac{1-2\alpha\gamma+\gamma}{\gamma(1-\alpha)} g_A - g_{RMP} \tag{53}$$

$$g_Y = \frac{1-2\alpha\gamma+\gamma}{\gamma(1-\alpha)} g_A - g_{RMP} \tag{54}$$

All these growth rates are a function of the growth in the stock of innovations and on the productivity in research, the latter is given by

$$g_{RMP} = g_A \frac{\left[\xi\phi_A A^{\phi_A} + \zeta A\left(T\frac{g_{A^*}}{g_A} - 1\right)\right]}{\left[\xi A^{\phi_A} + \zeta A\left(T-1\right)^{+}\right]} \tag{55}$$

Regarding GDP, we take (33) and simplify it. For this, is helpful to use the expressions for wages in both sectors:

$$\begin{aligned}
\bar{Y} &= Y + P_A\dot{A} \\
&= \frac{wH_Y}{1-\alpha} + wH_A \\
&= \frac{w}{1-\alpha}\Big(H_Y + (1-\alpha)H_A\Big) \\
&= \frac{w}{1-\alpha}\Big(H - \alpha H_A\Big) \\
&= \frac{(1-\alpha)\mu\left(\frac{\mu\alpha}{\bar{p}}\right)^{\frac{\alpha}{1-\alpha}} A^{\frac{1-\alpha\gamma}{\gamma(1-\alpha)}}}{1-\alpha}\Big(H - \alpha H_A\Big) \\
&= \mu\left(\frac{\mu\alpha}{\bar{p}}\right)^{\frac{\alpha}{1-\alpha}} A^{\frac{1-\alpha\gamma}{\gamma(1-\alpha)}}\Big(H - \alpha H_A\Big)
\end{aligned} \tag{56}$$

This shows that GDP level can be written solely as a combined function of the current stock of disembodied knowledge and aggregate human capital supplied to the market. We notice here, that human capital applied to the research sector displays an intertemporal trade-off. An increase in this variable decreases current GDP but will increase the stock of ideas and, consequently, future GDP.

6.1 The accounting effect

For notational convenience, we define $c_A = H_A/H$, the proportion of human capital allocated to research. The GDP growth rate is:

$$\begin{aligned} g_{\bar{Y}} &= g_w + \frac{\dot{H} - \alpha \dot{H}_A}{H - \alpha H_A} \\ &= \frac{1-\alpha\gamma}{\gamma - \alpha\gamma} g_A + \frac{\dot{H} - \alpha \dot{H}_A}{H - \alpha H_A} \\ &= \frac{1-\alpha\gamma}{\gamma - \alpha\gamma} g_A + \frac{\frac{\dot{H}}{H} - \alpha \frac{\dot{H}_A H_A}{H_A H}}{\frac{H}{H} - \alpha \frac{H_A}{H}} \\ &= \frac{1-\alpha\gamma}{\gamma - \alpha\gamma} g_A + \frac{g_H - \alpha c_A g_{H_A}}{1 - \alpha c_A} \end{aligned} \tag{57}$$

We have two sources of growth, disembodied knowledge and human capital.

Notice that if human capital in research grows at the same rate as total human capital, which would be the case of the economy on a balanced growth path, then the GDP growth rate would be

$$g_{\bar{Y}_{bgp}} = \frac{1-\alpha\gamma}{\gamma - \alpha\gamma} g_A + g_H \tag{58}$$

Although a balanced growth path in this model, can never exist while a demographic transition occurs, it is still useful to analyze the above result, as g_{H_A} cannot permanently diverge from g_H. It will be sometimes lower and sometimes higher, but a permanent divergence will mean we will eventually reach either $H_Y = 0$ or $H_A = 0$. The former is ruled out by the model because it requires $A = 0$, while the latter will be a temporary solution. We will have $g_A = 0$, but g_{RMP} will be positive due to catching up term, hence $g_{H_Y} < 0$. This decrease will ensure that, eventually, we will have again human capital employed in the research sector.

Total human capital H is defined as an integral on a product of the labor force population with time supplied to the labor market and individual human capital. To gain further insight, we may consider the approximation $g_H \approx g_{L_f} + g_{s^w} + g_h$ with L_f, s^w, h representing labor force, individual labor supply, and individual human capital respectively. Then,

$$g_{\bar{Y}_{bgp}} \approx \frac{1-\alpha\gamma}{\gamma - \alpha\gamma} g_A + g_{L_f} + g_{s^w} + g_h \tag{59}$$

Or, for the case of GDP per capita

$$g_{\bar{Y}_{bgppc}} \approx \frac{1-\alpha\gamma}{\gamma - \alpha\gamma} g_A + g_{L_f} + g_{s^w} + g_h - n$$

Where n represents the growth rate of the population. GDP per capita is affected directly by the difference in growth rates between the labor force and the population. To highlight the difference we use a measure of the dependency ratio with is total population over labor force: $dr = L/L_f$, and $g_{dr} = n - g_{L_f}$. Then:

$$g_{\bar{Y}_{bgppc}} \approx \frac{1-\alpha\gamma}{\gamma-\alpha\gamma} g_A + g_{s^w} + g_h - g_{dr} \tag{60}$$

This negative influence on GDP per capita by the deterioration of the dependency ratio is the accounting effect.[12] It suggests that the impact of demographic changes on the economy will be stronger in the period where the relative size of the labor force decreases to a new, lower, level. Once it stabilizes at this new level, the direct effect disappears.

We focused on the situation where $g_H = g_{H_A}$ but now we turn again to the general case. Starting with (57), using the approximation $g_{H_A} \approx g_{L_{f_A}} + g_{s^w} + g_h$ and with $L_{f_A} = c_A L_F$ denoting the population employed in the research sector, we get:

$$\begin{aligned} g_{\bar{Y}_{pc}} &= \frac{1-\alpha\gamma}{\gamma-\alpha\gamma} g_A + \frac{g_H - \alpha c_A g_{H_A}}{1-\alpha c_A} - n \\ &\approx \frac{1-\alpha\gamma}{\gamma-\alpha\gamma} g_A + g_{s^w} + g_h - \frac{\alpha c_A g_{c_A}}{1-\alpha c_A} - g_{dr} \end{aligned} \tag{61}$$

Which is the same expression as (60) with an extra term.

The above expressions have the advantage of highlighting the negative effect of aging on growth per capita, even abstracting from problems like social security sustainability and the policy responses they will require. Aging decreases growth per capita if there is not a sufficiently high increase in labor supply or human capital accumulation by consumers. What we termed as the behavioral effect of aging, the reaction of the consumer to a longer life expectancy, will show not only indirectly through g_A but directly through g_{s^w} and g_h, and has to be sufficiently positive in order to compensate for the deterioration of the dependency ratio. Otherwise, per capita growth will slow down.

In this analysis, we have been using an approximation for g_H, that relies on averages. In reality, g_H and consequently g_{H_A} will also depend on changes in the age composition of the labor force, which effect cannot be ignored, since both individual human capital and labor force are age-dependent. This means that the effect of the deterioration of the dependency ratio will not show in such a clear-cut way in the expression for GDP per capita growth.

It is not possible to pin down exactly analytical effects of the change in age composition on the GDP growth rate. However, we know that human capital is concave and will be decreasing on the latter part of the working phase of the life-cycle. Also, for many of parameterizations we tried, labor supply is concave on age, with a decrease at older ages. This means that, when the demographic transition unfolds and older cohorts start to have more weight in the population, there will be a relative increase of cohorts with lower human capital and lower labor supply. This will tend to make the aging effect on growth more negative than the one implied only by

[12] Gruescu (2007) does an analysis of the relation between the dependency ratio and GDP per capita growth for the Solow (1956) and Lucas (1988) models.

the dependency ratio unless people react to a longer life expectancy by working more and/or accumulating more human capital.

6.2 The impact of treating R&D expenses as investment

Since the analysis of population aging raises interest in transitional dynamics models, we want to highlight that for our horizontal innovation model, if we restricted the analysis to BGP, it would not make any difference whether we treated R&D expenses as an investment or not. However, the same cannot be said for a transitional dynamics analysis. To show this point, we compute the GDP growth rate, for an economy at the technological frontier and in the case of a BGP. If R&D expenses are considered intermediate consumption, expression (54) applies. If we follow The System of National Accounts 2008, the expression to use is (58). To make the exposition more clear we will assume that $\gamma = 1$. Also, we use $g_H = g_{H_A}$. For the economy at the frontier:

$$
\begin{aligned}
g_{Y^*} &= 2g_{A^*} - g_{RMP^*} \\
g_{\bar{Y}^*} &= g_{A^*} + g_{H^*}
\end{aligned}
$$

For that economy, the catching up term vanishes from expression (48). We have $g_{RMP^*} = \phi_A g_{A^*}$ and $g_{Y^*} = (2 - \phi_A)g_{A^*}$. Now, g_{Y^*} and $g_{\bar{Y}^*}$ still differ, but for a BGP analysis, which requires g_{A^*} to be constant, we will have $g_{A^*} = g_{H^*}/(1 - \phi_A)$.[13] Then, for the BGP case:

$$
\begin{aligned}
g_{Y^*} &= \frac{(2 - \phi_A)g_{H^*}}{1 - \phi_A} \\
g_{\bar{Y}^*} &= \frac{g_{H^*}}{1 - \phi_A} + g_{H^*} = \frac{(2 - \phi_A)g_{H^*}}{1 - \phi_A} \\
&\Rightarrow g_{Y^*} = g_{\bar{Y}^*}
\end{aligned}
$$

And, if we use the same approximation as before, the per capita growth rate is:

$$
g_{Y^*{}_{pc}} = g_{\bar{Y}^*_{pc}} \approx \frac{(2 - \phi_A)(g_h + g_{s^w} + g_{L_f})}{1 - \phi_A} - n = \frac{(2 - \phi_A)(g_h + g_{s^w})}{1 - \phi_A} + \frac{g_{L_f}}{1 - \phi_A} - g_{dr}
$$

However, in a transitional dynamics model, g_Y and $g_{\bar{Y}}$ are different, as we cannot use the implication that g_A has to be constant to simplify the expressions. The growth of A still depends on the growth of the labor force, and a change in the dependency ratio will still appear $g_{Y_{pc}}$, but $g_{Y_{pc}}$ and $g_{\bar{Y}_{pc}}$ will have different dynamics.

[13] This is a familiar result that appears first in Jones (1995), although in that paper, there is no human capital, just labor

6.3 An insight into growth policy

The growth rate of ideas can be written as a function of the proportion of human capital used in research.

$$
\begin{aligned}
g_A &= \frac{H_A RMP}{A} = (H - H_Y)\frac{RMP}{A} \\
&= \frac{r_k(1-\alpha)}{(1-t_p)(1-\alpha\gamma)\alpha}\frac{c_A}{(1-c_A)}
\end{aligned}
\tag{62}
$$

Naturally, an increase in the proportion of human capital employed in research increases the growth rate of ideas. This will lead to an increase in the growth rate of GDP per capita, in the case of expression (60). For the more general expression (61) we cannot exclude some transient effects of a different sign. Then, any policy that increases the equilibrium allocation c_A is growth enhancing and we arrive at the conclusion that the optimal taxation on profits is setting $t_p = 0$.[14] This is shown below.

Using the previous expression we can rewrite (58) as:

$$
g_{\bar{Y}_{bgp}} = \frac{r_k}{(1-t_p)\alpha\gamma}\frac{c_A}{(1-c_A)} + g_H
$$

We use now $c_A = 1 - H_Y/H$ to bring back H_Y into the equation because it depends in t_p. With the help of expression (31) we get:

$$
\begin{aligned}
g_{\bar{Y}_{bgp}} &= \frac{r_k}{(1-t_p)\alpha\gamma}\frac{1-H_Y/H}{H_Y/H} + g_H \\
&= \frac{r_k H}{(1-t_p)\alpha\gamma H_Y} - \frac{r_k}{(1-t_p)\alpha\gamma} + g_H \\
&= \frac{H(1-\alpha\gamma)RMP}{\gamma(1-\alpha)A} - \frac{r_k}{(1-t_p)\alpha\gamma} + g_H
\end{aligned}
$$

Then, assuming that feedback effects of t_p on H and g_H are small, we have

$$
\frac{\partial g_{\bar{Y}_{bgp}}}{\partial t_p} < 0
$$

Policies that increase g_A and g_H are growth enhancing. The growth rate of ideas increases if one increases the quantity of human capital allocated to research. This can be achieved by increasing c_A and this is where setting a low corporate tax will help. Intuitively, a reduction of the corporate tax increases the value of patents, making research more productive, and leading to a higher labor productivity.

Also, policies that increase the growth rate of aggregate human capital will promote growth. A policy to increase labor effort can only have a temporary effect, as labor effort, for obvious reasons, cannot grow without bound. An increase in individual human capital can have a lasting effect, because of the human capital externality we included in the model. It will increase

[14] For the general case, where $g_H \neq g_{H_A}$ we cannot obtain a non-ambiguous sign for the effect of t_p on growth, but our calculations suggest the same effect for the parameters and variables values we use in this thesis.

human capital in the future by making future time dedicated to studying more productive.

7 Discussion

In this paper, we built a horizontal innovation endogenous growth model and use it to study an economy facing demographic changes. The model is to be applied to a small open economy and research depends positively on the distance to the technological frontier. An assumption, made for simplicity, is that agents have myopic expectations. We introduced a government sector with enough detail to simulate changes in tax policy and in the pension system. Finally, we take into consideration new developments in national accounting that has implications on how GDP is calculated.

We obtained some relevant analytical results. The model can only display a balanced growth path in the case of a demographic steady-state, the case where the age composition of the population does not change. This is what we are interested in studying. Nevertheless is useful to discuss some of the results for this case.

With the change in national accounting conventions, the output of the research sector enters directly into GDP. We show that for a model of transitional dynamics, treating R//D expenses as an investment will lead to different dynamics. Also, the expression for GDP growth will depend directly, and not only via the growth of A, on the growth rate of aggregate human capital, which depends on the growth rate of labor supply and individual human capital. There are physical limits to the growth of labor supply, but not in the growth of individual human capital, because we introduced a human capital externality. This externality can increase the productivity of accumulating human capital without a theoretical limit. In this way, the model has two engines of growth: R&D and human capital. For the general case, the growth rate depends also on the difference in growth rates between total human capital and human capital employed in research.

GDP per capita growth is affected negatively by the difference in the rates of growth of labor force and of the total population in the period where the weight of the labor force decreases to a new, lower, level. The result we obtained here shows that the biggest impact on GDP per capita growth should be during the period where labor force grows at a lower rate than the population unless it is compensated by some other effect.

Naturally, growth depends positively on the proportion of human capital employed in research. This is evident for the theoretical BGP case although for the more general case there may exist some transient effects. This means that any policy that increases the proportion of human capital in research will be growth enhancing. Under some assumptions, a decrease in the corporate tax improves growth. For the same level of aggregate human capital, it decreases the human capital applied in the production of final goods. By decreasing the corporate tax, patents will be more valuable and research activities more profitable making the research sector to hire more human capital.

Regarding human capital, the other engine of growth, the partial equilibrium analysis in Guerra et al. (2018) suggests that a decrease in the labor tax income promotes both higher individual human capital and higher labor supply, an increase in the mandatory schooling has effects in labor supply but does not change lifetime human capital. An increase in the retirement age increases human capital and labor effort at the extensive margin. A decrease in the interest rate or an increase in the quality of education (measured by a lower δ_h) seem also positive but the former is not controlled by the government and manipulation of the latter is outside the scope of our model. However, it still needs to be seen if these results will remain in a general equilibrium setup.

Appendix

A Time derivative of aggregate consumer savings

The holdings of financial assets (savings) of the consumer are represented by $a_v(t)$. The differential equation governing its variation may be written in a simplified way as $\dot{a}_v(t) = (r + m_v(t))a_v(t) + x_v(t, S_v, R_v)$ in which we include in $x_v(t, S_v, R_v)$ all the remaining variables of that affect the flow of income of the consumers.

Aggregate savings are given by $\int_v^t L_v(t)a_v(t)dv$. We shall represent aggregate savings by $\bar{A}(t)$. Then, using Leibnitz's rule, the accumulation of $\bar{A}(t)$ is given by:

$$\dot{\bar{A}}(t) = L_t(t)a_t(t) + \int_v^t \dot{L}_v(t)a(t) + L_v(t)\dot{a}_v(t)dv$$

We define

$$L_v(t) = L_v(v)e^{-\int_v^t m_v(\theta)d\theta}$$

Then,

$$\dot{L}_v(t) = -m_v(t)L_v(v)e^{-\int_v^t m_v(\theta)d\theta} = -m_v(t)L_v(t) \tag{63}$$

And the accumulation of aggregate household savings becomes,

$$\begin{aligned}\dot{\bar{A}}(t) &= L_t(t)a_t(t) + \int_v^t \left[n(t)L_v(t) - (n(t) + m_v(t)L_v(t)\right]a_v(t) + L_v(t)\dot{a}_v(t)dv \\ &= L_t(t)a_t(t) + \int_v^t \left[n(t)L_v(t) - \big(n(t) + m_v(t)\big)L_v(t)\right]a_v(t) + \\ &+ L_v(t)\left[\big(r + m_v(t)\big)a_v(t) + x_v(t, S, T_R)\right]dv\end{aligned}$$

Collecting terms we arrive at

$$\begin{aligned}\dot{\bar{A}}(t) &= L_t(t)a_t(t) + \int_v^t rL_v(t)a_v(t) + L_v(t)x_v(t, S_v, R_v)dv \\ &= r\bar{A}(t) + L_t(t)a_t(t) + \int_v^t L_v(t)x_v(t, S_v, R_v)dv\end{aligned}$$

The mortality risk premium disappears at the aggregate level when we integrate cross-sectionally the generations. In our case, since there are no bequests, $(a_t(t) = 0)$ the expression simplifies to

$$\dot{\bar{A}}(t) = r\bar{A}(t) + \int_{v}^{t} L_v(t)x_v(t, S_v, R_v)dv \tag{64}$$

B Flows from the capital market to the consumer

Here we compute the consistency requirement between the net asset foreign stock position and its flow. It will determine the quantity, (that we call $d(\tau)_{*t}$), in the consumer's balance sheet that represents influences from the aggregate economy, other than the wage rate and the human capital externality.

We start with (42) and (44),

$$F_1(t) + F_2(t) = K(t) + P_A(t)A(t) + D(t) - \bar{A}(t) \tag{65}$$

The time derivative of this expression is

$$\dot{F}_1(t) + \dot{F}_2(t) = \dot{K}(t) + \dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) + \dot{D}(t) - \dot{\bar{A}}(t)$$

From (45) we have

$$\dot{F}_1(t) + \dot{F}_2(t) = r_k F_1(t) + rF_2(t) - NX(t)$$

Putting the two expressions together we get

$$\dot{K}(t) + \dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) + \dot{D}(t) - \dot{\bar{A}}(t) = r_k F_1(t) + rF_2(t) - NX(t)$$

Substituting in (64):

$$\dot{K}(t) + \dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) + \dot{D}(t) - r\bar{A}(t) - \int_{v}^{t} L_v(t)x_v(t, S_v, R_v)dv =$$
$$= r_k F_1(t) + rF_2(t) - NX(t)$$

We now substitute $NX(t)$ and by use of (38) we simplify to

$$\dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) + \dot{D}(t) - r\bar{A}(t) - \int_{v}^{t} L_v(t)x_v(t, S_v, R_v)dv =$$
$$= r_k F_1(t) + rF_2(t) - Y(t) + C(t) + \delta_x K(t) + G(t)$$

We use (21) to get rid of $\dot{D}(t)$

$$\dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) + r\left[D(t) - \bar{A}(t) - F_2(t)\right] - r_k F_1(t) + Y(t) - \delta_x K(t) =$$
$$= B(t) + \int_{v}^{t} L_v(t)x_v(t, S_v, R_v)dv + C(t) + G(t) \tag{66}$$

We now focus on simplifying the right hand side of the above expression. $\int_v^t L_v(t)x_v(t,S_v,R_v)dv$ is the cross-generational integration of the variables in the balance sheet of the consumer (see Guerra et al. (2018)) that are not the return on assets. It includes consumption, the lump sum tax, financial flows from the aggregate economy, the educational grant, net labor income and net pensions:

$$\int_v^t L_v(t)x_v(t,S_v,R_v)dv = -C(t) - Z_0(t) + \bar{d}(t) + E(t) + (1-z_l)w(t)H(t) + (1-z_p)P(t)$$

with $\bar{d}(t) = \int_v^t L_v(t)d(t)dv = d(t)L(t)$. $B(t)$ is given by (15). So the right hand side of (66) becomes:

$$\begin{aligned}
&Z_0(t) + z_l w(t)H(t) + t_p(t)\Pi(t) - (1-z_p(t))P(t) - E(t) - G(t) - C(t) - Z_0(t)+\\
&\bar{d}(t) + E(t) + (1-z_l)w(t)H(t) + (1-z_p)P(t) + C(t) + G(t)\\
=&t_p(t)\Pi(t) + \bar{d}(t) + w(t)H(t)
\end{aligned}$$

Plugging this back into (66)

$$\begin{aligned}
&\dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) + r\left[D(t) - \bar{A}(t) - F_2(t)\right] - r_kF_1(t) + Y(t) - \delta_x K(t) =\\
&= t_p(t)\Pi(t) + \bar{d}(t) + w(t)H(t)\\
\Leftrightarrow&\bar{d}(t) = \dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) + r\left[D(t) - \bar{A}(t) - F_2(t)\right] - r_kF_1(t) + Y(t) - \delta_x K(t)-\\
&\quad - t_p(t)\Pi(t) - w(t)H(t)
\end{aligned}$$

Now we can use the fact that $w(t)H(t) = w(t)H_Y(t) + w(t)H_A(t)$ and that, due to perfect competition in the final goods and research sector, there is zero profit in those sectors, to simplify further the above expression. From the zero profit condition we get,

$$\begin{aligned}
&Y(t) - w(t)H_Y(t) - \bar{p}A(t)X(t) = 0 \Leftrightarrow \bar{p}K(t) = Y(t) - w(t)H_Y(t)\\
&P_A(t)\dot{A}(t) - w(t)H_A(t) = 0
\end{aligned}$$

Then,

$$\bar{d}(t) = \dot{P}_A(t)A(t) + r\left[D(t) - \bar{A}(t) - F_2(t)\right] - r_kF_1(t) + \bar{p}K(t) - \delta_x K(t) - t_p(t)\Pi(t)$$

Using (65), we change the term inside square brackets:

$$\begin{aligned}
\bar{d}(t) &= \dot{P}_A(t)A(t) + r\left[F_1(t) - K(t) - P_A(t)A(t)\right] - r_kF_1(t) + \bar{p}K(t) - \delta_x K(t) - t_p(t)\Pi(t)\\
&= \dot{P}_A(t)A(t) - rP_A(t)A(t) - K(t)\left[r + \delta_x - \bar{p}\right] - F_1(t)\left[r_k - r\right] - t_p(t)\Pi(t)
\end{aligned}$$

From (18) we obtain the expression for the corporate tax revenue. It will be useful to use the following transformation:

$$\begin{aligned}
-t_p(t)\Pi(t) &= -t_p(t)\left(\pi(t)A(t) - P_A(t)\dot{A}(t)\right)\\
&= (1-t_p(t))\pi(t)A(t) - \pi(t)A(t) + t_p(t)P_A(t)\dot{A}(t)\\
&= \frac{r_k(1-t_p(t))\pi(t)A(t)}{r_k} - (1-\alpha\gamma)\bar{p}X(t)A(t) + t_p(t)P_A(t)\dot{A}(t)\\
&= r_kP_A(t)A(t) - (1-\alpha\gamma)\bar{p}K(t) + t_p(t)P_A(t)\dot{A}(t)
\end{aligned}$$

We use this result in the main expression

$$
\begin{aligned}
\bar{d}(t) = {} & \dot{P}_A(t)A(t) + t_p(t)P_A(t)\dot{A}(t) + P_A(t)A(t)\,[r_k - r] - K(t)\,[r + \delta_x - \bar{p} + (1-\alpha\gamma)\bar{p}] - \\
& - F_1(t)\,[r_k - r] \\
= {} & \dot{P}_A(t)A(t) + t_p(t)P_A(t)\dot{A}(t) + [r_k - r]\left[P_A(t)A(t) - F_1(t)\right] - \\
& - K(t)\,[r + \delta_x - \bar{p} + (1-\alpha\gamma)\bar{p}] \\
= {} & \dot{P}_A(t)A(t) + t_p(t)P_A(t)\dot{A}(t) + [r_k - r]\left[P_A(t)A(t) - F_1(t)\right] - K(t)\,[r + \delta_x - \alpha\gamma\bar{p}] \\
= {} & \dot{P}_A(t)A(t) + t_p(t)P_A(t)\dot{A}(t) + [r_k - r]\left[P_A(t)A(t) - F_1(t)\right] - K(t)\left[r + \delta_x - \alpha\gamma\frac{r_k + \delta_x}{\alpha\gamma}\right] \\
= {} & \dot{P}_A(t)A(t) + t_p(t)P_A(t)\dot{A}(t) + [r_k - r]\left[K(t) + P_A(t)A(t) - F_1(t)\right] \qquad (67)
\end{aligned}
$$

Therefore, the flow of income coming from the aggregate economy and affecting each consumer at a given point in time will be $d(t) = \bar{d}(t)/L(t)$.

Expression (67) contains the dividends from the monopolistic sector, the financial return on the secondary market for patents and the last term is the gain that arises from the fact that the consumer optimizes at rate r but will receive an extra return from applying at the rate r_k in the physical and incorporeal capital stock and in the F_1 part of the net foreign asset position, if the economy runs a net creditor position. If it is a net debtor, will only receive the extra return from the physical and incorporeal capital for the part not financed by non-residents.

The way we model the corporate tax means that at the current time, consumers receive a subsidy related to the investment in research. Notice that the residents own both capital stocks. The dividends and the gains/losses on the secondary market for patents are given by

$$
\begin{aligned}
& (1 - t_p(t))\left(\pi(t)A(t) - P_A(t)\dot{A}(t)\right) + \dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) \\
= & (1 - t_p(t))\pi(t)A(t) - P_A(t)\dot{A}(t) + t_p(t)P_A(t)\dot{A}(t) + \dot{P}_A(t)A(t) + P_A(t)\dot{A}(t) \\
= & (1 - t_p(t))\pi(t)A(t) + t_p(t)P_A(t)\dot{A}(t) + \dot{P}_A(t)A(t) \\
= & \frac{r_k(1 - t_p(t))\pi(t)A(t)}{r_k} + t_p(t)P_A(t)\dot{A}(t) + \dot{P}_A(t)A(t) \\
= & \frac{r_k(1 - t_p(t))\pi(t)A(t)}{r_k} + t_p(t)P_A(t)\dot{A}(t) + \dot{P}_A(t)A(t) \\
= & r_k P_A(t)A(t) + t_p(t)P_A(t)\dot{A}(t) + \dot{P}_A(t)A(t)
\end{aligned}
$$

The first term is already accounted for in the differential equation of individual assets in Guerra et al. (2018), so only the other two terms need to be considered in $d(t)$. Notice that if we didn't have any corporate taxes, the only adjustment we would need to do would be $\dot{P}_A(t)A(t)$, the price effect on the value of the stock of ideas.

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