Programa de Doctorado "Matemáticas"

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Local distribution of Rademacher series and function spaces

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Introduction

The study of the behavior in function spaces of Rademacher series

$$\sum_{k=1}^{\infty} a_k r_k,$$

where the Rademacher functions are

$$r_k(t) := \operatorname{sign} \sin(2^k \pi t), \qquad t \in [0,1],$$

is a classical problem that has attracted much attention.

In the case of the Lebesgue spaces $L^p([0,1])$, Khintchine proved in 1923 that given any $0 < p < \infty$, there exist constants $A_p, B_p > 0$ such that

$$A_p \Big(\sum_{k \geq 1} a_k^2 \Big)^{1/2} \leq \Big\| \sum_{k \geq 1} a_k r_k \Big\|_{L^p([0,1])} \leq B_p \Big(\sum_{k \geq 1} a_k^2 \Big)^{1/2}, \tag{1}$$

for $(a_k)_1^\infty \in \ell^2$, [19]. Note, for $p = \infty$, that

$$\Big\| \sum_{k \geq 1} a_k r_k \Big\|_{L^\infty([0,1])} = \sum_{k \geq 1} |a_k|.$$

Khintchine inequality has been extended to other function spaces. Rodin and Semenov proved in 1975 that, given a rearrangement invariant space X on $[0,1]$, the inequality

$$A_X \Big(\sum_{k \geq 1} a_k^2 \Big)^{1/2} \leq \Big\| \sum_{k \geq 1} a_k r_k \Big\|_X \leq B_X \Big(\sum_{k \geq 1} a_k^2 \Big)^{1/2}$$

holds for some constants $A_X, B_X > 0$ for all $(a_k)_1^\infty \in \ell^2$ if and only if

$$G \subset X,$$

where G is the closure of L^∞ in the Orlicz space L^{M_2} generated by the Young function $M_2(t) := \exp(t^2) - 1$, [26]. This result highlights the role of G as the minimal of all rearrangement invariant spaces where Khintchine inequality holds.

A result proved in 1979 independently by Rodin and Semenov and by Lindenstrauss and Tzafriri shows that the closed subspace generated by the Rademacher functions in a rearrangement invariant space X is complemented in X if and only if the embeddings $G \subset X$ and $G \subset X'$ hold, where X' is the associate space of X, [22, Theorem 2.b.4] and [27].

Rodin and Semenov's 1975 result mentioned above shows, for rearrangement invariant spaces between G and L^1, that the subspace generated by the Rademacher functions is isomorphic to ℓ^2. For function spaces X which are "close" to L^∞ (in the sense that they are interpolation spaces between L^∞ and G), the subspace generated by the Rademacher functions has been characterized by Astashkin in 2001 by means of the K–method of interpolation: there is a one–to–one correspondence between the interpolation spaces between ℓ^1 and ℓ^2 and the subspaces generated by the Rademacher functions in the interpolation spaces between L^∞ and G, [2].

The local versions of Khintchine inequality are one of the main subjects in this dissertation. By local version we refer to inequalities of Khintchine's type, but where the functions are restricted to a measurable set $E \subset [0,1]$ of positive measure. For example, let $E := (0, 1/2^n)$. From the fact that $r_{k+n}(x) = r_k(2^n x)$ for $x \in E$, and that (r_k) is an orthonormal system, we have

$$\Big(\int_E \Big|\sum_{k\ge n+1} a_k r_k(x)\Big|^2 \frac{dx}{m(E)}\Big)^{1/2} = \Big(\int_0^1 \Big|\sum_{k\ge 1} a_{k+n} r_k(x)\Big|^2 dx\Big)^{1/2} = \Big(\sum_{k\ge n+1} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

Can this be extended to the case when E is an arbitrary set of positive measure? Zygmund proved in 1930 the first local version of Khintchine inequality: there exist constants $A_2', B_2' > 0$ such that, given a set $E \subset [0,1]$ of positive measure, there exists $N = N(E)$ such that

$$A_2'\Big(\sum_{k\ge N} a_k^2\Big)^{1/2} \le \Big(\int_E \Big|\sum_{k\ge N} a_k r_k(t)\Big|^2 \frac{dt}{m(E)}\Big)^{1/2} \le B_2'\Big(\sum_{k\ge N} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$. This result was extended to the spaces L^p by Sagher and Zhou in 1990, who proved that for any $0 < p < \infty$, the inequality

$$A_p'\Big(\sum_{k\ge N} a_k^2\Big)^{1/2} \le \Big(\int_E \Big|\sum_{k\ge N} a_k r_k(t)\Big|^p \frac{dt}{m(E)}\Big)^{1/p} \le B_p'\Big(\sum_{k\ge N} a_k^2\Big)^{1/2} \qquad (2)$$

holds for $(a_k)_1^\infty \in \ell^2$, with constants $A'_p, B'_p > 0$ depending on p.

Sagher and Zhou also proved in 1996 a local version of Khintchine inequality for the Orlicz space L^{M_1}, generated by $M_1(t) := \exp(t) - 1$ (this space is also known as the space $L_{\exp}$ of functions of exponential integrability), which states that there exist constants $A'_{M_1}, B'_{M_1} > 0$ such that, given a set $E \subset [0,1]$ of positive measure, there exists $N = N(E)$ such that

$$A'_{M_1}\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2} \le \Big\| \sum_{k\ge N} a_k r_k \Big\|_{L^{M_1}(E, dt/m(E))} \le B'_{M_1}\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$. The norm in the space $L^{M_1}(E, dt/m(E))$ is

$$\|f\|_{L^{M_1}(E,dt/m(E))} := \inf\Big\{\lambda > 0 : \int_E \big(\exp(|f(t)|/\lambda) - 1\big)\frac{dt}{m(E)} \le 1\Big\}.$$

In view of the embeddings

$$L^\infty \subset G \subset L^{M_2} \subset L^{M_1},$$

and motivated by Rodin and Semenov's theorem, we have studied the extension of Sagher and Zhou's local result for L^{M_1} to the space L^{M_2}. This is presented in Chapter 1, where we prove the extension and consequently deduce the result above for all spaces L^{M_p} with $1 \le p \le 2$ (Theorem 1.9). In particular, we obtain the result of Sagher and Zhou with a simpler proof; originally it required the dyadic BMO norm of a function of the form $\exp(\sum a_k r_k)$.

The last section of Chapter 1 is devoted to extending a result by Sagher and Zhou related to the Walsh system. This is a complete, orthonormal system consisting of all finite products of Rademacher functions. Sagher and Zhou proved in 1990 that Khintchine inequality (1) also holds in L^p for the lacunary Walsh series, that is, given $0 < p < \infty$ and a lacunary Walsh subsequence (w_{n_k}) with $n_{k+1}/n_k \ge q > 1$ for all $k \ge 1$, there exist constants $A(p,q), B(p,q) > 0$ such that

$$A(p,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \le \Big(\int_0^1 \Big|\sum_{k=1}^{\infty} a_k w_{n_k}(t)\Big|^p dt\Big)^{1/p} \le B(p,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$, [29]. Sagher and Zhou also proved in 1990 a local version of the previous result for L^p,

$$A'(p,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \le \Big(\int_E \Big|\sum_{k=1}^{\infty} a_k w_{n_k}(t)\Big|^p \frac{dt}{m(E)}\Big)^{1/p} \le B'(p,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2},$$

for $n_k \geq N$ and $(a_k)_1^\infty \in \ell^2$. We show that the local version of Khintchine inequality for lacunary Walsh series also holds for the space L^{M_2} (Theorem 1.14). We also show that Rodin and Semenov's and Lindenstrauss and Tzafriri's theorems on the subspace generated by the Rademacher functions and its complementability in a rearrangement invariant space hold for lacunary Walsh series (Theorems 1.18 and 1.20).

Given a rearrangement invariant space X on $[0,1]$ with $G \subset X$, the question arises of studying the validity or not of a local version of Khintchine inequality for X. Related to this problem is that of giving a local version of the norm in a rearrangement invariant space X where an explicit expression of the norm is not available. These problems have been considered by Astashkin and Curbera, who proved in 2015 the following result, [5]. Let φ_X denote the fundamental function of X (that is, $\varphi_X(t) := \|\chi_{[0,t]}\|_X$, $0 \leq t \leq 1$), and for f a measurable function on a set $E \subset [0,1]$ of positive measure, denote

$$\|f\|_{X(E)} := \frac{1}{\varphi_X(m(E))}\|f\chi_E\|_X.$$

Then, the inequality

$$A'_X\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq N} a_k r_k\Big\|_{X(E)} \leq B'_X\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2}$$

holds, for some $N = N(E)$, constants $A'_X, B'_X > 0$ and all $(a_k)_1^\infty \in \ell^2$, if and only if the lower dilation index γ_X of X satisfies

$$\gamma_X > 0.$$

Since the lower dilation indexes of the spaces L^{M_1} and L^{M_2} are $\gamma_{L^{M_1}} = \gamma_{L^{M_2}} = 0$, the above result does not allow local versions of Khintchine inequality of this type for the spaces L^{M_1} and L^{M_2}. In Chapter 2 we give and study a different definition of local rearrangement invariant space, which we denote by $X|E$. The norm of the space $X|E$ coincides with the norms in the local versions of Khintchine inequality by Zygmund and Sagher and Zhou for $X = L^p$, L^{M_1} and L^{M_2}. With this definition, the inequality

$$A'_X\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k r_{k+N}\Big\|_{X|E} \leq B'_X\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2}$$

holds, for some constants $A'_X, B'_X > 0$ and all $(a_k)_1^\infty \in \ell^2$, for any rearrangement invariant space X with $G \subset X$ (Proposition 2.10).

In the case when $G \subset X$, the local version of Khintchine inequality for $X|E$ above and Rodin and Semenov's theorem imply, for some $N = N(E)$ and constants $C_1, C_2 > 0$ depending on X, that

$$C_1 \Big\| \sum_{k\geq 1} a_k r_k \Big\|_X \leq \Big\| \sum_{k\geq 1} a_k r_{k+N} \Big\|_{X|E} \leq C_2 \Big\| \sum_{k\geq 1} a_k r_k \Big\|_X \tag{3}$$

for $(a_k)_1^\infty \in \ell^2$.

In Chapter 3 we address the problem of extending inequality (3) to the case when $G \not\subseteq X$. The definition of the space $X|E$ turns out to be compatible with the notion of systems equivalent in distribution. This allows the following approach: we prove, for a set $E \subset [0, 1]$ of positive measure satisfying a certain condition, that there exists $N = N(E)$ such that the system of all Rademacher functions (r_k) on $([0, 1], m)$ is equivalent in distribution to (r_{k+N}) on $(E, m/m(E))$. This result implies (3) for any rearrangement invariant space X (Theorem 3.7), and allows to give a local version of Astashkin's theorem for interpolation spaces between L^∞ and G.

A different approach to extending Khintchine inequality to the local setting would be via the independence of the Rademacher functions on the measure space $(E, m/m(E))$. We give a partial result in this regard (Theorem 3.11).

We study when the local spaces $X|E$ and $X(E)$ coincide: precisely when the fundamental function $t \mapsto \varphi_X(t)$ and the norm of the dilation operator $t \mapsto h_X(t)$ are equivalent functions (Proposition 2.12).

The Rademacher functions have been considered in 2010 in the Cesàro spaces by Astashkin and Maligranda, [8]. For $1 \leq p < \infty$, the Cesàro space $\mathrm{Ces}(p)$ consists of all functions f on $[0, 1]$ such that

$$\|f\|_{\mathrm{Ces}(p)} = \Big(\int_0^1 \Big(\frac{1}{x} \int_0^x |f(t)|\, dt \Big)^p dx \Big)^{1/p} < \infty.$$

The previous results do not apply in this case, since these spaces are not rearrangement invariant. The case of the weighted Cesàro spaces $\mathrm{Ces}(\omega, p)$ has also been considered in [8]. For $\omega(x)$ a positive weight, the space $\mathrm{Ces}(\omega, p)$ consists of all functions f on $[0, 1]$ such that

$$\|f\|_{\mathrm{Ces}(\omega,p)} := \Big(\int_0^1 \Big(\frac{1}{\omega(x)} \int_0^x |f(t)|\, dt \Big)^p dx \Big)^{1/p} < \infty.$$

Astashkin and Maligranda proved in 2010, for $1 \leq p < \infty$, that the closed subspace generated by the Rademacher functions in $\mathrm{Ces}(p)$ is isomorphic to ℓ^2 [8, Theorem 1]. They also identified the norm of a Rademacher series in $\mathrm{Ces}(\omega, p)$ in the case when $p = \infty$ and ω is a quasiconcave weight [8, Theorem 2], and proved the non complementability of the subspace generated by the Rademacher functions in $\mathrm{Ces}(p)$ and in $\mathrm{Ces}(\omega, \infty)$, [8, Theorems 5 and 6].

In Chapter 4 we extend these results to the spaces $\mathrm{Ces}(\omega, p)$ with $1 \leq p \leq \infty$ and ω an arbitrary weight. We set different conditions on the weight ω and study their relation with the subspace generated by the Rademacher functions in $\mathrm{Ces}(\omega, p)$ (Proposition 4.8). For a weight ω satisfying certain natural conditions, we prove, for $1 \leq p < \infty$, that

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \asymp \Big(\sum_{n=0}^{\infty} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^{\infty}\|_2 \Big)^p \Big)^{1/p},$$

where

$$\omega_{p,n} := \int_{1/2^{n+1}}^{1/2^n} \Big(\frac{x}{\omega(x)} \Big)^p dx, \qquad n \geq 0.$$

(Theorem 4.6). We study the case when the Rademacher functions generate in $\mathrm{Ces}(\omega, p)$ a closed subspace isomorphic to ℓ^2 (Theorem 4.15), and show that this subspace is not complemented in $\mathrm{Ces}(\omega, p)$ for $1 \leq p \leq \infty$ and a very general weight ω (Theorem 4.10).

Some of the results contained in this memoir are included in the following papers:

- J. Carrillo–Alanís, *On local Khintchine inequalities for spaces of exponential integrability*. Proc. Amer. Math. Soc., **139**(8):2753–2757, 2011, [13].
- J. Carrillo–Alanís, *Rademacher functions in weighted Cesàro spaces*. Studia Math., **217**(1):19–40, 2013, [14].
- J. Carrillo–Alanís, *Local rearrangement invariant spaces and distribution of Rademacher series*. Positivity, to appear, [15].
- J. Carrillo–Alanís, *Lacunary Walsh series in rearrangement invariant function spaces*. In preparation.

Preliminaries

Rearrangement invariant spaces

Let (R, Σ, μ) be a measure space. We denote by $L^0(\mu) := L^0(R, \Sigma, \mu)$ the set of all μ–measurable functions $f : R \to [-\infty, +\infty]$. We will assume that $\mu(R)$ is finite and (R, Σ, μ) non–atomic.

Following the presentation by Zaanen, [32, Ch. 15], a Banach function space over (R, Σ, μ) is a linear subspace X of (classes of) measurable functions in $L^0(\mu)$, endowed with a complete norm $\| \cdot \|_X$, such that $g \in X$ and $|f| \leq |g|$ a.e. implies $f \in X$ and $\|f\|_X \leq \|g\|_X$. Note that other authors use more restrictive definitions of Banach function space. For example, Lindenstrauss and Tzafriri, [22], require X' being isomorphic to X^*, and Bennett and Sharpley, [10], require X having the Fatou property.

The associate space X' of a Banach function space X consists of all measurable functions $g \in L^0(\mu)$ for which the associate functional

$$\|g\|_{X'} := \sup \left\{ \left| \int_R f g \, d\mu \right| : \|f\|_X \leq 1 \right\}$$

is finite.

A Banach function space X is saturated if for every set E with $\mu(E) > 0$ there exists $F \subset E$ such that $\mu(F) > 0$ and $\chi_F \in X$. This property is equivalent to the associate functional $\| \cdot \|_{X'}$ being a norm in X'; see [32, Ch. 15, §68, Theorem 4].

For X Banach function space, the inclusion $X' \subset X^*$ always holds between the associate space X' and the dual Banach space X^*. A Banach function space has absolutely continuous norm when order bounded increasing sequences are norm convergent. For X a Banach function space, X' is isomorphic to X^* if and only if X has absolutely continuous norm.

We will denote the distribution function of $f \in L^0(\mu)$ by

$$\mu_f(\lambda) := \mu(\{x \in R : |f(x)| > \lambda\}), \qquad \lambda > 0.$$

A Banach function space X over (R,Σ,μ) is *rearrangement invariant* (r.i.) if $\mu_f = \mu_g$ and $f \in X$ implies $g \in X$ and $\|g\|_X = \|f\|_X$. The associate space X' of a rearrangement invariant space X is also a rearrangement invariant space.

The spaces $L^p := L^p(R,\Sigma,\mu)$ are the classical example of r.i. space. These spaces consist of all measurable functions $f \in L^0(\mu)$ such that the norm $\|f\|_p$ is finite, where

$$\|f\|_p := \Big(\int_R |f|^p \, d\mu \Big)^{1/p},$$

for $1 \leq p < \infty$, and

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in R} |f(x)|$$

for $p = \infty$. For $1 \leq p \leq \infty$, the associate space of L^p is $(L^p)' = L^q$, where p and q are conjugate exponents, that is,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For $1 \leq p < \infty$, since L^p has absolutely continuous norm, the associate space $(L^p)'$ coincides with the Banach dual space $(L^p)^*$.

The decreasing rearrangement of $f \in L^0(\mu)$ is the function defined by

$$f^*_\mu(t) := \inf\{\lambda > 0 : \mu_f(t) > \lambda\}, \qquad t \in [0, \mu(R)].$$

In the case when $R = [0,1]$ and $\mu = m$ is the Lebesgue measure, we will denote by f^* the decreasing rearrangement of f.

The following version of Hölder's inequality holds for r.i. spaces:

$$\int_R |fg| \, d\mu \leq \int_0^{\mu(R)} f^*_\mu(t) g^*_\mu(t) \, dt \leq \|f\|_X \|g\|_{X'}. \tag{4}$$

for $f \in X$ and $g \in X'$.

Let X and Y be Banach function spaces over (R,Σ,μ). The space X is continuously embedded into Y, which we denote by $X \subset Y$, if there exists a constant $C > 0$ such that

$$\|f\|_Y \leq C\|f\|_X, \qquad f \in X.$$

If $X \subset Y$, then $Y' \subset X'$, with the same embedding constant.

The second associate space of X is $X'' := (X')'$. The embedding $X \subset X''$ holds for any Banach function space X. The case when $X'' = X$ is related to the Fatou property. A Banach function space X satisfies the Fatou property if $f_n \in X$ with

$\|f_n\|_X \leq M$ for all $n \geq 1$ and $0 \leq f_n \leq f_{n+1} \nearrow f$ a.e. implies that $f \in X$ and $\|f\|_X = \sup_n \|f_n\|_X$. A Banach function space X satisfies the Fatou property if and only if X'' coincides with X.

For X an r.i. space on a finite measure space (R, Σ, μ), the continuous embeddings

$$L^\infty(R, \mu) \subset X \subset L^1(R, \mu)$$

hold. The closure of L^∞ in X will be denoted by X_0. The property $(X'')_0 = X_0$ holds.

The fundamental function and the dilation operators

Let X be an r.i. space over (R, Σ, μ). Since we assume that (R, μ) is non–atomic, for every $t \in [0, \mu(R)]$ there exists $E \subset R$ such that $\mu(E) = t$. The fundamental function of X is

$$\varphi_X(t) := \|\chi_E\|_X, \qquad 0 \leq t \leq \mu(R).$$

Since X is rearrangement invariant, φ_X is well–defined. The fundamental functions of X and of the associate space X' are related by means of the equality

$$\varphi_X(t)\varphi_{X'}(t) = t, \qquad 0 \leq t \leq \mu(R).$$

The lower fundamental index of X is

$$\gamma_X := \lim_{t \to 0^+} \frac{\log M_{\varphi_X}(t)}{\log t},$$

where

$$M_{\varphi_X}(t) := \sup_{0<s,st<\mu(R)} \frac{\varphi_X(st)}{\varphi_X(s)}.$$

Let X be an r.i. space over $([0, a], m)$, where $0 < a \leq \infty$ and m is the Lebesgue measure. For each $t > 0$ and $f \in L^0(m)$, define the dilation operator

$$(\sigma_t f)(s) := \begin{cases} f(ts), & 0 \leq s \leq a/t, \\ 0, & s > a/t. \end{cases} \tag{5}$$

Then, the operator $\sigma_t : X \to X$ is bounded. The dilation function h_X is defined by

$$h_X(t) := \|\sigma_{1/t}\|_{X \to X}, \qquad t > 0.$$

The dilation function satisfies

$$h_X(t) \leq \max\{1, t\}, \qquad t > 0,$$

and the relationship between h_X and $h_{X'}$ is given by the equality

$$h_X(t) = t \cdot h_{X'}(1/t), \qquad 0 < t < \infty.$$

Orlicz spaces

A Young function is a function of the form

$$\Phi(s) = \int_0^s \phi(u)\, du, \qquad s \geq 0,$$

where $\phi : [0,+\infty) \to [0,+\infty]$ is increasing, left–continuous and $\phi(0) = 0$. The function Φ is increasing, convex, with $\Phi(0) = 0$ and $\lim_{s\to\infty} \Phi(s) = \infty$.

The Orlicz space $L^{\Phi}(R,\Sigma,\mu)$ generated by a Young function Φ consists of all measurable functions $f \in L^0(\mu)$ for which the norm

$$\|f\|_{L^{\Phi}} := \inf\left\{\lambda > 0 : \int_R \Phi(|f|/\lambda)\, d\mu \leq 1\right\}$$

is finite.

For Φ a Young function, the Orlicz space L^{Φ} is rearrangement invariant. The fundamental function of L^{Φ} is given by

$$\varphi_{L^{\Phi}}(t) = \frac{1}{\Phi^{-1}(1/t)}, \qquad 0 < t < \mu(R). \tag{6}$$

Orlicz spaces include a number of different function spaces. For $\Phi(t) = t^p$, $p \geq 1$, the norm $\|f\|_{L^{\Phi}}$ coincides with the norm $\|f\|_p$ of the Lebesgue space L^p. The Young function $M_p(t) = \exp(t^p) - 1$, $p \geq 1$, generates the space L^{M_p} of functions of exponential integrability, which consists of all functions f on (R,μ) such that the norm

$$\|f\|_{L^{M_p}} := \inf\left\{\lambda > 0 : \int_R \big(\exp(|f|/\lambda)^p - 1\big)\, d\mu \leq 1\right\}$$

is finite. The spaces of exponential integrability are of particular interest in the context of the Rademacher functions.

For details on the theory of rearrangement invariant spaces, see [10], [21] and [22].

The K–method of interpolation

Let X_0 and X_1 be Banach spaces. The couple (X_0, X_1) is a Banach couple if there exists a Hausdorff topological vector space $\mathcal{X}$ such that $X_i \hookrightarrow \mathcal{X}$, for $i = 0, 1$. In this case, the spaces $X_0 + X_1 := \{f_0 + f_1 : f_0 \in X_0, f_1 \in X_1\}$ and $X_0 \cap X_1$, endowed with the norms

$$\|f\|_{X_0+X_1} := \inf\{\|f_0\|_{X_0} + \|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\},$$
$$\|f\|_{X_0\cap X_1} := \max\{\|f\|_{X_0}, \|f\|_{X_1}\},$$

are Banach spaces. A Banach space X for which the continuous embeddings $X_0 \cap X_1 \subset X \subset X_0 + X_1$ hold is called an intermediate space. An intermediate space between X_0 and X_1 is called an interpolation space if, for every linear operator $T : X_0 + X_1 \to X_0 + X_1$ such that $T : X_0 \to X_0$ and $T : X_1 \to X_1$ are continuous, then $T : X \to X$ is continuous.

One of the main results in the theory of interpolation establishes that interpolation spaces between $L^1([0,1])$ and $L^\infty([0,1])$ are (after renorming if necessary) rearrangement invariant, and that rearrangement invariant spaces satisfying the Fatou property or separable are interpolation spaces between $L^1([0,1])$ and $L^\infty([0,1])$ (for precise details, see [21, Chp. II, §4]). In this memoir we will mainly be concerned with rearrangement invariant spaces which are interpolation spaces between $L^1([0,1])$ and $L^\infty([0,1])$.

The K–method of interpolation provides a technique to construct interpolation spaces. The K–functional for a Banach couple (X_0, X_1) is defined as

$$K(f, t; X_0, X_1) := \inf \big\{ \|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1 \big\},$$

for $f \in X_0 + X_1$ and $t > 0$. The K–functional is a nonnegative, concave, increasing function of t.

A parameter of the K–method of interpolation is a Banach space F of sequences of real numbers, indexed by $\mathbb{Z}$, such that $\ell^\infty \cap \ell^\infty(2^{-k}) \subset F$, where $\ell^\infty(2^{-k})$ is the space of all sequences $(a_k)_{k\in\mathbb{Z}}$ such that $(a_k 2^{-k})_{k\in\mathbb{Z}} \in \ell^\infty$. For F a parameter of the K–method of interpolation, the space $(X_0, X_1)^K_F$ consists of all $x \in X_0 + X_1$ such that

$$(K(x, 2^k; X_0, X_1))_{k\in\mathbb{Z}} \in F.$$

The space $(X_0, X_1)^K_F$, endowed with the norm

$$\|x\| := \|(K(x, 2^k; X_0, X_1))_{k\in\mathbb{Z}}\|_F,$$

is an interpolation space between X_0 and X_1.

Further details on the theory of interpolation of operators can be found in [10], [12] and [21].

Notation

The symbol $\asymp$ will be used to denote an equivalence with multiplicative constants. For example,

$$\|f\|_X \asymp \|g\|_Y$$

stands for the fact that there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq \|g\|_Y \leq B\|f\|_X.$$

The dependence of the constants will be specified when necessary.

Chapter 1

Local norm inequalities for Rademacher series

In this chapter we consider the Rademacher functions,

$$r_k(t) := \operatorname{sign} \sin(2^k \pi t), \qquad t \in [0,1], \qquad k \geq 1.$$

Some classical results on the behavior of the Rademacher system (r_k) in function spaces are presented. Khintchine inequality states that, given any $0 < p < \infty$, the closed linear subspace $\operatorname{Rad}(L^p)$ generated by the Rademacher functions in L^p is isomorphic to ℓ^2 (Theorem 1.2). Let L^{M_2} be the Orlicz space generated by $M_2(t) := \exp(t^2) - 1$, and let $G := (L^{M_2})_0$ be the closure of L^∞ in L^{M_2}. A theorem of Rodin and Semenov shows that, for X an r.i. space on $[0,1]$, the condition $G \subset X$ is equivalent to $\operatorname{Rad}(X)$ being isomorphic to ℓ^2 (Theorem 1.3). Rodin and Semenov, and Lindenstrauss and Tzafriri, proved that for X an r.i. space on $[0,1]$, $\operatorname{Rad}(X)$ is complemented in X if and only if $G \subset X$ and $G \subset X'$ (Theorem 1.4).

Local versions of Khintchine inequality have been considered. By a local version we mean an inequality of the form of Khintchine inequality, but where the Rademacher functions are restricted to a set $E \subset [0,1]$ of positive measure. A result by Zygmund from 1930 shows that a local version of Khintchine inequality holds for the space L^2 (Theorem 1.5). Sagher and Zhou extended this result to L^p with $0 < p < \infty$ (Theorem 1.6) and to the space L^{M_1} (also known as $L_{\exp}$) of functions of exponential integrability (Theorem 1.7). We prove the corresponding local version for the space L^{M_2} (Theorem 1.9).

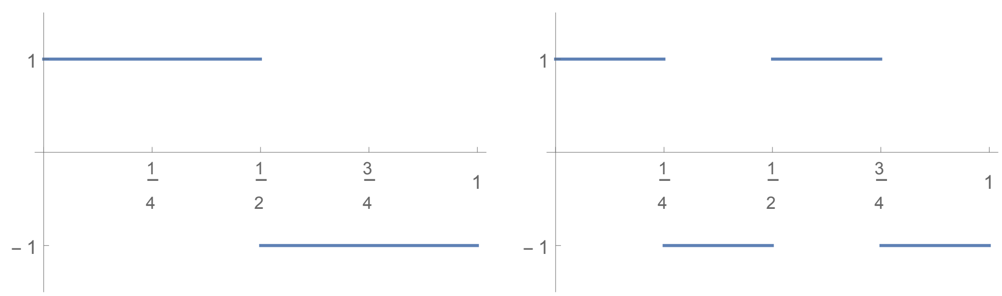
Next, we focus on the Walsh system (w_k), which consists of all finite products of Rademacher functions. The Walsh system is complete and not independent, contrary to the Rademacher system. However, the previous results for Rademacher functions still hold for lacunary subsequences of the Walsh system. Sagher and Zhou proved that Khintchine inequality and its local version hold in L^p, $0 < p < \infty$, for lacunary Walsh series (Theorem 1.12 and Theorem 1.13). We extend these local results to the space L^{M_2} (Theorem 1.14), which allows to prove a version of Rodin and Semenov's theorem for lacunary Walsh series (Theorem 1.18). Finally, we give a version for lacunary Walsh systems of the complementability result by Rodin and Semenov and by Lindenstrauss and Tzafriri (Theorem 1.20).

1.1 Rademacher series in rearrangement invariant spaces

The Rademacher system (r_k) is an orthonormal, independent, identically distributed system on $[0,1]$. Denote the dyadic intervals of order k by

$$I_j^k := \left(\frac{j-1}{2^k}, \frac{j}{2^k}\right), \qquad 1 \le j \le 2^k, \qquad k \ge 1.$$

The Rademacher function r_k is constant on the dyadic intervals of order k, and takes values 1 and -1. The graphic below shows the Rademacher functions r_1 and r_2.



The Rademacher system is not complete; for example, $\langle r_1 r_2, r_k \rangle = 0$ for all $k \ge 1$.

The sequences $(a_k)_1^\infty$ for which the Rademacher series $\sum_{k\ge1} a_k r_k$ converges were characterized by Rademacher [25] and Kolmogorov and Khintchine [20].

Theorem 1.1 (Kolmogorov and Khintchine; Rademacher)**.** *A Rademacher series*

$$\sum_{k\ge1} a_k r_k$$

converges a.e. on $[0,1]$ *if and only if* $(a_k)_1^\infty \in \ell^2$.

Denote by $\mathcal{R}$ the set of all Rademacher series that converge a.e. on $[0,1]$, that is,

$$\mathcal{R} := \Big\{ \sum_{k\geq 1} a_k r_k : (a_k)_1^\infty \in \ell^2 \Big\}.$$

For X a Banach function space on $[0,1]$, let $\mathrm{Rad}(X)$ be the closed linear subspace generated in X by the Rademacher functions. Describing the space $\mathrm{Rad}(X)$ is a classical problem. Since, for any r.i. space X on $[0,1]$, the Rademacher functions (r_k) are a basic sequence in X, we have $\mathrm{Rad}(X) = \mathcal{R} \cap X$ (see [3, Corollary 1.7], or [22, Proposition 2.c.1]).

From the fact that (r_k) is an orthonormal system on $[0,1]$, it follows that

$$\Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^2} = \Big(\sum_{k\geq 1} a_k^2 \Big)^{1/2},$$

and so $\mathrm{Rad}(L^2)$ is isometrically isomorphic to ℓ^2. On the other hand, for $X = L^\infty$,

$$\Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^\infty} = \sum_{k\geq 1} |a_k|,$$

that is, $\mathrm{Rad}(L^\infty)$ is isometrically isomorphic to ℓ^1. Khintchine inequality states, for $0 < p < \infty$, that the space $\mathrm{Rad}(L^p)$ is isomorphic to ℓ^2, [19].

Theorem 1.2 (Khintchine inequality)**.** *For each $0 < p < \infty$ there exist constants $A_p, B_p > 0$ such that*

$$A_p \Big(\sum_{k\geq 1} a_k^2 \Big)^{1/2} \leq \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^p} \leq B_p \Big(\sum_{k\geq 1} a_k^2 \Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

An important result in the study of the Rademacher system is a theorem by Rodin and Semenov that characterizes those r.i. spaces X on $[0,1]$ extending Khintchine inequality, that is, for which the space $\mathrm{Rad}(X)$ is isomorphic to ℓ^2, [26]. Let L^{M_2} be the Orlicz space generated by the Young function

$$M_2(t) := \exp(t^2) - 1, \qquad t \geq 0.$$

Let $G := (L^{M_2})_0$ denote the closure of L^∞ in L^{M_2}.

Theorem 1.3 (Rodin and Semenov)**.** *Let X be an r.i. space on $[0,1]$. The following conditions are equivalent.*

(i) There exist constants $A_X, B_X > 0$ such that

$$A_X\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k r_k\Big\|_X \leq B_X\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

(ii) The continuous embedding $G \subset X$ holds, that is, there exists a constant $C > 0$ such that

$$\|f\|_X \leq C\|f\|_{L^{M_2}}$$

for all $f \in L^\infty$.

The space G is also relevant in order to characterize the complementability of the subspace $\mathrm{Rad}(X)$. A closed subspace Y of a Banach function space X is complemented in X if there exists a bounded linear operator $P : X \to Y$ with $P^2 = P$. The following result was proved independently by Lindenstrauss and Tzafriri, [22, Theorem 2.b.4], and Rodin and Semenov, [27].

Theorem 1.4 (Lindenstrauss and Tzafriri; Rodin and Semenov)**.** *Let X be an r.i. space on $[0, 1]$. The following conditions are equivalent.*

(i) The space $\mathrm{Rad}(X)$ is complemented in X.

(ii) The continuous embeddings $G \subset X$ and $G \subset X'$ hold, that is, there exist constants $C, C' > 0$ such that

$$\|f\|_X \leq C\|f\|_{L^{M_2}}, \qquad \|f\|_{X'} \leq C'\|f\|_{L^{M_2}},$$

for all $f \in L^\infty$.

1.2 Local versions of Khintchine inequality I

This section is devoted to different results characterizing the "local norm" of a Rademacher series. The first result in this regard was given by Zygmund for $L^2([0, 1])$ in 1930 (see [33, Lemma V.8.3]).

Theorem 1.5 (Zygmund)**.** *Let $E \subset [0, 1]$ be a set of positive measure. For any $\lambda > 1$, there exists $N = N(E, \lambda)$ such that*

$$\lambda^{-1} m(E) \sum_{k\geq N} a_k^2 \leq \int_E \Big|\sum_{k\geq N} a_k r_k(t)\Big|^2 dt \leq \lambda m(E) \sum_{k\geq N} a_k^2 \tag{1.1}$$

for $(a_k)_1^\infty \in \ell^2$.

Inequality (1.1) can be written as

$$A\Big(\sum_{k\geq N} a_k^2\Big)^{1/2} \leq \Big(\int_E \Big|\sum_{k\geq N} a_k r_k\Big|^2 \frac{dt}{m(E)}\Big)^{1/2} \leq B\Big(\sum_{k\geq N} a_k^2\Big)^{1/2}, \tag{1.2}$$

for $(a_k)_1^\infty \in \ell^2$ and some constants $A, B > 0$. Motivated by the above expression, one can denote the local norm in L^p on a set E by

$$\|f\|_{L^p(E,dm/m(E))} := \Big(\int_E |f(t)|^p \frac{dt}{m(E)}\Big)^{1/p}, \tag{1.3}$$

for f a measurable function on E. With this setting, Theorem 1.5 is a local version of Khintchine inequality for $p = 2$. The following result by Sagher and Zhou extends (1.2) to the case $0 < p < \infty$, [28, Theorem 1].

Theorem 1.6 (Sagher and Zhou)**.** *For any $0 < p < \infty$, there exist constants $A'_p, B'_p > 0$ so that for any measurable set $E \subset [0,1]$ of positive measure, there exists an $N = N(E)$ such that*

$$A'_p\Big(\sum_{k=N}^\infty a_k^2\Big)^{1/2} \leq \Big(\int_E \Big|\sum_{k=N}^\infty a_k r_k(t)\Big|^p \frac{dt}{m(E)}\Big)^{1/p} \leq B'_p\Big(\sum_{k=N}^\infty a_k^2\Big)^{1/2}, \tag{1.4}$$

for $(a_k)_1^\infty \in \ell^2$.

The above result we refer to as the local version of Khintchine inequality in L^p. Sagher and Zhou also proved the local version of Khintchine inequality for the Orlicz space generated by $M_1(t) = \exp t - 1$, [30, Theorem 2]. This space is also known as the space $L_{\exp}$ of functions of exponential integrability (see [10, Section IV.6]). The norm in L^{M_1} is given by

$$\|f\|_{L^{M_1}} := \inf\Big\{\lambda > 0 : \int_0^1 \big(\exp(|f(t)|/\lambda) - 1\big)\, dt \leq 1\Big\}.$$

Theorem 1.7 (Sagher and Zhou)**.** *Let $E \subset [0,1]$ be a set of positive measure. Consider the local norm of L^{M_1} in E,*

$$\|f\|_{L^{M_1}(E,dt/m(E))} := \inf\Big\{\lambda > 0 : \int_E \big(\exp(|f(t)/\lambda|) - 1\big)\frac{dt}{m(E)} \leq 1\Big\}.$$

There exist constants $A'_{M_1}, B'_{M_1} > 0$ not depending on E and $N = N(E)$ such that

$$A'_{M_1}\Big(\sum_{k=N}^\infty a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq N} a_k r_k\Big\|_{L^{M_1}(E,dt/m(E))} \leq B'_{M_1}\Big(\sum_{k=N}^\infty a_k^2\Big)^{1/2} \tag{1.5}$$

for $(a_k)_1^\infty \in \ell^2$.

This result by Sagher and Zhou provides a definition of local space of exponential integrability for $p = 1$, which can be generalized as follows.

Let $L^{M_p}(E, dt/m(E))$ be the Orlicz space on $(E, dt/m(E))$ generated by the function $M_p := \exp(t^p) - 1$, for $1 \leq p < \infty$, that is, the space of all measurable functions f on E such that

$$\|f\|_{L^{M_p}(E,dt/m(E))} := \inf \left\{ \lambda > 0 : \int_E M_p(f(t)/\lambda) \frac{dt}{m(E)} \leq 1 \right\} < \infty. \tag{1.6}$$

Note that the inclusions

$$G \subset L^{M_2} \subset L^{M_1} \subset L^p \subset L^2, \qquad 2 < p < \infty,$$

show that Khintchine inequality for an r.i. space X is stronger as X gets closer to G. In view of the local versions of Khintchine inequality for L^2, L^p and L^{M_1}, Rodin and Semenov's theorem suggests considering the corresponding local result for the space L^{M_2}.

We will need the inequality

$$B_{2n} \leq \sqrt{n}, \qquad n \geq 1,$$

for the upper constant B_p in Khintchine inequality (see [33, Theorem V.8.4]). We give an estimate for the constant B'_p in (1.4), which is not explicitly computed in Sagher and Zhou's proof of Theorem 1.6.

Lemma 1.8. *For $n \geq 1$,*

$$B'_{2n} \leq 2^{1/2n}(1 + \sqrt{2})\sqrt{n}.$$

Proof. We follow the approach from [28]. For any set $E \subset [0, 1]$ of positive measure, there exists a set F which is a pairwise disjoint family of dyadic intervals such that $\chi_E \leq \chi_F$ a.e. and $m(F) \leq 2m(E)$. Set $F = F_1 \cup F_2$, where

$$F_1 = \bigcup_{j=1}^{s} I_{k_j}^{n_j}$$

is a union of finitely many intervals from F, satisfying $F_1 \cap F_2 = \emptyset$ and $m(F_2) \leq m(F)^2$. Set $N = \max\{n_j : j = 1, \ldots, s\} + 1$.

Then,

$$\Big(\int_E \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{2n} \frac{dt}{m(E)}\Big)^{1/2n} \le 2^{1/2n}\Big(\int_F \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{2n} \frac{dt}{m(F)}\Big)^{1/2n}$$

$$\le 2^{1/2n}\Big(\int_{F_1} \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{2n} \frac{dt}{m(F_1)}\Big)^{1/2n}$$

$$+ 2^{1/2n}\Big(\int_{F_2} \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{2n} \frac{dt}{\sqrt{m(F_2)}}\Big)^{1/2n}.$$

We find separately upper bounds for the integrals on F_1 and F_2.
Since $\sum_{k=N}^M a_k r_k$ is periodic with period $1/2^{N-1}$,

$$\Big(\int_{F_1} \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{2n} \frac{dt}{m(F_1)}\Big)^{1/2n} = \Big(\int_0^{m(F_1)} \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{2n} \frac{dt}{m(F_1)}\Big)^{1/2n}.$$

The set F_1 can be decomposed into a finite union of dyadic intervals of order $N-1$, and so $m(F_1) = p/2^{N-1}$ for some $p \ge 1$. By means of the change of variable $t = m(F_1)x$, since

$$r_k(x\, m(F_1)) = \operatorname{sign}(\sin(2^k \pi x p/2^{N-1})) = \operatorname{sign}(\sin(2^{k-N+1}\pi x)) = r_{k-N+1}(x),$$

it follows, from Khintchine inequality, that

$$\begin{aligned}\Big(\int_0^{m(F_1)} \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{2n} \frac{dt}{m(F_1)}\Big)^{1/2n} &= \Big(\int_0^1 \Big|\sum_{k=N}^M a_k r_{k-N+1}(x)\Big|^{2n} dx\Big)^{1/2n} \\ &\le B_{2n}\Big(\sum_{k=N}^M a_k^2\Big)^{1/2}.\end{aligned} \tag{1.7}$$

In order to bound the integral on F_2, from Cauchy–Schwarz inequality we have

$$\begin{aligned}&\int_{F_2} \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{2n} \frac{dt}{\sqrt{m(F_2)}} \\ &\le \Big(\int_0^1 \Big(\frac{\chi_{F_2}(t)}{\sqrt{m(F_2)}}\Big)^2 dt\Big)^{1/2}\Big(\int_0^1 \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{4n} dt\Big)^{1/2} \\ &= \Big(\int_0^1 \Big|\sum_{k=N}^M a_k r_k(t)\Big|^{4n} dt\Big)^{1/2},\end{aligned}$$

which, together with Khintchine inequality, yields

$$\Big(\int_{F_2}\Big|\sum_{k=N}^{M} a_k r_k(t)\Big|^{2n}\frac{dt}{\sqrt{m(F_2)}}\Big)^{1/2n} \le \Big(\int_0^1\Big|\sum_{k=N}^{M} a_k r_k(t)\Big|^{4n} dt\Big)^{1/4n} \le B_{4n}\Big(\sum_{k=N}^{M} a_k^2\Big)^{1/2}. \tag{1.8}$$

From (1.7) and (1.8), we have

$$B'_{2n} \le 2^{1/2n}(B_{2n}+B_{4n}).$$

From the inequality $B_{2n} \le \sqrt{n}$, it follows that

$$B'_{2n} \le 2^{1/2n}(1+\sqrt{2})\sqrt{n},$$

which concludes the proof. □

Theorem 1.9. *There exist constants $A'_{M_2}, B'_{M_2} > 0$ such that, for any set $E \subset [0,1]$ of positive measure, there exists $N = N(E)$ such that*

$$A'_{M_2}\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2} \le \Big\|\sum_{k\ge N} a_k r_k\Big\|_{L^{M_2}(E,dt/m(E))} \le B'_{M_2}\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2}, \tag{1.9}$$

for any $(a_k)_1^\infty \in \ell^2$.

Proof. Let $E \subset [0,1]$ be a set of positive measure, and let $N = N(E)$ be given by Theorem 1.6. The left–hand side inequality follows from Theorem 1.6 for $p = 1$ and from the continuous embedding $L^{M_2} \subset L^1$, which show that, for some constant $C_1 > 0$,

$$C_1 A'_1\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2} \le C_1\Big\|\sum_{k\ge N} a_k r_k\Big\|_{L^1(E,dm/m(E))} \le \Big\|\sum_{k\ge N} a_k r_k\Big\|_{L^{M_2}(E,dm/m(E))},$$

for $(a_k)_1^\infty \in \ell^2$.

In order to prove the right–hand side inequality, denote

$$R_N a := \sum_{k\ge N} a_k r_k, \qquad (a_k)_1^\infty \in \ell^2.$$

From the power series expansion of $M_2(t) = \exp t^2 - 1$ and Lemma 1.8, we have

$$\begin{aligned}\int_E \big(\exp|R_N(t)/\lambda|^2 - 1\big)\frac{dt}{m(E)} &= \sum_{n\geq 1}\frac{1}{n!\lambda^{2n}}\int_E\Big|\sum_{k\geq N} a_k r_k(t)\Big|^{2n}\frac{dt}{m(E)} \\ &\leq \sum_{n\geq 1}\frac{\big(2^{1/2n}(1+\sqrt{2})\sqrt{n}\big)^{2n}}{n!\lambda^{2n}}\|(a_k)_{k\geq N}\|_2^{2n}.\end{aligned}$$

Applying the asymptotic equivalence $n! \sim (2\pi n)^{1/2} n^n e^{-n}$, given by Stirling's formula, there exists an absolute constant $C_2 > 0$ such that

$$\begin{aligned}\int_E \big(\exp|R_N(t)/\lambda|^2 - 1\big)\frac{dt}{m(E)} &\leq C_2\sum_{n\geq 1}\frac{(1+\sqrt{2})^{2n}n^n}{n^{1/2}n^n e^{-n}\lambda^{2n}}\|(a_k)_{k\geq N}\|_2^{2n} \\ &\leq C_2\sum_{n\geq 1}\Big(\frac{(1+\sqrt{2})^2 e}{\lambda^2}\|(a_k)_{k\geq N}\|_2^2\Big)^n.\end{aligned}$$

The geometric series above converges for

$$\lambda > (1+\sqrt{2})\sqrt{e}\,\|(a_k)_{k\geq N}\|_2,$$

and so it follows that $R_N a \in L^{M_2}(E, dt/m(E))$. Furthermore, the inequality

$$C_2\sum_{n\geq 1}\Big(\frac{(1+\sqrt{2})^2 e}{\lambda^2}\|(a_k)_{k\geq N}\|_2^2\Big)^n \leq 1$$

holds if and only if

$$\lambda \geq \sqrt{1+C_2}(1+\sqrt{2})\sqrt{e}\,\|(a_k)_{k\geq N}\|_2.$$

Thus,

$$\begin{aligned}\Big\|\sum_{k\geq N} a_k r_k\Big\|_{L^{M_2}(E,dt/m(E))} &= \inf\Big\{\lambda > 0 : \int_E \big(\exp|R_N(t)/\lambda|^2 - 1\big)\frac{dt}{m(E)} \leq 1\Big\} \\ &\leq \inf\Big\{\lambda > 0 : C_2\sum_{n\geq 1}\Big(\frac{(1+\sqrt{2})^2 e}{\lambda^2}\|(a_k)_{k\geq N}\|_2^2\Big)^n \leq 1\Big\} \\ &= \sqrt{1+C_2}(1+\sqrt{2})\sqrt{e}\,\|(a_k)_{k\geq N}\|_2,\end{aligned}$$

which completes the proof. □

Corollary 1.10. *Let $1 \le p \le 2$. There exist constants $A'_{M_p}, B'_{M_p} > 0$ such that, for any set $E \subset [0,1]$ of positive measure, there exists $N = N(E)$ such that*

$$A'_{M_p}\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2} \le \Big\| \sum_{k\ge N} a_k r_k \Big\|_{L^{M_p}(E,dt/m(E))} \le B'_{M_p}\Big(\sum_{k=N}^{\infty} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

Remark 1.11. Corollary 1.10 provides an alternative proof of Theorem 1.7 by Sagher and Zhou, which originally required the dyadic BMO norm of a function of the form $\exp(\sum a_k r_k)$.

1.3 Lacunary Walsh series in exponential spaces

The Rademacher functions do not form a complete system in $L^2([0,1])$; it can be completed by adding all finite products of Rademacher functions. The system thus obtained is known as the Walsh system. We describe the Walsh functions (w_n) following the notation used by Lindenstrauss and Tzafriri [22, p. 104], that is,

$$w_1 := 1, \quad w_2 := r_1, \quad w_3 := r_2, \quad w_4 := r_1 r_2, \quad w_5 := r_3, \quad w_6 := r_1 r_3 \ldots$$

In general, given $k \in \mathbb{N}$, there exist $n \ge 1$ and $a_1, \ldots, a_n \in \{0,1\}$ such that

$$k = a_1 2^0 + a_2 2^1 + \ldots + a_n 2^{n-1}.$$

Then,

$$w_{k+1} := r_1^{a_1} \cdot \ldots \cdot r_n^{a_n}.$$

In particular,

$$r_{n+1} = w_{2^n+1}, \qquad n \ge 0.$$

The Walsh system (w_n) is complete and not independent, which establishes a major difference with the Rademacher system (r_k).

Sagher and Zhou proved a version of Khintchine inequality for lacunary Walsh systems, [29]. It is to be noted that Sagher and Zhou give for the Walsh functions a different numbering, namely, $w_0 = 1$, $w_1 = r_1$, $w_2 = r_2$, $w_3 = r_1 r_2$, $w_4 = r_3 \ldots$ This numbering was also considered by Paley [24] and Kashin and Saakyan [18].

Recall that, given $q > 1$, we say that a subsequence (w_{n_k}) of Walsh functions is q–lacunary if

$$\inf_{k\ge 1} \frac{n_{k+1}}{n_k} \ge q.$$

The Rademacher system is a lacunary subsystem of the Walsh system.

The result by Sagher and Zhou on lacunary Walsh systems is the following.

Theorem 1.12 (Sagher and Zhou). *Given $0 < p < \infty$ and $q > 1$, there exist constants $A(p,q), B(p,q) > 0$ such that, for any q–lacunary sequence (w_{n_k}) of Walsh functions, we have*

$$A(p,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k=1}^{\infty} a_k w_{n_k}\Big\|_{L^p} \leq B(p,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

Sagher and Zhou also proved a local version of Theorem 1.12, [29].

Theorem 1.13 (Sagher and Zhou). *Given $0 < p < \infty$ and $q > 1$, there exist constants $A'(p,q), B'(p,q) > 0$ such that for any set $E \subset [0,1]$ of positive measure, there exists $N = N(E,q)$ so that for any q–lacunary sequence (w_{n_k}) of Walsh functions with $n_k \geq N$ for all $k \geq 1$, we have*

$$A'(p,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k=1}^{\infty} a_k w_{n_k}\Big\|_{L^p(E,dm/m(E))} \leq B'(p,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

These results show a similarity between the behavior of the lacunary Walsh series and the Rademacher series. In this section we give some results in this regard for the space L^{M_2}.

The first one is a version of Theorem 1.9 for lacunary Walsh series. For this result we shall need the following estimates for the constants $B(p,q)$ in Theorem 1.12 and $B'(p,q)$ in Theorem 1.13.

For $n \geq 1$,

$$B(2n,q) \leq (1+m)^{1/2n} n^{1/2},$$
$$B'(2n,q) \leq (1+\sqrt{2})(2+2m)^{1/2n} n^{1/2},$$

where m is the least integer such that $q^m \geq 2$ when $1 < q < 2$, and $m = 0$ when $q \geq 2$. These inequalities are not explicitly stated in [29], but follow from the proofs of Theorem 1.12 and Theorem 1.13.

Theorem 1.14. *Let $q > 1$ and $E \subset [0,1]$ be a set of positive measure. Consider the space L^{M_2} of functions of square exponential integrability. There exist constants $A'(M_2,q)$, $B'(M_2,q) > 0$ and $N = N(E)$ such that, for any q–lacunary sequence (w_{n_k}) of Walsh functions with $n_k \geq N$ for all $k \geq 1$, we have*

$$A'(M_2,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k w_{n_k}\Big\|_{L^{M_2}(E,dm/m(E))} \leq B'(M_2,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

Proof. Let $N = N(E)$ be as in Theorem 1.13. The left–hand side inequality follows from the embedding $L^{M_2} \subset L^1$ and from Theorem 1.13 for $p = 1$, which show that, for some constant $C_1 > 0$,

$$C_1 A'(1,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \le C_1 \Big\| \sum_{k\ge 1} a_k w_{n_k} \Big\|_{L^1(E,dm/m(E))} \le \Big\| \sum_{k\ge 1} a_k w_{n_k} \Big\|_{L^{M_2}(E,dm/m(E))},$$

for any q–lacunary sequence (w_{n_k}) with $n_k \ge N$ and $(a_k)_1^\infty \in \ell^2$.

In order to prove the right–hand side inequality, let

$$f := \sum_{k\ge 1} a_k w_{n_k}.$$

We proceed as in the proof of Theorem 1.9, replacing Lemma 1.8 by the inequality

$$B'(2n,q) \le (1+\sqrt{2})(2+2m)^{1/2n} n^{1/2}.$$

From the power series expansion of $\exp(t^2) - 1$ and Theorem 1.13 for $p = 2n$, we have

$$\int_E \big(\exp|f(t)/\lambda|^2 - 1\big)\frac{dt}{m(E)} = \sum_{n\ge 1} \frac{1}{n!\lambda^{2n}} \int_E \Big|\sum_{k\ge 1} a_k w_{n_k}(t)\Big|^{2n} \frac{dt}{m(E)} \le \sum_{n\ge 1} \frac{B'(2n,q)^{2n}}{n!\lambda^{2n}} \|(a_k)_1^\infty\|_2^{2n}.$$

It follows, applying Stirling's formula, that for some absolute constant $C_2 > 0$,

$$\begin{aligned}
&\int_E \big(\exp|f(t)/\lambda|^2 - 1\big)\frac{dt}{m(E)} \\
&\quad \le \sum_{n\ge 1} \frac{\big((1+\sqrt{2})(2+2m)^{1/2n} n^{1/2}\big)^{2n}}{n!\lambda^{2n}} \|(a_k)_1^\infty\|_2^{2n} \\
&\quad = (2+2m)\sum_{n\ge 1} \frac{(1+\sqrt{2})^{2n} n^n}{n!\lambda^{2n}} \|(a_k)_1^\infty\|_2^{2n} \\
&\quad \le C_2(2+2m)\sum_{n\ge 1} \Big(\frac{(1+\sqrt{2})^2 e}{\lambda^2} \|(a_k)_1^\infty\|_2^2\Big)^n.
\end{aligned}$$

From this inequality it follows, as in the proof of Theorem 1.9, that there exists a constant $B'(M_2, q) > 0$ such that

$$\Big\| \sum_{k\geq 1} a_k w_{n_k} \Big\|_{L^{M_2}(E, dm/m(E))} \leq B'(M_2, q) \Big(\sum_{k=1}^{\infty} a_k^2 \Big)^{1/2},$$

and so the proof is complete. □

Remark 1.15. Note that m establishes the dependence between $B'(M_2, q)$ and q in Theorem 1.14. In particular, since $q \geq 2$ implies $m = 0$, the constant $B'(M_2, q)$ is the same for all $q \geq 2$.

In Theorem 1.18 below we give a version of Rodin and Semenov's theorem (Theorem 1.3) for lacunary Walsh series. For proving it we need a result by Astashkin on the selection of subsequences equivalent in distribution to the Rademacher system, see [3, Theorem 9.4].

Definition 1.16. Let (φ_k) and (ψ_k) be systems (sequences of measurable functions) on probability spaces (R, μ) and (S, ν), respectively. The system (φ_k) is majorized in distribution by the system (ψ_k) if there exists a constant $C > 0$ such that

$$\mu\Big(\Big\{x \in R : \Big| \sum_{k=1}^{m} a_k \varphi_k(x) \Big| > \lambda \Big\}\Big) \leq C\nu\Big(\Big\{t \in S : \Big| \sum_{k=1}^{m} a_k \psi_k(t) \Big| > C^{-1}\lambda \Big\}\Big)$$

for every $m \geq 1$, $a_1, \ldots, a_m \in \mathbb{R}$ and $\lambda > 0$. We denote this by $(\varphi_k) \prec (\psi_k)$. We specify the constant by writing $\prec_C$, and denote $((\varphi_k), R) \prec ((\psi_k), S)$ whenever it is necessary to remark the measure spaces involved.

The systems (φ_k) and (ψ_k) are equivalent in distribution if both $(\varphi_k) \prec (\psi_k)$ and $(\psi_k) \prec (\varphi_k)$ hold.

Theorem 1.17 (Astashkin)**.** *Let (R, μ) be a probability space. Given any orthonormal sequence (f_k) of random variables on (R, μ) such that $|f_k(x)| \leq D$ for almost every $x \in R$ and for all $k \geq 1$, there exists a subsequence $(\varphi_k) \subset (f_k)$ equivalent in distribution to the Rademacher system, with equivalence constants depending only on the uniform bound D.*

The version of Rodin and Semenov's theorem for lacunary Walsh series is the following.

Theorem 1.18. *Let X be an r.i. space on $[0, 1]$. The following conditions are equivalent.*

(i) The continuous embedding $G \subset X$ holds, that is, there exists a constant $C > 0$ such that

$$\|f\|_X \leq C\|f\|_{L^{M_2}}$$

for all $f \in L^\infty$.

(ii) For any $q > 1$, there exist constants $A(X,q), B(X,q) > 0$ such that

$$A(X,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k w_{n_k}\Big\|_X \leq B(X,q)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$, and any q–lacunary system (w_{n_k}) of Walsh functions.

(iii) There exist a sequence (n_k) and constants $A(X), B(X) > 0$ such that

$$A(X)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k w_{n_k}\Big\|_X \leq B(X)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

Proof. *(i) ⇒ (ii)* Assume that $G \subset X$. Since, for any r.i. space X, the continuous embedding $X \subset L^1$ holds, then *(ii)* follows from Theorem 1.12 for $p = 1$ and from Theorem 1.14 with $E = [0,1]$.

(ii) ⇒ (iii) It follows by considering the Rademacher system $(r_k) \subset (w_n)$.

(iii) ⇒ (i) To show *(i)* it suffices to assume that the right–hand side inequality in *(iii)* holds, that is, for some constant $B(X) > 0$,

$$\Big\|\sum_{k\geq 1} a_k w_{n_k}\Big\|_X \leq B(X)\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2},$$

for any $(a_k)_1^\infty \in \ell^2$. Applying Theorem 1.17 to the sequence (w_{n_k}), there exists a subsequence $(m_k) \subset (n_k)$ such that (w_{m_k}) is equivalent in distribution to the Rademacher system. Let

$$s_n := \frac{1}{\sqrt{n}}\sum_{k=1}^{n} w_{m_k}, \qquad v_n := \frac{1}{\sqrt{n}}\sum_{k=1}^{n} r_k.$$

From *(iii)*, we have $\|s_n\|_X \leq B(X)$, and from the fact that, for all $n \geq 1$, the functions v_n and s_n have equivalent distribution functions, it follows, from Proposition 2.9 below, that they are equivalent in norm. Hence, for some constant $C > 0$,

$$\|v_n\|_X \leq C\|s_n\|_X \leq C\,B(X),$$

and so $v_n \in X$ and v_n are uniformly bounded in norm. Following the steps of the proof of Theorem 1.3 by Rodin and Semenov (see [26, Theorem 6]), $v_n \in X$ with $\|v_n\|_X \leq C\, B(X)$ implies, via the Central Limit Theorem, that $G \subset X$. □

It follows from Theorem 1.18 that any lacunary Walsh sequence is basic on any r.i. space X with $G \subset X$. This result holds in arbitrary interpolation spaces between $L^1([0,1])$ and $L^\infty([0,1])$. We include a proof of this result, which is probably known, but for which we have not found any reference.

Proposition 1.19. *Let X be an r.i. space on $[0,1]$ which is an interpolation space between $L^1([0,1])$ and $L^\infty([0,1])$. Then, the Walsh system (w_k) is a basic sequence in X.*

In addition, if X is an exact interpolation space, then (w_k) is a monotone basic sequence in X.

Proof. For $n \geq 1$, consider the operator

$$A_n f := \sum_{j=1}^{2^n} \Big(\frac{1}{m(I_j^n)} \int_{I_j^n} f\, dm \Big) \chi_{I_j^n}, \qquad f \in L^1([0,1]).$$

We have that $\|A_n f\|_\infty \leq \|f\|_\infty$ for $f \in L^\infty([0,1])$ and $\|A_n f\|_1 \leq \|f\|_1$ for $f \in L^1([0,1])$. Since X is an interpolation space between L^1 and L^∞, the operators $A_n : X \to X$ are uniformly bounded, that is, $\|A_n\| \leq C$ for some constant $C > 0$ and $n \geq 1$.

For $1 \leq k, j \leq 2^n$ we have that w_k is constant on $\chi_{I_j^n}$. Thus,

$$A_n(w_k) = \sum_{j=1}^{2^n} \Big(\frac{1}{m(I_j^n)} \int_{I_j^n} w_k\, dm \Big) \chi_{I_j^n} = w_k.$$

On the other hand, noting that for $k \geq 2^n + 1$ and $1 \leq j \leq 2^n$,

$$\int_{I_j^n} w_k\, dm = 0,$$

it follows that $A_n(w_k) = 0$, for $k \geq 2^n + 1$. Thus, for any $m > n$ we have

$$C \Big\| \sum_{k=1}^{m} a_k w_k \Big\|_X \geq \Big\| A_n \Big(\sum_{k=1}^{m} a_k w_k \Big) \Big\|_X = \Big\| \sum_{k=1}^{n} a_k w_k \Big\|_X,$$

which concludes the proof. □

Let X be an r.i. space on $[0,1]$ which is an interpolation space between $L^1([0,1])$ and $L^\infty([0,1])$. Recall, from Theorem 1.4, that the closed linear space $\mathcal{R}\cap X$ generated by the Rademacher functions is complemented in X if an only if $G\subset X$ and $G\subset X'$. Next, we give a version of this result for lacunary Walsh series. Denote by $[w_{n_k}]_X$ the closed linear subspace generated in X by the Walsh functions (w_{n_k}). From Proposition 1.19, (w_{n_k}) is a basic sequence in X, and so $[w_{n_k}]_X$ consists of all Walsh series

$$\sum_{k\geq 1} a_k w_{n_k}$$

which belong to X. Denote by P the projection

$$f\mapsto P(f):=\sum_{k\geq 1}\langle w_{n_k},f\rangle w_{n_k}, \tag{1.10}$$

where

$$\langle w_{n_k},f\rangle=\int_0^1 w_{n_k}(t)f(t)\,dt.$$

Theorem 1.20. *Let X be an r.i. space on $[0,1]$ which is an interpolation space between $L^1([0,1])$ and $L^\infty([0,1])$. The following conditions are equivalent.*

(i) The continuous embeddings $G\subset X$ and $G\subset X'$ hold, that is, there exist constants $C,C'>0$ such that

$$\|f\|_X\leq C\|f\|_{L^{M_2}},\qquad \|f\|_{X'}\leq C'\|f\|_{L^{M_2}},$$

for all $f\in L^\infty$.

(ii) For any $q>1$ and any q–lacunary sequence (w_{n_k}) of Walsh functions, the space $[w_{n_k}]_X$ is complemented in X.

(iii) There exists $q>1$ and a q–lacunary sequence (w_{n_k}) of Walsh functions such that $[w_{n_k}]_X$ is complemented in X.

The proof follows the ideas of [22, Theorem 2.b.4] and [27].

Lemma 1.21. *Let X be an r.i. space on $[0,1]$ and (w_{n_k}) a q–lacunary sequence of Walsh functions. The following conditions are equivalent.*

(i) The operator $T:X\to\ell^2$ given by

$$Tf:=\big(\langle w_{n_k},f\rangle\big)_{k\geq 1} \tag{1.11}$$

is continuous.

(ii) The continuous embedding $G \subset X'$ holds.

Proof. *(i) ⇒ (ii)* Assume that $T : X \to \ell^2$ is continuous. Then, the adjoint operator $T' : \ell^2 \to X'$ is continuous. Denote by (e_k) the canonical basis of ℓ^2. Let $f \in X$. Then,

$$\langle T'e_k, f\rangle = \langle e_k, Tf\rangle = \langle w_{n_k}, f\rangle,$$

and so $T'e_k = w_{n_k}$. Hence, for $b = (b_k)_1^\infty \in \ell^2$, we have

$$T'b = T'\Big(\sum_{k=1}^{\infty} b_k e_k\Big) = \sum_{k=1}^{\infty} b_k\, T'(e_k) = \sum_{k=1}^{\infty} b_k w_{n_k}.$$

Together with the continuity of T', it follows that

$$\Big\|\sum_{k\geq 1} b_k w_{n_k}\Big\|_{X'} = \|T'(b_k)\|_{X'} \leq \|T'\|\|(b_k)\|_{\ell^2},$$

for all $(b_k) \in \ell^2$. This condition, as in the proof of Theorem 1.18, implies $G \subset X'$.

(ii) ⇒ (i) If (w_{n_k}) is a q–lacunary subsequence of Walsh functions and $G \subset X'$, we have from Theorem 1.18 applied to X' that there exists a constant $B(X', q) > 0$ such that, for $N \geq 1$, we have

$$\Big\|\sum_{k=1}^{N}\langle w_{n_k}, f\rangle w_{n_k}\Big\|_{X'} \leq B(X', q)\Big(\sum_{k=1}^{N}\langle w_{n_k}, f\rangle^2\Big)^{1/2}.$$

Let $f \in X$. Fix $N \geq 1$. Then, from Hölder's inequality for X and X',

$$\begin{aligned}\sum_{k=1}^{N}\langle w_{n_k}, f\rangle^2 &= \int_0^1 f(t)\Big(\sum_{k=1}^{N}\langle w_{n_k}, f\rangle w_{n_k}\Big)(t)\,dt \\ &\leq \|f\|_X\Big\|\sum_{k=1}^{N}\langle w_{n_k}, f\rangle w_{n_k}\Big\|_{X'} \\ &\leq B(X', q)\|f\|_X\Big(\sum_{k=1}^{N}\langle w_{n_k}, f\rangle^2\Big)^{1/2}.\end{aligned}$$

It follows that

$$\Big(\sum_{k=1}^{N}\langle w_{n_k}, f\rangle^2\Big)^{1/2} \leq B(X', q)\|f\|_X, \qquad f \in X,$$

that is, $\|Tf\|_{\ell^2} \leq B(X', q)\|f\|_X$, and so *(i)* is established. □

For $n \geq 1$ and any $1 \leq j, k \leq 2^n$, since w_k is constant on the dyadic intervals of order n, we have that $w_k(I_j^n)$ is well–defined and takes values 1 or -1. The following property appears in [22, Theorem 2.b.4]:

$$w_k(I_j^n) = w_j(I_k^n), \qquad 1 \leq j, k \leq 2^n,$$

but no detail of its proof is given. Next, we give a proof of this fact.

Lemma 1.22. *For $n \geq 1$ and $1 \leq j, k \leq 2^n$, we have*

$$w_k(I_j^n) = w_j(I_k^n).$$

Proof. The case when either $k = 1$ or $j = 1$ follows from the fact that $w_k(I_1^n) = 1$ and $w_1(I_k^n) = 1$ for $1 \leq k \leq 2^n$.

Fix $n \geq 1$ and $1 \leq j, k \leq 2^n - 1$. We will show that

$$w_{k+1}(I_{j+1}^n) = w_{j+1}(I_{k+1}^n).$$

Let

$$\begin{aligned} k &= a_1 2^0 + a_2 2^1 + \ldots + a_n 2^{n-1}, \qquad a_1, \ldots, a_n \in \{0, 1\}, \\ j &= b_1 2^0 + b_2 2^1 + \ldots + b_n 2^{n-1}, \qquad b_1, \ldots, b_n \in \{0, 1\}. \end{aligned}$$

Fix $x \in I_{j+1}^n$, and denote by $\{x\}$ the fractional part of x. From

$$r_i(x) = \operatorname{sign} \sin(2^i \pi x) = r_1(\{2^{i-1} x\}),$$

we have

$$\begin{aligned} w_{k+1}(x) &= (r_1(x))^{a_1} \cdot (r_2(x))^{a_2} \ldots (r_n(x))^{a_n} \\ &= (r_1(x))^{a_1} \cdot (r_1(\{2x\}))^{a_2} \ldots (r_1(\{2^{n-1} x\}))^{a_n}. \end{aligned} \tag{1.12}$$

Now we use the fact that $x \in I_{j+1}^n$ if and only if

$$x = \frac{b_n}{2} + \frac{b_{n-1}}{2^2} + \ldots + \frac{b_1}{2^n} + \varepsilon,$$

with $0 < \varepsilon < 1/2^n$. Then, for any $0 \leq i \leq n-1$,

$$\begin{aligned} \{2^i x\} &= \Big\{ 2^i \Big(\frac{b_n}{2} + \frac{b_{n-1}}{2^2} + \ldots + \frac{b_1}{2^n} + \varepsilon \Big) \Big\} \\ &= \Big\{ \frac{b_{n-i}}{2} + \frac{b_{n-i-1}}{2^2} + \ldots + \frac{b_1}{2^{n-i}} + 2^i \varepsilon \Big\}. \end{aligned}$$

It follows that

$$r_1(\{2^i x\}) = \begin{cases} 1 & \text{if } b_{n-i} = 0, \\ -1 & \text{if } b_{n-i} = 1. \end{cases}$$

Hence,

$$r_1(\{2^i x\}) = (-1)^{b_{n-i}}, \qquad 0 \le i \le n-1,$$

which together with (1.12) implies that

$$w_{k+1}(x) = (-1)^{a_1 \cdot b_n} \cdot (-1)^{a_2 \cdot b_{n-1}} \dots (-1)^{a_n \cdot b_1}.$$

The symmetry in a_i and b_i of the expression above implies that $w_{k+1}(I^n_{j+1}) = w_{j+1}(I^n_{k+1})$. □

We can now proceed to the proof of the main result.

Proof of Theorem 1.20. (i) ⇒ (iii) It follows from Theorem 1.4 considering the Rademacher system.

(ii) ⇒ (i) Follows as above.

(i) ⇒ (ii) Let (w_{n_k}) be a q–lacunary subsequence of Walsh functions. From the assumption that $G \subset X'$, Lemma 1.21 implies that the operator $T : X \to \ell^2$ in (1.11) is continuous, that is,

$$\Big(\sum_{k\ge1} \langle w_{n_k}, f \rangle^2 \Big)^{1/2} \le \|T\| \|f\|_X, \qquad f \in X.$$

On the other hand, from $G \subset X$ and Theorem 1.18 we have, for $P : X \to [w_{n_k}]_X$ the projection in (1.10),

$$\|Pf\|_X = \Big\| \sum_{k\ge1} \langle w_{n_k}, f \rangle w_{n_k} \Big\|_X \le B(X,q) \Big(\sum_{k\ge1} \langle w_{n_k}, f \rangle^2 \Big)^{1/2}.$$

It follows that

$$\|Pf\|_X \le B(X,q) \|T\| \|f\|_X, \qquad f \in X,$$

that is, the subspace $[w_{n_k}]_X$ is complemented in X.

(iii) ⇒ (i) We follow the ideas in the proof of the complementability result for Rademacher functions (Theorem 1.4) by Lindenstrauss and Tzafriri [22, Theorem 2.b.4]. Assume that (w_{n_k}) is a lacunary sequence with $q > 1$ such that $[w_{n_k}]_X$ is complemented in X. Then, there exists a bounded linear operator $Q : X \to [w_{n_k}]_X$ with $Q^2 = Q$. Since (w_{n_k}) is a basic sequence, there exists $(q_{n_k}) \subset X^*$ such that

$$Qf := \sum_{k\ge1} q_{n_k}(f) w_{n_k}, \qquad f \in X.$$

Let $n \geq 1$ and X_n be the linear space generated in X by the characteristic functions of the dyadic intervals of order n. From the linear independence of the Walsh functions and from the fact that w_k is constant on the dyadic intervals of order n for $1 \leq k \leq 2^n$, we have that X_n coincides with the linear space generated by $\{w_k : k = 1, \ldots, 2^n\}$, that is,

$$X_n = \big\langle \{\chi_{I_k^n} : k = 1, \ldots, 2^n\} \big\rangle = \big\langle \{w_k : k = 1, \ldots, 2^n\} \big\rangle.$$

Let $w_{m_1}, \ldots, w_{m_N}$ be the Walsh functions of the sequence (w_{n_k}) with order less or equal than n. Note that N and $m_1, \ldots m_N$ depend on n. Denote by Q_n the restriction of Q to X_n, $Q_n : X_n \to X_n$, that is,

$$Q_n(f) := \sum_{k=1}^{N} q_{m_k}(f) w_{m_k}, \qquad f \in X_n. \tag{1.13}$$

For $1 \leq j, k \leq 2^n$, denote

$$\theta_{j,k} := w_j(I_k^n).$$

Let $T_j : X_n \to X_n$ be the operator defined by

$$T_j(w_k) := \theta_{j,k} w_k.$$

Since X is r.i. and the distribution function of f and $T_j(f)$ coincide for $f \in X_n$, we have $\|T_j\| = 1$ for $1 \leq j \leq 2^n$.

Denote by $P_n : X_n \to X_n$ the restriction of the projection in (1.10) to X_n, that is,

$$P_n(f) := \sum_{k=1}^{N} \langle w_{m_k}, f \rangle w_{m_k}, \qquad f \in X_n.$$

We will prove that

$$P_n = \frac{1}{2^n} \sum_{j=1}^{2^n} T_j Q_n T_j. \tag{1.14}$$

From (1.10),

$$P_n(w_i) = \sum_{k=1}^{N} \langle w_{m_k}, w_i \rangle w_{m_k}.$$

Since $\langle w_{m_k}, w_i \rangle = \delta_{m_k,i}$, it follows that $P_n(w_i) = w_i$ for $i \in \{m_1, \ldots, m_N\}$. Otherwise, $P_n(w_i) = 0$.

On the other hand,

$$\begin{aligned}\frac{1}{2^n}\sum_{j=1}^{2^n} T_j Q_n T_j(w_i) &= \frac{1}{2^n}\sum_{j=1}^{2^n} T_j\Big(\sum_{k=1}^{N} q_{m_k}(\theta_{j,i}w_i)w_{m_k}\Big)\\ &= \sum_{k=1}^{N} q_{m_k}(w_i)\Big(\frac{1}{2^n}\sum_{j=1}^{2^n}\theta_{j,i}\theta_{j,m_k}\Big)w_{m_k}.\end{aligned}$$

Since Q_n is a projection, we have from (1.13) that $q_{m_k}(w_i) = \delta_{m_k,i}$. Thus, (1.14) follows from

$$\frac{1}{2^n}\sum_{j=1}^{2^n}\theta_{j,i}\theta_{j,m_k} = \delta_{i,m_k}.$$

Given integers k, n and m, we write $k = n \oplus m$ whenever the following relation holds:

$$w_k = w_n w_m.$$

Then, from Lemma 1.22,

$$\theta_{j,i}\theta_{j,m_k} = w_j(I_i^n)w_j(I_{m_k}^n) = w_i(I_j^n)w_{m_k}(I_j^n) = w_{i\oplus m_k}(I_j^n),$$

and so it follows that

$$\begin{aligned}\frac{1}{2^n}\sum_{j=1}^{2^n}\theta_{j,i}\theta_{j,m_k} &= \frac{1}{2^n}\sum_{j=1}^{2^n} w_{i\oplus m_k}(I_j^n)\\ &= \frac{1}{2^n}\sum_{j=1}^{2^n}\frac{1}{m(I_j^n)}\int_{I_j^n} w_{i\oplus m_k}\,dm\\ &= \int_0^1 w_{i\oplus m_k}\,dm = \delta_{i,m_k},\end{aligned}$$

which proves (1.14).

Let us see that the operators $P_n : X_n \to X_n$ have uniformly bounded norm (in

n). From (1.14), since $\|T_j\| = 1$, we have for any $f \in X_n$ that

$$\begin{aligned}
\|P_n f\|_{X_n} &= \Big\| \frac{1}{2^n} \sum_{j=1}^{2^n} T_j Q_n T_j f \Big\|_{X_n} \\
&\le \frac{1}{2^n} \sum_{j=1}^{2^n} \|T_j Q_n T_j f\|_{X_n} \\
&\le \frac{1}{2^n} \sum_{j=1}^{2^n} \|Q_n\| \|f\|_{X_n} \\
&\le \|Q\| \|f\|_{X_n},
\end{aligned}$$

that is, $\|P_n f\| \le \|Q\|$ for $n \ge 1$.

Now we show that $G \subset X'$ follows from the fact that $P_n : X_n \to X_n$ has uniformly bounded norm. Denote by $T_n : X_n \to \ell^2$ the operator given by

$$T_n f := (\langle w_{m_k}, f\rangle)_{k=1}^N, \qquad f \in X_n.$$

From $X \subset L^1$, there exists a constant $C_1 > 0$ such that $C_1 \|f\|_{L^1} \le \|f\|_X$ for all $f \in X$. Together with Theorem 1.12 for $p = 1$,

$$\begin{aligned}
C_1 A(1,q) \Big(\sum_{k=1}^N \langle w_{m_k}, f\rangle^2 \Big)^{1/2} &\le C_1 \Big\| \sum_{k=1}^N \langle w_{m_k}, f\rangle w_{m_k} \Big\|_{L^1} \\
&\le \Big\| \sum_{k=1}^N \langle w_{m_k}, f\rangle w_{m_k} \Big\|_X \\
&= \|P_n f\|_{X_n} \\
&\le \|Q\| \|f\|_X.
\end{aligned}$$

It follows, for some constant $C_2 > 0$, that

$$\|T_n f\|_2 \le C_2 \|f\|_{X_n}, \qquad f \in X_n,$$

for $n \ge 1$. Thus, the adjoint operator

$$T_n' : \ell^2 \to X_n'$$

is bounded with $\|T_n'\| \le C_2$, $n \ge 1$. Since, for any $(b_k) \in \ell^2$,

$$T_n'(b_k) = \sum_{k=1}^N b_k w_{m_k},$$

we have, as in the proof of Lemma 1.21, that

$$\Big\| \sum_{k=1}^{N} b_k w_{m_k} \Big\|_{X'} \leq \|T_n'\| \Big(\sum_{k=1}^{N} b_k^2 \Big)^{1/2} \leq C_2 \Big(\sum_{k=1}^{N} b_k^2 \Big)^{1/2}.$$

Since this inequality holds for all $n \geq 1$, we have

$$\Big\| \sum_{k \geq 1} b_k w_{n_k} \Big\|_{X'} \leq C_2 \Big(\sum_{k \geq 1} b_k^2 \Big)^{1/2}.$$

This inequality, together with the embedding $X' \subset L^1$ and Theorem 1.12, shows that the subspace $[w_{n_k}]_{X'}$ is isomorphic to ℓ^2. From Theorem 1.18, this is equivalent to $G \subset X'$.

Now we show that $G \subset X$. Let $n \geq 1$. For $f \in X_n$ and $g \in X_n'$,

$$\begin{aligned} \int_0^1 P_n(f)\, g\, dm &= \int_0^1 \Big(\sum_{k=1}^{N} \langle w_{m_k}, f \rangle w_{m_k} \Big) g\, dm \\ &= \sum_{k=1}^{N} \langle w_{m_k}, f \rangle \langle w_{m_k}, g \rangle \\ &= \int_0^1 f\, P_n(g)\, dm. \end{aligned}$$

Thus,

$$\Big| \int_0^1 f\, P_n(g)\, dm \Big| = \Big| \int_0^1 P_n(f)\, g\, dm \Big| \leq \|P_n(f)\|_X \|g\|_{X'} \leq \|Q\| \|f\|_X \|g\|_{X'}.$$

It follows that

$$\begin{aligned} \|P_n g\|_{X_n'} &= \sup \Big\{ \Big| \int_0^1 f\, P_n(g)\, dm \Big| : \|f\|_{X_n} \leq 1 \Big\} \\ &\leq \|Q\| \|g\|_{X_n'}. \end{aligned}$$

The argument above, together with X_n' in place of X_n, shows that $G \subset X''$ follows from the fact that $P_n : X_n' \to X_n'$ is uniformly bounded. Thus, for the separable parts we have that $G_0 \subset (X'')_0$. Since $G_0 = G$ and $(X'')_0 = X_0$, we have

$$G = G_0 \subset (X'')_0 = X_0 \subset X,$$

which concludes the proof. □

Chapter 2

Local rearrangement invariant spaces

In the previous chapter we have shown local results for Rademacher series and lacunary Walsh series. We consider the local norm for $E \subset [0,1]$ a set of positive measure in the spaces L^p,

$$\|f\|_{L^p(E,dm/m(E))} := \Big(\int_E |f(t)|^p \,\frac{dt}{m(E)} \Big)^{1/p},$$

and in the Orlicz spaces L^{M_p} for $1 \le p \le 2$,

$$\|f\|_{L^{M_p}(E,dt/m(E))} := \inf \Big\{ \lambda > 0 : \int_E M_p(f(t)/\lambda) \frac{dt}{m(E)} \le 1 \Big\}.$$

Let X be an r.i. space on $[0,1]$, and $E \subset [0,1]$ a set of positive measure. In this chapter we address the question of giving a definition of the "local norm" of X on E. This was done by Astashkin and Curbera in [5], by means of the fundamental function φ_X of the space X. The disadvantage of that definition is that only allows a local version of Khintchine inequality in the case when the lower dilation index γ_X is strictly positive (Theorem 2.11). We give a different definition of local r.i. space, which we denote by $X|E$. The situation has the difficulty that an explicit expression of the norm of X may not be available.

The space $X|E$ we define is isomorphic to X and compatible with the rearrangement of functions (Proposition 2.4) and with the notion of systems equivalent in distribution (Proposition 2.9). We give explicit expressions of the norm of the local space $X|E$ for the Lebesgue spaces L^p, the Orlicz spaces L^Φ, the spaces $L^{p,q}$, the

Lorentz space $\Lambda(X)$, the Marcinkiewicz space $M(X)$ and for the spaces generated by the K–method of interpolation.

The space $X|E$ generalizes the local norms of L^p and L^{M_p}. This allows us to show, for X an r.i. space on $[0,1]$ with $G \subset X$, that the closed linear subspace generated by the Rademacher functions (r_{k+N}) in $X|E$ is isomorphic to ℓ^2 (Proposition 2.10). The problem of giving a local version of Rodin and Semenov's theorem (Theorem 1.3) for $X|E$ will be addressed in Chapter 3.

2.1 Local rearrangement invariant spaces

Let $\mathcal{M}$ be the σ–algebra of all Lebesgue measurable sets of $[0,1]$, and $E \subset [0,1]$ a set of positive measure. Denote by $\mathcal{M}_E$ the σ–algebra

$$\mathcal{M}_E := \{A \cap E : A \text{ is a measurable set}, A \subset [0,1]\},$$

and by m_E the measure

$$m_E(A) := m(A \cap E)/m(E), \qquad A \in \mathcal{M}.$$

Note that the measure m_E is defined on $\mathcal{M}$. The restriction of m_E to $\mathcal{M}_E$ yields a probability space $(E, \mathcal{M}_E, m_E)$, which will be referred to as (E, m_E).

Let $\rho_E : E \to [0,1]$ be the map defined by

$$\rho_E(x) := m_E(E \cap [0,x]), \qquad x \in E.$$

Proposition 2.1. *Let $E \subset [0,1]$ be a set of positive measure. For $B \subset E$ with $B \in \mathcal{M}_E$ and $A \subset [0,1]$ with $A \in \mathcal{M}$, we have*

(i) $m_E(\rho_E^{-1}\rho_E(B)) = m_E(B)$.

(ii) $m_E(\rho_E^{-1}(A)) = m(A)$.

(iii) $m_E(B) = m(\rho_E(B))$.

(iv) There exist sets $A_1 \subset E$ with $A_1 \in \mathcal{M}_E$ and $A_2 \subset [0,1]$ with $A_2 \in \mathcal{M}$ with $m_E(A_1) = m(A_2) = 0$ for which the map

$$\rho_E : E \setminus A_1 \to [0,1] \setminus A_2$$

is bijective.

(v) For $B \in \mathcal{M}_E$ and f a measurable function on E,

$$\int_B f \, dm_E = \int_{\rho_E(B)} (f \circ \rho_E^{-1}) \, dm.$$

Proof. Consider $\rho\colon [0,1] \to [0,1]$, the extension of the mapping ρ_E to $[0,1]$,

$$\rho : x \in [0,1] \mapsto \rho(x) := m_E([0,x]) = \frac{m(E \cap [0,x])}{m(E)}, \quad x \in [0,1].$$

It is continuous and (since $\rho(0) = 0$ and $\rho(1) = 1$) surjective. A constancy set of ρ is a set $D_z := \rho^{-1}(z) \subset [0,1]$ consisting of more than one element, with $z \in [0,1]$. Since ρ is increasing, the constancy sets are closed intervals. There are at most countably many constancy sets, $D_{z_n} = [x_n, y_n] = \rho^{-1}(z_n) \subset [0,1]$, with $z_n \in [0,1]$, $n \geq 1$. Set

$$D := \bigcup_{n \geq 1} D_{z_n}. \tag{2.1}$$

(i) Let $B \subset [0,1]$. For $x \in \rho^{-1}(\rho(B)) \setminus B$, we have $\rho(x) \in \rho(B)$ and $x \notin B$, so there exists $y \in B$, $y \neq x$, with $\rho(y) = \rho(x)$. Hence $x, y \in D_{z_n}$ for some n. It follows that

$$\rho^{-1}\rho(B) \setminus B \subseteq \bigcup_{B \cap D_{z_n} \neq \emptyset} D_{z_n},$$

and so, for $B \in \mathcal{M}_E$,

$$\rho_E^{-1}\rho_E(B) \setminus B \subseteq \bigcup_{B \cap D_{z_n} \neq \emptyset} D_{z_n} \cap E.$$

Hence, *(i)* follows from the fact that

$$m_E(D_{z_n}) = m_E([x_n, y_n]) = m_E([0, y_n]) - m_E([0, x_n)) = \rho_E(y_n) - \rho_E(x_n) = 0.$$

(ii) Consider a set of the form $A = (a, b]$. The sets $\rho^{-1}(a)$ and $\rho^{-1}(b)$ are either closed intervals or single points. Let z_a and z_b be the right endpoints of $\rho^{-1}(a)$ and $\rho^{-1}(b)$, respectively. Then $\rho^{-1}((a,b]) = (z_a, z_b]$. Thus,

$$\rho_E^{-1}((a,b]) = \rho^{-1}((a,b]) \cap E = (z_a, z_b] \cap E = ([0, z_b] \cap E) \setminus ([0, z_a] \cap E).$$

Since $z_a < z_b$,

$$m_E(\rho_E^{-1}(a,b]) = m_E([0, z_b] \cap E) - m_E([0, z_a] \cap E) = \rho(z_b) - \rho(z_a) = b - a.$$

Hence, the measures $A \mapsto m(A)$ and $A \mapsto m_E(\rho_E^{-1}(A))$ coincide on the semi–ring consisting of finite unions of half open–closed intervals. From the Hahn's extension theorem, both measures agree the Borel sets. Consider now a Lebesgue measurable set $A \subset [0,1]$. Then $A = B \cup N$, where B is a Borel set and $m(N) = 0$, and so

$$m_E(\rho_E^{-1}(B \cup N)) = m_E(\rho_E^{-1}(B) \cup \rho_E^{-1}(N)) = m_E(\rho_E^{-1}(B)) + m_E(\rho_E^{-1}(N)).$$

Let $\varepsilon > 0$, and choose an open set G with $N \subset G$ and $m(G) < \varepsilon$. Then, we have $m_E(\rho_E^{-1}(G)) = m(G) < \varepsilon$, and

$$m_E(\rho_E^{-1}(N)) \leq m_E(\rho_E^{-1}(G)) < \varepsilon.$$

It follows that $m_E(\rho_E^{-1}(N)) = 0$. Thus,

$$m_E(\rho_E^{-1}(B \cup N)) = m_E(\rho_E^{-1}(B)) = m(B) = m(B \cup N).$$

(iii) It follows from *(i)* and *(ii)*. Given $B \in \mathcal{M}_E$, set $A = \rho_E(B)$. Then,

$$m(\rho_E(B)) = m(A) = m_E(\rho_E^{-1}(A)) = m_E(\rho_E^{-1}\rho_E(B)) = m_E(B).$$

(iv) Let $A_1 := D \cap E$ and $A_2 := [0,1] \setminus \rho_E(E \setminus A_1)$ for the set D in (2.1). If $\rho_E(x) = \rho_E(y)$, then $x, y \in D$, and so the restriction of ρ_E to $E \setminus A_1$ is injective. Surjectivity follows from $[0,1]\setminus A_2 = \rho_E(E\setminus A_1)$. We also have $m_E(A_1) \leq m_E(D) = 0$ and, from *(iii)*,

$$m(A_2) = m([0,1]) - m(\rho_E(E \setminus A_1)) = 1 - m_E(E \setminus A_1) = 0.$$

(v) If follows from the Lebesgue measure m on $[0,1]$ being the image measure of m_E on E via the map ρ_E. □

Let $f\colon E \to \mathbb{R}$ be a measurable function and X an r.i. space on $[0,1]$. From Proposition 2.1,

$$f \circ \rho_E^{-1} : [0,1] \to \mathbb{R}$$

is well–defined as a class of functions. This allows us to define the space $X|E$ as follows.

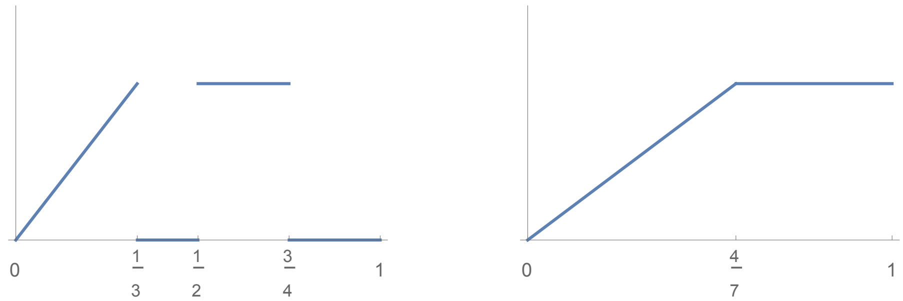
Definition 2.2. Let X be a Banach function space on $[0,1]$, and $E \subset [0,1]$ a set of positive measure. The space $X|E$ consists of all functions $f \in L^0(E, m_E)$ such that $f \circ \rho_E^{-1} \in X$, that is,

$$X|E := \left\{ f \in L^0(E, m_E) : f \circ \rho_E^{-1} \in X \right\}.$$

The norm in the space $X|E$ is

$$\|f\|_{X|E} := \|f \circ \rho_E^{-1}\|_X, \qquad f \in X|E.$$

Consider the set $E = (0, 1/3) \cup (1/2, 3/4)$. The graphic below shows a function f on E and the corresponding function $f \circ \rho_E^{-1}$ on $[0, 1]$.



Note that the intervals of definition of f are proportional to the intervals of definition of $f \circ \rho_E^{-1}$, which relates the space $X|E$ with the dilation operator.

Definition 2.3. Let $E \subset [0, 1]$ be a set of positive measure. For f a measurable function on E, denote by $(m_E)_f$ its distribution function on (E, m_E), that is,

$$(m_E)_f(\lambda) := m_E(\{x \in E : |f(x)| > \lambda\}), \quad \lambda > 0.$$

The decreasing rearrangement of f on (E, m_E) is

$$f^*_{m_E}(x) := m(\{\lambda > 0 : (m_E)_f(\lambda) > x\}), \quad x \in [0, 1].$$

Let $E \subset [0, 1]$ be a set of positive measure and f a measurable function on E. We will denote by f^* the decreasing rearrangement in $([0, 1], m)$ of the function $f\chi_E$ that coincides with f in E and vanishes in $[0, 1] \setminus E$.

Recall the dilation operator σ_t in (5),

$$(\sigma_t f)(s) = f(ts), \qquad 0 \leq s, st \leq 1,$$

for f a measurable function on $[0, 1]$.

Proposition 2.4. *Let X be an r.i. space on $[0, 1]$ and $E \subset [0, 1]$ a set of positive measure. The following assertions hold.*

(i) For f a measurable function on E,

$$(f \circ \rho_E^{-1})^* = f^*_{m_E} = \sigma_{m(E)}(f^*),$$

where $\sigma_{m(E)}$ is the dilation operator by $m(E)$. Consequently,

$$\|f\|_{X|E} = \|f^*_{m_E}\|_X = \|\sigma_{m(E)}(f^*)\|_X.$$

(ii) The spaces $X|E$ and X are isometrically isomorphic via the mapping

$$f \in X|E \mapsto f \circ \rho_E^{-1} \in X.$$

(iii) $X|E$ is rearrangement invariant.

(iv) The fundamental functions of $X|E$ and X coincide.

(v) The associate space of $X|E$ is $X'|E$, with equality of norms.

Proof. *(i)* It suffices to prove *(i)* for characteristic functions. The general case follows considering step functions and a limiting argument. Let $B \in \mathcal{M}_E$. From Proposition 2.1,

$$(\chi_B \circ \rho_E^{-1})^* = (\chi_{\rho_E(B)})^* = \chi_{[0,m(\rho_E(B))]} = \chi_{[0,m_E(B)]}.$$

On the other hand,

$$(m_E)_{\chi_B}(\lambda) = m_E(\{x \in E : \chi_B(x) > \lambda\}) = \begin{cases} 0, & 1 < \lambda, \\ m_E(B), & 0 \le \lambda \le 1. \end{cases}$$

Hence,

$$(\chi_B)^*_{m_E} = \chi_{[0,m_E(B)]},$$

whereas, for $x \in [0,1]$,

$$\sigma_{m(E)}(\chi_B^*)(x) = \chi_B^*(m(E)x) = \chi_{[0,m(B)]}(m(E)x) = \chi_{[0,m_E(B)]}(x).$$

(ii) It follows from the definition of $X|E$, part *(i)* and Proposition 2.1 *(iv)*.

(iii) Consider equimeasurable functions f, g on E. From *(ii)*, since X is r.i.,

$$\|f\|_{X|E} = \|f^*_{m_E}\|_X = \|g^*_{m_E}\|_X = \|g\|_{X|E}.$$

(iv) For $0 < t < 1$, let $x \in (0,1)$ such that $m_E([0,x)) = t$. From Proposition 2.1, we have $m(\rho_E([0,x))) = m_E([0,x)) = t$, and so

$$\varphi_{X|E}(t) = \|\chi_{[0,x)}\|_{X|E} = \|\chi_{[0,x)} \circ \rho_E^{-1}\|_X = \|\chi_{\rho_E([0,x))}\|_X = \varphi_X(t).$$

(v) We show that $(X|E)' = X'|E$. For $f \in X$,

$$\|f \circ \rho_E\|_{X|E} = \|f \circ \rho_E \circ \rho_E^{-1}\|_X = \|f\|_X.$$

Hence, using Proposition 2.1 *(v)*,

$$\begin{aligned}\|g\|_{(X|E)'} &= \sup\Big\{\int_E |h(x)g(x)|\frac{dx}{m(E)} : \|h\|_{X|E} \le 1\Big\} \\ &= \sup\Big\{\int_0^1 |h(\rho_E^{-1}(t))\, g(\rho_E^{-1}(t))|\, dt : \|h\circ\rho_E^{-1}\|_X \le 1\Big\}.\end{aligned}$$

From the fact that X and $X|E$ are isometric, the last supremum coincides with the supremum on all functions $f \in X$ with $\|f\|_X \le 1$. Thus,

$$\|g\|_{(X|E)'} = \sup\Big\{\int_0^1 |f(t)\, g(\rho_E^{-1}(t))|\, dt : \|f\|_X \le 1\Big\} = \|g\circ\rho_E^{-1}\|_{X'} = \|g\|_{X'|E},$$

which completes the proof. □

Now we give, for some classical r.i. spaces X, the explicit expression of the corresponding local norm of $X|E$.

Example 2.5. In this example we consider the Lebesgue spaces L^p and the Orlicz spaces L^Φ generated by a Young function Φ. We show that $L^p|E$ coincides with the local space considered by Zygmund (see Theorem 1.5) and by Sagher and Zhou (see Theorem 1.6). We also show that $L^\Phi|E$ coincides with the local spaces considered by Sagher and Zhou (see Theorem 1.7) and by Carrillo–Alanís (see Theorem 1.9).

Let $E \subset [0,1]$ be a set of positive measure. From Proposition 2.1,

$$\|f\|_{L^p|E} = \Big(\int_0^1 |f\circ\rho_E^{-1}(t)|^p dt\Big)^{1/p} = \Big(\int_E |f(x)|^p \frac{dx}{m(E)}\Big)^{1/p},$$

for $1 \le p < \infty$.

In the case of an Orlicz space L^Φ, we have

$$\begin{aligned}\|f\|_{L^\Phi|E} &= \inf\Big\{\lambda > 0 : \int_0^1 \Phi(|f\circ\rho_E^{-1}(t)|/\lambda)\, dt \le 1\Big\} \\ &= \inf\Big\{\lambda > 0 : \int_E \Phi(|f(x)|/\lambda)\frac{dx}{m(E)} \le 1\Big\}.\end{aligned}$$

Example 2.6. In the case of the space $L^{p,q}$,

$$\|f\|_{L^{p,q}|E} = \Big(\int_0^1 \big(t^{1/p}(f\circ\rho_E^{-1})^*(t)\big)^q \frac{dt}{t}\Big)^{1/q} = \Big(\int_0^1 \big(t^{1/p} f^*_{m_E}(t)\big)^q \frac{dt}{t}\Big)^{1/q}.$$

Example 2.7. Let us see that the Lorentz space $\Lambda(X)|E$ and the Marcinkiewicz space $M(X)|E$ coincide, respectively, with the spaces $\Lambda(X|E)$ and $M(X|E)$. Since $\varphi_X = \varphi_{X|E}$,

$$\begin{aligned}\|f\|_{\Lambda(X)|E} &= \|f \circ \rho_E^{-1}\|_{\Lambda(X)} \\ &= \int_0^1 (f \circ \rho_E^{-1})^*(t)\varphi_X'(t)\,dt \\ &= \int_0^1 f^*_{m_E}(t)\varphi_{X|E}'(t)\,dt \\ &= \|f\|_{\Lambda(X|E)}.\end{aligned}$$

We also have, from Proposition 2.4,

$$\begin{aligned}\|f\|_{M(X)|E} &= \|f \circ \rho_E^{-1}\|_{M(X)} \\ &= \sup_{0\le t\le 1} \varphi_X(t)\frac{1}{t}\int_0^t (f \circ \rho_E^{-1})^*(s)\,ds \\ &= \sup_{0\le t\le 1} \varphi_{X|E}(t)\frac{1}{t}\int_0^t f^*_{m_E}(s)\,ds \\ &= \|f\|_{M(X|E)}.\end{aligned}$$

Example 2.8. In this example we show that, in the sense specified below, the construction of the local space $X|E$ is compatible with the K–method of interpolation.

Let (X_0, X_1) be a Banach couple on $[0,1]$, and

$$X = (X_0, X_1)^K_F$$

an interpolation space between X_0 and X_1, generated by a discrete parameter F of the K–method of interpolation.

Since $f \mapsto f \circ \rho_E^{-1}$ is an isometric isomorphism from $X|E$ onto X, we have that

$$\begin{aligned}&K(f,t;X_0|E, X_1|E) \\ &\quad= \inf\left\{\|f_0\|_{X_0|E} + t\|f_1\|_{X_1|E} : f = f_0 + f_1 \in X_0|E + X_1|E\right\} \\ &\quad= \inf\left\{\|f_0'\|_{X_0} + t\|f_1'\|_{X_1} : f \circ \rho_E^{-1} = f_0' + f_1' \in X_0 + X_1\right\} \\ &\quad= K(f \circ \rho_E^{-1}, t; X_0, X_1).\end{aligned}$$

Hence,

$$\|f\|_{(X_0|E, X_1|E)^K_F} = \|f \circ \rho_E^{-1}\|_{(X_0,X_1)^K_F} = \|f \circ \rho_E^{-1}\|_X = \|f\|_{X|E},$$

that is, the space $X|E$ coincides with $(X_0|E, X_1|E)^K_F$.

Recall, from Definition 1.16, that a system of random variables (φ_k) on a probability space (R, μ) is majorized in distribution by a system (ψ_k) on a probability space (S, ν) if there exists a constant $C > 0$ such that

$$\mu\Big(\Big\{x \in R : \Big|\sum_{k=1}^{m} a_k \varphi_k(x)\Big| > \lambda\Big\}\Big) \leq C\nu\Big(\Big\{t \in S : \Big|\sum_{k=1}^{m} a_k \psi_k(t)\Big| > C^{-1}\lambda\Big\}\Big).$$

This situation is denoted by $(\varphi_k) \prec (\psi_k)$. The systems (φ_k) and (ψ_k) are equivalent in distribution if $(\varphi_k) \prec (\psi_k)$ and $(\psi_k) \prec (\varphi_k)$.

Next we show that the construction of the local space $X|E$ is compatible with the notion of systems equivalent in distribution.

Proposition 2.9. *Let X be an r.i. space on $[0,1]$, $E \subset [0,1]$ a set of positive measure. Let (φ_k) and (ψ_k) be systems on $([0,1], m)$ and (E, m_E), respectively. If (φ_k) and (ψ_k) are equivalent in distribution with constants $(\psi_k) \prec_{C_1} (\varphi_k)$ and $(\varphi_k) \prec_{C_2} (\psi_k)$, then*

$$\frac{1}{C_1^2}\Big\|\sum_{k=1}^{n} a_k \psi_k\Big\|_{X|E} \leq \Big\|\sum_{k=1}^{n} a_k \varphi_k\Big\|_X \leq C_2^2\Big\|\sum_{k=1}^{n} a_k \psi_k\Big\|_{X|E}$$

for all $n \geq 1$ and $a_1, \dots, a_n \in \mathbb{R}$.

Proof. Assume that

$$\begin{aligned} &m\Big(\Big\{t \in [0,1] : \Big|\sum_{k=1}^{n} a_k \varphi_k(t)\Big| > \lambda\Big\}\Big) \\ &\qquad \leq C m_E\Big(\Big\{x \in E : \Big|\sum_{k=1}^{n} a_k \psi_k(x)\Big| > C^{-1}\lambda\Big\}\Big) \end{aligned} \tag{2.2}$$

for $\lambda > 0$, $n \geq 1$ and $a_1, \dots, a_n \in \mathbb{R}$. Denote

$$f = \sum_{k=1}^{n} a_k \varphi_k \qquad g = \sum_{k=1}^{n} a_k \psi_k,$$

so that (2.2) is

$$m_f(\lambda) \leq C(m_E)_{Cg}(\lambda), \quad \lambda > 0.$$

Hence,

$$\begin{aligned} m(\{\lambda > 0 : m_f(\lambda) > t\}) &\leq m(\{\lambda > 0 : C(m_E)_{Cg}(\lambda) > t\}) \\ &= m(\{\lambda > 0 : (m_E)_{Cg}(\lambda) > t/C\}). \end{aligned}$$

It follows, for $0 \leq t \leq 1$, and the dilation operator $\sigma_{1/C}$, that

$$f_m^*(t) \leq (Cg)_{m_E}^*(t/C) = C \cdot \sigma_{1/C}(g_{m_E}^*)(t).$$

From Proposition 2.4 *(i)* we have, for $h_X(t) = \|\sigma_t\|_{X\to X}$,

$$\begin{aligned}\|f\|_X &\leq C\|\sigma_{1/C}(g_{m_E}^*)\|_X \\ &\leq C\|\sigma_{1/C}\|_{X\to X}\|g_{m_E}^*\|_X \\ &= Ch_X(C)\|g\|_{X|E}.\end{aligned}$$

Since $h_X(t) \leq \max(1, t)$, it follows that

$$\Big\|\sum_{k=1}^n a_k\varphi_k\Big\|_X \leq C^2\Big\|\sum_{k=1}^n a_k\psi_k\Big\|_{X|E}.$$

The opposite inequality follows analogously. □

2.2 Local versions of Khintchine inequality II

Next we focus on the local versions of Khintchine inequality. The definition of $X|E$ allows us to give the following result, which generalizes the local results for L^p and L^{M_p} in the previous chapter. It also applies to other r.i. spaces X than the ones considered in Example 2.5 and Example 2.7, provided that $G \subset X$.

The following change of notation will prove to be meaningful in the next chapter. We will write the tail of a Rademacher series

$$\sum_{k\geq N} a_k r_k$$

in the following way,

$$\sum_{k\geq 1} a_k r_{k+N}.$$

Proposition 2.10. *Let X be an r.i. space on $[0,1]$ with $G \subset X$. There exist constants $A'_X, B'_X > 0$ such that, for every set $E \subset [0,1]$ with positive measure, there exists $N = N(E)$ for which*

$$A'_X\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k r_{k+N}\Big\|_{X|E} \leq B'_X\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

Proof. From the fact that any r.i. space X on $[0,1]$ is embedded into L^1, we have

$$C_1\|f\|_{L^1|E} \le \|f\|_{X|E}, \qquad f \in X|E,$$

for some constant $C_1 > 0$. From Theorem 1.6 for $p = 1$, there exists N such that

$$C_1 A_1'\Big(\sum_{k\ge 1} a_k^2\Big)^{1/2} \le C_1\Big\|\sum_{k\ge 1} a_k r_{k+N}\Big\|_{L^1|E} \le \Big\|\sum_{k\ge 1} a_k r_{k+N}\Big\|_{X|E}$$

for all $(a_k)_1^\infty \in \ell^2$.

In order to prove the right–hand side inequality, since $G \subset X$, we have that, for some constant $C_2 > 0$,

$$\|f\|_{X|E} \le C_2\|f\|_{L^{M_2}|E}, \qquad f \in L^\infty|E.$$

In particular, for all M, it follows from the local version of Khintchine inequality for L^{M_2} (Theorem 1.9) that, for some N depending on E,

$$\Big\|\sum_{k=1}^{M} a_k r_{k+N}\Big\|_{X|E} \le C_2\Big\|\sum_{k=1}^{M} a_k r_{k+N}\Big\|_{L^{M_2}|E} \le C_2 B_{M_2}'\Big(\sum_{k=1}^{M} a_k^2\Big)^{1/2}.$$

which completes the proof. □

According to Astashkin and Curbera, an r.i. space X with fundamental function φ_X satisfies the local version of Khintchine inequality if there exist constants $A_X^\varphi, B_X^\varphi > 0$ such that for any measurable set $E \subset [0,1]$ with positive measure, there exists $N = N(E)$ such that

$$\begin{aligned} A_X^\varphi\,\varphi_X(m(E))\Big(\sum_{k\ge 1} a_k^2\Big)^{1/2} &\le \Big\|\chi_E\sum_{k\ge 1} a_k r_{k+N}\Big\|_X \\ &\le B_X^\varphi\,\varphi_X(m(E))\Big(\sum_{k\ge 1} a_k^2\Big)^{1/2}. \end{aligned} \tag{2.3}$$

The spaces X for which the local version of Khintchine inequality holds have been characterized in [5, Theorem 4.2].

Theorem 2.11 (Astashkin and Curbera)**.** *Let X be an r.i. space on $[0,1]$ with $X \ne L^\infty$. The following conditions are equivalent:*

(i) Inequality (2.3) holds.

(ii) The fundamental index of X satisfies $\gamma_{\varphi_X} > 0$.

The fact that the constants in the local versions of Khintchine inequality in the previous chapter are independent of the set E suggests considering the space $X(E)$ of all measurable functions f on E for which the norm

$$\|f\|_{X(E)} := \frac{1}{\varphi_X(m(E))}\|f\chi_E\|_X$$

is finite. Inequality (2.3) is then equivalent to

$$A_X^{\varphi}\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k r_{k+N}\Big\|_{X(E)} \leq B_X^{\varphi}\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2}. \tag{2.4}$$

For $X = L^p$, with $1 \leq p < \infty$, since $\varphi_{L^p}(t) = t^{1/p}$, we have

$$\begin{aligned}\|f\|_{X|E} &= \Big(\int_E |f(t)|^p \frac{dt}{m(E)}\Big)^{1/p} \\ &= \frac{1}{\varphi_X(m(E))}\Big(\int_0^1 \chi_E(t)|f(t)|^p\Big)^{1/p} \\ &= \|f\|_{X(E)}.\end{aligned}$$

Thus, for $X = L^p$, the spaces $X(E)$ and $X|E$ coincide, and so Theorem 2.11 is equivalent to Theorem 1.6 and Proposition 2.10 for $X = L^p$. On the other hand, since the fundamental index of the space L^{M_2} is $\gamma_{L^{M_2}} = 0$, the norm

$$\Big\|\sum_{k\geq 1} a_k r_{k+N}\Big\|_{L^{M_2}(E)}$$

is not equivalent to $\|(a_k)_1^\infty\|_2$ for any N. Since Theorem 1.9 shows that

$$A'_{M_2}\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k r_{k+N}\Big\|_{L^{M_2}|E} \leq B'_{M_2}\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2}$$

for some constants $A'_{M_2}, B'_{M_2} > 0$ and $N = N(E)$, it follows in general that the spaces $X|E$ and $X(E)$ are not isomorphic.

Next, we prove that the equivalence between the norm of the dilation operator h_X and the fundamental function φ_X is related to $X(E)$ being isomorphic to $X|E$. This is natural, since the space $X|E$ is related to the norm of dilation operators and the norm of $X(E)$ depends directly on the fundamental function of X.

Recall the definition of the lower fundamental index γ_X of an r.i. space X over (R,μ),

$$\gamma_X := \lim_{t\to 0^+} \frac{\log M_{\varphi_X}(t)}{\log t},$$

where

$$M_{\varphi_X}(t) := \sup_{0<s,st<\mu(R)} \frac{\varphi_X(st)}{\varphi_X(s)}.$$

Proposition 2.12. *Let X be an r.i. space on $[0,1]$ with the Fatou property, and X' be its associate space. The following assertions hold.*

(i) For every set of positive measure $E \subset [0,1]$ the continuous embedding $X|E \subset X(E)$ holds, that is,

$$\|f\|_{X(E)} \le C\|f\|_{X|E}, \quad f \in X|E,$$

for some $C>0$ if and only if

$$h_X(t) \le C\varphi_X(t), \quad 0 \le t \le 1.$$

(ii) For every set of positive measure $E \subset [0,1]$ the continuous embedding $X(E) \subset X|E$ holds, that is,

$$\|f\|_{X|E} \le C\|f\|_{X(E)}, \quad f \in X(E),$$

for some $C>0$ if and only if

$$h_{X'}(t) \le C\varphi_{X'}(t), \quad 0 \le t \le 1.$$

(iii) Let $E \subset [0,1]$ be a set of positive measure. The spaces $X|E$ and $X(E)$ are isomorphic if and only if there exists a constant $C>0$ such that

$$C^{-1}\varphi_X(t) \le h_X(t) \le C\varphi_X(t), \qquad C^{-1}\varphi_{X'}(t) \le h_{X'}(t) \le C\varphi_{X'}(t),$$

for $0 \le t \le 1$.

Proof. *(i)* Suppose that $h_X \le C\varphi_X$. Set $t = m(E)$ and let $f \in X|E$. From Proposition 2.4, we have $(f\circ\rho_E^{-1})^* = \sigma_t(f\chi_E)^*$. Then,

$$\begin{aligned}\|f\chi_E\|_X &= \|(f\chi_E)^*\|_X = \|\sigma_{1/t}\sigma_t(f\chi_E)^*\|_X \\ &\le h_X(t)\,\|\sigma_t(f\chi_E)^*\|_X = h_X(t)\,\|(f\circ\rho_E^{-1})^*\|_X \\ &\le C\varphi_X(t)\,\|f\|_{X|E}.\end{aligned}$$

Conversely, let $0 \le t \le 1$ and choose $E = [0,t]$. The inequality

$$\|f\|_{X([0,t])} \le C\|f\|_{X|[0,t]}$$

is equivalent, for $f \in X|[0,t]$, to

$$\frac{1}{\varphi_X(t)}\|f\|_X \le C\|f\|_{X|[0,t]}.$$

From Proposition 2.4 it follows, for $f \in X|[0,t]$, that

$$\|f\|_X \le C\varphi_X(t)\|\sigma_t(f)\|_X.$$

Let $g \in X$. Then, $\sigma_{1/t}g \in X|[0,t]$, and

$$\|\sigma_{1/t}(g)\|_X \le C\varphi_X(t)\|\sigma_t\sigma_{1/t}(g)\|_X = C\varphi_X(t)\|g\|_X.$$

Taking supremum over all $g \in X$ with $\|g\|_X \le 1$ we arrive at

$$h_X(t) \le C\varphi_X(t).$$

(ii) Applying part *(i)* with X' in place of X we have that $h_{X'}(t) \le C\varphi_{X'}(t)$ if and only if $\|f\|_{X'|E} \le \|f\|_{X'(E)}$. From Proposition 2.4 we have $(X|E)' = X'|E$. It is straightforward to verify that $(X(E))' = X'(E)$. Hence, $h_{X'}(t) \le C\varphi_{X'}(t)$ holds if and only if $\|f\|_{(X|E)'} \le C\|f\|_{(X(E))'}$. The equivalence to $\|f\|_{X(E)} \le C\|f\|_{X|E}$ follows from the assumption of X satisfying the Fatou property, which implies that $X'' = X$, and from the fact that, for any r.i. spaces X and Y with the Fatou property, $X \subset Y$ is equivalent to $Y' \subset X'$.

(iii) It is a consequence of *(i)*, *(ii)* and the inequality

$$\varphi_X(t) = \|\chi_{(0,t)}\|_X = \|\sigma_{1/t}\chi_{(0,1)}\|_X \le \varphi_X(1)h_X(t).$$

The corresponding inequality for X' is symmetric. □

In the case when $X = L^\infty$, we have $\varphi_{L^\infty}(t) = 1$ for $0 < t \le 1$. Thus, given any $f \in L^0(E, m_E)$,

$$\|f\|_{L^\infty|E} = \operatorname*{ess\,sup}_{t\in[0,1]}(f \circ \rho_E^{-1})(t) = \operatorname*{ess\,sup}_{x\in E} f(x) = \|f\|_{L^\infty(E)},$$

that is, the spaces $L^\infty|E$ and $L^\infty(E)$ coincide with equality of norms.

In the next result we show that, given any r.i. space on $[0,1]$, the condition $\gamma_X > 0$ is necessary for $X|E$ and $X(E)$ to be isomorphic.

Corollary 2.13. *Let X be an r.i. space on $[0,1]$, $X \neq L^\infty$, and $E \subset [0,1]$ a set of positive measure. If the spaces $X|E$ and $X(E)$ are isomorphic, then the fundamental functions φ_X and $\varphi_{X'}$ are both equivalent to multiplicative functions. In particular, $\gamma_X > 0$ and $\gamma_{X'} > 0$.*

Proof. Assume that $X|E$ and $X(E)$ are isomorphic. Let $0 < s, t \leq 1$. From Proposition 2.12 *(i)*,

$$\varphi_X(ts) = \|\chi_{[0,st]}\|_X = \|\sigma_{1/t}\chi_{[0,s]}\|_X \leq h_X(t)\varphi_X(s) \leq C\varphi_X(t)\varphi_X(s).$$

For the opposite inequality, set $u = st$. Recall that $\varphi_X(s)\varphi_{X'}(s) = s$ (see [10, Theorem II.5.2]) and $h_X(s) = sh_{X'}(1/s)$ (see [10, Proposition III.5.11]). This, together with $h_{X'}(s) \leq C\varphi_{X'}(s)$ yields

$$h(1/s)\varphi_X(s) \leq C.$$

Then,

$$\begin{aligned}\varphi_X(t)\varphi_X(s) &= \varphi_X(u/s)\varphi_X(s) = \|\chi_{[0,u/s]}\|_X\varphi_X(s) \\ &= \|\sigma_s\chi_{[0,u]}\|_X\varphi_X(s) \leq h_X(1/s)\varphi_X(u)\varphi_X(s) \\ &\leq C\varphi_X(st).\end{aligned}$$

Hence, φ_X is equivalent to a multiplicative function.

The relation $\varphi_X(s)\varphi_{X'}(s) = s$, together with φ_X being equivalent to a multiplicative function, implies that $\varphi_{X'}$ is also equivalent to a multiplicative function.

We complete the proof showing that if φ_X is equivalent to a multiplicative function, then $\gamma_{\varphi_X} > 0$. From $\varphi_X(st) \geq C\varphi_X(t)\varphi_X(s)$ it follows that

$$M_{\varphi_X}(t) \geq C\varphi_X(t).$$

Choose $t_0 \in (0,1)$ such that $C\varphi_X(t_0) < 1$ (which exists if $X \neq L^\infty$). Since

$$\varphi_X(t_0^n) \geq C^{n-1}\varphi_X(t_0)^n, \qquad n \geq 1,$$

we have

$$\begin{aligned}\gamma_X &= \lim_{t\to 0^+}\frac{\log M_X(t)}{\log t} \geq \lim_{t\to 0^+}\frac{\log C\varphi_X(t)}{\log t} \\ &= \lim_{n\to\infty}\frac{\log C\varphi_X(t_0^n)}{\log t_0^n} \geq \lim_{n\to\infty}\frac{\log(C^n\varphi_X(t_0)^n)}{\log t_0^n} \\ &= \frac{\log C\varphi_X(t_0)}{\log t_0} > 0,\end{aligned}$$

which completes the proof. □

Chapter 3

Local distribution of Rademacher series

Consider an r.i. space X on $[0,1]$ with $G \subset X$. Proposition 2.10 shows that, for any measurable set E of positive measure, there exists $N = N(E)$ such that

$$A'_X \Big(\sum_{k \geq 1} a_k^2 \Big)^{1/2} \leq \Big\| \sum_{k \geq N} a_k r_{k+N} \Big\|_{X|E} \leq B'_X \Big(\sum_{k \geq 1} a_k^2 \Big)^{1/2}, \qquad (3.1)$$

for some constants A'_X, B'_X and for $(a_k)_1^\infty \in \ell^2$. On the other hand, we have from Theorem 1.3 that

$$A_X \Big(\sum_{k \geq 1} a_k^2 \Big)^{1/2} \leq \Big\| \sum_{k \geq N} a_k r_{k+N} \Big\|_{X} \leq B_X \Big(\sum_{k \geq 1} a_k^2 \Big)^{1/2},$$

for some constants A_X, B_X and for $(a_k)_1^\infty \in \ell^2$. It follows that, for any r.i. space X with $G \subset X$, there exist constants $C_{1,X}, C_{2,X}$ such that

$$C_{1,X} \Big\| \sum_{k \geq 1} a_k r_k \Big\|_X \leq \Big\| \sum_{k \geq 1} a_k r_{k+N} \Big\|_{X|E} \leq C_{2,X} \Big\| \sum_{k \geq 1} a_k r_k \Big\|_X, \qquad (3.2)$$

for $(a_k)_1^\infty \in \ell^2$. The question arises whether (3.2) still holds for r.i. spaces X for which $G \not\subseteq X$.

Our strategy to address this problem focuses on using the distribution function of the Rademacher series rather than on the norm of the spaces. Suppose that E is a finite union of dyadic intervals of order N. From the dilation properties of the Rademacher functions, we have

$$m_E\Big(\Big\{ x \in E : \Big| \sum_{k=1}^{M} a_k r_{k+N}(x) \Big| > \lambda \Big\} \Big) = m\Big(\Big\{ t \in [0,1] : \Big| \sum_{k=1}^{M} a_k r_k(t) \Big| > \lambda \Big\} \Big), \quad (3.3)$$

for all $\lambda > 0$, that is, (r_{k+N}) on (E, m_E) is equivalent in distribution to (r_k) on $([0,1], m)$. Together with Proposition 2.9, we have that, for any r.i. space X,

$$\Big\| \sum_{k\geq 1} a_k r_{k+N} \Big\|_{X|E} = \Big\| \sum_{k\geq 1} a_k r_k \Big\|_X. \tag{3.4}$$

In particular, it follows that (3.4) holds whenever E is a finite union of dyadic intervals and X is any r.i. space (regardless of the embedding $G \subset X$). The same argument shows that Proposition 2.10 (and the other local results in Chapter 2) can be deduced from (3.3) for E a finite union of dyadic intervals.

In this chapter we prove that the Rademacher system (r_k) on $([0,1], m)$ is equivalent in distribution to (r_{k+N}) on (E, m_E) for sets E satisfying a certain property (condition (3.8) below), which includes the open sets and the Peano–Jordan measurable sets (Proposition 3.4 and Proposition 3.5). The general case of an arbitrary set of positive measure remains an open problem.

The equivalence in distribution between (r_k) on $([0,1], m)$ and (r_{k+N}) on (E, m_E) allows us to extend (3.4) (with inequalities) to any r.i. space X, provided that E satisfies condition (3.8) below in Proposition 3.5. We also give a result on the independence of (r_{k+N}) on (E, m_E), which involves a version of the independence of random variables where multiplicative constants are allowed (Theorem 3.11).

We will use the K–functional in order to study the distribution of Rademacher series. The K–method of interpolation provides a description of the space generated by the Rademacher functions on the interpolation spaces X between G and L^∞. Let $a = (a_k)_1^\infty \in \ell^2$ and

$$Ra := \sum_{k\geq 1} a_k r_k.$$

From Theorem 1.3, there exist constants $A_G, B_G > 0$ such that

$$A_G \|a\|_2 \leq \|Ra\|_G \leq B_G \|a\|_2.$$

Together with the fact that

$$\|Ra\|_{L^\infty} = \|a\|_1,$$

it implies that

$$\begin{aligned} K(Ra, t; L^\infty, G) &= \inf \left\{ \|f\|_{L^\infty} + t\|g\|_G : Ra = f + g \in L^\infty + G \right\} \\ &\leq \inf \left\{ \|Rb\|_{L^\infty} + t\|Rc\|_G : Ra = Rb + Rc \in L^\infty + G \right\} \\ &\leq \inf \left\{ \|b\|_{\ell^1} + tB_G\|c\|_{\ell^2} : a = b + c \in \ell^1 + \ell^2 \right\} \\ &\leq B_G K(a, t; \ell^1, \ell^2). \end{aligned}$$

In [2] Astashkin establishes a one–to–one correspondence between the r.i. function spaces which are "close"to L^∞ (in the sense that they are interpolation spaces between L^∞ and G) and the sequence spaces generated by the Rademacher functions in these spaces (which correspond to interpolation spaces with respect to ℓ^1 and ℓ^2). The situation is thoroughly described in [3, Chapter 4].

The following result by Astashkin, [2, Theorem 1.2], establishes the equivalence between $K(Ra, t; L^\infty, G)$ and $K(a, t; \ell^1, \ell^2)$; that is, it shows that when calculating the infimum appearing in $K(Ra, t; L^\infty, G)$, it suffices to consider decompositions of the form

$$Ra = Rb + Rc, \qquad b \in \ell^1,\ c \in \ell^2.$$

Denote

$$\kappa_a(t) := K(a, t; \ell^1, \ell^2), \qquad t > 0. \tag{3.5}$$

Theorem 3.1 (Astashkin)**.** *There exist constants $C_1, C_2 > 0$ such that*

$$C_1 \kappa_a(t) \leq K(Ra, t; L^\infty, G) \leq C_2 \kappa_a(t), \qquad t > 0,$$

for $a = (a_k)_1^\infty \in \ell^2$.

The properties of Rademacher functions in r.i. spaces imply that, given an r.i. space X, the sequence space F_X defined by

$$\|(a_k)_1^\infty\|_{F_X} := \Big\| \sum_{k=1}^\infty a_k r_k \Big\|_X,$$

is an r.i. space which is an interpolation space between ℓ^1 and ℓ^2.

From Theorem 3.1, using the K–method of interpolation with a parameter space, the following result is deduced; [2, Theorem 1.4].

Theorem 3.2 (Astashkin)**.** *Let $S = (\ell^1, \ell^2)_F^K$ be an interpolation space between ℓ^1 and ℓ^2 given by a discrete parameter F via the K–method of interpolation. Then, there exist constants $A_F^K, B_F^K > 0$, such that*

$$A_F^K \|(a_k)_1^\infty\|_S \leq \Big\| \sum_{k\geq 1} a_k r_k \Big\|_X \leq B_F^K \|(a_k)_1^\infty\|_S,$$

where $X = (L^\infty, G)_F^K$ is the interpolation space between L^∞ and G given by the parameter F.

Since all interpolation function spaces between L^∞ and G, and all interpolation sequence spaces between ℓ^1 and ℓ^2 can be obtained via the K–method of interpolation for some parameter space (as they are K–monotone couples), the result above establishes a correspondence between all interpolation spaces between L^∞ and G and all interpolation spaces between ℓ^1 and ℓ^2.

The functional κ_a in (3.5) allows to describe a number of properties of the Rademacher series. Montgomery–Smith proved, for some $C > 0$, that

$$\begin{aligned} m\Big(\Big\{t \in [0,1] : \sum_{k\geq 1} a_k r_k(t) > \kappa_a(\lambda)\Big\}\Big) &\leq \exp(-\lambda^2/2), \\ m\Big(\Big\{t \in [0,1] : \sum_{k\geq 1} a_k r_k(t) > \kappa_a(\lambda)/C\Big\}\Big) &\geq \frac{1}{C}\exp(-C\lambda^2), \end{aligned} \tag{3.6}$$

for $a = (a_k)_1^\infty \in \ell^2$ and $\lambda \geq 0$, [23]. Hitczenko proved the equivalence

$$C_1\kappa_a(\sqrt{p}) \leq \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^p([0,1])} \leq C_2\kappa_a(\sqrt{p}), \tag{3.7}$$

for some constants $C_1, C_2 > 0$ and $p \geq 1$, [16]. These estimates are more accurate than Khintchine inequality. Astashkin proved that (3.7) characterizes those systems equivalent in distribution to (r_k) on $[0,1]$, [1, Theorem 1].

Theorem 3.3 (Astashkin)**.** *Let (φ_k) be a system of random variables on a probability space (R,μ). The system (φ_k) is equivalent in distribution to (r_k) if and only if there exists $C > 0$ such that*

$$C^{-1}\kappa_a(\sqrt{p}) \leq \Big\| \sum_{k=1}^m a_k\varphi_k \Big\|_{L^p(R,\mu)} \leq C\kappa_a(\sqrt{p}),$$

for $1 \leq p < \infty$, $m \geq 1$, and $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$.

It is to be remarked that the left–hand side inequality in Theorem 3.3,

$$C^{-1}\kappa_a(\sqrt{p}) \leq \Big\| \sum_{k=1}^m a_k\varphi_k \Big\|_{L^p(R,\mu)},$$

is equivalent to $(r_k) \prec (\varphi_k)$, that is, (r_k) being majorized in distribution by (φ_k), and the right–hand side inequality,

$$\Big\| \sum_{k=1}^m a_k\varphi_k \Big\|_{L^p(R,\mu)} \leq C\kappa_a(\sqrt{p}),$$

is equivalent to (φ_k) being majorized in distribution by (r_k).

3.1 Distribution of Rademacher series on an open set

We start showing, given any set $E \subset [0,1]$ of positive measure, that, for some $N = N(E)$, the system (r_{k+N}) on (E, m_E) is majorized in distribution by (r_k) on $([0,1], m)$.

Proposition 3.4. *Let $E \subset [0,1]$ be a set of positive measure. There exists N such that*

$$(r_{k+N}, E) \prec (r_k, [0,1]).$$

Proof. Let $a = (a_k)_1^\infty \in \ell^2$ and $t > 0$, and denote, for $N \geq 1$,

$$R_N a := \sum_{k \geq 1} a_k r_{k+N}.$$

From the definition of the K–functional κ_a it follows, for $\varepsilon > 0$, that there exist $b = (b_k)_1^\infty \in \ell^1$, $c = (c_k)_1^\infty \in \ell^2$ such that $a = b + c$ and

$$\kappa_a(\sqrt{t}) + \varepsilon \geq \|b\|_1 + \sqrt{t}\|c\|_2.$$

From Theorem 1.6 for $L^p|E$ with $p = t$ and Lemma 1.8, there exist an absolute constant $C > 0$ and $N = N(E)$ such that

$$\|R_N c\|_{L^t|E} \leq C\sqrt{t}\|c\|_2, \qquad c \in \ell^2.$$

Hence,

$$\begin{aligned} \|R_N a\|_{L^t|E} &\leq \|R_N b\|_{L^t|E} + \|R_N c\|_{L^t|E} \\ &\leq \|b\|_1 + C\sqrt{t}\|c\|_2 \\ &\leq C\kappa_a(\sqrt{t}) + C\varepsilon. \end{aligned}$$

Thus, there exists a constant $C > 0$ such that, for $a \in \ell^2$ and $t > 0$,

$$\Big\| \sum_{k \geq 1} a_k r_{k+N} \Big\|_{L^t|E} \leq C\kappa_a(\sqrt{t}).$$

According to Theorem 3.3, the inequality above is equivalent to $((r_{k+N}), E) \prec ((r_k), [0,1])$. $\square$

It remains to show the reverse majorization, namely

$$(r_k, [0,1]) \prec (r_{k+N}, E).$$

In this regard, it is interesting to record a recent result by Astashkin, [4], showing that if a set $E \subset [0,1]$ has the property that $m(E \cap (a,b)) > 0$ for any interval $(a,b) \subset [0,1]$, then, for some constant $\gamma = \gamma(E) > 0$ and all sequences $(a_k)_1^\infty \in \ell^2$,

$$\int_E \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| \, dt \geq \gamma \Big(\sum_{k=1}^{\infty} a_k^2 \Big)^{1/2}.$$

We address the problem of majorization for a particular family of sets E in the next result.

Proposition 3.5. *Let $E \subset [0,1]$ be a set of positive measure. Assume that there exists a constant $C > 0$ such that, for almost every point $x \in E$, there exists an interval J_x such that $x \in J_x$ and, for every interval $I \subset J_x$,*

$$\frac{m(E \cap I)}{m(I)} \geq C. \tag{3.8}$$

Then, there exists N such that

$$(r_k, [0,1]) \prec (r_{k+N}, E), \tag{3.9}$$

where the majorization constant is $C/2$.

Proof. In order to establish (3.9), we have to prove, for some $N \in \mathbb{N}$ and some constant $\beta > 0$, that

$$m_E(A) = \frac{m(E \cap A)}{m(E)} \geq \beta m(A), \tag{3.10}$$

where the set A is given, for $\lambda > 0$, $M \geq 1$ and $a_1, \ldots, a_M \in \mathbb{R}$, by

$$A := \Big\{ x \in [0,1] : \Big| \sum_{k=1}^{M} a_k r_{k+N}(x) \Big| > \lambda \Big\}. \tag{3.11}$$

From the inner regularity of the Lebesgue measure, there exists a compact set K with $K \subset E$ and $m(E) \leq 2m(K)$. Then,

$$m_E(A) = \frac{m(E \cap A)}{m(E)} \geq \frac{m(K \cap A)}{2m(K)} = \frac{1}{2} m_K(A),$$

and so, in order to prove (3.10), we can assume that E is compact.

Consider a finite covering of E,

$$E \subset J_{x_1} \cup \ldots \cup J_{x_s},$$

where $x_1, \ldots, x_s \in E$ and the sets J_{x_i} are dyadic intervals satisfying condition (3.8) for every interval $I \subset J_{x_i}$ and $1 \le i \le s$. Let N be the maximum among the orders of the dyadic intervals $J_{x_1}, \ldots, J_{x_s}$. Then, each dyadic interval I_k^{N+M} of order $N+M$ satisfies either $I_k^{N+M} \subset J_{x_i}$ for some i, $1 \le i \le s$, or $I_k^{N+M} \cap J_{x_i} = \emptyset$ for all i, $1 \le i \le s$.

Since the Rademacher functions r_k with $1 \le k \le N+M$ are constant on the dyadic intervals of order $N+M$, the set A in (3.11) consists of a finite union of dyadic intervals of order $N+M$. Let us denote

$$A = \bigcup_{j=1}^{t} I_{k_j}^{N+M}, \qquad 1 \le k_1 < k_2 < \ldots < k_t \le 2^{N+M}.$$

Then,

$$m(E \cap A) = m\Big(E \cap \bigcup_{j=1}^{t} I_{k_j}^{N+M} \cap \bigcup_{i=1}^{s} J_{x_i}\Big) = \sum_{i=1}^{s} \sum_{I_{k_j}^{N+M} \subset J_{x_i}} m(E \cap I_{k_j}^{N+M}).$$

From condition (3.8) on the set E, we have

$$m(E \cap I_{k_j}^{N+M}) \ge C m(I_{k_j}^{N+M})$$

for all dyadic intervals $I_{k_j}^{N+M}$ such that $I_{k_j}^{N+M} \subset J_{x_i}$ for some i. Thus,

$$\begin{aligned} m(E \cap A) &\ge C \sum_{i=1}^{s} \sum_{I_{k_j}^{N+M} \subset J_{x_i}} m(I_{k_j}^{N+M}) \\ &= C m\Big(\bigcup_{j=1}^{t} I_{k_j}^{N+M} \cap \bigcup_{i=1}^{s} J_{x_i}\Big) \\ &= C m\Big(A \cap \bigcup_{i=1}^{s} J_{x_i}\Big). \end{aligned}$$

From the choice of N, the set $\cup_{i=1}^{s} J_{x_i}$ is a finite union of dyadic intervals of order N, and so, from (3.3) with $\cup_{i=1}^{s} J_{x_i}$ in place of E, we have

$$m\Big(\Big\{x \in \bigcup_{i=1}^{s} J_{x_i} : \Big|\sum_{k=1}^{M} a_k r_{k+N}(x)\Big| > \lambda\Big\}\Big)$$
$$= m\Big(\Big\{t \in [0,1] : \Big|\sum_{k=1}^{M} a_k r_k(t)\Big| > \lambda\Big\}\Big) m\Big(\bigcup_{i=1}^{s} J_{x_i}\Big).$$

Thus,

$$m\Big(A \cap \bigcup_{j=1}^{s} J_{x_j}\Big) = m(A) m\Big(\bigcup_{j=1}^{s} J_{x_j}\Big) \geq m(A) m(E),$$

and so it follows that

$$m(E \cap A) \geq C m(E) m(A).$$

Hence, in the general case,

$$m(E \cap A) \geq \frac{C}{2} m(E) m(A).$$

which completes the proof. □

Remark 3.6. We discuss condition (3.8) on the set E.

(i) Inequality (3.8) holds with $C = 1$ for sets E which are an union of intervals (open, closed or arbitrary). In particular, this is the case of open sets.

(ii) Inequality (3.8) also holds for sets of the form $E = G \cup Z$ or $E = G \setminus Z$, where G is an open set and $m(Z) = 0$. In particular, Proposition 3.5 applies to Peano–Jordan measurable sets.

(iii) We give an example of a set E satisfying the conclusion of Proposition 3.5 but not condition (3.8). Let $E = K_1 \cup K_2$, where K_1 is a "fat" Cantor set contained in $[0, 1/2]$, and

$$K_2 = \{x + 1/2 : x \in [0, 1/2] \setminus K_1\}.$$

Observe that, for every $x \in K_1$ and for every interval J with $x \in J$ there exists an interval $I \subset J$ such that $I \cap K_1 = \emptyset$. Thus, condition (3.8) is not satisfied. On the other hand, from the fact that $\sum a_k r_{k+1}$ is periodic with period $1/2$, it follows that

$$m\Big(\Big\{x \in [0,1] : \Big|\sum_{k\geq 1} a_k r_k(x)\Big| > \lambda\Big\}\Big) = m_E\Big(\Big\{x \in E : \Big|\sum_{k\geq 1} a_k r_{k+1}(x)\Big| > \lambda\Big\}\Big),$$

for $(a_k)_1^\infty \in \ell^2$ and $\lambda > 0$.

The important feature in this example is that the set E modulo $1/2$ is in fact the interval $[0,1/2]$. The same construction can be applied to sets E such that, for some $n \geq 1$, the set E modulo $1/2^n$ is, but for measure zero, an open set.

Now we discuss some consequences of the equivalence in distribution between (r_{k+N}) on (E, m_E) and (r_k) on $([0,1], m)$. We will consider the case when E is an open set. Nevertheless, the following results hold for any set E satisfying condition (3.8) in Proposition 3.5 (recall that Proposition 3.4 holds for arbitrary measurable sets).

We start showing that the equivalence in (3.2) can be extended to any r.i. space X under the assumption of E being an open set. It is to be remarked that, whereas Proposition 2.10 provides inequality (3.2) with constants depending on X, the next result gives the same inequality with absolute constants.

Theorem 3.7. *Let X be an r.i. space on $[0,1]$ and $E \subset [0,1]$ be a non–empty open set. There exists $N = N(E)$ such that if $\sum_{k=1}^{\infty} a_k r_k \in X$, then*

$$C_1 \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X \leq \Big\| \sum_{k=1}^{\infty} a_k r_{N+k} \Big\|_{X|E} \leq C_2 \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X,$$

where $C_1, C_2 > 0$ are absolute constants.

Proof. Proposition 3.4 gives

$$(r_{k+N}, E) \prec (r_k, [0,1])$$

with an absolute constant for all sets E. On the other hand, open sets satisfy condition (3.8) with $C = 1$, and so Proposition 3.5 gives

$$(r_k, [0,1]) \prec (r_{k+N}, E)$$

with the same constant, $C/2 = 1/2$, for all open sets. The result follows from Proposition 2.9. □

Next, we prove the reciprocal of Proposition 2.10, which gives a local version of Rodin and Semenov's theorem (Theorem 1.3).

Proposition 3.8. *Let X be an r.i. space on $[0,1]$ and G the closure of L^∞ in L^{M_2}. The following conditions are equivalent.*

(i) The continuous embedding $G \subset X$ holds, that is, there exists a constant $C > 0$ such that

$$\|f\|_X \leq C\|f\|_{L^{M_2}}$$

for all $f \in L^\infty$.

(ii) There exist constants $A'_X, B'_X > 0$ such that, for every non–empty open set E, there exists $N = N(E)$ such that

$$A'_X\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k r_{k+N}\Big\|_{X|E} \leq B'_X\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

(iii) There exist constants $A'_X, B'_X > 0$ and a non–empty open set E such that, for some N,

$$A'_X\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2} \leq \Big\|\sum_{k\geq 1} a_k r_{k+N}\Big\|_{X|E} \leq B'_X\Big(\sum_{k\geq 1} a_k^2\Big)^{1/2},$$

for $(a_k)_1^\infty \in \ell^2$.

Proof. From Proposition 2.10, we have that *(i)* implies *(ii)*, and it is clear that *(ii)* implies *(iii)*.

Assume now that *(iii)* holds for an open set E and for some N. From Theorem 3.7, there exist constants C_1 and C_2 such that

$$C_1\Big\|\sum_{k=1}^\infty a_k r_{N+k}\Big\|_{X|E} \leq \Big\|\sum_{k=1}^\infty a_k r_k\Big\|_X \leq C_2\Big\|\sum_{k=1}^\infty a_k r_{N+k}\Big\|_{X|E},$$

for any $(a_k)_1^\infty \in \ell^2$. From *(iii)*,

$$C_1 A'_X\Big(\sum_{k=1}^\infty a_k^2\Big)^{1/2} \leq \Big\|\sum_{k=1}^\infty a_k r_k\Big\|_X \leq C_2 B'_X\Big(\sum_{k=1}^\infty a_k^2\Big)^{1/2},$$

which implies, together with Theorem 1.3, that $G \subset X$. □

Example 3.9. We consider two examples of r.i. spaces not covered by Proposition 2.10 where Theorem 3.7 applies.

(i) Let L^{M_p} be the Orlicz space generated by $M_p(t) = \exp(t^p)-1$, with $1 \leq p < \infty$. For $1 \leq p \leq 2$, the Rademacher functions generate in L^{M_p} a subspace isomorphic to

ℓ^2. For $p > 2$, the situation is described by Rodin and Semenov in [26]. Let $\ell_{q,\infty}$ be the space of all sequences $(a_k)_1^\infty$ such that the norm

$$\|(a_k)_1^\infty\|_{\ell_{q,\infty}} := \sup_{n\geq 1} n^{1/q-1} \sum_{k=1}^{n} a_k^*$$

is finite, where q is the conjugate exponent to p and $(a_k^*)_1^\infty$ denotes the decreasing rearrangement of $(|a_k|)_1^\infty$. Then, there exist constants $A_{M_p}, B_{M_p} > 0$ such that

$$A_{M_p}\|(a_k)_1^\infty\|_{\ell_{q,\infty}} \leq \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^{M_p}} \leq B_{M_p}\|(a_k)_1^\infty\|_{\ell_{q,\infty}}, \tag{3.12}$$

with $1/p + 1/q = 1$.

From Theorem 3.7, we have, in the case $p > 2$, that there exist constants $C_{1,p}, C_{2,p} > 0$ such that, for any non–empty open set E, there exists $N = N(E)$ such that

$$C_{1,p}\Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^{M_p}} \leq \Big\| \sum_{k\geq 1} a_k r_{k+N} \Big\|_{L^{M_p}|E} \leq C_{2,p}\Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^{M_p}},$$

and so

$$\inf \Big\{ \lambda > 0 : \int_E \Big(\exp \Big| \frac{1}{\lambda} \sum_{k\geq 1} a_k r_{k+N}(x) \Big|^p - 1 \Big) \frac{dx}{m(E)} \leq 1 \Big\} \asymp \sup_{n\geq 1} n^{1/q-1} \sum_{k=1}^{n} a_k^*,$$

with constants depending only on p.

(ii) Let $S = (\ell^1, \ell^2)_F^K$ be an interpolation space between ℓ^1 and ℓ^2 given by a discrete parameter F, and $X = (L^\infty, G)_F^K$ be the corresponding interpolation space between L^∞ and G. From Theorem 3.2 by Astashkin, there exist constants $A_F^K, B_F^K > 0$ such that

$$A_F^K \|(a_k)_1^\infty\|_S \leq \Big\| \sum_{k\geq 1} a_k r_k \Big\|_X \leq B_F^K \|(a_k)_1^\infty\|_S.$$

From Theorem 3.7, we have, for E a non–empty open set, that there exist constants $A_F'^K, B_F'^K > 0$ such that

$$A_F'^K \|(a_k)_1^\infty\|_S \leq \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{X|E} \leq B_F'^K \|(a_k)_1^\infty\|_S,$$

for some N depending on E.

3.2 Local independence of Rademacher functions

One of the key properties of the Rademacher system (r_k) on $[0,1]$ is that of independence: for $n \geq 1$, $i_1, \dots, i_n \geq 1$ and $\delta_1, \dots, \delta_k \in \{1, -1\}$, we have

$$m\Big(\bigcap_{k=1}^{n} [r_{i_k} = \delta_k]\Big) = \prod_{k=1}^{n} m([r_{i_k} = \delta_k]),$$

where

$$[r_k = \delta_k] := \{x \in [0,1] : r_k(x) = \delta_k\}.$$

A large number of results (and their proofs) rely on the independence of the Rademacher system. For example, the proofs of (3.6) by Montgomery–Smith, [23], and (3.7) by Hitczenko, [30], (see also [3, Theorem 2.1 and Theorem 2.2]).

Given a set $E \subset [0,1]$ of positive measure, the question arises of the independence of the Rademacher system on (E, m_E). Considering the set $E = [0, 1/3]$ we see that this is not the case (since $1/3$ is not dyadic). A further possibility to consider would be that, for some $N \geq 1$, the system (r_{k+N}) is independent on (E, m_E). Neither this is the case.

Note that, if we denote (as in the proof of Proposition 3.5),

$$A := \Big\{x \in [0,1] : \Big|\sum_{k=1}^{M} a_k r_{k+N}(x)\Big| > \lambda\Big\},$$

then the equivalence in distribution between (r_k) on $([0,1], m)$ and (r_{k+N}) on (E, m_E) holds if and only if there exists a constant $C \geq 1$ such that

$$\frac{1}{C} m(E) m(A) \leq m(E \cap A) \leq C m(E) m(A) \tag{3.13}$$

for all $M \geq 1$, $\lambda \geq 0$ and $(a_k)_1^\infty \in \ell^2$, that is, the set E is independent (up to a multiplicative constant) to all dyadic intervals of order greater or equal than N, for some N. Inequality (3.13) is a weaker version of independence of sets where a multiplicative constant is allowed. This motivates the definition of C–independence below.

For a probability space $(\Omega, \Sigma, \mathbb{P})$, a measurable function $X : \Omega \to \mathbb{R}$ and a Borel set $B \subset \mathbb{R}$, denote

$$[X \in B] := X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}.$$

Definition 3.10. We say that a system $(X_k)_{k\in J}$ of random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$ is C–independent, with $C \geq 1$, if the inequalities

$$\frac{1}{C}\,\mathbb{P}\Big(\bigcap_{j=1}^{m}[X_{k_j}\in B_j]\Big) \leq \prod_{j=1}^{m}\mathbb{P}([X_{k_j}\in B_j]) \leq C\,\mathbb{P}\Big(\bigcap_{j=1}^{m}[X_{k_j}\in B_j]\Big)$$

hold for any $m \geq 1$, $k_1, \ldots, k_m \in J$ and $B_1, \ldots, B_m$ Borel sets on $\mathbb{R}$.

For $C = 1$, the 1–independence coincides with the classical independence of random variables.

The interest of C–independence is that, if given a set $E \subset [0,1]$ of positive measure there exists $N = N(E)$ such that the system (r_{k+N}) is C–independent over (E, m_E), that is,

$$\frac{1}{C}m_E\Big(\bigcap_{j=1}^{k}[r_{i_j}=\delta_j]\Big) \leq \prod_{j=1}^{k}m_E([r_{i_j}=\delta_j]) \leq Cm_E\Big(\bigcap_{j=1}^{k}[r_{i_j}=\delta_j]\Big),$$

for $k \geq 1$, $N \leq i_1 < \ldots < i_k$ and $\delta_1, \ldots, \delta_k = \pm 1$, then the local versions on E of inequalities (3.6),

$$m_E\Big(\Big\{x\in E : \sum_{k\geq 1} a_k r_{k+N}(x) > \kappa_a(\lambda)\Big\}\Big) \leq C\exp(-\lambda^2/2),$$

and (3.7),

$$C_1\,\kappa_a(\sqrt{p}) \leq \Big\|\sum_{k\geq 1} a_k r_{k+N}\Big\|_{L^p|E} \leq C_2\,\kappa_a(\sqrt{p}),$$

would be available. Note, from Theorem 3.3, that this local version of (3.7) would imply Proposition 3.5 for any set E of positive measure.

Next we present a partial result on C–independence showing, under some conditions on the set E, that given any $\lambda > 1$, the Rademacher system (r_{k+N}) on (E, m_E) is λ–independent, for some $N \geq 1$ depending on E and λ.

Given an open set $G \subset [0,1]$, we will denote

$$G_n := \bigcup\big\{I_j^{n-1} : I_j^{n-1} \subset G, 1 \leq j \leq 2^{n-1}\big\},$$

that is, G_n is the union of all dyadic intervals of order less than n contained in G.

Theorem 3.11. *Let $E \subset [0,1]$ be a Peano–Jordan measurable set of positive measure. Assume, for $G := \operatorname{int}(E)$, that*

$$\sum_{n\geq 1} m(E \setminus G_n) < \infty. \tag{3.14}$$

Then, for any $\lambda > 1$, there exists $N = N(E,\lambda)$ such that the system of random variables (r_{k+N}) on (E, m_E) is λ–independent, that is,

$$\frac{1}{\lambda} m_E\Big(\bigcap_{j=1}^{k} [r_{i_j} = \delta_j]\Big) \leq \prod_{j=1}^{k} m_E([r_{i_j} = \delta_j]) \leq \lambda m_E\Big(\bigcap_{j=1}^{k} [r_{i_j} = \delta_j]\Big),$$

for $k \geq 1$, $N \leq i_1 < \ldots < i_k$ and $\delta_1, \ldots, \delta_k = \pm 1$,

Proof. The case when E is a finite union of dyadic intervals follows from (3.3) with $\lambda = 1$. We assume that E is not a finite union of dyadic intervals.

Since E is Peano–Jordan measurable, we have $E = \operatorname{int}(E) \sqcup (E \cap \partial E)$, with $m(\partial E) = 0$, and so we can assume that E is open and $m(E) = m(\overline{E})$.

Step 1. We will show that, given any $\lambda > 1$, there exists $N = N(E, \lambda)$ such that

$$\frac{1}{\lambda}\frac{1}{2^k} \leq m_E\Big(\bigcap_{j=1}^{k} [r_{i_j} = \delta_j]\Big) \leq \lambda \frac{1}{2^k}, \tag{3.15}$$

for $k \geq 1$, $N < i_1 < \ldots < i_k$ and $\delta_1, \ldots, \delta_k = \pm 1$.

Let us denote, for $n \geq 1$,

$$\begin{aligned} \mathcal{I}_n &:= \{I_j^n : I_j^n \subset E, 1 \leq j \leq 2^n\}, \\ \mathcal{J}_n &:= \{I_j^n : I_j^n \cap E \neq \emptyset, I_j^n \cap E^c \neq \emptyset, 1 \leq j \leq 2^n\}, \end{aligned}$$

and let $M_n := \operatorname{card}(\mathcal{I}_n)$, $P_n := \operatorname{card}(\mathcal{J}_n)$.

Since E is open, it is equal to the union of all dyadic intervals that it contains, except for a set of dyadic points. Then,

$$m(E) = \lim_{n\to+\infty} m\Big(\bigcup_{I \in \mathcal{I}_n} I\Big) = \lim_{n\to+\infty} \frac{M_n}{2^n}.$$

In a similar way,

$$m(\overline{E}) = \lim_{n\to+\infty} m\Big(\bigcup_{I \in \mathcal{I}_n \cup \mathcal{J}_n} I\Big) = \lim_{n\to+\infty} \frac{M_n + P_n}{2^n}.$$

Together with the assumption that E is a Peano–Jordan measurable set, we have

$$\lim_{n\to+\infty} \frac{M_n + P_n}{M_n} = \frac{m(\overline{E})}{m(E)} = 1.$$

Hence, given any $\lambda > 1$, there exist $N = N(E, \lambda)$ such that

$$\frac{1}{\lambda} \leq \frac{M_n + P_n}{M_n} \leq \lambda, \tag{3.16}$$

for all $n \geq N$.

Let $k \geq 1$ and $N < i_1 < \ldots < i_k$. We have that

$$\frac{M_{i_1-1}}{2^{i_1-1}} = \sum_{I\in\mathcal{I}_{i_1-1}} m(I) \leq m(E),$$

and

$$m(E) \leq \sum_{I\in\mathcal{I}_{i_1-1}} m(I) + \sum_{J\in\mathcal{J}_{i_1-1}} m(J) = \frac{M_{i_1-1} + P_{i_1-1}}{2^{i_1-1}}.$$

Thus,

$$\frac{M_{i_1-1}}{2^{i_1-1}} \leq m(E) \leq \frac{M_{i_1-1} + P_{i_1-1}}{2^{i_1-1}}. \tag{3.17}$$

On the other hand, a direct computation (or the independence of the Rademacher functions on $[0, 1]$) shows that

$$m\Big(\bigcap_{j=1}^{k}[r_{i_j} = \delta_j]\Big) = \frac{1}{2^k}.$$

The characteristic function χ_A of the set $A := \bigcap_{j=1}^{k}[r_{i_j} = \delta_j]$ is periodic, with period $1/2^{i_1-1}$. Thus, for any dyadic interval I of order $i_1 - 1$,

$$m\Big(I \cap \bigcap_{j=1}^{k}[r_{i_j} = \delta_j]\Big) = \frac{1/2^k}{2^{i_1-1}} = \frac{1}{2^{k+i_1-1}}.$$

It follows, as in (3.17), that

$$\frac{M_{i_1-1}}{2^{k+i_1-1}} \leq m\Big(E \cap \bigcap_{j=1}^{k}[r_{i_j} = \delta_j]\Big) \leq \frac{M_{i_1-1} + P_{i_1-1}}{2^{k+i_1-1}}. \tag{3.18}$$

Recall that

$$m_E\Big(\bigcap_{j=1}^{k}[r_{i_j}=\delta_j]\Big)=\frac{1}{m(E)}m\Big(E\cap\bigcap_{j=1}^{k}[r_{i_j}=\delta_j]\Big).$$

Then, from (3.17) and (3.18),

$$\frac{M_{i_1-1}}{M_{i_1-1}+P_{i_1-1}}\frac{1}{2^k}\leq m_E\Big(\bigcap_{j=1}^{k}[r_{i_j}=\delta_j]\Big)\leq\frac{M_{i_1-1}+P_{i_1-1}}{M_{i_1-1}}\frac{1}{2^k}.$$

Taking into account (3.16) it follows that, given any $\lambda>1$, there exists $N=N(E,\lambda)$ such that

$$\frac{1}{\lambda}\frac{1}{2^k}\leq m_E\Big(\bigcap_{j=1}^{k}[r_{i_j}=\delta_j]\Big)\leq\lambda\frac{1}{2^k},$$

for $k\geq 1$ and $N<i_1<\ldots<i_k$.

Step 2. Now we show that given any $\lambda>1$, there exists $N=N(E,\lambda)$ such that

$$\frac{1}{\lambda}\frac{1}{2^k}\leq\prod_{j=1}^{k}m_E([r_{i_j}=\delta_j])\leq\lambda\frac{1}{2^k},\tag{3.19}$$

for $k\geq 1$, $N\leq i_1<\ldots<i_k$ and $\delta_1,\ldots,\delta_k=\pm 1$.

Let $n\geq 1$ and $\delta=\pm 1$. Consider the set $[r_n=\delta]$, and let $\varepsilon_n\in[-1,1]$ be such that

$$m_E([r_n=\delta])=\frac{m(E\cap[r_n=\delta])}{m(E)}=\frac{1+\varepsilon_n}{2}.$$

Since G_n is an union of dyadic intervals with order less or equal than $n-1$, we have

$$m_{G_n}([r_n=\delta])=\frac{m(G_n\cap[r_n=\delta])}{m(G_n)}=\frac{1}{2},$$

which allows to obtain explicitly the value of ε_n. Let

$$F_n:=E\setminus G_n.$$

Then, we have

$$\begin{aligned} m_E([r_n = \delta]) &= \frac{1}{m(E)}\big(m(G_n \cap [r_n = \delta]) + m(F_n \cap [r_n = \delta])\big) \\ &= \frac{m(G_n)}{m(E)} m_{G_n}([r_n = \delta]) + \frac{m(F_n)}{m(E)} m_{F_n}([r_n = \delta]) \\ &= \frac{m(G_n)}{m(E)} \frac{1}{2} + \frac{m(F_n)}{m(E)} m_{F_n}([r_n = \delta]) \\ &= \frac{1}{2}\Big(1 - \frac{m(F_n)}{m(E)} + 2\frac{m(F_n)}{m(E)} m_{F_n}([r_n = \delta])\Big) \\ &= \frac{1}{2}(1 + \varepsilon_n), \end{aligned}$$

for

$$\varepsilon_n := \frac{m(F_n)}{m(E)}\big(2m_{F_n}([r_n = \delta]) - 1\big).$$

The product in (3.19) can be written as

$$\prod_{j=1}^{k} m_E([r_{i_j} = \delta_j]) = \frac{1}{2^k}\prod_{j=1}^{k}(1 + \varepsilon_{i_j}),$$

and so, in order to prove (3.19), we will show that given any $\lambda > 1$, there exists $N = N(E, \lambda)$ such that, for all $k \geq 1$ and $N < i_1 < \ldots < i_k$,

$$\frac{1}{\lambda} \leq \prod_{j=1}^{k}(1 + \varepsilon_{i_j}) \leq \lambda. \tag{3.20}$$

Let $\lambda > 1$, and consider $\delta \in (0, 1)$ such that

$$\exp(\delta(1 + \delta)) < \lambda.$$

From the fact that, for all $n \geq 1$,

$$|\varepsilon_n| = \frac{m(F_n)}{m(E)}\big|2m_{F_n}([r_n = \delta]) - 1\big| \leq \frac{m(F_n)}{m(E)} = \frac{m(E \setminus G_n)}{m(E)},$$

and the assumption in (3.14), it follows that

$$\sum_{n \geq 1} |\varepsilon_n| < \infty.$$

Then, given $\delta > 0$, there exists $N = N(E, \lambda)$ such that

$$\sum_{n=N}^{\infty} |\varepsilon_n| < \delta,$$

and, for all $n \geq N$ with $\varepsilon_n \neq 0$,

$$1 - \delta < \frac{\log(1 + |\varepsilon_n|)}{|\varepsilon_n|} < 1 + \delta.$$

Hence, given $k \geq 1$ and $i_1, \ldots, i_k$ with $N < i_1 < \ldots < i_k$, we have

$$\begin{aligned}\frac{1}{\lambda} &\leq \exp\Big((1-\delta)\sum_{j=1}^{k} |\varepsilon_{i_j}|\Big) \leq \exp\Big(\sum_{j=1}^{k} \log(1 + |\varepsilon_{i_j}|)\Big) \\ &\leq \exp\Big((1+\delta)\sum_{j=1}^{k} |\varepsilon_{i_j}|\Big) \leq \exp(\delta(1+\delta)) \leq \lambda,\end{aligned}$$

and so follows (3.20).

The result follows from (3.15) and (3.19) using $\lambda^{1/2}$ instead of λ. □

Remark 3.12. Condition (3.14) is satisfied in the case when E is an interval. Assume that $E = (a, b)$, and denote, for $n \geq 1$, $G_n = (a_n, b_n)$. Then, from the definition of G_n, we have $a_n - a < 1/2^{n-1}$ and $b - b_n < 1/2^{n-1}$, and so

$$\sum_{n\geq 1} m(E \setminus G_n) = \sum_{n\geq 1} \big((b-a) + (b_n - a_n)\big) \leq \sum_{n\geq 1} \frac{1}{2^n}.$$

The same argument shows that any finite union of dyadic intervals satisfies condition (3.14).

Chapter 4

Rademacher functions in weighted Cesàro spaces

The Cesàro function spaces $\mathrm{Ces}(p)$ are defined as the set of all measurable functions $f \in L^0([0,1])$ such that

$$\|f\|_{\mathrm{Ces}(p)} := \Big(\int_0^1 \Big(\frac{1}{x} \int_0^x |f(t)| dt \Big)^p dx \Big)^{1/p} < \infty, \qquad 1 \le p < \infty,$$
$$\|f\|_{\mathrm{Ces}(\infty)} := \sup_{0<x\le 1} \frac{1}{x} \int_0^x |f(t)| dt < \infty, \qquad p = \infty.$$

These spaces have been thoroughly studied in [6] and [7]. They are the continuous counterpart to the classical Cesàro sequence spaces, which have been studied in detail; see, for example, [11], and the references in [17]. Functional and geometrical properties of $\mathrm{Ces}(p)$ have been studied: duality and reflexivity; isomorphic copies of classical sequence and function spaces; type and cotype; fixed point, Dunford-Pettis, Banach-Saks, and Radon-Nikodym properties; see [6], [7], [9], [17].

More recently, weighted Cesàro function spaces have been considered. In [17] their dual space has been identified. For $\omega(x)$ a weight, i.e., a measurable function with $0 < \omega(x) < \infty$ a.e., and $1 \le p \le \infty$, the weighted Cesàro spaces $\mathrm{Ces}(\omega, p)$ are defined as the set of all measurable functions $f \in L^0([0,1])$ such that

$$\|f\|_{\mathrm{Ces}(\omega,p)} := \Big(\int_0^1 \Big(\frac{1}{\omega(x)} \int_0^x |f(t)| dt \Big)^p dx \Big)^{1/p} < \infty, \qquad 1 \le p < \infty,$$
$$\|f\|_{\mathrm{Ces}(\omega,\infty)} := \sup_{0\le x\le 1} \frac{1}{\omega(x)} \int_0^x |f(t)| dt < \infty, \qquad p = \infty.$$

Recall that

$$\mathcal{R} := \Big\{ \sum_{k\geq 1} a_k r_k : (a_k)_1^\infty \in \ell^2 \Big\}.$$

Then, $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is the linear subspace generated by the Rademacher system in $\mathrm{Ces}(\omega, p)$. For the "unweighted" case $\omega(x) = x$ and for $1 \leq p < \infty$, Astashkin and Maligranda showed that $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to ℓ^2, that is,

$$\Big\| \sum_{k\geq 1} a_k r_k \Big\|_{\mathrm{Ces}(p)} \asymp \|(a_k)_1^\infty\|_2, \tag{4.1}$$

with constants depending only on p, [8, Theorem 1].

For the case when $p = \infty$ and $\omega(x)$ a quasiconcave weight, (that is, $\omega(0) = 0$, $\omega(x)$ is non-decreasing, and $\omega(x)/x$ is non-increasing) it was also shown in [8, Theorem 2] that

$$\Big\| \sum_{k=1}^m a_k r_k \Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \|(a_k)_1^m\|_2 + \max_{1\leq n\leq m} \frac{2^{-n}}{\omega(2^{-n})} \Big| \sum_{k=1}^n a_k \Big|.$$

The above expression is equivalent to $\|(a_k)_1^\infty\|_2$ if and only if

$$\omega(x) \geq C x \log_2^{1/2}(2/x), \qquad 0 < x \leq 1,$$

for some constant $C > 0$ [8, Theorem 3].

In this chapter we study, by means of conditions on $\omega(x)$ and p, the behavior of the Rademacher functions (r_k) in the spaces $\mathrm{Ces}(\omega, p)$. We compute, under certain condition on the weight $\omega(x)$, the norm in $\mathrm{Ces}(\omega, p)$ of a Rademacher series showing, for $1 \leq p < \infty$, that

$$\Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \asymp \Big(\sum_{n=0}^\infty \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^\infty\|_2 \Big)^p \Big)^{1/p},$$

and, for $p = \infty$, that

$$\Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \sup_{n\geq 0} \omega_{\infty,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^\infty\|_2 \Big),$$

where, for $J_n = (1/2^{n+1}, 1/2^n)$, $n \geq 0$, we have

$$\omega_{p,n} := \int_{J_n} \Big(\frac{x}{\omega(x)} \Big)^p dx, \quad \omega_{\infty,n} := \sup_{x\in J_n} \frac{x}{\omega(x)}.$$

These equivalences allow to describe $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$, studying when (r_k) is a basic sequence, studying the complementability of $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ in $\mathrm{Ces}(\omega, p)$, and studying the extremal cases when the individual Rademacher functions do not belong to $\mathrm{Ces}(\omega, p)$ and $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ consists only on certain Rademacher polynomials.

We also consider the case when $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to ℓ^2. By means of determining the norm in $\mathrm{Ces}(\omega, p)$ of the decreasing rearrangement of a Rademacher series, we prove that

$$\frac{x}{\omega(x)} \log_2^{1/2}(2/x) \in L^p([0,1])$$

is a sufficient condition for $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ being isomorphic to ℓ^2, for all $1 \leq p \leq \infty$, which is necessary in the case $p = \infty$, and "almost" necessary for $1 \leq p < \infty$.

4.1 Conditions on the weight

We start discussing several conditions on the weight $\omega(x)$ related to the behavior of the Rademacher series in $\mathrm{Ces}(\omega, p)$.

Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$. We will write the norm in $\mathrm{Ces}(\omega, p)$ in the following way:

$$\|f\|_{\mathrm{Ces}(\omega,p)} = \Big(\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \Big(\frac{1}{x}\int_0^x |f(t)|\, dt\Big)^p dx \Big)^{1/p}, \qquad 1 \leq p < \infty,$$
$$\|f\|_{\mathrm{Ces}(\omega,\infty)} = \sup_{0<x\leq 1} \Big(\frac{x}{\omega(x)}\Big) \frac{1}{x}\int_0^x |f(t)|\, dt, \qquad p = \infty.$$

We say that the weight $\omega(x)$ satisfies condition (P1) for p, with $1 \leq p \leq \infty$, if for all $n \geq 0$,

$$\begin{aligned} \omega_{p,n} &:= \int_{J_n} \Big(\frac{x}{\omega(x)}\Big)^p dx < \infty, \qquad 1 \leq p < \infty, \\ \omega_{\infty,n} &:= \sup_{x \in J_n} \frac{x}{\omega(x)} < \infty, \qquad p = \infty, \end{aligned} \tag{P1}$$

where $J_n := (1/2^{n+1}, 1/2^n)$, for $n \geq 0$. Note that, since $\omega(x)$ is finite a.e. we have that $\omega_{p,n} > 0$, for $n \geq 0$.

Recall from the Preliminaries that a Banach function space X is saturated if for every set E with $m(E) > 0$ there exists $F \subset E$ such that $m(F) > 0$ and $\chi_F \in X$ (see [32, p. 454]). This property is equivalent to the associate functional $\|\cdot\|_{X'}$ being a norm in the associate space X'; see [32, Ch. 15, §68, Theorem 4].

Proposition 4.1. *Let* $1 \le p \le \infty$ *and* $\omega(x)$ *be a weight on* $[0,1]$. *If condition* (P1) *is satisfied, then the space* $\mathrm{Ces}(\omega,p)$ *has a saturated norm. In particular, the associate functional* $\|\cdot\|_{\mathrm{Ces}(\omega,p)'}$ *is a norm in* $\mathrm{Ces}(\omega,p)'$.

Proof. Since the average of χ_{J_n} on $[0,x]$ vanishes for $0 < x < 1/2^{n+1}$ and it is less or equal than 1 for $1/2^{n+1} \le x \le 1$, for $1 \le p < \infty$ we have

$$\|\chi_{J_n}\|^p_{\mathrm{Ces}(\omega,p)} \le \int_{1/2^{n+1}}^1 \Big(\frac{x}{\omega(x)}\Big)^p dx = \sum_{k=0}^n \omega_{p,k}.$$

Analogously, for $p = \infty$,

$$\|\chi_{J_n}\|_{\mathrm{Ces}(\omega,\infty)} \le \sup_{0\le k\le n} \omega_{\infty,k}.$$

It follows that $\chi_{J_n} \in \mathrm{Ces}(\omega,p)$ for all $n \ge 0$ and $1 \le p \le \infty$.

Let $E \subset [0,1]$ be a set of positive measure. There exists J_n such that $m(E\cap J_n) > 0$. Noting that

$$\|\chi_{E\cap J_n}\|_{\mathrm{Ces}(\omega,p)} \le \|\chi_{J_n}\|_{\mathrm{Ces}(\omega,p)},$$

we deduce that $\mathrm{Ces}(\omega,p)$ is saturated. □

We say that the weight $\omega(x)$ satisfies condition (P2) for p, with $1 \le p \le \infty$, if $x/\omega(x) \in L^p([0,1])$, that is,

$$\begin{aligned} &\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p dx < \infty, \qquad 1 \le p < \infty,\\ &\sup_{0\le x\le 1} \frac{x}{\omega(x)} < \infty, \qquad p = \infty. \end{aligned} \tag{P2}$$

Note that condition (P2) is equivalent to $r_k \in \mathrm{Ces}(\omega,p)$ for all $k \ge 1$. Condition (P2) can be written via the coefficients $\omega_{p,n}$. Namely, it is equivalent to

$$\begin{aligned} &\sum_{n=0}^\infty \omega_{p,n} < \infty, \qquad 1 \le p < \infty,\\ &\sup_{n\ge 0} \omega_{\infty,n} < \infty, \qquad p = \infty. \end{aligned}$$

We say that the weight $\omega(x)$ satisfies condition (P3) for p, with $1 \le p \le \infty$, if

$$\begin{aligned} &\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2}(2/x)\, dx < \infty, \qquad 1 \le p < \infty,\\ &\sup_{0< x\le 1} \frac{x}{\omega(x)} \log_2^{1/2}(2/x) < \infty, \qquad p = \infty. \end{aligned} \tag{P3}$$

Since, for $x \in J_n$,

$$(n+1)^{1/2} \le \log_2^{1/2}(2/x) \le (n+2)^{1/2},$$

condition (P3) is equivalent to

$$\sum_{n=0}^{\infty} \omega_{p,n}(n+1)^{p/2} < \infty, \qquad 1 \le p < \infty,$$
$$\sup_{n\ge 0} \omega_{\infty,n}(n+1)^{1/2} < \infty, \qquad p = \infty.$$

In the case when $\omega(x)$ is a non–decreasing function, condition (P3) can be stated in terms of the Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}$, see [10, §4.6]. Namely, it is equivalent to

$$\frac{1}{\omega(x)} \in L^{p/(p+1),p}(\log L)^{1/2}.$$

In particular, condition (P3) holds for

$$\frac{1}{\omega(x)} \in L^{r,s}([0,1])$$

in the case when $p/(p+1) < r$ and $1 < s \le \infty$ or $r = p/(p+1)$ and $1 \le s < p$.

We say that the weight $\omega(x)$ satisfies condition (P4*) for p, with $1 \le p \le \infty$, if there exists $C > 0$ such that

$$\sup_{n\ge 0} \frac{\omega_{p,n+1}}{\omega_{p,n}} \le C. \tag{P4*}$$

Condition (P4*) is a particular case of a more general condition.

We say that $\omega(x)$ satisfies condition (P4) for p, with $1 \le p \le \infty$, if there exists $C > 0$ and $M \ge 1$ such that for every $n \ge 1$ there exists $n' \ge 1$ with

$$0 < n - n' \le M \quad \text{and} \quad \sup_{n\ge 1} \frac{\omega_{p,n}}{\omega_{p,n'}} \le C. \tag{P4}$$

Remark 4.2. Condition (P4) holds in the following situations.

(i) If $\omega(x)$ is a quasiconcave function, then it satisfies (P4*). Since $1/\omega(x)$ is non-increasing it follows that $\omega_{p,0}$ is finite; since $x/\omega(x)$ is non-decreasing, we have $\omega_{p,n+1}/\omega_{p,n} \le 1$ for $n \ge 0$.

(ii) If $\omega(x)$ is non-increasing, we have $\omega_{p,n+1} \le \omega_{p,n}/2$. Hence, (P4*) holds provided that $\omega_{p,0}$ is finite.

(iii) If $x/\omega(x)$ is non-increasing, then condition (P4*) depends on the slope of the function $x/\omega(x)$. In particular, it holds for $1 \leq p \leq \infty$ when $\omega(x)$ satisfies, for some $C > 0$,

$$\sup_{n\geq 0} \frac{\omega(1/2^n)}{\omega(1/2^{n+1})} \leq C.$$

(iv) A weight $\omega(x)$ has the doubling property if there exists a positive constant C such that $\omega(I) \leq C\omega(2I)$ for every interval I, where $2I$ denotes the interval with the same center as I and twice its radius, and

$$\omega(I) = \int_I \omega(x)\, dx.$$

If $(x/\omega(x))^p$ has the doubling property, then condition (P4*) is satisfied. Namely, since $J_{n+1} \subset 2J_n$, we have

$$\int_{J_{n+1}} \Big(\frac{x}{\omega(x)}\Big)^p dx \leq \int_{2J_n} \Big(\frac{x}{\omega(x)}\Big)^p dx \leq C \int_{J_n} \Big(\frac{x}{\omega(x)}\Big)^p dx.$$

Hence, $\omega_{p,n+1} \leq C\omega_{p,n}$. In particular, $(x/\omega(x))^p$ has the doubling property if it belongs to the Muckenhoupt weight class A_r for some $1 < r < \infty$.

We say that the weight $\omega(x)$ satisfies condition (P5) for p, with $1 \leq p \leq \infty$, if there exists a constant $C > 0$ such that for every $m \geq 0$,

$$\begin{aligned} &\sum_{n=m}^{\infty} \omega_{p,n} \leq C\omega_{p,m}, \qquad 1 \leq p < \infty, \\ &\sup_{n\geq m} \omega_{\infty,n} \leq C\omega_{\infty,m}, \qquad p = \infty. \end{aligned} \tag{P5}$$

Condition (P5) is satisfied whenever $\omega(x)$ is quasiconcave. In this case, $x/\omega(x)$ is non–decreasing, and so

$$\begin{aligned} \sum_{n=m}^{\infty} \omega_{p,n} &= \int_0^{1/2^m} \Big(\frac{x}{\omega(x)}\Big)^p dx \\ &= \int_0^{1/2^{m+1}} \Big(\frac{x}{\omega(x)}\Big)^p dx + \int_{1/2^{m+1}}^{1/2^m} \Big(\frac{x}{\omega(x)}\Big)^p dx \\ &\leq \int_{1/2^{m+1}}^{1/2^m} \Big(\frac{x}{\omega(x)}\Big)^p dx + \int_{1/2^{m+1}}^{1/2^m} \Big(\frac{x}{\omega(x)}\Big)^p dx \\ &= 2\omega_{p,m}. \end{aligned}$$

Remark 4.3. Note that, since for all $n \geq 0$,

$$\int_{J_n} \Big(\frac{x}{\omega(x)}\Big)^p dx \leq \int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p dx \leq \int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2}(2/x) dx,$$

condition (P3) implies condition (P2), and condition (P2) implies condition (P1).

We also have that (P5) implies (P4*) and (P2). This follows from the fact that, if condition (P5) is satisfied, we have

$$\omega_{p,m+1} \leq \sum_{n=m}^{\infty} \omega_{p,n} \leq C\omega_{p,m},$$

for all $m \geq 1$, and

$$\sum_{n=0}^{\infty} \omega_{p,n} \leq C\omega_{p,0},$$

Finally, note that condition (P4*) implies condition (P4), with $M = 1$.

4.2 Rademacher functions in $\mathrm{Ces}(\omega, p)$

In this section we study the space $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$. The following sequence space $\mathcal{R}(\omega, p)$ is useful to describe the norm of a Rademacher series in $\mathrm{Ces}(\omega, p)$.

Definition 4.4. Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0, 1]$. Assume that condition (P1) holds. Let $\mathcal{R}(\omega, p)$ be the space of all sequences $(a_k)_1^\infty \in \ell^2$ such that, for $1 \leq p < \infty$,

$$\|(a_k)_1^\infty\|_{\mathcal{R}(\omega,p)} := \Big(\sum_{n=0}^{\infty} \omega_{p,n}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{n+2}^\infty\|_2\Big)^p\Big)^{1/p} < \infty,$$

and, for $p = \infty$,

$$\|(a_k)_1^\infty\|_{\mathcal{R}(\omega,\infty)} := \sup_{n\geq 0} \omega_{\infty,n}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{n+2}^\infty\|_2\Big) < \infty.$$

The space $\mathcal{R}(\omega, p)$ with the norm $\|\cdot\|_{\mathcal{R}(\omega,p)}$ is a Banach space.

Proposition 2.10, gives the equivalence

$$A'_X\|(a_k)_N^\infty\|_2 \leq \Big\|\sum_{k\geq N} a_k r_k\Big\|_{X|E} \leq B'_X\|(a_k)_N^\infty\|_2$$

for X an r.i. space with $G \subset X$, with N depending on E. Next we give a similar result for E a dyadic interval and a Rademacher series of the form $\sum_{k\geq 1} a_k r_k$. Recall that the dyadic intervals of order n are

$$I_j^n := \Big(\frac{j-1}{2^n}, \frac{j}{2^n}\Big), \qquad 1 \leq j \leq 2^n.$$

Denote the sign of a Rademacher function r_k on I_j^n by

$$\varepsilon_{k,j}^n := \operatorname{sign}(r_k(I_j^n)),$$

for $1 \leq j \leq 2^n$ and $1 \leq k \leq n$.

Lemma 4.5. *Let X be an r.i. space on $[0,1]$ with $G \subset X$. Given a dyadic interval I_j^n, with $n \geq 1$ and $1 \leq j \leq 2^n$, we have*

$$\Big\| \sum_{k\geq 1} a_k r_k \Big\|_{X|I_j^n} \asymp \Big(\Big| \sum_{k=1}^{n} \varepsilon_{k,j}^n a_k \Big| + \|(a_k)_{n+1}^\infty\|_2 \Big),$$

for $(a_k)_1^\infty \in \ell^2$, with constants depending on X.

In particular, for $X = L^1$ and $I_j^n = (0, 1/2^n)$, we have

$$\frac{1}{3\sqrt{2}} \Big(\Big| \sum_{k=1}^{n} a_k \Big| + \|(a_k)_{n+1}^\infty\|_2 \Big) \leq \frac{1}{1/2^n} \int_0^{1/2^n} \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| \, dt$$
$$\leq \Big| \sum_{k=1}^{n} a_k \Big| + \|(a_k)_{n+1}^\infty\|_2.$$

Proof. Recall that for any r.i. space X on $[0,1]$, we have $L^\infty \subset X \subset L^1$. Thus, there exist constants $C_1, C_2 > 0$ such that $\|f\|_X \leq C_1 \|f\|_{L^\infty}$ for all $f \in L^\infty$, and $\|f\|_{L^1} \leq C_2 \|f\|_X$ for all $f \in X$.

From (3.4) and from Theorem 1.3, since $G \subset X$,

$$\begin{aligned}
\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{X|I_j^n} &\leq \Big\| \sum_{k=1}^{n} a_k r_k \Big\|_{X|I_j^n} + \Big\| \sum_{k=n+1}^{\infty} a_k r_k \Big\|_{X|I_j^n} \\
&\leq C_1 \Big\| \sum_{k=1}^{n} a_k r_k \Big\|_{L^\infty|I_j^n} + \Big\| \sum_{k=1}^{\infty} a_{k+n} r_k \Big\|_X \\
&\leq C_1 \Big| \sum_{k=1}^{n} \varepsilon_{k,j}^n a_k \Big| + B_X \|(a_k)_{n+1}^\infty\|_2.
\end{aligned}$$

The upper inequality follows for a constant $B := \max\{C_1, B_X\}$.

Concerning the lower bound, we will combine two inequalities. The first one relies on the fact that, for $k \geq n+1$, the integral of r_k on I_j^n vanishes. Thus,

$$\begin{aligned} C_2 \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{X|I_j^n} &\geq \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^1|I_j^n} \\ &\geq \Big| \frac{1}{m(I_j^n)} \int_{I_j^n} \sum_{k=1}^{\infty} a_k r_k(t)dt \Big| \\ &= \Big| \sum_{k=1}^{n} \varepsilon_{k,j}^n a_k \Big|. \end{aligned} \tag{4.2}$$

On the other hand, from the inverse triangle inequality and Khintchine inequality for $L^1([0,1])$ with the optimal constant $1/\sqrt{2}$ (see [31]) it follows that

$$\begin{aligned} C_2 \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{X|I_j^n} &\geq \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{L^1|I_j^n} \\ &= \frac{1}{m(I_j^n)} \int_{I_j^n} \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| dt \\ &\geq \frac{1}{m(I_j^n)} \int_{I_j^n} \Big(\Big| \sum_{k=n+1}^{\infty} a_k r_k(t) \Big| - \Big| \sum_{k=1}^{n} a_k r_k(t) \Big| \Big) dt \\ &\geq \frac{1}{\sqrt{2}} \|(a_k)_{n+1}^{\infty}\|_2 - \Big| \sum_{k=1}^{n} \varepsilon_{k,j}^n a_k \Big|. \end{aligned} \tag{4.3}$$

From (4.2) and (4.3) it follows that

$$3\, C_2 \Big\| \sum_{k\geq 1} a_k r_k \Big\|_{X|I_j^n} \geq \Big| \sum_{k=1}^{n} \varepsilon_{k,j} a_k \Big| + \frac{1}{\sqrt{2}} \|(a_k)_{n+1}^{\infty}\|_2,$$

which completes the proof. □

For $(a_k)_1^{\infty} \in \ell^2$, we denote

$$\begin{aligned} A_0 &:= \|(a_k)_1^{\infty}\|_2 \\ A_n &:= \Big| \sum_{k=1}^{n} a_k \Big| + \|(a_k)_{n+1}^{\infty}\|_2, \qquad n \geq 1. \end{aligned} \tag{4.4}$$

Theorem 4.6. *Let $1 \le p \le \infty$ and $\omega(x)$ be a weight on $[0,1]$. Assume that condition* (P4) *holds. Then, the space $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is isomorphic to $\mathcal{R}(\omega,p)$ with equivalent norms. Consequently, $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is a Banach space.*

In particular, for $(a_k)_1^\infty \in \mathcal{R}(\omega,p)$ and $1 \le p < \infty$, we have

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \asymp \Big(\sum_{n=0}^{\infty} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^\infty\|_2 \Big)^p \Big)^{1/p},$$

and for $p = \infty$,

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \sup_{n\ge 0} \omega_{\infty,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^\infty\|_2 \Big),$$

with constants depending on p and $\omega(x)$.

Proof. Let $1 \le p < \infty$, $n \ge 0$ and $x \in J_n = (1/2^{n+1}, 1/2^n)$. From Lemma 4.5, we have

$$\begin{aligned} \frac{1}{6\sqrt{2}} A_{n+1} &\le \frac{1}{1/2^n} \int_0^{1/2^{n+1}} \Big| \sum_{k\ge 1} a_k r_k(t) \Big| \, dt \\ &\le \frac{1}{x} \int_0^x \Big| \sum_{k\ge 1} a_k r_k(t) \Big| \, dt \\ &\le \frac{1}{1/2^{n+1}} \int_0^{1/2^n} \Big| \sum_{k\ge 1} a_k r_k(t) \Big| \, dt \\ &\le 2A_n. \end{aligned}$$

Thus, for $n \ge 0$,

$$\frac{1}{6\sqrt{2}} A_{n+1} \le \frac{1}{x} \int_0^x \Big| \sum_{k\ge 1} a_k r_k(t) \Big| \, dt \le 2A_n, \quad x \in J_n. \tag{4.5}$$

By splitting the interval $[0,1]$ into the intervals J_n, using (4.5) we have

$$\begin{aligned} \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)}^p &= \sum_{n=0}^{\infty} \int_{J_n} \Big(\frac{x}{\omega(x)} \Big)^p \Big(\frac{1}{x} \int_0^x \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| dt \Big)^p dx \\ &\ge \frac{1}{(6\sqrt{2})^p} \sum_{n=0}^{\infty} \omega_{p,n} A_{n+1}^p, \end{aligned} \tag{4.6}$$

and

$$\Big\|\sum_{k=1}^{\infty}a_kr_k\Big\|^p_{\mathrm{Ces}(\omega,p)}\leq 2^p\sum_{n=0}^{\infty}\omega_{p,n}A_n^p. \tag{4.7}$$

Condition (P4) allows to show that the lower bound in (4.6) and the upper bound in (4.7) are equivalent. Assume that there exist $C>0$ and $M\geq 1$, such that for every $n\geq 1$ there exists $n'\geq 1$ with $0<n-n'\leq M$ and

$$\sup_{n\geq 1}\frac{\omega_{p,n}}{\omega_{p,n'}}\leq C.$$

From (4.7) and the inequality $\omega_{p,n}\leq C\omega_{p,n'}$ provided by (P4),

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty}a_kr_k\Big\|^p_{\mathrm{Ces}(\omega,p)}&\leq 2^p\sum_{n=0}^{\infty}\omega_{p,n}\,A_n^p\\&\leq 2^p\omega_{p,0}\,A_0^p+2^pC\sum_{n=1}^{\infty}\omega_{p,n'}\,A_n^p.\end{aligned}\tag{4.8}$$

From Cauchy-Schwarz inequality and the fact that $n-n'<M$, we have

$$\begin{aligned}A_n&=\Big|\sum_{k=1}^{n}a_k\Big|+\|(a_k)_{n+1}^{\infty}\|_2\\&\leq\Big|\sum_{k=1}^{n'+1}a_k\Big|+\Big|\sum_{k=n'+2}^{n}a_k\Big|+\|(a_k)_{n+1}^{\infty}\|_2\\&\leq\Big|\sum_{k=1}^{n'+1}a_k\Big|+(n-n'-1)^{1/2}\|(a_k)_{n'+2}^{n}\|_2+\|(a_k)_{n+1}^{\infty}\|_2\\&\leq\Big|\sum_{k=1}^{n'+1}a_k\Big|+(M-1)^{1/2}\|(a_k)_{n+1}^{\infty}\|_2+\|(a_k)_{n+1}^{\infty}\|_2\\&\leq 2(M-1)^{1/2}A_{n'+1}.\end{aligned}\tag{4.9}$$

Noting that $A_0\leq A_1$, inequalities (4.8) and (4.9) yield

$$\Big\|\sum_{k=1}^{\infty}a_kr_k\Big\|^p_{\mathrm{Ces}(\omega,p)}\leq 2^p\omega_{p,0}\,A_1^p+4^pC(M-1)^{p/2}\sum_{n=1}^{\infty}\omega_{p,n'}\,A_{n'+1}^p. \tag{4.10}$$

Assume that $n'_1 = n'_2$ for some $n_1, n_2 \in \mathbb{N}$ with $n_1 > n_2$. Then, since $0 < n - n' \leq M$ for all $n \geq 1$,

$$0 < n_1 - n_2 < n_1 - n'_2 = n_1 - n'_1 \leq M,$$

and so, for each $m \geq 0$, there are at most M indexes $n \geq 1$ such that $n' = m$. Hence,

$$\sum_{n=1}^{\infty} \omega_{p,n'} A^p_{n'+1} \leq M \sum_{n=1}^{\infty} \omega_{p,n} A^p_{n+1}. \tag{4.11}$$

From (4.10) and (4.11),

$$\begin{aligned}\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|^p_{\mathrm{Ces}(\omega,p)} &\leq 2^p \omega_{p,0}\, A^p_1 + 4^p C (M-1)^{p/2} M \sum_{n=1}^{\infty} \omega_{p,n} A^p_{n+1} \\ &\leq B^p_{\omega,p} \sum_{n=0}^{\infty} \omega_{p,n} A^p_{n+1},\end{aligned}$$

where $B^p_{\omega,p} := \max\{2^p \omega_{p,0}, 4^p C (M-1)^{p/2} M\}$.

Hence, for $A_{\omega,p} = 1/6\sqrt{2}$,

$$A_{\omega,p} \Big(\sum_{n=0}^{\infty} \omega_{p,n} A^p_{n+1} \Big)^{1/p} \leq \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \leq B_{\omega,p} \Big(\sum_{n=0}^{\infty} \omega_{p,n} A^p_{n+1} \Big)^{1/p}.$$

Completeness of $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ follows since it is isomorphic to $\mathcal{R}(\omega, p)$.

The proof in the case $p = \infty$ is completely analogous. □

Motivated by Theorem 4.6 we consider when (r_k) is a basic sequence in $\mathrm{Ces}(\omega, p)$.

Corollary 4.7. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0, 1]$. If condition (P5) is satisfied, then (r_k) is a basic sequence in $\mathrm{Ces}(\omega, p)$.*

Proof. Suppose that $1 \leq p < \infty$. Let $m_1 < m_2$. Since condition (P5) implies condition (P4), Theorem 4.6 is available. Thus, for some constant $B_{\omega,p} > 0$,

$$\begin{aligned}\Big\| \sum_{k=1}^{m_1} a_k r_k \Big\|^p_{\mathrm{Ces}(\omega,p)} &\leq B^p_{\omega,p} \sum_{n=0}^{\infty} \omega_{p,n} A^p_{n+1} \\ &= B^p_{\omega,p} \Big(\sum_{n=0}^{m_1-2} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)^{m_1}_{n+2}\|_2 \Big)^p + \Big| \sum_{k=1}^{m_1} a_k \Big|^p \sum_{n=m_1-1}^{\infty} \omega_{p,n} \Big),\end{aligned}$$

and, for $A_{\omega,p} = 1/6\sqrt{2}$,

$$\begin{aligned}
\Big\| \sum_{k=1}^{m_2} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)}^p &\geq A_{\omega,p}^p \sum_{n=0}^{\infty} \omega_{p,n} A_{n+1}^p \\
&= A_{\omega,p}^p \Bigg(\sum_{n=0}^{m_1-2} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^{m_2}\|_2 \Big)^p \\
&\qquad + \sum_{n=m_1-1}^{\infty} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^{m_2}\|_2 \Big)^p \Bigg) \\
&\geq A_{\omega,p}^p \Bigg(\sum_{n=0}^{m_1-2} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^{m_1}\|_2 \Big)^p + \omega_{p,m_1-1} \Big| \sum_{k=1}^{m_1} a_k \Big|^p \Bigg),
\end{aligned}$$

From condition (P5), there exists a constant $C > 0$ such that, for every $m \geq 0$,

$$\sum_{n=m}^{\infty} \omega_{p,n} \leq C \omega_{p,m}.$$

It follows that

$$\begin{aligned}
\Big\| \sum_{k=1}^{m_1} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} &\leq \\
&\leq B_{\omega,p} \Bigg(\sum_{n=0}^{m_1-2} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^{m_1}\|_2 \Big)^p + \Big| \sum_{k=1}^{m_1} a_k \Big|^p C \omega_{p,m_1-1} \Bigg)^{1/p} \\
&\leq C^{1/p} \, \frac{B_{\omega,p}}{A_{\omega,p}} \Big\| \sum_{k=1}^{m_2} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)},
\end{aligned}$$

which proves that (r_k) is a basic sequence.

The proof in the case $p = \infty$ is analogous. □

For $\omega(x)$ a quasiconcave function, conditions (P5) and (P4) are satisfied, and so Theorem 4.6 and Corollary 4.7 describe the behavior of the Rademacher series in $\mathrm{Ces}(\omega, p)$. Next, we isolate conditions (P2) and (P1), studying the situation when either of them fails, which allows to identify their role in the case when $\omega(x)$ is not quasiconcave (recall that (P2) implies (P1)). Condition (P3) will be considered in Section 4.4, where we show that it is related to $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ being isomorphic to ℓ^2.

Let $\mathcal{P}$ be the space of all Rademacher polynomials, and set $\mathcal{P}^0 := \cup_{m\geq 1}\mathcal{P}^0_m$, where, for $m \geq 1$,

$$\mathcal{P}^0_m := \Big\{ \sum_{k=1}^m a_k r_k : a_k \in \mathbb{R}, \text{ with } \sum_{k=1}^m a_k = 0 \Big\}.$$

Proposition 4.8. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$.*

(i) Condition (P2) *holds, that is,*

$$\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p dx < \infty, \qquad 1 \leq p < \infty,$$
$$\sup_{0\leq x\leq 1} \frac{x}{\omega(x)} < \infty, \qquad p = \infty,$$

if and only if $\mathcal{P} \subset \mathcal{R} \cap \mathrm{Ces}(\omega, p)$.

(ii) Assume that condition (P1) *holds, that is, $\omega_{p,n} < \infty$ for all n. If condition* (P2) *is not satisfied, then*

$$\mathcal{P} \cap \mathrm{Ces}(\omega, p) = \mathcal{P}^0.$$

In this case, $r_k \notin \mathrm{Ces}(\omega, p)$ for all $k \geq 1$.

(iii) Assume that condition (P1) *fails. If $\omega_{p,m} = \infty$ and $\omega_{p,n}$ is finite for $0 \leq n \leq m-1$, then*

$$\mathcal{P}^0_m \subset \mathcal{R} \cap \mathrm{Ces}(\omega, p) \subset \mathcal{P}^0_{m+1}.$$

Moreover, $\mathcal{R} \cap \mathrm{Ces}(\omega, p) = \mathcal{P}^0_{m+1}$ if and only if

$$\int_{J_m} \Big(\frac{x - 1/2^{m+1}}{\omega(x)}\Big)^p dx < \infty, \qquad 1 \leq p < \infty,$$
$$\sup_{x\in J_m} \frac{x - 1/2^{m+1}}{\omega(x)} < \infty, \qquad p = \infty.$$

Otherwise, $\mathcal{R} \cap \mathrm{Ces}(\omega, p) = \mathcal{P}^0_m$.

(iv) If $\omega_{p,0} = \infty$, then $\mathcal{R} \cap \mathrm{Ces}(\omega, p) = \{0\}$.

Proof. We suppose that $1 \le p < \infty$. Recall the definition of A_n in (4.4). For a Rademacher polynomial $\sum_{k=1}^m a_k r_k$, we have that

$$A_n = \Big|\sum_{k=1}^m a_k\Big|, \qquad n \ge m.$$

From (4.7), we have

$$\Big\|\sum_{k=1}^m a_k r_k\Big\|_{\mathrm{Ces}(\omega,p)} \le 2\Big(\sum_{n=1}^{m-1} \omega_{p,n} A_n^p + \Big|\sum_{k=1}^m a_k\Big|^p \sum_{n=m}^\infty \omega_{p,n}\Big)^{1/p}, \tag{4.12}$$

whereas the version of (4.6) for Rademacher polynomials is

$$\Big\|\sum_{k=1}^m a_k r_k\Big\|_{\mathrm{Ces}(\omega,p)} \ge \frac{1}{6\sqrt{2}}\Big(\sum_{n=1}^{m-1} \omega_{p,n} A_{n+1}^p + \Big|\sum_{k=1}^m a_k\Big|^p \sum_{n=m}^\infty \omega_{p,n}\Big)^{1/p}. \tag{4.13}$$

(i) Noting that, for all k,

$$\|r_k\|_{\mathrm{Ces}(\omega,p)} = \int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p\, dx,$$

we have that condition (P2) holds if $r_k \in \mathcal{R} \cap \mathrm{Ces}(\omega,p)$ for all $k \ge 1$. Conversely, since (P2) is equivalent to

$$\sum_{n=0}^\infty \omega_{p,n} < \infty,$$

from (4.12) it follows that $\mathcal{P} \subset \mathcal{R} \cap \mathrm{Ces}(\omega,p)$.

(ii) Since $\sum_{k=1}^m a_k r_k \in \mathcal{P}^0$ implies that

$$\sum_{k=1}^m a_k = 0,$$

from (4.12) we have that

$$\mathcal{P}^0 \subset \mathcal{P} \cap \mathrm{Ces}(\omega,p).$$

On the other hand, if (P2) fails, then

$$\sum_{n=0}^\infty \omega_{p,n} = \infty.$$

From (4.13), the space $\mathrm{Ces}(\omega,p)$ only contains Rademacher polynomials of the form $\sum_{k=1}^{m} a_k r_k$ with

$$\sum_{k=1}^{m} a_k = 0.$$

(iii) Assume that $\omega_{p,m} = \infty$. The inclusion

$$\mathcal{P}_m^0 \subset \mathcal{R} \cap \mathrm{Ces}(\omega,p)$$

follows from (4.12) and the fact that $\sum_{k=1}^{m} a_k r_k \in \mathcal{P}_m^0$, implies

$$\sum_{k=1}^{m} a_k = 0.$$

From (4.6), we have

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)}^p \geq \frac{1}{(6\sqrt{2})^p} \omega_{p,m} A_{m+1}^p.$$

Since $\omega_{p,m} = \infty$, if we have that

$$\sum_{k=1}^{\infty} a_k r_k \in \mathrm{Ces}(\omega,p),$$

then it necessarily follows that

$$A_{m+1} = \Big| \sum_{k=1}^{m+1} a_k \Big| + \|(a_k)_{m+2}^{\infty}\|_2 = 0,$$

that is, $\mathcal{R} \cap \mathrm{Ces}(\omega,p) \subset \mathcal{P}_{m+1}^0$.

Set

$$\sum_{k=1}^{m+1} a_k r_k \in \mathcal{P}_{m+1}^0 \setminus \mathcal{P}_m^0,$$

where $a_k = 1$, for $1 \leq k \leq m$, and $a_{m+1} = -m$. Noting that the inclusions

$$\mathcal{P}_m^0 \subset \mathcal{R} \cap \mathrm{Ces}(\omega,p) \subset \mathcal{P}_{m+1}^0$$

involve finite dimensional vector spaces, we have that $\mathcal{R} \cap \mathrm{Ces}(\omega,p) = \mathcal{P}_{m+1}^0$ if and only if

$$\sum_{k=1}^{m+1} a_k r_k \in \mathrm{Ces}(\omega,p);$$

otherwise, $\mathcal{R} \cap \mathrm{Ces}(\omega, p) = \mathcal{P}_m^0$.

Since $r_k(x) = 1$ for $1 \le k \le m+1$ and $x \in (0, 1/2^{m+1})$, we have

$$\frac{1}{x}\int_0^x \Big|\sum_{k=1}^{m+1} a_k r_k(t)\Big| dt = \Big|\sum_{k=1}^{m+1} a_k\Big| = 0.$$

On the other hand, for $x \in J_m = (1/2^{m+1}, 1/2^m)$, we have $r_k(x) = 1$ for $1 \le k \le m$, and $r_{m+1}(x) = -1$. Thus,

$$\int_0^x \Big|\sum_{k=1}^{m+1} a_k r_k(t)\Big| dt = 2m(x - 1/2^{m+1}).$$

It follows that

$$\begin{aligned} \Big\|\sum_{k=1}^{m+1} a_k r_k\Big\|_{\mathrm{Ces}(\omega,p)}^p &= \sum_{n\ge 0}\int_{J_n}\Big(\frac{x}{\omega(x)}\Big)^p\Big(\frac{1}{x}\int_0^x \Big|\sum_{k=1}^{m+1} a_k r_k(t)\Big|\, dt\Big)^p dx \\ &= (2m)^p \int_{J_m}\Big(\frac{x - 1/2^{m+1}}{\omega(x)}\Big)^p dx \\ &\quad + \sum_{n=0}^{m-1}\int_{J_n}\Big(\frac{x}{\omega(x)}\Big)^p\Big(\frac{1}{x}\int_0^x \Big|\sum_{k=1}^{m+1} a_k r_k(t)\Big|\, dt\Big)^p dx. \end{aligned} \tag{4.14}$$

Since $a_1 = \ldots = a_m = 1$ and $a_{m+1} = -m$, we have for $0 \le n \le m$ that

$$A_n = \Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{n+2}^\infty\|_2 = n + (m - n + m^2)^{1/2}.$$

Thus, there exist constants $C_1, C_2 > 0$, depending only on m, such that for $0 \le n \le m-1$,

$$C_1 \le A_{n+1} \le 2A_n \le C_2.$$

Together with (4.5), this inequality yields, for $x \in J_n$ with $0 \le n \le m-1$,

$$\frac{C_1}{6\sqrt{2}} \le \frac{1}{x}\int_0^x \Big|\sum_{k=1}^{m+1} a_k r_k(t)\Big|\, dt \le C_2.$$

From (4.14), we have that

$$\Big\|\sum_{k=1}^{m+1} a_k r_k\Big\|_{\mathrm{Ces}(\omega,p)}^p \asymp (2m)^p \int_{J_m}\Big(\frac{x - 1/2^{m+1}}{\omega(x)}\Big)^p dx + \sum_{n=0}^{m-1}\int_{J_n}\Big(\frac{x}{\omega(x)}\Big)^p dx.$$

Since $\omega_{p,n}$ is finite for $0 \leq n \leq m-1$, it follows that

$$\sum_{k=1}^{m+1} a_k r_k \in \mathrm{Ces}(\omega, p)$$

is equivalent to

$$\int_{J_m} \Big(\frac{x - 1/2^{m+1}}{\omega(x)}\Big)^p dx < \infty.$$

(iv) It follows from *(iii)* and from the fact that $\mathcal{P}_1^0 = \{0\}$.

The proof in the case $p = \infty$ is analogous. □

4.3 Complementability

Next, we consider the problem of the complementability of $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ in $\mathrm{Ces}(\omega, p)$. In [8, Theorem 4 and Theorem 6] it was proved, for $1 \leq p < \infty$ and $\omega(x) = x$, that $\mathcal{R} \cap \mathrm{Ces}(x, p)$ is not complemented in $\mathrm{Ces}(x, p)$, and, for $\omega(x)$ a quasiconcave function, that $\mathcal{R} \cap \mathrm{Ces}(\omega, \infty)$ is not complemented in $\mathrm{Ces}(\omega, \infty)$. We extend these results to spaces $\mathrm{Ces}(\omega, p)$ with $1 \leq p \leq \infty$ under the sole assumption that (r_k) is a basic sequence in $\mathrm{Ces}(\omega, p)$. In particular, this result applies for $\omega(x)$ a quasiconcave weight, and for the power weights $\omega(x) = x^\lambda$ with $\lambda < 1 + 1/p$ (see Example 4.12 below).

We need the following lemma, which is related to the study of when $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to ℓ^2 (see Section 4.4). Recall, for $\omega(x)$ a weight with

$$\omega_{p,0} = \int_{1/2}^1 \Big(\frac{x}{\omega(x)}\Big)^p dx = \infty,$$

that from Proposition 4.8 we have $\mathcal{R} \cap \mathrm{Ces}(\omega, p) = \{0\}$.

Lemma 4.9. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0, 1]$. Assume that $\omega_{p,0}$ is finite. There exists a constant $A_{\omega,p} > 0$ such that,*

$$A_{\omega,p} \|(a_k)_1^\infty\|_2 \leq \Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)},$$

provided that $\sum_{k=1}^\infty a_k r_k \in \mathrm{Ces}(\omega, p)$.

Proof. Let $1 \le p < \infty$, and recall that $J_0 = (1/2, 1)$. From Lemma 4.5,

$$\begin{aligned}\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} &\ge \Big(\int_{J_0} \Big(\frac{x}{\omega(x)} \Big)^p \Big(\frac{1}{x} \int_0^x \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| \, dt \Big)^p dx \Big)^{1/p} \\ &\ge \Big(\int_{J_0} \Big(\frac{x}{\omega(x)} \Big)^p dx \Big)^{1/p} \int_0^{1/2} \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| \, dt \\ &\ge \omega_{p,0}^{1/p} \frac{1}{6\sqrt{2}} \big(|a_1| + \|(a_k)_{k=2}^{\infty}\|_2 \big) \\ &\ge \frac{\omega_{p,0}^{1/p}}{6\sqrt{2}} \|(a_k)_1^{\infty}\|_2\end{aligned}$$

The case $p = \infty$ is analogous. □

The proof of the next result follows, with suitable and necessary adaptations, the steps of the case when $p = \infty$ and $\omega(x)$ is quasiconcave proved in [8, Theorem 4].

Theorem 4.10. *Let $1 \le p \le \infty$ and $\omega(x)$ be a weight on $[0, 1]$. Assume that (r_k) is a basic sequence in $\mathrm{Ces}(\omega, p)$. Then, the space $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is not complemented in $\mathrm{Ces}(\omega, p)$.*

Proof. Let $1 \le p < \infty$. Since (r_k) is a basic sequence in $\mathrm{Ces}(\omega, p)$, we have, for all $k \ge 1$, that $r_k \in \mathrm{Ces}(\omega, p)$. Thus, condition (P2) is satisfied. From Proposition 4.1, $\mathrm{Ces}(\omega, p)$ has a saturated norm, and so $\mathrm{Ces}(\omega, p)'$ is a normed space. It also follows from condition (P2) that $L^\infty([0, 1]) \subset \mathrm{Ces}(\omega, p)$. Hence, $\mathrm{Ces}(\omega, p)' \subset L^1([0, 1])$.

Suppose that there exists a projection P from $\mathrm{Ces}(\omega, p)$ onto $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$. Then,

$$Pf = \sum_{n \ge 1} \phi_n(f) r_n, \tag{4.15}$$

for some $\phi_n \in \mathrm{Ces}(\omega, p)^*$. Since $\mathrm{Ces}(\omega, p)$ has absolutely continuous norm for $1 \le p < \infty$, we have $\mathrm{Ces}(\omega, p)^* = \mathrm{Ces}(\omega, p)'$, and so

$$Pf = \sum_{n=1}^{\infty} \Big(\int_0^1 g_n(t) f(t) \, dt \Big) r_n, \qquad f \in \mathrm{Ces}(\omega, p), \tag{4.16}$$

for some $g_n \in \mathrm{Ces}(\omega, p)' \subset L^1([0, 1])$. Since P is a projection,

$$\langle g_i, r_j \rangle = \int_0^1 g_i(t) r_j(t) \, dt = \delta_{ij}.$$

Since $L^\infty([0,1]) \subset \mathrm{Ces}(\omega, p)$ we have, from (4.16),

$$\sum_{n=1}^{\infty} \Big(\int_0^1 g_n(t) f(t)\, dt \Big)^2 < \infty, \qquad f \in L^\infty([0,1]).$$

This implies that (g_n) converges weakly to zero in $L^1([0,1])$ and, by the Dunford–Pettis criterion for weak compactness in $L^1([0,1])$, the sequence (g_n) is uniformly integrable. Thus, there exists $h \in (0,1)$ and n_0 such that, for $n \geq n_0$,

$$\Big| \int_h^1 g_n(t) r_n(t)\, dt \Big| > \frac{1}{2}. \tag{4.17}$$

To see this, assume that (4.17) is not true. Then, there exists $(g_{n_i}) \subset (g_n)$ such that

$$\Big| \int_{1/i}^1 g_{n_i}(t) r_{n_i}(t)\, dt \Big| \leq \frac{1}{2}.$$

Since $\langle g_i, r_j \rangle = \delta_{i,j}$, we have

$$\int_0^{1/i} |g_{n_i}(t)|\, dt \geq \Big| \int_0^{1/i} g_{n_i}(t) r_{n_i}(t)\, dt \Big| \geq \frac{1}{2},$$

for $i \geq 1$, which contradicts the fact that (g_n) is uniformly integrable.

Let us see that there exists a constant $C > 0$, depending on $\omega(x)$ and h, such that the inequality

$$\|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,p)} \leq C \|f\|_{L^1([h,1])} \tag{4.18}$$

holds for $f \in L^1([0,1])$. To see this,

$$\begin{aligned} \|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,p)} &= \Big(\int_0^1 \Big(\frac{1}{\omega(x)} \int_0^x |f(t)| \chi_{[h,1]}(t)\, dt \Big)^p dx \Big)^{1/p} \\ &= \Big(\int_h^1 \Big(\frac{1}{\omega(x)} \int_0^x |f(t)| \chi_{[h,1]}(t)\, dt \Big)^p dx \Big)^{1/p} \\ &\leq \Big(\int_h^1 \frac{dx}{\omega(x)^p} \Big)^{1/p} \|f\|_{L^1([h,1])}. \end{aligned}$$

The finiteness of the integral above follows from condition (P2).

Define

$$P_h(f) := P(f\chi_{[h,1]}).$$

Then, the operator

$$P_h\colon L^1([h,1]) \to L^1([0,1])$$

is bounded. To see this, from Khintchine inequality in $L^1([0,1])$, we have

$$\|P_h f\|_{L^1([0,1])} = \|P(f\chi_{[h,1]})\|_{L^1([0,1])} \leq \big\|\big(\langle f\chi_{[h,1]}, g_n\rangle\big)_{n=1}^{\infty}\big\|_{\ell^2}.$$

From Lemma 4.9,

$$A_{\omega,p}\big\|\big(\langle f\chi_{[h,1]}, g_n\rangle\big)_{n=1}^{\infty}\big\|_{\ell^2} \leq \|P(f\chi_{[h,1]})\|_{\mathrm{Ces}(\omega,p)}.$$

From the fact that P is a bounded operator and (4.18), it follows that

$$\|P(f\chi_{[h,1]})\|_{\mathrm{Ces}(\omega,p)} \leq \|P\|\|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,p)} \leq C\|P\|\|f\|_{L^1([h,1])}.$$

So, we have

$$\|P_h f\|_{L^1([0,1])} \leq \frac{C}{A_{\omega,p}}\|P\|\|f\|_{L^1([h,1])},$$

that is, $P_h\colon L^1([h,1]) \to L^1([0,1])$ is bounded.

Since P_h factors through a reflexive space, it is weakly compact. Thus, from the fact that $L^1([h,1])$ has the Dunford-Pettis property and $r_n\chi_{[h,1]}$ tends weakly to zero in $L^1([0,1])$, it follows that

$$\|P_h(r_n\chi_{[h,1]})\|_{L^1([0,1])} \to 0$$

as $n \to \infty$. On the other hand, from (4.17) and Khintchine inequality, it follows, for $n \geq n_0$, that

$$\begin{aligned}\|P_h(r_n\chi_{[h,1]})\|_{L^1([0,1])} &\geq A_1\Big(\sum_{k=1}^{\infty}\Big(\int_h^1 g_k(t)r_n(t)\,dt\Big)^2\Big)^{1/2}\\ &\geq A_1\Big|\int_h^1 g_n(t)r_n(t)\,dt\Big| > \frac{A_1}{2},\end{aligned}$$

which gives a contradiction.

For the case $p=\infty$, since $\mathrm{Ces}(\omega,\infty)$ is not separable, the situation is different. However, we will see that for f in the separable part of $\mathrm{Ces}(\omega,\infty)$, denoted by $\mathrm{Ces}(\omega,\infty)_0$, we still have the projection P represented as in (4.16), that is,

$$Pf = \sum_{n=1}^{\infty}\Big(\int_0^1 g_n(t)f(t)\,dt\Big)r_n, \qquad f \in \mathrm{Ces}(\omega,\infty)_0,$$

with $g_n \in \mathrm{Ces}(\omega,\infty)' \subset L^1([0,1])$ and $\langle g_i, r_j\rangle = \delta_{ij}$. To see this, recall that we have the decomposition

$$\mathrm{Ces}(\omega,\infty)^* = \mathrm{Ces}(\omega,\infty)' \oplus (\mathrm{Ces}(\omega,\infty)')^d,$$

see [32, Ch. 15, §70, Theorem 2], where $(\mathrm{Ces}(\omega,\infty)')^d$ is the space of all singular bounded linear functionals on $\mathrm{Ces}(\omega,\infty)$. It follows, for $\phi_n \in \mathrm{Ces}(\omega,\infty)^*$ in (4.15), that

$$\phi_n = \psi_n + \theta_n, \qquad n \geq 1,$$

where $\psi_n \in \mathrm{Ces}(\omega,\infty)'$ and $\theta_n \in (\mathrm{Ces}(\omega,\infty)')^d$. In particular,

$$\theta_n(f) = 0, \quad f \in \mathrm{Ces}(\omega,\infty)_0,$$

and, for some $g_n \in \mathrm{Ces}(\omega,\infty)' \subset L^1([0,1])$,

$$\psi_n(f) = \int_0^1 f(t)g_n(t)dt, \qquad f \in \mathrm{Ces}(\omega,\infty).$$

Note that, since we do not necessarily have $r_k \in \mathrm{Ces}(\omega,\infty)_0$, it does not follow immediately that $\langle g_i, r_j\rangle = \delta_{ij}$. Since condition (P2) is satisfied, and $r_k - \chi_{[0,1]} = 0$ on $[0, 1/2^k]$, it follows that $r_k - \chi_{[0,1]} \in \mathrm{Ces}(\omega,\infty)_0$. Fix $n \geq 1$. Then, we have $\theta_n(r_k - \chi_{[0,1]}) = 0$, that is,

$$\theta_n(r_k) = \theta_n(\chi_{[0,1]}), \quad k \geq 1.$$

Since P is a projection,

$$\begin{aligned} \phi_n(r_n) &= \psi_n(r_n) + \theta_n(r_n) = 1, \\ \phi_n(r_k) &= \psi_n(r_k) + \theta_n(r_k) = 0, \quad k \neq n. \end{aligned} \tag{4.19}$$

Hence, for $k > n$, we have $\theta_n(\chi_{[0,1]}) = -\psi_n(r_k)$. Moreover, since $g_n \in L^1([0,1])$,

$$\lim_{k\to\infty} \psi_n(r_k) = \lim_{k\to\infty} \int_0^1 g_n(t)r_k(t)dt = 0.$$

Thus, $\theta_n(r_k) = \theta_n(\chi_{[0,1]}) = 0$ for all $k \geq 1$, which together with (4.19) implies that $\langle g_i, r_j\rangle = \delta_{ij}$.

The proof then follows the same steps as in the case $1 \leq p < \infty$, noting that the inequality

$$\|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,\infty)} \leq C\|f\|_{L^1([h,1])}$$

follows from

$$\|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,\infty)} \leq \sup_{h\leq x\leq 1} \frac{1}{\omega(x)}\|f\|_{L^1([h,1])}$$

together with condition (P2). $\square$

From Theorem 4.10 and Corollary 4.7, we have the following.

Corollary 4.11. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$.*

(i) If condition (P5) *holds, then $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is not complemented in $\mathrm{Ces}(\omega, p)$.*

(ii) In particular, if $\omega(x)$ is quasiconcave, then $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is not complemented in $\mathrm{Ces}(\omega, p)$.

We end this section considering the Cesàro spaces $\mathrm{Ces}(x^\lambda, p)$ corresponding to power weights $\omega(x) = x^\lambda$, for $\lambda \in \mathbb{R}$.

Example 4.12. Let $1 \leq p < \infty$ and consider $\mathrm{Ces}(x^\lambda, p)$ for $\lambda \in \mathbb{R}$, that is,

$$\|f\|_{\mathrm{Ces}(x^\lambda,p)} = \Big(\int_0^1 \Big(\frac{1}{x^\lambda} \int_0^x |f(t)|\, dt \Big)^p dx \Big)^{1/p}.$$

Set $\delta := p(1-\lambda) + 1$. We have, for $\delta \neq 0$ and $n \geq 0$,

$$\omega_{p,n} := \int_{J_n} \Big(\frac{x}{\omega(x)} \Big)^p dx = \int_{1/2^{n+1}}^{1/2^n} x^{(1-\lambda)p}\, dx = \frac{1}{\delta}\Big(\frac{1}{2^{n\delta}} - \frac{1}{2^{(n+1)\delta}} \Big) = \frac{1}{\delta}\Big(1 - \frac{1}{2^\delta}\Big) \frac{1}{2^{n\delta}},$$

and for $\delta = 0$, that is, $p(1-\lambda) = -1$,

$$\omega_{p,n} := \int_{J_n} \Big(\frac{x}{\omega(x)} \Big)^p dx = \int_{1/2^{n+1}}^{1/2^n} x^{-1}\, dx = \Big(\ln \frac{1}{2^n} - \ln \frac{1}{2^{(n+1)}} \Big) = \ln 2.$$

In both cases,

$$\frac{\omega_{p,n+1}}{\omega_{p,n}} = 2^{-\delta},$$

and so (P4*) holds for arbitrary $\lambda \in \mathbb{R}$ and $1 \leq p < \infty$. From Theorem 4.6 it follows that

$$\Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{\mathrm{Ces}(x^\lambda,p)} \asymp \Big(\sum_{n=0}^\infty \frac{1}{2^{n\delta}} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^\infty\|_2 \Big)^p \Big)^{1/p}. \tag{4.20}$$

Suppose $\delta > 0$, that is, $\lambda < 1 + 1/p$. Then, since

$$\omega_{p,n} = \frac{1}{\delta}\Big(1 - \frac{1}{2^\delta}\Big)\frac{1}{2^{n\delta}},$$

we have

$$\sum_{n \geq m} \omega_{p,n} = \frac{1}{\delta}\Big(1 - \frac{1}{2^\delta}\Big) \sum_{n \geq m} \frac{1}{2^{n\delta}} = \frac{1}{\delta}\Big(1 - \frac{1}{2^\delta}\Big)\Big(1 - \frac{1}{2^\delta}\Big)^{-1} \frac{1}{2^{m\delta}} = \Big(1 - \frac{1}{2^\delta}\Big)^{-1} \omega_{m,p},$$

and so condition (P5) is satisfied. From Corollary 4.7, we have that (r_k) is a basic sequence in $\mathrm{Ces}(\omega, p)$, and $\mathcal{R} \cap \mathrm{Ces}(x^\lambda, p)$ is not complemented in $\mathrm{Ces}(x^\lambda, p)$. From Cauchy-Schwarz inequality, we have

$$\Big|\sum_{k=1}^{n} a_k\Big| + \|(a_k)_{n+1}^{\infty}\|_2 \leq 2n^{1/2}\|(a_k)_1^{\infty}\|_2.$$

Hence, from (4.20) we have, for a constant $C > 0$ depending on λ and p,

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{Ces}(x^\lambda,p)} \leq C\Big(\sum_{n=0}^{\infty} \frac{n^{p/2}}{2^{n\delta}}\Big)^{1/p}\|(a_k)_1^{\infty}\|_2.$$

The previous series converges, as $\delta > 0$. Together with Lemma 4.9, we have, for some constants $A_{\lambda,p}, B_{\lambda,p} > 0$,

$$A_{\lambda,p}\|(a_k)_1^{\infty}\|_2 \leq \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{Ces}(x^\lambda,p)} \leq B_{\lambda,p}\|(a_k)_1^{\infty}\|_2.$$

Thus, $\mathcal{R} \cap \mathrm{Ces}(x^\lambda, p)$ is isomorphic to ℓ^2 for $1 \leq p < \infty$ and $\lambda < 1 + 1/p$. This was proved in [8] in the case $1 \leq p < \infty$ and $\lambda = 1$.

Suppose now that $\delta \leq 0$, that is, $\lambda \geq 1 + 1/p$. In this case, condition (P2) fails, since the integral

$$\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p dx = \int_0^1 \frac{dx}{x^{(\lambda-1)p}}$$

is not finite for $(\lambda - 1)p \geq 1$. It follows that $\mathrm{Ces}(x^\lambda, p)$ does not contain the single Rademacher functions, and from Proposition 4.8, it only contains among the Rademacher polynomials those of the form

$$\sum_{k=1}^{m} a_k r_k, \text{ with } \sum_{k=1}^{m} a_k = 0.$$

But there are also infinite Rademacher series in $\mathrm{Ces}(x^\lambda, p)$. To see this, let, for example, $\delta = 0$, that is, $\lambda = 1 + 1/p$. In this case, (4.20) becomes

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{Ces}(x^{1+1/p},p)} \asymp \Big(\sum_{n=0}^{\infty}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{n+2}^{\infty}\|_2\Big)^p\Big)^{1/p}.$$

Set

$$a_{3k} := 1/k^2, \qquad a_{3k+1} := a_{3k+2} := -1/2k^2, \qquad k \geq 0.$$

Then, for $n \geq 1$ and some constant $C > 0$,

$$\Big|\sum_{k=1}^{n} a_k\Big| \leq \frac{1}{n^2}, \quad \|(a_k)_n^\infty\|_2 \leq \frac{C}{n^{3/2}}.$$

Thus, $\sum_{k=1}^\infty a_k r_k \in \mathrm{Ces}(x^{1+1/p}, p)$.

In the case $p = \infty$, we have, for $\lambda \leq 1$,

$$\omega_{\infty,n} := \sup_{x \in J_n} \frac{x}{\omega(x)} = \sup_{x \in J_n} x^{1-\lambda} = 2^{n(\lambda-1)},$$

and for $\lambda > 1$,

$$\omega_{\infty,n} = \sup_{x \in J_n} x^{1-\lambda} = 2^{(n+1)(\lambda-1)}.$$

In both cases, since

$$\frac{\omega_{\infty,n+1}}{\omega_{\infty,n}} = 2^{\lambda-1},$$

condition (P4*) holds. Then, we have the equivalence

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{Ces}(x^\lambda,\infty)} \asymp \sup_{n\geq 0} 2^{n(\lambda-1)}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{n+2}^\infty\|_2\Big).$$

For $\lambda < 1$ it follows, as in the case $1 \leq p < \infty$, that $\mathrm{Ces}(x^\lambda, \infty)$ is isomorphic to ℓ^2, and $\mathcal{R} \cap \mathrm{Ces}(x^\lambda, \infty)$ is not complemented in $\mathrm{Ces}(x^\lambda, \infty)$.

For $\lambda = 1$ we have

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{Ces}(x,\infty)} \asymp \sup_{n\geq 0} \Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{n+2}^\infty\|_2\Big),$$

which shows that $\mathrm{Ces}(x, \infty)$ is not isomorphic to ℓ^2.

For $\lambda > 1$, condition (P2) is not satisfied, since

$$\sup_{0<x<1} \frac{x}{\omega(x)} = \sup_{0<x<1} \frac{1}{x^{\lambda-1}} = \infty,$$

and so $r_k \notin \mathrm{Ces}(x^\lambda, \infty)$ for all $k \geq 1$.

Remark 4.13. The previous example shows, for power weights $\omega(x) = x^\lambda$, that condition (P2) is equivalent to $\mathcal{R} \cap \mathrm{Ces}(x^\lambda, p)$ being isomorphic to ℓ^2. This equivalence is not true in general, as can be seen by considering

$$\omega(x) = x^2 \log_2^{3/2}(2/x).$$

For $p = 1$ and $n \geq 0$, since

$$\int_{J_n} \frac{1}{x}\, dx = \ln 2$$

and

$$\frac{1}{(n+2)^{3/2}} \leq \frac{1}{\log_2^{3/2}(2/x)} \leq \frac{1}{(n+1)^{3/2}}, \qquad x \in J_n,$$

we have

$$\omega_{1,n} = \int_{J_n} \frac{dx}{x \log_2^{3/2}(2/x)} \asymp \frac{1}{(n+1)^{3/2}}.$$

Thus, condition (P2) is satisfied.

To see that $\mathcal{R} \cap \mathrm{Ces}(\omega, 1)$ is not isomorphic to ℓ^2, let $a_k = 1/\sqrt{k}$, for $1 \leq k \leq N$. Then, we have

$$\|(a_k)_1^N\|_2 = \Big(\sum_{k=1}^{N} \frac{1}{k}\Big)^{1/2} \asymp \log_2^{1/2} N.$$

On the other hand, since

$$\Big|\sum_{k=1}^{n+1} a_k\Big| = \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} \asymp (n+1)^{1/2},$$

from Theorem 4.6 it follows that, for some constant $A_{\omega,1} > 0$,

$$\Big\|\sum_{k=1}^{N} a_k r_k\Big\|_{\mathrm{Ces}(\omega,1)} \geq A_{\omega,1} \sum_{n=0}^{N-1} \frac{1}{(n+1)^{3/2}} \Big|\sum_{k=1}^{n+1} a_k\Big| \asymp \log_2 N,$$

Hence, $\mathcal{R} \cap \mathrm{Ces}(\omega, 1)$ is not isomorphic to ℓ^2.

4.4 $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ isomorphic to ℓ^2

In this section we study the situation when $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to ℓ^2. In Example 4.12 it was shown, for power weights $\omega(x) = x^\lambda$, that $\mathcal{R} \cap \mathrm{Ces}(x^\lambda, p)$ is isomorphic to ℓ^2 precisely when $\lambda < 1 + 1/p$ and $1 \leq p \leq \infty$, generalizing Theorem 1 of [8], where it is shown that that $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to ℓ^2 when $\omega(x) = x$ and $1 \leq p < \infty$.

For $p = \infty$ it was shown in Theorem 3 of [8], for $\omega(x)$ a quasiconcave function, that $\mathcal{R} \cap \mathrm{Ces}(\omega, \infty)$ is isomorphic to ℓ^2 if and only if $\omega(x) \geq cx \log_2^{1/2}(2/x)$. Note that this last condition is precisely condition (P3) for $p = \infty$.

We prove, for every $1 \le p \le \infty$, that condition (P3) suffices for $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ being isomorphic to ℓ^2, thus removing the need of quasiconcavity. However, while condition (P3) is necessary when $p = \infty$, it is not necessary when $1 \le p < \infty$, even though it is very close to being so, as it will be shown, by considering decreasing rearrangements of Rademacher series.

Theorem 4.14. *Let $1 \le p \le \infty$ and $\omega(x)$ be a weight on $[0,1]$. Condition (P3) holds if and only if there exist constants $A_{\omega,p}, B_{\omega,p} > 0$ such that*

$$A_{\omega,p}\|(a_k)_1^\infty\|_2 \le \Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,p)} \le B_{\omega,p}\|(a_k)_1^\infty\|_2, \tag{4.21}$$

for $(a_k)_1^\infty \in \ell^2$.

Proof. Assume that condition (P3) holds, that is,

$$\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2}(2/x)\, dx < \infty, \qquad 1 \le p < \infty,$$
$$\sup_{0<x\le 1} \frac{x}{\omega(x)} \log_2^{1/2}(2/x) < \infty, \qquad p = \infty,$$

From Lemma 4.9 and the general inequality

$$\int_0^x |f(t)|\, dt \le \int_0^x f^*(t)\, dt, \tag{4.22}$$

we have

$$A_{\omega,p}\|(a_k)_1^\infty\|_2 \le \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \le \Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,p)}.$$

To prove the right–hand side inequality of (4.21), let L^{M_2} be the Orlicz space generated by $M_2(t) := \exp(t^2) - 1$. Recall from (6) that the fundamental function of its associate space $(L^{M_2})'$ is given by

$$\varphi_{(L^{M_2})'}(x) = x \log_2^{1/2}(2/x).$$

Thus, for $0 < x \le 1$ and $(a_k)_1^\infty \in \ell^2$, since L^{M_2} is rearrangement invariant,

$$\begin{aligned}\frac{1}{x}\int_0^x \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^*(t)\, dt &\le \frac{1}{x}\|\chi_{(0,x)}\|_{(L^{M_2})'} \Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{L^{M_2}} \\ &\le B \log_2^{1/2}(2/x)\|(a_k)_1^\infty\|_2,\end{aligned}$$

where $B = B_{L^{M_2}}$. Hence, for $1 \le p < \infty$,

$$\Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,p)} \le B \Big(\int_0^1 \Big(\frac{x}{\omega(x)} \Big)^p \log_2^{p/2}(2/x)\, dx \Big)^{1/p} \|(a_k)_1^\infty\|_2;$$

whereas for $p = \infty$,

$$\Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,\infty)} \le B \sup_{0<x\le 1} \frac{x}{\omega(x)} \log_2^{1/2}(2/x) \|(a_k)_1^\infty\|_2.$$

Condition (P3) is precisely the finiteness of the integral or the supremum above.

For the converse, the cases $1 \le p < \infty$ and $p = \infty$ are different.

Let $1 \le p < \infty$, and assume that inequality (4.21) holds. Let

$$v_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} r_k.$$

By our assumption we have, for $n \ge 1$

$$\|v_n^*\|_{\mathrm{Ces}(\omega,p)} \le B_{\omega,p} \|(1/\sqrt{n})_1^n\|_2 = B_{\omega,p}.$$

Via the Central Limit Theorem (as can be seen in the proof of [26, Theorem 6], see also [22, Theorem 2.b.4]) we have, for some $C > 0$, that

$$\log_2^{1/2}(2/x) \le C \lim_{n\to\infty} v_n^*(x), \qquad 0 < x \le 1.$$

Hence, noting that the average of f^* on $[0,x]$ is greater than $f^*(x)$ for any measurable function f and $0 < x \le 1$,

$$\begin{aligned}
\int_0^1 \Big(\frac{x}{\omega(x)} \Big)^p \log_2^{p/2}(2/x)\, dx &\le C^p \int_0^1 \Big(\frac{x}{\omega(x)} \Big)^p (\lim_{n\to\infty} v_n^*(x))^p dx \\
&= C^p \lim_{n\to\infty} \int_0^1 \Big(\frac{x}{\omega(x)} \Big)^p v_n^*(x)^p dx \\
&\le C^p \lim_{n\to\infty} \int_0^1 \Big(\frac{x}{\omega(x)} \Big)^p \Big(\frac{1}{x} \int_0^x v_n^*(s)\, ds \Big)^p dx \\
&= C^p \lim_{n\to\infty} \|v_n^*\|_{\mathrm{Ces}(\omega,p)} \le C^p B_{\omega,p}.
\end{aligned}$$

Thus, condition (P3) is satisfied.

Now we consider the case $p = \infty$. Assume that the equivalence

$$\Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \|(a_k)_1^\infty\|_2$$

holds with constants depending on the weight $\omega(x)$. In particular, this implies that $r_k \in \mathrm{Ces}(\omega, \infty)$, $k \geq 1$, and so all the coefficients $\omega_{\infty,n}$ are finite. Thus, if (P3) does not hold, we have

$$\sup_{n \geq 0} \omega_{\infty,n}(n+1)^{1/2} = \infty,$$

and so there exists a subsequence $(n_j)_{j=1}^\infty$ such that

$$\lim_{j \to \infty} \omega_{\infty,n_j}(n_j+1)^{1/2} = \infty.$$

Set

$$a_k^j = (n_j+1)^{-1/2}, \qquad 1 \leq k \leq n_j + 1,$$
$$a_k^j = 0, \qquad k \geq n_j + 2.$$

We have that $\|(a_k^j)_{k=1}^\infty\|_2 = 1$ for $j \geq 1$. From the corresponding version of (4.6) for $p = \infty$ we have, for a constant $A_{\omega,\infty} > 0$,

$$\begin{aligned}
\Big\| \Big(\sum_{k=1}^{\infty} a_k^j r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,\infty)} &\geq \Big\| \sum_{k=1}^{\infty} a_k^j r_k \Big\|_{\mathrm{Ces}(\omega,\infty)} \\
&\geq A_{\omega,\infty}\, \omega_{\infty,n_j} \Big| \sum_{k=1}^{n_j+1} a_k^j \Big| \\
&= A_{\omega,\infty}\, \omega_{\infty,n_j} \Big| \sum_{k=1}^{n_j+1} (n_j+1)^{-1/2} \Big| \\
&= A_{\omega,\infty}\, \omega_{\infty,n_j} (n_j+1)^{1/2},
\end{aligned}$$

which letting $j \to \infty$ yields a contradiction. □

The inequality

$$\Big\| \sum_{k \geq 1} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \leq \Big\| \Big(\sum_{k \geq 1} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,p)}$$

holds, but the norms in $\mathrm{Ces}(\omega,p)$ of a Rademacher series $\sum_{k=1}^{\infty} a_k r_k$ and its decreasing rearrangement $(\sum_{k=1}^{\infty} a_k r_k)^*$ need not be equivalent. To see this, consider $\omega(x) = x^{1+1/p}$. From Proposition 4.8 we have that $r_1 - r_2 \in \mathrm{Ces}(\omega,p)$. On the other hand, $(r_1-r_2)^* \notin \mathrm{Ces}(\omega,p)$, since $(r_1-r_2)^* = 2\chi_{[0,1/2]}$. This example, together with the following theorem, suggests that condition (P3),

$$\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2}(2/x)\,dx < \infty, \qquad 1 \le p < \infty,$$
$$\sup_{0<x\le 1} \frac{x}{\omega(x)} \log_2^{1/2}(2/x) < \infty, \qquad p = \infty,$$

is stronger than $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ being isomorphic to ℓ^2.

Theorem 4.15. *Let $\omega(x)$ be a weight on $[0,1]$.*

(i) Let $1 \le p < \infty$.

a) If condition (P3) holds, then $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is isomorphic to ℓ^2.

b) If $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is isomorphic to ℓ^2, then for every ε with $0 < \varepsilon < p/2$ we have

$$\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2-\varepsilon}(2/x)dx < \infty.$$

(ii) For $p = \infty$, the space $\mathcal{R} \cap \mathrm{Ces}(\omega,\infty)$ is isomorphic to ℓ^2 if and only if condition (P3) holds.

Proof. *(i)* If condition (P3) holds, from Theorem 4.14 and Lemma 4.9 we have

$$\begin{aligned} A_{\omega,p}\|(a_k)_1^\infty\|_2 &\le \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \\ &\le \Big\| \Big(\sum_{k=1}^{\infty} a_k r_k\Big)^* \Big\|_{\mathrm{Ces}(\omega,p)} \\ &\le B_{\omega,p}\|(a_k)_1^\infty\|_2, \end{aligned}$$

which proves *a)*.

To prove *b)*, let $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ be isomorphic to ℓ^2. In particular, $\omega_{p,n}$ is finite for $n \ge 0$. Suppose, for some $0 < \varepsilon < p/2$, that

$$\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2-\varepsilon}(2/x)\,dx = \infty.$$

Since, for $x \in J_n = (1/2^{n+1}, 1/2^n)$, we have

$$n + 1 \le \log_2(2/x) \le n + 2,$$

it follows that

$$\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2-\varepsilon}(2/x)\, dx = \sum_{n\ge 0} \int_{J_n} \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2-\varepsilon}(2/x)\, dx$$
$$\asymp \sum_{n\ge 0} \int_{J_n} \Big(\frac{x}{\omega(x)}\Big)^p (n+1)^{p/2-\varepsilon}\, dx.$$

Thus,

$$\sum_{n=0}^{\infty} \omega_{p,n}(n+1)^{p/2-\varepsilon} = \infty.$$

Set

$$a_k = k^{-1/2-\varepsilon/p}, \qquad k \ge 1.$$

We have $(a_k)_1^\infty \in \ell^2$. On the other hand, from (4.6) it follows the inequality

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)}^p \ge \frac{1}{(6\sqrt{2})^p} \sum_{n=0}^{\infty} \omega_{p,n} \Big| \sum_{k=1}^{n+1} a_k \Big|^p,$$

which together with the fact that

$$\Big| \sum_{k=1}^{n+1} \frac{1}{k^{1/2+\varepsilon/p}} \Big|^p \asymp (n+1)^{p/2-\varepsilon}$$

implies that $\sum_{k=1}^{\infty} a_k r_k \notin \mathrm{Ces}(\omega,p)$. This gives a contradiction.

(ii) If (P3) is satisfied, the equivalence

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \|(a_k)_1^\infty\|_2 \tag{4.23}$$

follows as in the case $1 \le p < \infty$.

Conversely, assume that (4.23) holds. In particular, this implies that $\omega_{\infty,n}$ is finite, for all $n \ge 0$. Suppose that

$$\sup_{0<x\le 1} \frac{x}{\omega(x)} \log_2^{1/2}(2/x) = \infty,$$

that is,

$$\sup_{n\geq 0}\sup_{x\in J_n}\frac{x}{\omega(x)}\log_2^{1/2}(2/x)=\infty.$$

From the fact that, for $x\in J_n=(1/2^{n+1},1/2^n)$, we have

$$n+1\leq \log_2(2/x)\leq n+2,$$

it follows that

$$\sup_{n\geq 0}\omega_{\infty,n}(n+1)^{1/2}=\infty.$$

Thus, there exists a sequence n_j such that

$$\lim_{j\to\infty}\omega_{\infty,n_j}(n_j+1)^{1/2}=\infty.$$

Set

$$\begin{aligned}&a_k^j=(n_j+1)^{-1/2}, \qquad 1\leq k\leq n_j+1,\\ &a_k^j=0, \qquad k\geq n_j+2.\end{aligned}$$

We have that $\|(a_k^j)_{k=1}^\infty\|_2=1$ for $j\geq 1$. From Theorem 4.6, we have, for some constant $A_{\omega,\infty}>0$,

$$\Big\|\sum_{k=1}^\infty a_k^j r_k\Big\|_{\mathrm{Ces}(\omega,\infty)}\geq A_{\omega,\infty}\,\omega_{\infty,n_j}\Big|\sum_{k=1}^{n_j+1}a_k^j\Big|=A_{\omega,\infty}\,\omega_{\infty,n_j}(n_j+1)^{1/2},$$

which letting $j\to\infty$ yields a contradiction. □

Corollary 4.16. *Let $1\leq p\leq\infty$ and $\omega(x)$ be a weight on $[0,1]$. Suppose that $\omega(x)$ satisfies condition* (P3)*. Then,*

(i) The sequence (r_k) is basic in $\mathrm{Ces}(\omega,p)$.

(ii) The space $\mathcal{R}\cap\mathrm{Ces}(\omega,p)$ is not complemented in $\mathrm{Ces}(\omega,p)$.

(iii) For $(a_k)_1^\infty\in\ell^2$, the series $\sum_{k=1}^\infty a_k r_k$ converges unconditionally.

We end this section giving an equivalent expression for the norm of $(\sum_{k=1}^\infty a_k r_k)^*$ in $\mathrm{Ces}(\omega,p)$. For this, we need the following result, which follows from the proof of [3, Corollary 8.1], with suitable modifications.

For $(a_k)_1^\infty\in\ell^2$, recall that $(a_k^*)_1^\infty$ is the decreasing rearrangement of the sequence $(|a_k|)_1^\infty$.

Lemma 4.17. *For $(a_k)_1^\infty \in \ell^2$ and $0 < x \le 1$, the equivalence*

$$\frac{1}{x}\int_0^x \Big(\sum_{k=1}^\infty a_k r_k\Big)^*(t)\,dt \asymp \sum_{k=1}^{[\log_2(2/x)]} a_k^* + \log_2^{1/2}(2/x)\|(a_k^*)_{[\log_2(2/x)]+1}^\infty\|_2$$

holds with absolute constants.

Since $[\log_2(2/x)] = n+1$ when $x \in J_n$, it follows from the previous lemma that

$$\frac{1}{x}\int_0^x \Big(\sum_{k=1}^\infty a_k r_k\Big)^*(t)\,dt \asymp \sum_{k=1}^{n+1} a_k^* + (n+1)^{1/2}\|(a_k^*)_{n+2}^\infty\|_2, \quad x \in J_n.$$

This allows us to obtain an analogous result to Theorem 4.6 for the decreasing rearrangement of a Rademacher series. Note, since the upper and lower bounds in the equivalence above are the same, that condition (P4) is not necessary.

Theorem 4.18. *Let $\omega(x)$ be a weight on $[0,1]$.*

For $1 \le p < \infty$, we have

$$\Big\|\Big(\sum_{k=1}^\infty a_k r_k\Big)^*\Big\|_{\mathrm{Ces}(\omega,p)} \asymp \Big(\sum_{n\ge 0}\omega_{p,n}\Big(\sum_{k=1}^{n+1} a_k^* + (n+1)^{1/2}\|(a_k^*)_{n+2}^\infty\|_2\Big)^p\Big)^{1/p}.$$

For $p = \infty$,

$$\Big\|\Big(\sum_{k=1}^\infty a_k r_k\Big)^*\Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \sup_{n\ge 0}\omega_{p,n}\Big(\sum_{k=1}^{n+1} a_k^* + (n+1)^{1/2}\|(a_k^*)_{n+2}^\infty\|_2\Big).$$

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