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SURVEY PAPER

FRACTIONAL INTEGRALS AND DERIVATIVES: MAPPING PROPERTIES

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Abstract

This survey is aimed at the audience of readers interested in the information on mapping properties of various forms of fractional integration operators, including multidimensional ones, in a large scale of various known function spaces.

As is well known, the fractional integrals defined in this or other forms improve in some sense the properties of the functions, at least locally, while fractional derivatives to the contrary worsen them. With the development of functional analysis this simple fact led to a number of important results on the mapping properties of fractional integrals in various function spaces.

In the one-dimensional case we consider both Riemann-Liouville and Liouville forms of fractional integrals and derivatives. In the multidimensional case we consider in particular mixed Liouville fractional integrals, Riesz fractional integrals of elliptic and hyperbolic type and hypersingular integrals. Among the function spaces considered in this survey, the reader can find Hölder spaces, Lebesgue spaces, Morrey spaces, Grand spaces and also weighted and/or variable exponent versions.

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Key Words and Phrases: mapping properties, fractional integral, Riesz potential, hypersingular integrals, fractional derivatives, Lebesgue spaces, variable exponent Lebesgue spaces, Hölder spaces, variable exponent Hölder spaces, Grand spaces, Morrey spaces

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1. **Introduction: Pre-history**

This survey is aimed at the audience of readers interested in the information on the mapping properties of various forms of fractional integration operators, including multidimensional ones, in a large scale of various known function spaces.

As is well known, the fractional integrals defined in this or other forms improve in some sense the properties of the functions, at least locally, while fractional derivatives to the contrary worsen them. With the development of functional analysis this simple fact led to a number of important results on the mapping properties of fractional integrals in various function spaces. By mapping properties we mean the following: given a certain space of functions X, how can we characterize a possible choice of another space of functions Y so that the fractional integral I^α is bounded from X to Y? An ideal situation would be to choose such a space Y which coincides with the range of $I^\alpha(X)$. The first results in this directions are due to G.H. Hardy and J.E. Littlewood. They proved the following two important results for the integral

$$I^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,\mathrm{d}t.$$

The first concerns *integrability properties of functions* and the second is related to *continuity properties of functions*. The first result states that I^α_{a+} is bounded from $L^p(a,b)$ to $L^q(a,b)$, where $-\infty \le a < b \le \infty$, $0 < \alpha < 1/p$ and $1/q = 1/p - \alpha$ (obviously one can take $1/q \ge 1/p - \alpha$ when both a and b are finite and only $1/q = 1/p - \alpha$ when one of them is infinite; the "pre-limiting case" $1/q > 1/p - \alpha$ was treated by them earlier, see [30, 31]). The second result concerns fractional integrals of functions in Hölder space

$$H^\lambda_0(a,b) = \left\{ f \in H^\lambda(a,b) : f(a) = 0 \right\},$$

where

$$H^\lambda(a,b) = \left\{ f : |f(x+h) - f(x)| \le ch^\lambda \right\}$$

with $-\infty < a < b < \infty$ and $0 < \lambda < 1$. They proved that I^α_{a+} is a bounded operator from H^λ_0 to $H^{\lambda+\alpha}_0$ provided that $\lambda + \alpha < 1$.

These two basic results had an extremely important impact on the development of various fields in mathematical analysis in general, and led to many generalizations in applications.

Note that a typical difference between the mapping properties of fractional integration with respect to spaces of integrable or continuous functions is the following. Let X_γ be a set of function spaces depending on the parameter γ (Lebesgue spaces L^p, $\gamma = p$, being a standard example). Then the mapping property

$$I^\alpha(X_\gamma) \subset X_\nu,$$

holds with some $\nu = \nu(\gamma, \alpha)$, but usually $I^\alpha(X_\gamma) \neq X_\nu$ whatever ν one takes. In the case X_γ is a space of continuous functions the situation when $I^\alpha(X_\gamma) = X_\nu$, to the contrary is very typical, see for instance (2.7), (2.8) and Theorem 2.3.

During almost a century after that, dozens of functions spaces were introduced and studied, influenced by various applications. Of course in the related development of the fractional calculus, there naturally appeared a inter-mixture of both of the results of Hardy and Littlewood, when both the integrability and continuity properties are involved (in the case of, e.g., Nikolsky and Besov spaces).

In this brief survey it is impossible to overview all the known mapping properties of fractional integrals in a big variety of spaces, specially in the case of many variables where there is a big number of various notions of fractional integrals and the branch theory of function spaces, but we will survey the more important results, at least from our point of view. We will pay a special attention to the mapping properties of fractional integrals in the spaces with variable exponents, such as $L^{p(\cdot)}$, $H^{\lambda(\cdot)}$ and others. For the variable exponents, the theory of which was extensively developed during the last two decades, we refer for instance to the books [8, 11, 53, 54]. We overview results in the real analysis and do not touch fractional integrals in complex analysis, in particular fractional integrals in spaces of analytic functions. For some results in complex analysis case, we refer to the book [46] and [96, §23.2]. In the sequel we use the notation adopted in the book [96].

2. **Constant exponent case**

Everywhere in the sequel the weighted Lebesgue space $L^p(\Omega, w)$ is used in form

$$L^p(\Omega, w) = \left\{ f : \int\limits_\Omega |f(x)|^p w(x) dx < \infty \right\}$$

with the naturally introduced norm.

2.1. **On improving integrability properties of functions.**

2.1.1. **One-dimensional case.** The first generalization is due to G.H. Hardy and J.E. Littlewood themselves. It concerns integrability properties of functions with power weights and reads as follows, where $\rho^\mu := (x - a)^\mu$.

THEOREM 2.1 ([30]). *Let* $\alpha > 0$, $-\infty < a < b \leq \infty$, $p \geq 1$, $\mu < p-1$, $0 < \alpha < m + 1/p$, $q = p/[1-(\alpha - m)p]$; *and* $0 \leq m \leq \alpha$ *if* $p \neq 1$ *and* $0 < m \leq \alpha$ *if* $p = 1$. *Then the operator* I^{α}_{a+} *is bounded from* $L^p([a,b],\rho^{\mu})$ *into* $L^q\left([a,b],\rho^{(\mu/\rho - m)q}\right)$.

The case of power weight $\rho(x) = |x-d|^{\mu}$ with $a < d \leq b$, important for some applications, can be found in [96].

The corresponding mapping results, non-weighted and also with power weights, for the Erdélyi-Kober fractional integrals (see [96, p. 322]) maybe found in [14, 47]. Note that within the frameworks of power weights the $L^p \to L^q$-mapping properties of Erdélyi-Kober operators are immediately reduced to such properties of the usual fractional integrals by means of the obvious change of variables. We do not touch here generalizations of fractional integrals where the kernel $(x-y)^{\alpha-1}$ is replaced by some special function. In particular, when it is the Gauss hypergeometric function, some results of such a kind can be found in [96, §9.2 and §23.2], and for the case with a G- or H-function in the kernel, there are analogous ones in Kiryakova [46] (for example, Th.5.1.3), based on the Hardy-Littlewood general result ([96, Th.1.5]).

The above results concern the case $p < 1/\alpha$. When $0 < \alpha - 1/p < 1$, and a and b are finite, then $I^{\alpha}_{a+} : L^p(a,b) \longrightarrow H^{\alpha - \frac{1}{p}}(a,b)$ which was known to Hardy and Littlewood, see [30]. A generalization of such a fact to the case of power weights was obtained in Karapetyants and Rubin [37] and can be also found in the book [96, Th.3.8].

In the intermediate case $p = 1/\alpha$ the fractional integral of L^p-functions in general do not belong to L^{∞} but are known to belong to the space BMO of functions of bounded mean oscillation. This fact is well-known and in the multidimensional case for the Riesz fractional integrals over $\mathbb{R}^n$ was noted in [108]. For a direct proof in the one-dimensional case on a finite interval we refer to [76]. Certainly, $I^{\alpha}(L^p) \neq BMO$. In [38, 36, 39] there were found spaces more narrow than BMO. Namely, for $\lambda > 0$ let

$$X_{\gamma} = \left\{ f : \|f\|_{X_{\gamma}} < \infty \right\}, \quad \|f\|_{X_{\gamma}} = \sup_{r \leq 1} r^{-\gamma} \|f\|_{L^r},$$

then $I^{\alpha} : L^p \to X_{\gamma}$ is bounded for $\gamma \geq 1/p'$, and is not bounded if $\gamma < 1/p'$. Note that $X_{\gamma} \not\subseteq BMO$ and $BMO \not\subseteq X_{\gamma}$ when $0 < \gamma < 1$. In [38, 39] more narrow spaces of such a kind maybe also found.

Note also the famous result of G.H. Hardy and J.E. Littlewood [30] for the operator

$$A^{\alpha} f(x) := \frac{1}{x^{\alpha}} \frac{1}{\Gamma(\alpha)} \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}} \, \mathrm{d}t, \quad 0 < \alpha < 1,$$

stating that $\|A^\alpha f\|_{L^p(\mathbb{R}_+)} \leq K\|f\|_{L^p(\mathbb{R}_+)}$, $1 < p < \infty$. This is a particular case of the Schur theorem for integral operators with a kernel homogeneous of degree -1 (see references in the sequel for the multidimensional case). Observe that the above inequality holds with sharp constant $K = (\Gamma(1/p')/\Gamma(\alpha + 1/p'))^p$.

2.2. **On improving integrability properties: the multidimensional case.**

2.2.1. **The case of Riesz fractional integral.** A direct extension of the Hardy-Littlewood theorem concerns the iterated fractional integral

$$I^{\bar{\alpha}}_{++}f(x) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int\limits_{-\infty}^{x_1}\int\limits_{-\infty}^{x_2} \frac{f(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2}{(x_1-y_1)^{1-\alpha_1}(x_2-y_2)^{1-\alpha_2}}, \quad x=(x_1,x_2), \tag{2.1}$$

where $\bar{\alpha} = (\alpha_1, \alpha_2)$, $\alpha_i > 0$, $i = 1, 2$. It is natural to study such integrals in the mix norm spaces $L^{\bar{p}}(\mathbb{R}^2)$ defined by the norm

$$\|f\|_{\bar{p}} = \left\{ \int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}} |f(y_1,y_2)|^{p_2}\,\mathrm{d}y_2 \right)^{\frac{p_1}{p_2}} \mathrm{d}y_1 \right\}^{\frac{1}{p_1}}.$$

The following theorem is valid, see [96, Theorem 24.1].

Theorem 2.2. *Let $1 \leq p_i < \infty$, $1 \leq q_i < \infty$. The operator $I^{\bar{\alpha}}_{++}$ is bounded from $L^{\bar{p}}(\mathbb{R}^2)$ into $L^{\bar{q}}(\mathbb{R}^2)$ if and only if*

$$1 < p_i < 1/\alpha_i, \quad 1/q_i = 1/p_i - \alpha_i, \quad i = 1, 2.$$

Let

$$I^\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y, \quad 0 < \alpha < n, \tag{2.2}$$

be the Riesz fractional integral (we do not write the standard normalizing constant). As a generalization of the one-dimensional Hardy-Littlewood result, we have

$$\|I^a f\|_q \leq C\|f\|, \quad 1 < p < n/\alpha, \quad 1/q = 1/p - \alpha/n, \tag{2.3}$$

which was proved by S.L. Sobolev [105] and is known as the *Sobolev inequality* and the exponent q defined in (2.3) is known as the *Sobolev exponent.* The proof may be found in various books, see for instance [106]. Observe that this n-dimensional result may be derived (see [72]) from the one-dimensional result of G.H. Hardy and J.E. Littlewood, since $|x-y|^n \geq$

$n^{\frac{n}{2}} \prod_{i=1}^{n} |x_i - y_i|$ which is nothing else but the statement that the geometric mean is dominated by the arithmetic mean. The weighted version of the Sobolev inequality with power weights of the form $|x|^\beta$ (or $(1+|x|)^\beta$) is due to E. Stein and G. Weiss [107]. We mention also the boundedness

$$\left\| \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \,\mathrm{d}y \right\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n)}, \quad \alpha > 0, \quad 1 < p < \infty$$

which is a particular case of $L^p \to L^p$-boundedness for multidimensional integral operators with a kernel homogeneous of degree $-n$. For the latter we refer to [41], see also its presentation in the book [42]. Note also that the sharp constant for the last inequality is known, see [91].

In the limiting case where $\alpha p = n$, the Riesz potential I^α maps $L^p(\mathbb{R}^n)$ into the space BMO of functions of finite mean oscillations

$$\|I^a f\|_{BMO} \leq C\|f\|_p, \tag{2.4}$$

where

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \,\mathrm{d}x, \ f_B = \frac{1}{|B|} \int_B f(x) \,\mathrm{d}x$$

is the integral average of the function f, and the operator I^α on functions $f \in L^p$ is treated as a continuation from a dense set in L^p.

There is also known the complete characterization of weights $w(x)$ such that

$$\|wI^\alpha f\|_q \leq c\|wf\|_p \tag{2.5}$$

known as the *Muckenhoupt-Wheeden condition* obtained in [66], denoted by $A_{p,q}$ and defined by the condition

$$\sup_{B\subset\mathbb{R}^n} \left(\frac{1}{|B|} \int_B w(x)^q \,\mathrm{d}x \right)^{\frac{1}{q}} \left(\frac{1}{|B|} \int_B w(x)^{-p'} \,\mathrm{d}x \right)^{\frac{1}{p'}}.$$

It is known that

$$w \in A_{pq} \equiv w^{-p'} \in A_{1+\frac{p'}{q}},$$

where A_p is a more widely known Muckenhoupt class defined by the condition

$$\sup_{B\subset R^n} \left(\frac{1}{|B|} \int_B w(t) \,\mathrm{d}t \right) \left(\frac{1}{|B|} \int_B w(t)^{-\frac{1}{p-1}} \,\mathrm{d}t \right)^{p-1} \leq c.$$

In the applications the case of radial weights $w = w(|x - x_0|)$, $x \in \mathbb{R}^n$ is important. Note that for such weights the Muckenhoupt condition is reduced to the easily checkable condition

$$\left(\int_0^r t^{n-1} w(t)\,\mathrm{d}t\right)\left(\int_0^r t^{n-1} w(t)^{-\frac{1}{p-1}}\,\mathrm{d}t\right)^{p-1} \le Cr^{np}.$$

Note also that the A_p condition is guaranteed by the assumptions

$$\int_0^r t^{n-1} w(t)\,\mathrm{d}t \le cr^n w(r), \quad \int_0^r t^{n-1} w(t)^{-\frac{1}{p-1}}\,\mathrm{d}t \le Cr^n w(r)^{-\frac{1}{p-1}},$$

known as *Zygmund-type conditions*. The power weight $w = |x|^\gamma$ is in A_{pq} with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ if and only if

$$\alpha - \frac{n}{p} < \gamma < \frac{n}{p'}.$$

The Sobolev theorem with such a weight was proved by E. Stein and G. Weiss [107]. For the so-called two-weight estimates

$$\|wI^\alpha f\|_q \le C\|vf\|_p\,, \tag{2.6}$$

we refer to [103, 104].

The first generalization for Orlicz spaces is due to R. O'Neil [69], for more general and more precise result we refer to [6].

We do not touch in detail results for fractional integrals in Lorentz spaces, the reader can find them for instance in [3, 7, 28, 49, 48, 57], see also references therein.

2.2.2. **Other forms of multidimensional fractional integration: hyperbolic and parabolic fractional integrals.** The hyperbolic potential is defined by

$$I^\alpha_\square \varphi = \frac{1}{H_n(\alpha)} \int_{K^+_+} \frac{\varphi(x-y)}{r^{n-\alpha}(y)}\,\mathrm{d}y, \quad \alpha > n-2,$$

where $K^+_+ = \left\{x : x_1^2 \ge x_2^2 + \cdots + x_n^2, x_1 \ge 0\right\}$, $r(y)$ is the so-called Lorentz distance $r(y) = \sqrt{x_1^2 + \cdots +_p^2 - x_{p+1}^2 - \cdots - x_n^2}$ and the normalizing constant $H_n(\alpha)$ is given by $H_n(\alpha) = 2^{\alpha-1}\pi^{-1+n/2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+2-n}{2}\right)$. Like the Riesz fractional integral, these operators also have a semigroup property $I^\alpha_\square I^\beta_\square \varphi = I^{\alpha+\beta}_\square \varphi$. The result on the $L^p \to L^q$ boundedness of the operator $I^\alpha_\square$ in the case $n = 2$, with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2}$ may be found in [96, p. 562], see also [44, 45].

The *parabolic fractional integral*, known also as the *heat fractional integral* is defined by

$$(H^\alpha\varphi)(x,t) = \frac{1}{\Gamma(\alpha/2)}\int\limits_0^\infty \tau^{\frac{\alpha}{2}-1}(W_\tau\varphi)(x,t-\tau)\,\mathrm{d}\tau,$$

where $W_\tau\varphi$ is the Gauss–Weierstrass operator

$$(W_\tau\varphi)(x,t) = (4\pi\tau)^{-n/2}\int\limits_{\mathbb{R}^n} \exp\left(-|\xi|^2/(4\tau)\right)\varphi(x-\xi,t)\,\mathrm{d}\xi,$$

and it also enjoys the semigroup property. The operator H^α is $L^p(\mathbb{R}^{n+1}) \to L^q(\mathbb{R}^{n+1})$ bounded when $1 < p < (n+2)/\alpha$ and $1/q = 1/p - \alpha/n + 2$, see [16]. Mapping properties in Lebesgue spaces were obtained in [79, 58], and for Morrey type spaces – in [23, 24, 25, 20, 26, 21].

2.3. **On improving continuity properties of functions.**

2.3.1. **One-dimensional case.** As mentioned in Introduction,

$$\left\|I^\alpha_{0+}f\right\|_{H^{\lambda+\alpha}} \le C\|f\|_{H^\lambda}, \quad \lambda > 0, \quad \lambda+\alpha < 1$$

for all $f \in H^\lambda_0([0,a])$, $0 < a < \infty$ which was proved in [30]. Moreover, it proves to be that

$$I^\alpha(H^\lambda_0) = H^{\lambda+\alpha}_0, \tag{2.7}$$

see also details of the proof in [96, Th.13.13]. Even more, it holds with power type weights where

$$H^\lambda_0(\varrho) = \left\{f : \varrho f \in H^\lambda_0\right\}$$

and $\varrho = \prod_{k=1}^n |x-x_k|^{\mu_k}$, $0 = x_1 < x_2 < \cdots < x_n = a$ and $0 \le \mu_1 < \lambda+1$, $\alpha < \mu < \lambda+1$, $k = 2,\cdots,n$ as proved in [78] and presented in [96, Th. 13.13] (for a shorter proof see [43]). Note also that for the case $\lambda = 0$ the following is true:

$$I^\alpha : C([0,a]) \longrightarrow H^\alpha([0,a])$$

but the inverse statement for the fractional derivative does not hold. There holds also the corresponding statements for Hölder spaces on $(0,\infty)$ if these spaces are considered with a weight fixed to infinity. We will mention to such a kind of statements in a more general multidimensional case, but refer to [96, §5.2] for some one-dimensional results.

Within the frameworks of continuous functions, more precise results are also known in terms of modulus of continuity. Let $H^\omega_0 = H^\omega_0([0,a])$ be a generalized Hölder space of functions $f(x)$ with a given non-decreasing positive dominant $\omega(h)$, $h > 0$ of their modulus of continuity:

$$\omega(f,h) \le C\omega(h),$$

where $\omega(f,h) = \sup_{|x-y|<h} |f(x) - f(y)|$ and such that $f(0) = 0$. This is a Banach space with respect to the norm

$$\|f\|_{H_0^\omega} = \sup_{h>0} \frac{\omega(f,h)}{\omega(h)}.$$

The following statements are valid:

THEOREM 2.3. *Let $0 < \alpha < 1$ and $\omega(t)$ satisfy the conditions $\int\limits_0^h \frac{\omega(t)}{t}\,\mathrm{d}t \leq C\omega(h)$ and $\int\limits_h^a \frac{\omega(t)}{2-\alpha}\,\mathrm{d}t \leq C\frac{\omega(h)}{h^{1-\alpha}}$. Then the operator I_{a+}^α maps the space H_0^ω isomorphically onto the space $H_0^{w_\alpha}$ where $w_\alpha(h) = h^\alpha w(h)$.*

The proof is given originally in [67] and presented in [96, §13.6]. Similar results hold also for generalized Hölder spaces $H_0^\omega(\varrho)$ with power weights:

$$I_{0+}^\alpha \left(H_0^\omega(\varrho)\right) = H_0^\omega(\varrho), \tag{2.8}$$

see [96, §13.6].

We refer also to [40], where the assumptions on ω and ϱ for the validity of (2.8) are given in easy to check numerical inequalities in terms of the so-called Matuzsewska-Orlicz indices.

2.3.2. **Multidimensional case.** In the multidimensional case similar statements for the fractional type integrals

$$I_\Omega^\alpha f(x) = \int\limits_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y$$

over an open set $\Omega \subseteq \mathbb{R}^n$, with a characterization of the range $I_\Omega^\alpha(H_0^\omega)$, where the index 0 means that one considers functions vanishing on the boundary $\partial\Omega$ of Ω, seem to be studied only in the case $\Omega = \mathbb{R}^n$, in the weighted setting. This is due to the fact that $\mathbb{R}^n$ may be mapped onto a unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ by means of the so-called stereographic projection, see [90, §6.2] on this projection (check also p. 596 on this survey). Moreover, under this projection the Riesz potential operator I^α over $\mathbb{R}^n$ is reduced, up to some weights, to potential operator over $\mathbb{S}^n$. In other words, the study of Hölder type behavior of functions on $\mathbb{R}^n$ is reduced to the case of functions on a compact set. This approach was used in a number of papers. We do not enter into the analysis of the results obtained there, but refer to [115, 116, 85, 84, 113, 110], see also references therein. For an arbitrary bounded set Ω there are only known mapping properties of the type:

$$I_\Omega^\alpha : H_0^\omega(\Omega) \longrightarrow H^{\omega_\alpha}(\Omega)$$

as proved in [12] under some additional assumptions on the geometry of Ω and in [93] in the general case. Note also that in [93] such a statement was proved in a general setting of quasi-metric measure spaces instead of $\mathbb{R}^n$.

3. **Fractional integrals in non-standard function spaces**

In this section we mainly consider mapping properties of integrable functions, but also touch mapping properties of continuous functions in § 3.3.

Studies of various operators of harmonic analysis, including fractional order operators, are nowadays well-known in the "variable exponent setting". The latter means that the parameters defining the operator and/or the space (which usually are constant), may vary from point to point. Nowadays there exists a vast field of research known as *Variable Exponent Analysis*, we refer to the books [8, 11, 53, 54]. In this survey we concentrate ourselves on mapping properties of fractional integrals.

Note that in this section we will mainly deal with mapping properties of a fractional operator from a given space X to some other space Y, but not onto Y. The results of the type "onto" are known only in the case where α is constant.

3.1. **Variable Exponent Lebesgue Spaces.**

3.1.1. **One-dimensional Case.** We will define variable exponent Lebesgue spaces in general in the multidimensional case, but in this section we deal with one-dimensional results which have some specific one-dimensional nature.

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p : \Omega \to \mathbb{R}_+$ be a measurable function such that $p_- := \inf_{x\in\Omega} p(x) \geq 1$ and $p_+ := \sup_{x\in\Omega} p(x) < \infty$. The space $L^{p(\cdot)}(\Omega)$ is defined as the space of functions with the finite norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{f(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\}. \tag{3.1}$$

We will often refer to the conditions

$$|p(x) - p(y)| \leq \frac{A}{\ln\frac{1}{|x-y|}},\ x, y \in \Omega,\ |x-y| \leq \frac{1}{2} \tag{3.2}$$

and to

$$|p(x) - p(\infty)| \leq \frac{A}{\ln(e + |x|)},\ x \in \Omega,$$

known as the *local log-condition* and *decay condition*, respectively. The latter is imposed when Ω is unbounded.

First we note that the case of constant p but variable $\alpha(x)$ was studied in [87], where under some assumptions on $\alpha(x)$ and p the continuous embedding

$$\mathbb{D}_+^{\alpha(\cdot)} I_+^{\alpha(\cdot)}(L^p(\mathbb{R})) \hookrightarrow L^p(\mathbb{R})$$

was obtained. Here,

$$I_+^{\alpha(\cdot)} f(x) = \frac{1}{\Gamma(\alpha(x))} \int_{-\infty}^{x} \frac{f(t)}{(t-x)^{1-\alpha(x)}} \, \mathrm{d}t$$

and

$$\mathbb{D}_+^{\alpha(\cdot)} f(x) = \frac{\alpha(x)}{\Gamma(1-\alpha(x))} \int_{-\infty}^{x} \frac{f(x)-f(t)}{(x-t)^{1+\alpha(x)}} \, \mathrm{d}t$$

and $0 < \inf \alpha(x)$, $\sup \alpha(x) < 1$.

In [75], under the assumption of log-continuity of the variable exponent, it was obtained a characterization of the range of the one-dimensional Riemann-Liouville fractional integral operators $(0 < \alpha < 1)$

$$I_{a+}^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} \, \mathrm{d}t, \; I_{b-}^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}} \, \mathrm{d}t \quad (3.3)$$

over weighted Lebesgue spaces $L^{p(\cdot)}[(a,b),\rho]$ in terms of convergence of the corresponding hypersingular integrals $\mathbb{D}_{a+}^{\alpha} f$ defined by

$$\mathbb{D}_{a+}^{\alpha} f(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}} \, \mathrm{d}t.$$

It was shown that the ranges of the operators (3.3) coincide under some natural assumptions. Necessary and sufficient conditions for a function f to belong to this range are also given. It was also proved that the range coincide with a Sobolev type space $L^{\alpha,p(\cdot)}[(a,b),\rho]$, defined as the space of restrictions onto Ω of functions in the space of Bessel potentials, viz.

$$L^{\alpha,p(\cdot)}[(a,b),\rho] = \mathcal{B}^{\alpha}\left[L^{\tilde{p}(\cdot)}(\mathbb{R},\tilde{\rho})\right]\Bigg|_{(a,b)},$$

where $\tilde{p}$ and $\tilde{\rho}$ are appropriate extensions of p and ρ respectively, see the details in [75].

Due to the one-sided nature of the Riemann-Liouville integrals (variable limit of integration), in contrast to the case of constant limits like in the case of potential type fractional integrals, it is possible to use weaker assumptions in the variable exponent $p(x)$.

We say that an exponent p belongs to the class $\mathcal{P}_{-}^{\log}(I)$ if there exists a positive constant c_1 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < x - y \leq 1/2$, the inequality

$$p(x) \leq p(y) + \frac{c_1}{\ln(1/(x-y))} \tag{3.4}$$

holds. Further, we say that p belongs to $\mathcal{P}_{+}^{\log}(I)$ if there exists a positive constant c_2 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < y - x \leq 1/2$, the inequality

$$p(x) \leq p(y) + \frac{c_2}{\ln(1/(y-x))} \tag{3.5}$$

holds.

We assume that $I = [0, b)$, where $0 < b \leq \infty$. Let

$$I_{0+}^{\alpha(\cdot)} f(x) = \int_0^x f(t)(x-t)^{\alpha(x)-1} dt, \qquad x \in (0, b),$$

$$I_{b-}^{\alpha(\cdot)} f(x) = \int_x^b f(t)(t-x)^{\alpha(x)-1} dt, \qquad x \in (0, b),$$

where $0 < \alpha(x) < 1$.

The following results regarding, see [62, 102], the aforementioned operators were obtained when $I = \mathbb{R}_+$.

THEOREM 3.1. *Let $I = \mathbb{R}_+$ and let $1 < p_-(I) \leq p_+(I) < \infty$.*

(a) *Let $p \in \mathcal{P}_{+}^{\log}(I)$. Suppose that there exists a positive constant a such that $p \in \mathcal{P}_\infty((a, \infty))$. Suppose that α is a constant on I, $0 < \alpha < \frac{1}{p_+(I)}$ and $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then $I_{-}^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

(b) *Suppose that $p \in \mathcal{P}_{-}^{\log}(I)$. Let α be a constant on I, $0 < \alpha < \frac{1}{p_+(I)}$ and let $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Suppose that $p \in \mathcal{P}_\infty((a, \infty))$ for some positive number a. Then $I_{0+}^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Similar results for finite bounded intervals are also known, viz.

THEOREM 3.2. *Let $I := [0, b]$ be a bounded interval and $1 < p_-(I) \leq p_+(I) < \infty$.*

(a) *Assume that $p \in \mathcal{P}_{+}^{\log}(I)$, $0 < \alpha_-(I)$ and that $(\alpha p)_+(I) < 1$. Suppose that $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then $I_{b-}^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

(b) *Let $p \in \mathcal{P}_{-}^{\log}(I)$. Suppose that $0 < \alpha_-(I)$. Assume also that $(\alpha p)_+(I) < 1$ and $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then $I_{0+}^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

The conditions $\mathcal{P}_-^{\log}$ and $\mathcal{P}_+^{\log}$ are really weaker: in [102] the reader can find examples of the exponents satisfying these conditions for which the fractional integral with constant limits of integration is not bounded in $L^{p(\cdot)}([0,b])$.

3.1.2. **Multidimensional Case.** In this subsection and in the following, we use the following notation:

$$I^{\alpha(\cdot)}f(x) = \frac{1}{\gamma_n(\alpha(x))}\int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\,\mathrm{d}y, \quad \alpha(x)>0, \tag{3.6}$$

for the Riesz potential of variable order preassuming that $\alpha(x)$ nowhere vanishes. If $\inf \alpha(x) > 0$ and $\sup \alpha(x) < n$, the factor $\frac{1}{\gamma_n(\alpha(x))}$ is inessential for the study of the mapping properties of the operator; recall that in the case of constant α the presence of this factor was important for the validity of the semigroup property $I^\alpha I^\beta = I^{\alpha+\beta}$), which is no more expected for variable orders.

An interesting question relates to the admission of the order $\alpha(x)$ which may be degenerate at some points. Then we have to study mapping properties of $I^{\alpha(\cdot)}$ in these or other function spaces, taking into account the degeneracy of the order $\alpha(x)$. Note that $\frac{1}{\gamma_n(\alpha)} \to 0$ as $\alpha \to 0$, so that the presence of normalizing factor $\frac{1}{\gamma_n(\alpha(x))}$ equivalent to $\frac{\alpha(x)}{|\mathbb{S}^{n-1}|}$ as $\alpha(x)\to 0$, is of importance when we admit a possibility for $\alpha(x)$ to degenerate. Then we expect that the operator with this normalizing factor will behave as the identity operator at the points of degeneracy.

Similarly, the corresponding variable order hypersingular integral (written for the case $0<\alpha(x)<1$) is:

$$\mathbb{D}^{\alpha(\cdot)}f(x) = \int\limits_{\mathbb{R}^n} \frac{f(x)-f(x-y)}{|y|^{n+\alpha(x)}}\,\mathrm{d}y,$$

where for simplicity we omit the normalizing factor.

Fractional integrals in variable exponent Lebesgue spaces. The first known result for the fractional integral concerns bounded sets (see [89, 88]) in $\mathbb{R}^n$ and reads as follows:

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and p satisfy* (3.1). *Let also $\inf_{x\in\Omega}\alpha(x) > 0$ and $\sup_{x\in\Omega}\alpha(x)p(x) < n$. Then the operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ with $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$.*

The proof of this theorem in the above stated form may be found in [53, Theorem 2.50].

It is known that such a mapping property is no more valid for variable $\alpha(x)$ when Ω is unbounded (as explicitly shown in [32]), but remains valid when α is constant under the additional decay condition (3.2) on $p(x)$ as proved in [5].

However, Sobolev type theorem on $\mathbb{R}^n$ with variable $\alpha(x)$ holds in a modified form, either with additional weight related to infinity

$$\left\|(1+|x|)^{-\gamma(x)} I^{\alpha(\cdot)} f\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\gamma(x) = A_\infty \alpha(x)\left[1 - \frac{\alpha(x)}{n}\right]$ and A_∞ comes from the decay condition, or in terms of algebraic sum of spaces:

$$\left\|I^{\alpha(\cdot)} f\right\|_{L^{q(\cdot)}(\mathbb{R}^n)+L^{q_\infty(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $1/q_\infty(x) = 1/p(\infty) - \alpha(x)/n$. The first can be found in [55] and the second in [54, Theorem 13.46] where it was given in a more general form.

In the limiting case where $\alpha(x)p(x) = n$, there holds a generalization of (2.4) to the case of bounded domains, we refer for details to [94, 95].

The weighted case. For various goals in applications, similar weighted estimates are of importance. As an extension of Theorem 3.3, the following result is valid, see [53, Corollary 2.66].

THEOREM 3.4. *Let Ω be bounded, p satisfy* (3.1), $\inf_{x\in\Omega}\alpha(x) > 0$, $\sup_{x\in\Omega}\alpha(x)p(x) < n$. *The operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(\Omega,\varrho)$ to the space $L^{q(\cdot)}(\Omega,\varrho)$ with the weight*

$$\varrho(x) = |x-x_0|^\gamma \ln^\beta \frac{D}{|x-x_0|},$$

where $1/q(x) = 1/p(x) - \alpha(x)/n$, $D > \operatorname{diam}\Omega, x_0 \in \Omega$ and $\beta \in \mathbb{R}$, if

$$\alpha(x_0) - \frac{n}{p(x_0)} < \gamma < \frac{n}{p'(x_0)}.$$

Such a weighted mapping property holds not only for power logarithmic weights, but for a more general class of weights. We do not go into details, but refer the reader to [83, 97, 101] and [53, Theorem 2.64].

3.2. Morrey Spaces.

The Morrey spaces $\mathcal{L}^{p,\alpha}$ are known to be defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0}\left(\frac{1}{r^\lambda}\int\limits_{B(x,r)} |f(y)|^p\, dy\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad 0 < \lambda < n,$$

where $B(x,r) := \{y \in \mathbb{R}^n : |y-x| < r\}$. We do not touch the so-called Campanato spaces which are extensions of the Morrey spaces to the case when $\lambda \geq n$. Correspondingly, there are known mapping properties for fractional integrals separately for $\lambda < n$ and $\lambda > n$. A unifying approach for the Campanato spaces is natural in the variable exponent setting, this is an open problem. For such spaces we refer to the books [15, 56, 117], see also the survey [74]. The mapping properties for the Riesz fractional integral within the frameworks of $\mathcal{L}^{p,\lambda}$-spaces were first obtained by S. Spanne with the Sobolev exponent $1/q = 1/p - n/\alpha$, this result was published in J. Peetre [70]. A stronger result with a better exponent $1/q = 1/p - n/(\alpha - \lambda)$ in the range $1 < p < (n-\lambda)/\alpha$ is due to D.R. Adams [1].

Let $1 < p < \infty$, $0 < \lambda < n$. By dilation arguments it is easy to show that if $\|I^\alpha f\|_{\mathcal{L}^{q,\mu}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)}$ for some $q \in (1,\infty)$ and $\mu \in (0,n)$, then $1 < p < (n-\lambda)/\alpha$, $\mu < n/p' + \lambda/p + \alpha$ and $\alpha + (n-\mu)/q = (n-\lambda)/p$. The choice $\mu = \lambda$ corresponds to the Adams result.

There are also known Spanne type results on mapping properties of the Riesz potential operator in the so-called *generalized Morrey spaces*, see [22, 29] and reference therein. We touch such generalized Morrey spaces in Section 3.2.1 in a more general setting of variable exponents. Weighted estimates for potential operators in Morrey spaces are less studied even in the case of classical Morrey spaces $L^{p,\lambda}$.

There are results on weighted estimates of potential operators in Morrey type spaces, where it was initially supposed that the weight is in the Muckenhoupt class. Such an assumption on the weight is unnatural for the Morrey spaces because the class A_p contains weights such that the maximal operator is certainly not bounded in $L^{p,\lambda}$, for every $\lambda > 0$. Indeed, in N. Samko [80] in the one-dimensional case it was shown that the Hilbert transform is bounded in $L^{p,\lambda}$ if and only if

$$\frac{\lambda - 1}{p} < \lambda < \frac{1}{p'} + \frac{\lambda}{p},$$

which is the Muckenhoupt interval $(\frac{-1}{p}, \frac{1}{p})$ shifted to the right by $\frac{\lambda}{p}$. This clearly shows that the class of weights of Muckenhoupt-Wheeden type for Morrey spaces should essentially depend on λ and of course must be different from A_{pq} class.

For weighted estimates of the Riesz potential in Morrey spaces, free of the assumption of $w \in A_p$, we refer to N. Samko [71, 81, 82].

Finally, we refer to [13] and [4, Chapter 6] with respect to fractional integrals in Besov spaces.

3.2.1. **Variable Exponent Case.** One of the possible definition of variable exponent Morrey spaces is given by the norm

$$\|f\|_{p(\cdot),\lambda(\cdot)} = \sup_{x\in\Omega,\ r>0} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)}.$$

An extension of the Adams result (obtained in [2]) on fractional integrals in Morrey spaces to the variable exponent case, for bounded domains $\Omega \subset \mathbb{R}^n$ has the form

$$\left\| I^{\alpha(\cdot)} f \right\|_{q(\cdot),\lambda(\cdot)} \leq C\|f\|_{p(\cdot),\lambda(\cdot)},$$

where $I^{\alpha(\cdot)} f(x) = \int\limits_\Omega \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy$, $1/q(x) = 1/p(x) - \alpha(x)/(n-\lambda(x))$, under the assumption that $\inf \alpha(x) > 0$ and $\inf[n - \lambda(x) - \alpha(x)p(x)] > 0$ and the usual log conditions on the $\alpha(x)$ and $p(x)$. Similar result in a more general setting of quasimetric measure spaces was proved in [50].

The generalized Morrey spaces with variable exponents are known in the literature in two forms: the spaces $\mathcal{L}^{p(\cdot),\varphi(\cdot)}$ and $\mathcal{M}^{p(\cdot),\omega(\cdot)}$ defined respectively by

$$\|f\|_{\mathcal{L}^{p(\cdot),\varphi(\cdot)}} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}$$

and

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}.$$

For Morrey spaces with constant p but a general function $\omega(x,r)$ defining the Morrey space, such results under these or those assumptions were obtained in [18, 19, 64, 68].

Mapping properties of fractional integrals in generalized Morrey spaces $\mathcal{M}^{p(\cdot),\omega(\cdot)}$ in the general case may be found in [27].

For mapping properties in the so-called Musielak-Orlicz spaces we refer to [65]. In the case of Orlicz-Morrey spaces more general results were obtained in [9], including weak-type statements, and in [10] for vanishing Orlicz-Morrey spaces.

3.3. **Hölder Type Spaces.**

3.3.1. **One-dimensional case.** Let

$$I_{0+}^{\alpha(\cdot)} f(x) = \frac{1}{\Gamma(\alpha(x))} \int\limits_0^x f(y)(x-y)^{\alpha(x)-1}\, \mathrm{d}y, \quad \alpha(x) > 0,$$

where $0 < x < a$ and $H^{\lambda(\cdot)} = H^{\lambda(\cdot)}([0,a])$ be the variable exponent Hölder space defined by the semi-norm

$$[f]_{H^{\lambda(\cdot)}} = \sup_{x,x+h\in[0,a],h>0} \frac{|f(x+h)-f(x)|}{h^{\lambda(x)}}. \tag{3.7}$$

Let also $H_0^{\lambda(\cdot)} = \{f : f \in H^{\lambda(\cdot)} \text{ and } f(0)=0\}$. As a partial generalization of (2.7) the following is valid. Let $\alpha \in H^1([0,a])$, $\inf_{x\in[0,a]} \lambda(x) > 0$ and $\sup_{x\in[0,a]} \lambda(x) + \alpha(x) < 1$. Then the operator $I_{0+}^{\alpha(\cdot)}$ maps the space $H_0^{\lambda(\cdot)}$ to $H_0^{\lambda(\cdot)+\alpha(\cdot)}$. This statement is contained in the main theorem of the paper [77, p. 782]. In the case where $\alpha = const$ but $\lambda(x)$ remains variable, such a generalization holds in a full form, i.e. $I_{0+}^{\alpha}(H_0^{\lambda(\cdot)}) = H_0^{\lambda(\cdot)+\alpha(\cdot)}$ as proved in [34, 35].

3.3.2. **Multidimensional case. Spherical fractional integration in Hölder spaces with variable exponent.** In the presentation in this section we partially follow [92]. Let $x, \sigma \in \mathbb{S}^{n-1}$ and $f(\sigma)$ be a function defined on $\mathbb{S}^{n-1}$. To introduce the spherical fractional integral of Riesz type, we may try just to copy the construction (3.6) and introduce the spherical potential operator of variable order of the function f directly as

$$\mathfrak{I}^{\alpha(\cdot)} f(x) = \frac{1}{\gamma_{n-1}(\alpha(x))} \int_{\mathbb{S}^{n-1}} \frac{f(\sigma)}{|x-\sigma|^{n-1-\alpha(x)}}\, \mathrm{d}\sigma, \quad x \in \mathbb{S}^{n-1}, \tag{3.8}$$

where $\mathrm{d}\sigma$ stands for the surface measure on $\mathbb{S}^{n-1}$ and we assume that $0 < \alpha(x) < n-1$.

It is known that the space $\mathbb{R}^n$ may be one-to-one transformed onto the n-dimensional sphere via the stereographic projection. Under this projection the spatial potential over $\mathbb{R}^n$ transforms into exactly the spherical potential over $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, up to some weight function. The stereographic projection maps the sphere $\mathbb{S}^n$ onto the space $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ via the change of variables in $\mathbb{R}^{n+1}$: $\xi = s(x) = \{s_1(x), s_2(x), ..., s_{n+1}(x)\}$, where

$$s_k(x) = \frac{2x_k}{1+|x|^2},\ k = 1,2,...,n \quad \text{and} \quad s_{n+1}(x) = \frac{|x|^2-1}{|x|^2+1},$$

$x \in \mathbb{R}^{n+1}$, $|x| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$ (see [63]). There hold the formulas: $|x-y| = \frac{2|\sigma-\xi|}{|\sigma-e_{n+1}|\cdot|\xi-e_{n+1}|}$, $dy = \frac{2^n}{|\sigma-e_{n+1}|^{2n}}\, d\sigma$, where $e_{n+1} = (0,0,...,0,1)$, which imply the relation

$$\int_{\mathbb{R}^n} \frac{\varphi(y)\, dy}{|x-y|^{n-\alpha(x)}} = 2^{\widetilde{\alpha}(\xi)} |\xi - e_{n+1}|^{n-\widetilde{\alpha}(\xi)} \int_{\mathbb{S}^n} \frac{\varphi_*(\sigma)\, d\sigma}{|\xi-\sigma|^{n-\widetilde{\alpha}(\xi)}} \tag{3.9}$$

where $\varphi_*(\sigma) = \frac{\varphi[s^{-1}(\sigma)]}{|\sigma-e_{n+1}|^{n+\widetilde{\alpha}(\xi)}}$ and $\widetilde{\alpha}(\xi) = \alpha[s^{-1}(\xi)]$.

Therefore, via the stereographic projection we can transfer many properties of spatial fractional integrals to the case of similar spherical integrals.

Problems arising in the case where $\alpha(x) = 0$ on some set (of measure zero), were first resolved for the spherical fractional operator (3.8) in [86] in the setting of variable exponent Hölder spaces $H^{\lambda(\cdot)}(\mathbb{S}^{n-1})$. The case of non-vanishing orders $\alpha(x)$ was earlier studied in [111, 112, 114] (where there was also studied the case of generalized Hölder spaces).

In [86] complex values of $\alpha(x)$ were also admitted. This more general setting together with degeneracy of $\alpha(x)$ led to a certain exclusion of purely imaginary orders $\alpha(x) = i\theta(x)$:

$$\max_{x\in\mathbb{S}^{n-1}} |\arg\alpha(x)| < \frac{\pi}{2} - \varepsilon, \quad \text{for some} \quad \varepsilon > 0. \tag{3.10}$$

Under this assumption in [86] there was proved, in particular, the statement given in Theorem 3.5, below. In that theorem, the operator $\alpha(x)\mathfrak{I}^{\alpha(\cdot)}$ at the points x_0, where $\alpha(x_0) = 0$, is interpreted as the limit $\alpha(x_0)\mathfrak{I}^{\alpha(x_0)} = \lim_{\beta\to 0}\beta\mathfrak{I}^{\beta}$. As is well known, such a limit in the case of spatial fractional integrals is the identity operator, up to a constant factor. The same holds in the case of spherical integrals.

THEOREM 3.5. *Let $\alpha \in \mathrm{Lip}(\mathbb{S}^{n-1})$ and the set $\{x \in \mathbb{S}^{n-1} : \Re\alpha(x) = 0\}$ have measure zero. The operator $\alpha(x)\mathfrak{I}^{\alpha(\cdot)}$ acts boundedly from the space $H^{\lambda(\cdot)}(\mathbb{S}^{n-1})$ into the space $H^{\lambda(\cdot)+\alpha(\cdot)}(\mathbb{S}^{n-1},\alpha)$, if $\sup_{x\in\mathbb{S}^{n-1}}[\lambda(x)+\Re\alpha(x)] < 1$.*

The above mentioned tendency to the identity operator is obviously reflected in this theorem: at the points where $\alpha(x) = 0$ there is stated only the conservation of the smoothness properties of the function f, but in general when $\Re\alpha(x) \to 0$, the limiting operator, under condition (3.10), is a singular integral operator of Calderón-Zygmund type, also preserving the smoothness properties, in general).

It is worthwhile noticing that for the corresponding spherical fractional differentiation operator

$$\mathfrak{D}^{\alpha(\cdot)}f(x) = \lim_{\varepsilon\to 0} \int\limits_{\substack{\mathbb{S}^{n-1} \\ |x-\sigma|\geq\varepsilon}} \frac{f(\sigma) - f(x)}{|x-\sigma|^{n-1+\alpha(x)}}d\sigma, \quad x \in \mathbb{S}^{n-1}, \tag{3.11}$$

where $0 < \Re\alpha(x) < 1$, a symmetrical statement holds on mapping of $H^{\lambda(\cdot)}(\mathbb{S}^{n-1})$ into $H^{\lambda(\cdot)-\alpha(\cdot)}(\mathbb{S}^{n-1})$ under the assumption that

$$\min_{x\in\mathbb{S}^{n-1}} \Re\alpha(x) > 0, \quad \max_{x\in\mathbb{S}^{n-1}} \Re\alpha(x) < 1, \quad \text{and} \quad \min_{x\in\mathbb{S}^{n-1}} \Re[\lambda(x) - \alpha(x)] > 0,$$

see Theorem 3.13 in [86] (for simplicity, we do not touch the degeneracy cases in this result).

In fact, in [86] there was obtained a more general statement on mapping properties within the frameworks of generalized Hölder spaces, defined by a prescribed dominant of the continuity modulus, see details in [86].

From the above statement for spherical fractional integrals, one can derive corresponding results for spatial fractional operators via relations of type (3.9).

3.4. **Grand Spaces.** The grand Lebesgue spaces, also known as Iwaniec-Sbordonne spaces, are denoted by $L^{p),\theta}(\Omega)$ and defined as the set of all measurable functions in the bounded set Ω for which

$$\|f\|_{L^{p),\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^\theta}{|\Omega|} \int_\Omega |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}$$

is finite. These spaces were introduced in [33] in the case of $\theta = 1$ and with general θ was given in [17]. A definition of grand Lebesgue spaces tailored for unbounded sets was given in [98, 99]. A more general approach to grand Lebesgue spaces on unbounded sets was suggested in [109].

In [59] it was studied with some detail the mapping properties of the fractional integral in the one-dimensional case. For example it was shown the dependence of the boundedness of the operator with the parameter θ, namely, it was shown that I^α is $\left(L^{p),\theta_1}([0,1]) \to L^{q),\theta_2}([0,1])\right)$-bounded when $\theta_2 > \theta_1(1+\alpha q)$ and $q = p/(1-\alpha p)$. A more general theorem was also given relating the boundedness of the operator in weighted grand Lebesgue spaces with Muckenhoupt weights, see [59] for details.

Under the approach of [109] the corresponding $\left(L^{p)}(\mathbb{R}^n) \to L^{q)}(\mathbb{R}^n)\right)$-boundedness of the Riesz fractional operator was proved in [100], where there was given an inversion of the operator I^α in vanishing grand Lebesgue spaces.

It is noteworthy to mention that the idea of grandification was also applied to the Morrey spaces. In [60, 61] it was introduced the grand Morrey space on quasi-metric measure spaces with doubling measure and it was obtained boundedness results for the generalized fractional integral operator $\mathcal{J}^\alpha$ defined by

$$\mathcal{J}^\alpha f(x) = \int_X \frac{f(y)}{\rho(x,y)^{\gamma-\alpha}} d\mu,$$

namely, it was shown that $\mathcal{J}^\alpha : \mathcal{L}^{p),\theta_1,\lambda}(X,\mu) \to \mathcal{L}^{q),\theta_2,\lambda}(X,\mu)$ is bounded when $\theta_2 = [1 + \frac{\alpha q}{(1-\lambda)\gamma}]\theta_1$, $0 < \alpha < \frac{(1-\lambda)\gamma}{p}$, $0 \le \lambda < 1/\gamma$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-\lambda)\gamma}$ and

$\|f\|_{\mathcal{L}^{p),\theta,\lambda}} = \sup_{\varepsilon,x,r}\left(\frac{\varepsilon^{\theta}}{r^{\gamma\lambda}}\int\limits_{B(x,r)}|f|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}}$. In the aforementioned paper the grandification process was applied only to the integrability parameter, but in [73] it was already applied to both parameters defining the Morrey space, the space $L^{p),\lambda)}_{\theta,\alpha}(\Omega)$ was introduced in the following way

$$\|f\|_{L^{p),\lambda)}_{\theta,\alpha}(\Omega)} := \sup_{0<\varepsilon<\max\{p-1,\frac{\lambda}{\alpha}\}} \varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_{L^{p-\varepsilon,\lambda-\alpha\varepsilon}(\Omega)}.$$

In [52] it was obtained the boundedness of Riesz type potential operators both in the framework of homogeneous and also in the nonhomogeneous cases in generalized grand Morrey spaces, see [52] for details.

A new function space unifying the grand Lebesgue spaces and the variable exponent Lebesgue spaces was introduced in [51] and denoted by *grand variable exponent Lebesgue spaces* with the notation $L^{p(\cdot),\theta}$. The authors obtain Sobolev type theorem for fractional integrals in a subspace of $L^{p(\cdot),\theta}$.

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