MOBI ALGEBRA AS AN ABSTRACTION TO THE UNIT INTERVAL AND ITS COMPARISON TO RINGS

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ABSTRACT. We introduce a new algebraic structure, called mobi algebra, consisting of three constants and one ternary operation. The canonical example of a mobi algebra is the unit interval with the three constants 0, 1 and 1/2 and the ternary operation given by the formula $x-yx+yz$. We study some of its properties and prove that every unitary ring with one half uniquely determines and is uniquely determined by a mobi algebra with one double. Another algebraic structure, called involutive medial monoid (IMM), is considered to establish the passage between rings and mobi algebras.

1. INTRODUCTION

The unit interval $I=[0,1]$ is equipped with several algebraic structures. It is a monoid $(I,\cdot,1)$ with $\cdot$ the usual multiplication. It is a midpoint algebra $(I,\oplus)$ with the mean operation $x\oplus y=\frac{x+y}{2}$. It admits an involution $\overline{x}=1-x$ thus creating a dual monoid $(I,\circ,0)$ with $x\circ y=x+y-xy$. The two monoid structures are isomorphic and related by the formulas

$$\begin{aligned}\overline{x\cdot y} &= \overline{y}\circ\overline{x}\\ \overline{x\circ y} &= \overline{y}\cdot\overline{x}.\end{aligned}$$

In our search for an axiomatization of the unit interval as a monoid together with a mean operation and an involution we have discovered a more fundamental structure. From this more fundamental structure we can derive all the other algebraic operations. This new structure is called mobi algebra and it consists of a ternary operation together with three constants. In the case of the unit interval the three constants are 0, $\frac{1}{2}$, 1, while the ternary operation is $p(x,y,z)=x-yx+yz$. The

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other operations are now derived operations and they are obtained as

$$\begin{aligned}\overline{x} &= p(1,x,0)\\ x\cdot y &= p(0,x,y)\\ x\oplus y &= p(x,\frac{1}{2},y)\\ x\circ y &= p(x,y,1).\end{aligned}$$

The canonical example of a mobi algebra is thus the unit interval of the real numbers. In the same way, the unit interval of an ordered field is an example of a mobi algebra. More generally, if R is a unitary ring in which the element 2 is invertible, and if $A\subseteq R$ is any subset containing 0, 2^{-1}, 1 and being closed under the formula $p(x,y,z)=x-yx+yz$, then the structure $(A,p,0,2^{-1},1)$ is a mobi algebra (Definition 2.1).

The idea of abstracting the unit interval is not new and has been considered extensively in the literature (see e.g. [2] and the references therein). One of the reasons that have convinced us about the significance of the mobi algebra structure is its deep connection with unitary rings (see diagram (41) at the end). We will use the expression *a ring with one half* to refer to a unitary ring in which the element 2 is invertible. And we will use the expression *a mobi algebra with one double* to refer to a mobi algebra $(A,p,0,½,1)$ in which the element ½ is invertible (in the sense of the derived multiplication $x\cdot y=p(0,x,y)$). The equivalence that is proved here, between mobi algebras with one double and unitary rings with one half (Theorem 7.3), is in some sense comparable to the equivalence between boolean rings and boolean algebras. Although in a boolean ring the element 2 is not invertible.

This paper is part of an ongoing work which aims at an axiomatic characterization of spaces with geodesic paths between any two of its points. The interest in the study of geodesic paths from an algebraic point of view has started with an observation on weakly Mal'cev categories [5]. We observed that a certain ternary operation, $q=q(x,y,z)$, arising in the context of weakly Mal'tsev quasi-varieties [6], could be used to describe the path of a particle moving in space along a geodesic curve from a point x to a point z at an instant y. This idea is in some sense related with the approach taken in [1] to the treatment of affine spaces. There, the ternary operation $q(x,y,z)$ is the fourth vertex of a parallelogram determined by the other three. In our case, a binary operation is obtained by fixing y to a value that positions the particle at half way from x to z. The study of this binary operation was a first step in our investigation [3].

This paper is organized as follows. In Section 2 we give the definition of a mobi algebra and present some basic properties. Examples are given in Section 3 and in the Appendix. In Section 4, we observe that a mobi algebra gives rise to several other structures. Namely, two monoid structures $(A,\circ,0)$, $(A,\cdot,1)$, dual to each other, one midpoint algebra

$(A, \oplus)$ and an involution $\overline{()}\colon A \to A$. These derived operations satisfy some axioms and form what we call an involutive medial monoid, IMM for short. The importance of this structure is highlighted in Section 5 where its relation to unitary rings is shown. In Section 6 we characterize those IMM that are obtained from a mobi algebra. Our main result is Theorem 7.3.

2. Definition and basic properties

A mobi algebra consists of three constants and one ternary operation. The ternary operation $p(x, y, z)$ is thought of as being the point, in a path linking x to z, which lies at a location y in-between. This image might provide some intuition and the reader is invited to keep in mind, for simplicity, either one of the two formulas $p = (1 - y)x + yz$ or $p = x + y(z - x)$.

The motivating example is the unit interval $[0, 1]$, with the three constants $0, \frac{1}{2}, 1$ and the formula $p(x, y, z) = x + yz - yx$. Note that we will consistently distinguish between the element ½, which is only used as a symbol, and the real number $\frac{1}{2} \in [0, 1]$.

Definition 2.1. *A mobi algebra is a system $(A, p, 0, ½, 1)$, in which A is a set, p is a ternary operation and 0, ½ and 1 are elements of A, that satisfies the following axioms:*

- **(A1)** $p(1, ½, 0) = ½$
- **(A2)** $p(0, a, 1) = a$
- **(A3)** $p(a, b, a) = a$
- **(A4)** $p(a, 0, b) = a$
- **(A5)** $p(a, 1, b) = b$
- **(A6)** $p(a, ½, b) = p(a', ½, b) \implies a = a'$
- **(A7)** $p(a, p(c_1, c_2, c_3), b) = p(p(a, c_1, b), c_2, p(a, c_3, b))$
- **(A8)** $p(p(a_1, c, b_1), ½, p(a_2, c, b_2)) = p(p(a_1, ½, a_2), c, p(b_1, ½, b_2))$

The ternary operation p is not associative, not even partially. Nevertheless, it verifies several properties that involve an interaction of p with itself, as exemplified in the next proposition.

Proposition 2.1. *Let $(A, p, 0, ½, 1)$ be a mobi algebra. It follows that:*

$$p(a, p(0, c, d), b) = p(a, c, p(a, d, b)) \tag{1}$$
$$p(a, p(1, c, d), b) = p(b, c, p(a, d, b)) \tag{2}$$
$$p(a, p(c, d, 0), b) = p(p(a, c, b), d, a) \tag{3}$$
$$p(a, p(c, d, 1), b) = p(p(a, c, b), d, b) \tag{4}$$

Proof. These properties are obtained directly from axiom **(A7)** and the use of **(A4)** or **(A5)**. □

Fixing some other elements in the previous relations, we get further properties. For example, from (2), we get the following result.

Corollary 2.2. *If* $(A, p, 1, 1/2, 0)$ *is a mobi algebra, then:*

$$p(a, p(1, c, 0), b) = p(b, c, a) \quad (5)$$

$$p(a, 1/2, b) = p(b, 1/2, a) \quad (6)$$

$$p(1, p(1, c, 0), 0) = c \quad (7)$$

Proof. These properties are an immediate consequence of (2) and, respectively, axioms**(A4)**, **(A1)** and **(A2)**. □

We finish this section with three more properties of a mobi algebra structure.

Proposition 2.3. *Let* $(A, p, 0, 1/2, 1)$ *be a mobi algebra. It follows that:*

$$p(p(a_1, c, b_1), 1/2, p(a_2, c, b_2)) = p(p(a_2, c, b_1), 1/2, p(a_1, c, b_2)) \quad (8)$$

$$p(p(a, c, b), 1/2, p(b, c, a)) = p(a, 1/2, b) \quad (9)$$

$$p(p(a, p(1, c, 0), b), 1/2, p(a, c, d)) = p(a, 1/2, p(b, c, d)). \quad (10)$$

Proof. Using **(A8)**, (6) and again **(A8)**, we get:

$$\begin{aligned} p(p(a_1, c, b_1), 1/2, p(a_2, c, b_2)) &= p(p(a_1, 1/2, a_2), c, p(b_1, 1/2, b_2)) \\ &= p(p(a_2, 1/2, a_1), c, p(b_1, 1/2, b_2)) \\ &= p(p(a_2, c, b_1), 1/2, p(a_1, c, b_2)). \end{aligned}$$

Property (9) is just a particular case of (8) if we consider **(A3)**.
Using **(A7)**,**(A4)**, **(A5)**, **(A8)**, (6) and **(A3)**, we get:

$$\begin{aligned} & p(p(a, p(1, c, 0), b), 1/2, p(a, c, d)) \\ = \; & p(p(p(a, 1, b), c, p(a, 0, b)), 1/2, p(a, c, d)) \\ = \; & p(p(b, c, a), 1/2, p(a, c, d)) \\ = \; & p(p(b, 1/2, a), c, p(a, 1/2, d)) \\ = \; & p(p(a, 1/2, b), c, p(a, 1/2, d)) \\ = \; & p(p(a, c, a), 1/2, p(b, c, d)) \\ = \; & p(a, 1/2, p(b, c, d)). \end{aligned}$$

□

Some of these basic properties will be used to derive the structure introduced in section 4 (involutive medial monoid) as an intermediate step in the comparison with rings. For the moment we observe some examples.

3. Examples

In this section we give examples of mobi algebras by presenting a set A, a ternary operation $p(x, y, z) \in A$, for all $x, y, z \in A$, and three constants $0, 1/2, 1$ in A.

Our prototyping example is clearly the unit interval. We observe that it can be defined with the usual ternary operation

$$p(a, b, c) = (1 - b)a + bc$$

with one half as the element ½ but it can also be considered in other forms like those of Examples 2 to 5. Note that Examples 1 and 2 are isomorphic via the bijection $x \mapsto \frac{x}{2-x}$. More in general, the bijection $x \mapsto \frac{x}{(\alpha-1)+(2-\alpha)x}$ induces an automorphism leading to $½ = \frac{1}{\alpha}$, $\alpha \in]1, +\infty[$ (see Example 3). Example 4 is isomorphic, via the bijective correspondence $x \mapsto 2x - 1$, to Example 1. Via $x \mapsto \frac{1}{x}$ we easily observe that Example 5 is isomorphic to Example 1.

All examples have a mobi algebra structure $(A, p, 0, ½, 1)$, as given in Definition 2.1, where the symbols 0, ½ and 1 are explicit.

Example 1: $(A, p, 0, \frac{1}{2}, 1)$, with $A = [0, 1]$ and, for all $a, b, c \in A$,

$$p(a, b, c) = (1 - b)a + bc.$$

Example 2: $(A, p, 0, \frac{1}{3}, 1)$, with $A = [0, 1]$ and

$$p(a, b, c) = \frac{a - ab + ac + 2bc + abc}{1 + b + c + 2ab - bc}.$$

Example 3: For any value of $\alpha \in]1, +\infty[$ we have $(A, p, 0, \frac{1}{\alpha}, 1)$, with $A = [0, 1]$ and

$$p(a, b, c) = \frac{a - ab + (\alpha - 2)ac + (\alpha - 1)bc + (\alpha - 2)^2 abc}{1 + (\alpha - 2)(b + c - bc) + (\alpha - 1)(\alpha - 2)ab}.$$

Example 4: $(A, p, -1, 0, 1)$, with $A = [-1, 1]$ and

$$p(a, b, c) = \frac{a(1 - b) + c(1 + b)}{2}.$$

Example 5: $(A, p, +\infty, 2, 1)$, with $A = [1, +\infty]$ and

$$p(a, b, c) = \frac{abc}{a - c + bc}.$$

Example 6: Excluding the trivial case, where $0 = ½ = 1$, the smallest mobi algebra is the set of constants $A = \{0, ½, 1\}$ with the operation p defined as in the following tables.

p(-,0,-)	0	½	1
0	0	0	0
½	½	½	½
1	1	1	1

p(-,½,-)	0	½	1
0	0	1	½
½	1	½	0
1	½	0	1

p(-,1,-)	0	½	1
0	0	½	1
½	0	½	1
1	0	½	1

Isomorphically, we can consider the correspondence $(0, ½, 1)$ with $(-1, 0, 1)$, $(0, \frac{1}{2}, 1)$, $(0, 1, 2)$ or $(0, 2, 1)$ obtaining thus some natural structures on the given sets.

Example 7: Considering the correspondence $(0, ½, 1) \mapsto (0, 2, 1)$, the previous example may be generalized as follows. For every number $n \in \mathbb{N}$, we have $(A, p, 0, n + 1, 1)$ with $A = \{0, 1, \ldots, 2n\}$ and $p(a, b, c) = a - ba + bc \mod 2n + 1$.

Example 8: The dyadic numbers as a subset of the unit interval, with the same structure as in Example 1. Clearly, if a, b, c are dyadic numbers then $p(a, b, c)$ is dyadic.

Example 9: $(A, p, 0, \frac{1}{2}, 1)$ with A as $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and

$$p(x, y, z) = (1 - y)x + yz.$$

Example 10: An example of a mobi algebra in $\mathbb{R}^2$ is $(A, p, 0, ½, 1)$ with

$$A = \left\{(x, y) \in \mathbb{R}^2 \colon |y| \leq x \leq 1 - |y|\right\}$$

$$\begin{aligned} p(a, b, c) = (&a_1 - b_1a_1 - b_2a_2 + b_1c_1 + b_2c_2, \\ &a_2 - b_1a_2 - b_2a_1 + b_1c_2 + b_2c_1) \end{aligned}$$

and the three constants $½ = \left(\frac{1}{2}, 0\right)$; $1 = (1, 0)$; $0 = (0, 0)$.

Example 11: The previous example may be generalized, for any $K \in \mathbb{R}$, by defining

$$\begin{aligned} p(a, b, c) = ((1 - b_1)a_1 + b_1c_1 \quad &+ \quad Kb_2(c_2 - a_2), \\ (1 - b_1)a_2 + b_1c_2 \quad &+ \quad b_2(c_1 - a_1)). \end{aligned}$$

With $½ = \left(\frac{1}{2}, 0\right)$, $1 = (1, 0)$ and $0 = (0, 0)$, $(A, p, 0, ½, 1)$ is a mobi algebra on $A = \mathbb{R}^2$ and, for $K \in \mathbb{R}_0^+$, on

$$A = \left\{(x, y) \in \mathbb{R}^2 \colon \sqrt{K}|y| \leq x \leq 1 - \sqrt{K}|y|\right\}.$$

This example is obtained by identifying the plane with the ring of 2 by 2 matrices of the form

$$(x, y) \mapsto \begin{pmatrix} x & k_1\, y \\ k_2\, y & x \end{pmatrix},$$

letting $K = k_1\, k_2$ and computing $p(a, b, c)$ as $(1 - b)a + bc$.

Example 12: Any unitary ring R in which the element 2 is invertible gives an example of a mobi algebra $(R, p, 0, 2^{-1}, 1)$ with $p(x, y, z) = (1 - y)x + yz$ (Theorem 7.2).

Example 13: Any subset S of a unitary ring (in which 2 is an invertible element) that is closed under the formula $p(x, y, z) = (1-y)x+yz$ and contains the three constants 0, 2^{-1} and 1 gives a mobi algebra $(S, p, 0, 2^{-1}, 1)$.

Example 14: Let $(A, +, \cdot, 0, 1)$ be a semiring such that 2 is invertible (i.e. $\exists\ 2^{-1} : 2 \cdot 2^{-1} = 1 = 2^{-1} \cdot 2$) and it exists $B \subseteq A$ with the following properties:

i) $1, 2^{-1}, 0 \in B$;

ii) $\forall a, b, c \in B, \exists\ 1!$ solution $p = p(a, b, c) \in B$ to the equation $p + b \cdot a = a + b \cdot c$,

then the system $(B, p, 0, 2^{-1}, 1)$ is a mobi Algebra. The same arguments that are used in the case of rings (Theorem 7.2) are valid here by rearranging the terms in order to avoid negative terms.

Example 15: Every finite mobi is uniquely determined by (and uniquely determines) a unitary ring in which 2 is invertible. Indeed, let $(A, p, 0, \frac{1}{2}, 1)$ be a finite mobi algebra and consider the function $h : A \to A$ such that $h(x) = p(0, \frac{1}{2}, x)$. By axiom **(A6)**, h is injective. Now, as A is a finite set, h is also surjective. So h is a bijection and, in particular, there exists $h^{-1}(1)$ which is a solution to the equation $p(0, \frac{1}{2}, x) = 1$. In other words, $h^{-1}(1)$ is $\frac{1}{2}^{-1}$ and so Theorem 7.3 holds.

Example 16: Example 1 can be extended to the context of an ordered field.

Note that Example 7 above illustrates the fact that every finite mobi must have an odd number of elements.

4. Derived operations and IMM algebras

A closer look to the propositions of Section 2 suggests that some properties of mobi algebras can be suitably expressed in terms of a unary operation "$\overline{()}$" and binary operations "$\cdot$", "$\circ$" and "$\oplus$" defined as follows.

Definition 4.1. *Let $(A, p, 0, \frac{1}{2}, 1)$ be a mobi algebra. We define:*

$$\overline{a} = p(1, a, 0) \tag{11}$$

$$a \cdot b = p(0, a, b) \tag{12}$$

$$a \oplus b = p(a, \tfrac{1}{2}, b) \tag{13}$$

$$a \circ b = p(a, b, 1). \tag{14}$$

In the first example of previous section, with $p(a, b, c) = (1-b)a+bc$ on $A = [0, 1]$, these operations have the following form:

$$\begin{aligned} \overline{a} &= 1 - a \\ a \cdot b &= ab \\ a \oplus b &= \frac{a+b}{2} \\ a \circ b &= a + b - ab. \end{aligned}$$

In Example 11, the derived operations are:

$$\begin{aligned} \overline{(x, y)} &= (1 - x, -y) \\ (x, y) \cdot (x', y') &= (xx' + Kyy', xy' + yx') \\ (x, y) \oplus (x', y') &= (\frac{x + x'}{2}, \frac{y + y'}{2}) \\ (x, y) \circ (x', y') &= (x + x' - x'x - K\, y'y, y + y' - x'y - y'x). \end{aligned}$$

In particular, complex multiplication is obtained as $\cdot$, if letting $A = \mathbb{R}^2 \cong \mathbb{C}$ and $K = -1$.

From property (5) and (7), as well as axioms **(A5)** and **(A7)**, we immediately find that:

$$\begin{aligned} \overline{\overline{a}} &= a & (15)\\ \overline{1} &= 0 & (16)\\ p(b,c,a) &= p(a,\overline{c},b) & (17)\\ \overline{p(a,c,b)} &= p(\overline{a},c,\overline{b}). & (18) \end{aligned}$$

These relations show, in particular, that $\cdot$ and $\circ$ are dual operations in the sense that:

$$\begin{aligned} \overline{a\cdot b} &= \overline{b}\circ\overline{a} & (19)\\ \overline{a\circ b} &= \overline{b}\cdot\overline{a}. & (20) \end{aligned}$$

This is why, we will leave out the operation $\circ$ in the rest of the article, except for noting that (8) gives the following relation between the three binary operations:

$$(a\circ b)\oplus(b\cdot a) = b\oplus a. \qquad (21)$$

It is also worth noting that, **(A2)** implies that

$$½ = \overline{1}\oplus 1. \qquad (22)$$

Using (10), we can relate the ternary operation p of any mobi algebra with the derived operations through the relation:

$$½\cdot p(a,b,c) = (\overline{b}\cdot a)\oplus(b\cdot c). \qquad (23)$$

This property will be at the bottom line of Section 7 for comparing a mobi algebra with rings.

Before that, we show, in Proposition 4.1 below, that every mobi algebra gives rise to a new structure, that we call involutive medial monoid (IMM), presented in the following Definition .

Definition 4.2. *An IMM algebra is a system* $(B,\overline{()},\oplus,\cdot,1)$*, in which* B *is a set,* $\overline{()}$ *is an unary operation,* $\oplus$ *and* $\cdot$ *are binary operations and* 1 *is an element of* B*, that satisfies the following axioms:*

- **(B1)** $a\oplus a = a$
- **(B2)** $a\oplus b = b\oplus a$
- **(B3)** $(a\oplus b)\oplus(c\oplus d) = (a\oplus c)\oplus(b\oplus d)$
- **(B4)** $a\cdot(b\cdot c) = (a\cdot b)\cdot c$
- **(B5)** $a\cdot 1 = a = 1\cdot a$
- **(B6)** $a\cdot(b\oplus c) = (a\cdot b)\oplus(a\cdot c), \quad (a\oplus b)\cdot c = (a\cdot c)\oplus(b\cdot c)$
- **(B7)** $\overline{\overline{a}} = a$
- **(B8)** $\overline{a\oplus b} = \overline{a}\oplus\overline{b}$
- **(B9)** $a\cdot\overline{1} = \overline{1} = \overline{1}\cdot a$
- **(B10)** $\overline{a}\oplus a = \overline{1}\oplus 1$

The name IMM is chosen to highlight the existence of an involution, the presence of the medial law **(B3)**[4] satisfied by $\oplus$ and the fact that $(B, \cdot, 1)$ is a monoid.

Proposition 4.1. *If $(A, p, 0, 1/2, 1)$ is a mobi algebra and if $\overline{()}$, $\oplus$, and $\cdot$ are defined as in (11), (12) and (13), then $(A, \overline{()}, \oplus, \cdot, 1)$ is an IMM algebra in which $\overline{1} = 0$ and $\overline{1} \oplus 1 = 1/2$.*

Proof. All axioms of an IMM are easily proved using the axioms of a mobi algebra and the properties presented in Section 2. Indeed: **(B1)** is a particular case of **(A3)**; **(B2)** is just a rewriting of (6); **(B3)** is a consequence of **(A8)**; **(B4)** follows from **(A4)** and **(A7)**, like (1). The first equality in **(B5)** is **(A2)** and the second comes from **(A5)**. Left-distributivity in **(B6)** is deduced from **(A3)** and **(A8)** and right-distributivity from **(A7)**, as follows:

$$\begin{aligned} a \cdot (b \oplus c) &= p(0, a, p(b, 1/2, c)) = p(p(0, 1/2, 0), a, p(b, 1/2, c)) \\ &= p(p(0, a, b), 1/2, p(0, a, c)) = (a \cdot b) \oplus (a \cdot c); \\ (a \oplus b) \cdot c &= p(0, p(a, 1/2, b), c) = p(p(0, a, c), 1/2, p(0, b, c)) \\ &= (a \cdot c) \oplus (b \cdot c). \end{aligned}$$

(B7) is just a rewriting of (7); **(B8)** is a consequence of **(A7)**, while **(B9)** can be proved through **(A3)**, **(A4)** and **(A5)**:

$$\begin{aligned} a \cdot \overline{1} &= p(0, a, p(1, 1, 0) = p(0, a, 0) = 0 = \overline{1} \\ \overline{1} \cdot a &= p(0, p(1, 1, 0), a) = p(0, 0, a) = 0 = \overline{1}. \end{aligned}$$

Using **(A1)**, **(A2)**, **(A3)** and **(A8)**, we can prove **(B10)**:

$$\begin{aligned} \overline{a} \oplus a &= p(p(1, a, 0), 1/2, a) = p(p(1, a, 0), 1/2, p(0, a, 1)) \\ &= p(p(1, 1/2, 0), a, p(0, 1/2, 1)) = p(1/2, a, 1/2) = 1/2. \end{aligned}$$

□

A finite example of an IMM algebra which is not obtained from a mobi algebra is $(A, \overline{()}, \oplus, \cdot, 1)$ with $A = \{0, 1/2, 1\}$ and the operations $\oplus$ and $\cdot$ as in the following tables.

$\oplus$	0	½	1
0	0	½	½
½	½	½	½
1	½	½	1

$\cdot$	0	½	1
0	0	0	0
½	0	½	½
1	0	½	1

It is clear that $\oplus$ is not cancellative.

We finish this section with some properties of an IMM algebra.

Proposition 4.2. *Let $(B, \overline{()}, \oplus, \cdot, 1)$ be a IMM algebra. It follows that:*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus (a \oplus c) \tag{24}$$
$$\overline{\overline{1} \oplus 1} = \overline{1} \oplus 1 \tag{25}$$
$$\overline{a} = a \Rightarrow a = \overline{1} \oplus 1 \tag{26}$$
$$(\overline{1} \oplus 1) \cdot a = \overline{1} \oplus a \tag{27}$$
$$(\overline{1} \oplus 1) \cdot a = a \cdot (\overline{1} \oplus 1). \tag{28}$$

Proof. (24) is a direct consequence of **(B1)** and **(B3)**; (25) of **(B2)**, **(B7)** and **(B8)**; and (26) is a consequence of **(B1)** and **(B10)**. To prove (27) and (28), we use **(B6)**, **(B5)** and **(B9)**:

$$(\overline{1} \oplus 1) \cdot a = (\overline{1} \cdot a) \oplus (1 \cdot a) = \overline{1} \oplus a$$
$$a \cdot (\overline{1} \oplus 1) = (a \cdot \overline{1}) \oplus (a \cdot 1) = \overline{1} \oplus a.$$

□

These are the main properties that will be used in the following sections.

5. IMM algebras and unitary rings

We are now going to see how, in an IMM algebra, the operation $\oplus$, under the existence of an inverse (in the sense of $\cdot$) to the element $\overline{1} \oplus 1$, gives rise to the additive structure of a unitary ring with one half. Let us begin by recalling that, in a monoid $(A, \cdot, 1)$, if an element admits an inverse, the inverse is unique. Indeed, suppose that $x \cdot a = 1 = a \cdot x$ and $a \cdot x' = 1 = x' \cdot a$, then

$$x = x \cdot 1 = x \cdot (a \cdot x') = (x \cdot a) \cdot x' = 1 \cdot x' = x'.$$

As usual, the inverse of a is denoted a^{-1}. So, in an IMM algebra $(A, \overline{()}, \oplus, \cdot, 1)$, when the equation $(\overline{1} \oplus 1) \cdot x = 1$ has a solution, its unique solution is denoted $(\overline{1} \oplus 1)^{-1}$. The fact that $\overline{1} \oplus 1$ is central in the monoid $(A, \cdot, 1)$, as expressed in (28), implies that its inverse, when it exists, is also central:

$$x \cdot (\overline{1} \oplus 1)^{-1} = (\overline{1} \oplus 1)^{-1} \cdot x, \quad \forall x \in A. \tag{29}$$

Indeed, if $x \cdot a = a \cdot x$ then $x \cdot a^{-1} = a \cdot a^{-1} \cdot x \cdot a^{-1} = a^{-1} \cdot x \cdot a^{-1} \cdot a = a^{-1} \cdot x$.

As we will see, the following proposition is essential to find the symmetric elements in the induced ring.

Proposition 5.1. *If $(A, \overline{()}, \oplus, \cdot, 1)$ is an IMM algebra and if $\overline{1} \oplus 1$ admits an inverse, then:*

$$\overline{1} \oplus (\overline{1} \oplus 1)^{-1} = 1. \tag{30}$$

Proof. This Property follows directly from (27). □

Before presenting the main results of this section through the Theorems 5.2 and 5.3, we enumerate the axioms of a ring in Definition 5.1 below so we can refer to them in the subsequent demonstrations.

Definition 5.1. *A unitary ring is a system* $(R, +, \cdot, 0, 1)$ *that satisfies the following axioms:*

(R1) $(a + b) + c = a + (b + c)$
(R2) $a + b = b + a$
(R3) $a + 0 = a$
(R4) $\forall a, \exists -a : -a + a = 0$
(R5) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
(R6) $a \cdot 1 = a = 1 \cdot a$
(R7) $a \cdot (b + c) = (a \cdot b) + (a \cdot c), \quad (a + b) \cdot c = (a \cdot c) + (b \cdot c)$

Immediate consequences of the axioms are the following properties:

$$a \cdot 0 = \quad 0 \quad = 0 \cdot a \tag{31}$$

$$a + b = a' + b \;\Rightarrow\; a = a'. \tag{32}$$

The main result of this section shows the connection between an IMM algebra and a ring. It claims that an IMM algebra $(A, \overline{()}, \oplus, \cdot, 1)$ determines a unique structure of a ring on its underlying set, if, and only if, its element $(\overline{1} \oplus 1)$ is invertible, i.e. that $(\overline{1} \oplus 1)^{-1}$ exists.

Theorem 5.2. *Let* $(A, \overline{()}, \oplus, \cdot, 1)$ *be an IMM algebra. The following three affirmations are equivalent.*

(*i*) *The equation* $(\overline{1} \oplus 1) \cdot x = 1$ *has a solution in* A*;*
(*ii*) *There is a unique unitary ring* $(A, +, \cdot, \overline{1}, 1)$ *such that:*

$$a + b = (1 + 1) \cdot (a \oplus b)$$

(*iii*) *There is a unique unitary ring* $(A, +, \cdot, \overline{1}, 1)$ *such that:*

$$a \oplus b \;=\; (\overline{1} \oplus 1) \cdot (a + b).$$

Proof. In order to present this proof in a concise way, we will use the following notations:

$$\begin{aligned} ½ &= \overline{1} \oplus 1 \\ \mathit{2} &= (\overline{1} \oplus 1)^{-1}. \end{aligned}$$

To prove that (*i*) implies (*ii*), we observe that $½ \cdot \mathit{2} = 1$ and define

$$\begin{aligned} a + b &= \mathit{2} \cdot (a \oplus b) \\ -a &= \overline{\mathit{2}} \cdot a. \end{aligned}$$

To prove that the structure $(A, +, \cdot, \overline{1}, 1)$ is a ring, we will prove axioms **(R1)** to **(R7)** above. For **(R1)**, we begin with a particular case of the

IMM axiom **(B3)** and, using also **(B2)**, **(B5)**, **(B6)** and property (27), we get:

$$\begin{aligned}
& (a \oplus b) \oplus (\overline{1} \oplus c) = (a \oplus \overline{1}) \oplus (b \oplus c) \\
\Longrightarrow \quad & (a \oplus b) \oplus (1/2 \cdot c) = (1/2 \cdot a) \oplus (b \oplus c) \\
\Longrightarrow \quad & (1/2 \cdot 2 \cdot (a \oplus b)) \oplus (1/2 \cdot c) = (1/2 \cdot a) \oplus (1/2 \cdot 2 \cdot (b \oplus c)) \\
\Longrightarrow \quad & 1/2 \cdot (2 \cdot (a \oplus b) \oplus c) = 1/2 \cdot (a \oplus (2 \cdot (b \oplus c))) \\
\Longrightarrow \quad & 2 \cdot 2 \cdot 1/2 \cdot ((a + b) \oplus c) = 2 \cdot 2 \cdot 1/2 \cdot (a \oplus (b + c)) \\
\Longrightarrow \quad & (a + b) + c = a + (b + c).
\end{aligned}$$

(R2) is a direct consequence of **(B2)**, and **(R3)** follows from the statement $0 = \overline{1}$ and the use of property (27). To prove **(R4)**, we use **(B6)** and **(B9)**, after remarking that (30), together with **(B8)**, implies $1 \oplus \overline{2} = \overline{1}$:

$$-a + a = 2 \cdot ((\overline{2} \cdot a) \oplus a) = 2 \cdot (\overline{2} \oplus 1) \cdot a = 2 \cdot \overline{1} \cdot a = \overline{1}.$$

(R5) and **(R6)** are guaranteed by **(B4)** and **(B5)**. Finally **(R7)** is deduced from **(B6)** with the use of (29).

This proves existence of a ring induced by an IMM when $(\overline{1} \oplus 1)^{-1}$ exists. To prove uniqueness, we just need to show that in any ring $(A, +', \cdot, \overline{1}, 1)$ such that $a +' b = (1 +' 1) \cdot (a \oplus b)$, we have $(1 +' 1) = (\overline{1} \oplus 1)^{-1}$. Indeed:

$$\begin{aligned}
a +' b = (1 +' 1) \cdot (a \oplus b) \quad & \Longrightarrow \quad \overline{1} +' 1 = (1 +' 1) \cdot (\overline{1} \oplus 1) \\
& \Longrightarrow \quad 1 = (1 +' 1) \cdot (\overline{1} \oplus 1) \\
& \Longrightarrow \quad 1 +' 1 = (\overline{1} \oplus 1)^{-1}.
\end{aligned}$$

This also shows directly that (*ii*) implies (*iii*) because the inverse of $1 + 1$ is $\overline{1} \oplus 1$. Now, as $1 + 1$ exists in any ring and $1 \oplus 1 = 1$, we also have that (*iii*) implies (*i*). □

The previous proposition creates the question of characterizing those rings that come from an IMM algebra. Theorem 5.3 tell us that a ring $(R, +, \cdot, 0, 1)$ is determined by an IMM algebra structure if and only if its element $(1 + 1)$ is invertible.

Theorem 5.3. *Let* $(R, +, \cdot, 0, 1)$ *be a unitary ring. The following three affirmations are equivalent.*

(*i*) *The element* $1 + 1$ *admits an inverse in R;*

(*ii*) *There is a unique IMM algebra* $(R, \overline{()}, \oplus, \cdot, 1)$ *such that:*

$$\begin{aligned}
\overline{a} &= 1 - a \\
a \oplus b &= (\overline{1} \oplus 1) \cdot (a + b).
\end{aligned}$$

(*iii*) *There is a unique IMM algebra* $(R, \overline{()}, \oplus, \cdot, 1)$ *such that:*

$$\begin{aligned}
\overline{a} &= 1 - a \\
a + b &= (1 + 1) \cdot (a \oplus b).
\end{aligned}$$

Proof. To prove that (i) implies (ii), we define

$$a \oplus b = (1+1)^{-1} \cdot (a+b).$$

To prove that the structure $(A, \overline{()}, \oplus, \cdot, 1)$ is an IMM algebra, we will deduce axioms **(B1)** to **(B10)** of Definition 4.2. **(B1)** is satisfied because $a+a = (1+1)\cdot a$. **(B2)** is a consequence of the commutativity of $+$. The medial law **(B3)** may be proved using the associativity of $+$ and the distributivity of $\cdot$ over $+$:

$$\begin{aligned}
(a \oplus b) \oplus (c \oplus d) &= (1+1)^{-1} \cdot ((1+1)^{-1}(a+b) + (1+1)^{-1}(c+d)) \\
&= (1+1)^{-1} \cdot (1+1)^{-1} \cdot ((a+b)+(c+d)) \\
&= (1+1)^{-1} \cdot (1+1)^{-1} \cdot ((a+c)+(b+d)) \\
&= (a \oplus c) \oplus (b \oplus d).
\end{aligned}$$

(B4) and **(B5)** are guaranteed by **(R5)** and **(R6)** while **(B6)** is guaranteed by **(R7)** because $(1+1)^{-1}$ commutes with all the elements of the ring. **(B7)** is a consequence of $1-(1-a) = a$. A proof of **(B8)** goes like this:

$$\begin{aligned}
\overline{a \oplus b} &= 1 - (1+1)^{-1}(a+b) \\
&= (1+1)^{-1}(1+1) - (1+1)^{-1}(a+b) \\
&= (1+1)^{-1}((1+1) - (a+b)) \\
&= (1+1)^{-1}((1-a) + (1-b)) \\
&= \overline{a} \oplus \overline{b}.
\end{aligned}$$

(B9) is obvious because $\overline{1} = 0$. Finally, to prove **(B10)**, we observe that

$$\overline{a} \oplus a = (1+1)^{-1}(1-a+a) = (1+1)^{-1}$$

and consequently $\overline{1} \oplus 1 = (1+1)^{-1}$. This proves existence of an IMM induced by a ring when $(1+1)^{-1}$ exists. To prove uniqueness, we just need to show that in any IMM $(A, \overline{()}, \oplus', \cdot, 1)$ such that $a \oplus' b = (\overline{1} \oplus' 1) \cdot (a+b)$, we have $(\overline{1} \oplus' 1) = (1+1)^{-1}$. Indeed:

$$\begin{aligned}
a \oplus' b = (\overline{1} \oplus' 1) \cdot (a+b) \quad &\Longrightarrow \quad 1 \oplus' 1 = (\overline{1} \oplus' 1) \cdot (1+1) \\
&\Longrightarrow \quad 1 = (\overline{1} \oplus' 1) \cdot (1+1) \\
&\Longrightarrow \quad \overline{1} \oplus' 1 = (1+1)^{-1}.
\end{aligned}$$

This also shows directly that (ii) implies (iii) because the inverse of $\overline{1} \oplus 1$ is $1+1$. Now, as $\overline{1} \oplus 1$ exists in any IMM algebra and $\overline{1} + 1 = 1$, we also have that (iii) implies (i).

□

6. Mobi algebras and IMM algebras

We have seen the comparison between IMM algebras and rings. We have also seen that a mobi algebra gives rise to an IMM algebra (Proposition 4.1). It remains to answer the question on whether an IMM algebra is obtained from a mobi algebra. Due to axiom **(A6)**, the subalgebra $(A, \oplus)$, of an IMM algebra which is induced by a mobi algebra, is a midpoint algebra [2, 3]. In other words, the operation $\oplus$ is cancellative. We present in the Appendix two examples of a IMM algebra in which $\oplus$ is not cancellative (IMM 2 and IMM 3). Note that the existence of $(\overline{1} \oplus 1)^{-1}$ is sufficient to imply that $\oplus$ is cancellative but it is not necessary.

An IMM algebra in which the operation $\oplus$ is cancellative will be called an IMM* algebra. Therefore, the question, that will be answered in Theorem 6.2 below, is to determine what extra conditions on an IMM* algebra are needed to certify that it comes from a mobi algebra. On an IMM* algebra, some axioms of IMM algebras may be deduced from the others using the cancellation of $\oplus$, thus we decided to present it as an independent algebraic structure.

Definition 6.1. *An IMM* algebra is a system $(C, \overline{()}, \oplus, \cdot, 1)$, in which C is a set, $\overline{()}$ is an unary operation, $\oplus$ and $\cdot$ are binary operations and 1 is an element of C, that satisfies the following axioms:*

- **(C1)** $a \oplus a = a$
- **(C2)** $a \oplus b = b \oplus a$
- **(C3)** $a \oplus b = a' \oplus b \implies a = a'$
- **(C4)** $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$
- **(C5)** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- **(C6)** $a \cdot 1 = a = 1 \cdot a$
- **(C7)** $a \cdot (b \oplus c) = (a \cdot b) \oplus (a \cdot c), \quad (a \oplus b) \cdot c = (a \cdot c) \oplus (b \cdot c)$
- **(C8)** $a \cdot \overline{1} = \overline{1} = \overline{1} \cdot a$
- **(C9)** $\overline{a} \oplus a = \overline{1} \oplus 1$

We now observe that, in particular, every IMM* algebra is a IMM algebra.

Proposition 6.1. *Let $(C, \overline{()}, \oplus, \cdot, 1)$ be a IMM* algebra. It follows that:*

$$\overline{\overline{a}} = a \tag{33}$$

$$\overline{a \oplus b} = \overline{a} \oplus \overline{b} \tag{34}$$

$$b \oplus a = \overline{1} \oplus 1 \implies b = \overline{a} \tag{35}$$

$$\overline{1} \oplus x = (\overline{b} \cdot a) \oplus (b \cdot c) \implies \overline{1} \oplus \overline{x} = (\overline{b} \cdot \overline{a}) \oplus (b \cdot \overline{c}). \tag{36}$$

Proof. Using **(C2)** and **(C9)**, we have $\overline{\overline{a}} \oplus \overline{a} = \overline{1} \oplus 1$ and $a \oplus \overline{a} = \overline{1} \oplus 1$ which, by **(C3)**, implies (33). Using **(C4)**, **(C9)** and **(C1)** we find that

$$(\overline{a} \oplus \overline{b}) \oplus (a \oplus b) = (\overline{a} \oplus a) \oplus (\overline{b} \oplus b) = (\overline{1} \oplus 1) \oplus (\overline{1} \oplus 1) = \overline{1} \oplus 1$$

which implies (34) by cancellation, because $\overline{(a \oplus b)} \oplus (a \oplus b) = \overline{1} \oplus 1$. (35) is again a consequence of **(C3)** and **(C9)**. To prove (36), suppose that

$$\begin{aligned}\overline{1} \oplus x &= (\overline{b} \cdot a) \oplus (b \cdot c), \quad \text{and} \\ \overline{1} \oplus y &= (\overline{b} \cdot \overline{a}) \oplus (b \cdot \overline{c}).\end{aligned}$$

Then, we have:

$$\begin{aligned}\overline{1} \oplus (x \oplus y) &= (\overline{1} \oplus \overline{1}) \oplus (x \oplus y) \\ &= (\overline{1} \oplus x) \oplus (\overline{1} \oplus y) \\ &= ((\overline{b} \cdot a) \oplus (b \cdot c)) \oplus ((\overline{b} \cdot \overline{a}) \oplus (b \cdot \overline{c})) \\ &= ((\overline{b} \cdot a) \oplus (\overline{b} \cdot \overline{a})) \oplus ((b \cdot c) \oplus (b \cdot \overline{c})) \\ &= (\overline{b} \cdot (a \oplus \overline{a})) \oplus (b \cdot (c \oplus \overline{c})) \\ &= (\overline{b} \cdot (1 \oplus \overline{1})) \oplus (b \cdot (1 \oplus \overline{1})) \\ &= (\overline{b} \oplus b) \cdot (1 \oplus \overline{1}) \\ &= (\overline{1} \oplus 1) \cdot (1 \oplus \overline{1}) \\ &= \overline{1} \oplus (\overline{1} \oplus 1).\end{aligned}$$

So, from **(C3)**, we conclude that $x \oplus y = (\overline{1} \oplus 1)$ which means, using (35), that $y = \overline{x}$. □

We can now state the main result of this section.

Theorem 6.2. *Let $(A, \overline{()}, \oplus, \cdot, 1)$ be an IMM* algebra. The following two affirmations are equivalent.*

(i) *For each $a, b, c \in A$, the equation*

$$\overline{1} \oplus x = (\overline{b} \cdot a) \oplus (b \cdot c) \tag{37}$$

has a solution x in A;

(ii) *There is a unique mobi algebra $(A, p, \overline{1}, \overline{1} \oplus 1, 1)$ such that:*

$$\overline{1} \oplus p(a, b, c) = (\overline{b} \cdot a) \oplus (b \cdot c). \tag{38}$$

Proof. (ii) implies (i) because, for each $a, b, c \in A$, $p(a, b, c)$ exists if (ii) is true and is therefore the solution of the equation (37). To prove that (i) implies (ii), we will deduce axioms **(A1)** to **(A8)** of Definition 2.1. This is facilitated by the fact that $\oplus$ is cancellative which means that if we find an element d of A such that $\overline{1} \oplus d = (\overline{b} \cdot a) \oplus (b \cdot c)$ then, we can conclude that $p(a, b, c) = d$. To prove **(A1)**, we observe that

$$(\overline{(\overline{1} \oplus 1)} \cdot 1) \oplus ((\overline{1} \oplus 1) \cdot \overline{1}) = (\overline{1} \oplus 1) \oplus \overline{1} = \overline{1} \oplus (\overline{1} \oplus 1)$$

which means that $p(1, \overline{1} \oplus 1, \overline{1}) = \overline{1} \oplus 1$. In a similar way, **(A2)** to **(A5)** are satisfied:

$$\begin{aligned}(\overline{a} \cdot \overline{1}) \oplus (a \cdot 1) = \overline{1} \oplus a &\implies p(\overline{1}, a, 1) = a \\ (\overline{b} \cdot a) \oplus (b \cdot a) = (\overline{b} \oplus b) \cdot a = \overline{1} \oplus a &\implies p(a, b, a) = a \\ (\overline{\overline{1}} \cdot a) \oplus (\overline{1} \cdot b) = (1 \cdot a) \oplus \overline{1} = \overline{1} \oplus a &\implies p(a, \overline{1}, b) = a \\ (\overline{1} \cdot a) \oplus (1 \cdot b) = \overline{1} \oplus b &\implies p(a, 1, b) = b.\end{aligned}$$

(A6) is guaranteed by **(C3)**. To prove **(A7)**, we first observe that

$$\begin{aligned}
\overline{1} \oplus p(c_1, c_2, c_3) &= \overline{c_2} \cdot c_1 \oplus c_2 \cdot c_3 \\
\overline{1} \oplus p(a, c_1, b) &= \overline{c_1} \cdot a \oplus c_1 \cdot b \\
\overline{1} \oplus p(a, c_3, b) &= \overline{c_3} \cdot a \oplus c_3 \cdot b
\end{aligned}$$

and, using (36):

$$\overline{1} \oplus \overline{p(c_1, c_2, c_3)} = \overline{c_2} \cdot \overline{c_1} \oplus c_2 \cdot \overline{c_3}.$$

Then, we use these relations, as well as the axioms of an IMM*, to transform an obvious identity into **(A7)**:

$$\begin{aligned}
& (\overline{c_2} \cdot \overline{c_1} \cdot a \oplus c_2 \cdot \overline{c_3} \cdot a) \oplus (\overline{c_2} \cdot c_1 \cdot b \oplus c_2 \cdot c_3 \cdot b) = \\
& (\overline{c_2} \cdot \overline{c_1} \cdot a \oplus c_2 \cdot \overline{c_3} \cdot a) \oplus (\overline{c_2} \cdot c_1 \cdot b \oplus c_2 \cdot c_3 \cdot b) \\
\Leftrightarrow\ & (\overline{c_2} \cdot \overline{c_1} \cdot a \oplus \overline{c_2} \cdot c_1 \cdot b) \oplus (c_2 \cdot \overline{c_3} \cdot a \oplus c_2 \cdot c_3 \cdot b) = \\
& (\overline{c_2} \cdot \overline{c_1} \cdot a \oplus c_2 \cdot \overline{c_3} \cdot a) \oplus (\overline{c_2} \cdot c_1 \cdot b \oplus c_2 \cdot c_3 \cdot b) \\
\Leftrightarrow\ & \overline{c_2} \cdot (\overline{c_1} \cdot a \oplus c_1 \cdot b) \oplus c_2 \cdot (\overline{c_3} \cdot a \oplus c_3 \cdot b) = \\
& (\overline{c_2} \cdot \overline{c_1} \oplus c_2 \cdot \overline{c_3}) \cdot a \oplus (\overline{c_2} \cdot c_1 \oplus c_2 \cdot c_3) \cdot b \\
\Leftrightarrow\ & \overline{c_2} \cdot (\overline{1} \oplus p(a, c_1, b)) \oplus c_2 \cdot (\overline{1} \oplus p(a, c_3, b)) = \\
& (\overline{1} \oplus \overline{p(c_1, c_2, c_3)}) \cdot a \oplus (\overline{1} \oplus p(c_1, c_2, c_3)) \cdot b \\
\Leftrightarrow\ & (\overline{1} \oplus \overline{c_2} \cdot p(a, c_1, b)) \oplus (\overline{1} \oplus c_2 \cdot p(a, c_3, b)) = \\
& (\overline{1} \oplus \overline{p(c_1, c_2, c_3)} \cdot a) \oplus (\overline{1} \oplus p(c_1, c_2, c_3) \cdot b) \\
\Leftrightarrow\ & \overline{1} \oplus (\overline{c_2} \cdot p(a, c_1, b) \oplus c_2 \cdot p(a, c_3, b)) = \\
& \overline{1} \oplus (\overline{p(c_1, c_2, c_3)} \cdot a \oplus p(c_1, c_2, c_3) \cdot b) \\
\Leftrightarrow\ & p(p(a, c_1, b), c_2, p(a, c_3, b)) = p(a, p(c_1, c_2, c_3), b).
\end{aligned}$$

The proof of **(A8)** is similar. To write the proof in a concise way, let us use the notation $\overline{1} \oplus 1 = ½$ and recall that (28) reads $½ \cdot a = a \cdot ½$ for all a, and (25) means $\overline{½} = ½$.

$$\begin{aligned}
& (\overline{c} \cdot ½ \cdot a_1 \oplus c \cdot ½ \cdot b_1) \oplus (\overline{c} \cdot ½ \cdot a_2 \oplus c \cdot ½ \cdot b_2) = \\
& (½ \cdot \overline{c} \cdot a_1 \oplus ½ \cdot c \cdot b_1) \oplus (½ \cdot \overline{c} \cdot a_2 \oplus ½ \cdot c \cdot b_2) \\
\Leftrightarrow\ & (\overline{c} \cdot ½ \cdot a_1 \oplus \overline{c} \cdot ½ \cdot a_2) \oplus (c \cdot ½ \cdot b_1 \oplus c \cdot ½ \cdot b_2) = \\
& (½ \cdot \overline{c} \cdot a_1 \oplus ½ \cdot c \cdot b_1) \oplus (½ \cdot \overline{c} \cdot a_2 \oplus ½ \cdot c \cdot b_2) \\
\Leftrightarrow\ & \overline{c} \cdot (½ \cdot a_1 \oplus ½ \cdot a_2) \oplus c \cdot (½ \cdot b_1 \oplus ½ \cdot b_2) = \\
& ½ \cdot (\overline{c} \cdot a_1 \oplus c \cdot b_1) \oplus ½ \cdot (\overline{c} \cdot a_2 \oplus c \cdot b_2) \\
\Leftrightarrow\ & \overline{c} \cdot (\overline{1} \oplus p(a_1, ½, a_2)) \oplus c \cdot (\overline{1} \oplus p(b_1, ½, b_2)) = \\
& ½ \cdot (\overline{1} \oplus p(a_1, c, b_1)) \oplus ½ \cdot (\overline{1} \oplus p(a_2, c, b_2)) \\
\Leftrightarrow\ & \overline{1} \oplus (\overline{c} \cdot p(a_1, ½, a_2) \oplus c \cdot p(b_1, ½, b_2)) = \\
& \overline{1} \oplus (½ \cdot p(a_1, c, b_1) \oplus ½ \cdot p(a_2, c, b_2)) \\
\Leftrightarrow\ & p(p(a_1, ½, a_2), c, p(b_1, ½, b_2)) = p(p(a_1, c, b_1), ½, p(a_2, c, b_2)).
\end{aligned}$$

□

An IMM algebra in which $\overline{1} \oplus 1$ is invertible is an IMM* algebra. Moreover, $x = (\overline{1} \oplus 1)^{-1} \cdot ((\overline{b} \cdot a) \oplus (b \cdot c))$ is a solution to (37). Hence it gives rise to a mobi algebra.

Corollary 6.3. *Let* $(A, \overline{()}, \oplus, \cdot, 1)$ *be an IMM algebra (or an IMM* algebra). If the element* $\overline{1} \oplus 1$ *is invertible, then there exists a unique mobi algebra* $(A, p, \overline{1}, \overline{1} \oplus 1, 1)$ *such that:*

$$p(a, b, c) = (\overline{1} \oplus 1)^{-1} \cdot ((\overline{b} \cdot a) \oplus (b \cdot c)). \tag{39}$$

Proof. It is a consequence of Theorem 6.2 and property (27). □

The previous results show the connection between mobi algebras and IMM algebras, or IMM* algebras. The case when the monoid part $(A, \cdot, 1)$ of an IMM algebra is a commutative monoid has an interesting reflection on the axiom **(A8)**. Instead of having it restricted to the element ½,

$$p(p(a_1, c, b_1), ½, p(a_2, c, b_2)) = p(p(a_1, ½, a_2), c, p(b_1, ½, b_2))$$

it holds for an arbitrary element as shown in the following proposition.

Proposition 6.4. *Let* $(A, \overline{()}, \oplus, \cdot, 1)$ *be an IMM* algebra satisfying condition (i) of Theorem 6.2 and suppose that* $(A, p, \overline{1}, \overline{1} \oplus 1, 1)$ *is its corresponding mobi algebra. The monoid* $(A, \cdot, 1)$ *is a commutative monoid if and only if*

$$p(p(a_1, c, b_1), d, p(a_2, c, b_2)) = p(p(a_1, d, a_2), c, p(b_1, d, b_2)) \tag{40}$$

for all $a_1, b_1, a_2, b_2, c, d \in A$.

Proof. If (40) is a property of the mobi algebra, then:

$$\begin{aligned} a \cdot b &= p(0, a, b) \\ &= p(p(0, b, 0), a, p(0, b, 1)) \\ &= p(p(0, a, 0), b, p(0, a, 1)) \\ &= p(0, b, a) \\ &= b \cdot a. \end{aligned}$$

Conversely and considering (38), if $a \cdot b = b \cdot a$, for all $a, b \in A$ in the IMM*, then:

$$\begin{aligned}
& ((\overline{d} \cdot \overline{c} \cdot a_1) \oplus (\overline{d} \cdot c \cdot b_1)) \oplus ((d \cdot \overline{c} \cdot a_2) \oplus (d \cdot c \cdot b_2)) = \\
& ((\overline{c} \cdot \overline{d} \cdot a_1) \oplus (c \cdot \overline{d} \cdot b_1)) \oplus ((\overline{c} \cdot d \cdot a_2) \oplus (c \cdot d \cdot b_2)) \\
\Longrightarrow \quad & ((\overline{d} \cdot \overline{c} \cdot a_1) \oplus (\overline{d} \cdot c \cdot b_1)) \oplus ((d \cdot \overline{c} \cdot a_2) \oplus (d \cdot c \cdot b_2)) = \\
& ((\overline{c} \cdot \overline{d} \cdot a_1) \oplus (\overline{c} \cdot d \cdot a_2)) \oplus ((c \cdot \overline{d} \cdot b_1) \oplus (c \cdot d \cdot b_2)) \\
\Longrightarrow \quad & (\overline{d} \cdot ((\overline{c} \cdot a_1) \oplus (c \cdot b_1))) \oplus (d \cdot ((\overline{c} \cdot a_2) \oplus (c \cdot b_2))) = \\
& (\overline{c} \cdot ((\overline{d} \cdot a_1) \oplus (d \cdot a_2))) \oplus (c \cdot ((\overline{d} \cdot b_1) \oplus (d \cdot b_2))) \\
\Longrightarrow \quad & (\overline{d} \cdot (\overline{1} \oplus p(a_1, c, b_1))) \oplus (d \cdot (\overline{1} \oplus p(a_2, c, b_2))) = \\
& (\overline{c} \cdot (\overline{1} \oplus p(a_1, d, a_2))) \oplus (c \cdot (\overline{1} \oplus p(b_1, d, b_2))) \\
\Longrightarrow \quad & (\overline{1} \oplus (\overline{d} \cdot p(a_1, c, b_1))) \oplus (\overline{1} \oplus (d \cdot p(a_2, c, b_2))) = \\
& (\overline{1} \oplus (\overline{c} \cdot p(a_1, d, a_2))) \oplus (\overline{1} \oplus (c \cdot p(b_1, d, b_2))) \\
\Longrightarrow \quad & \overline{1} \oplus ((\overline{d} \cdot p(a_1, c, b_1)) \oplus (d \cdot p(a_2, c, b_2))) = \\
& \overline{1} \oplus ((\overline{c} \cdot p(a_1, d, a_2)) \oplus (c \cdot p(b_1, d, b_2))) \\
\Longrightarrow \quad & \overline{1} \oplus (\overline{1} \oplus p(p(a_1, c, b_1), d, p(a_2, c, b_2))) = \\
& \overline{1} \oplus (\overline{1} \oplus p(p(a_1, d, a_2), c, p(b_1, d, b_2))) \\
\Longrightarrow \quad & p(p(a_1, c, b_1), d, p(a_2, c, b_2)) = p(p(a_1, d, a_2), c, p(b_1, d, b_2)).
\end{aligned}$$

□

7. Mobi algebras and unitary rings with one half

We have seen the passage from mobi algebras to IMM(*) algebras and back (Proposition 4.1 and Theorem 6.2), as well as the passage from IMM algebras to unitary rings and back (Theorem 5.2 and Theorem 5.3). Here we make explicit the fact that there is a straightforward connection between mobi algebras in which ½ is invertible and unitary rings in which 2 is invertible.

Theorem 7.1. *Let $(A, p, 0, \frac{1}{2}, 1)$ be a mobi algebra on the set A and let $2 \in A$ be a distinguished element on that set.*
The structure $(A, +, \cdot, 0, 1)$, with $a \cdot b = p(0, a, b)$ and $a + b = 2 \cdot p(a, \frac{1}{2}, b)$, is a unitary ring if and only if 2 is the inverse of ½.

Proof. If $(A, +, \cdot, 0, 1)$ is a unitary ring, with $a + b = 2 \cdot p(a, \frac{1}{2}, b)$, then, in particular, $0 + 1 = 2 \cdot p(0, \frac{1}{2}, 1)$ which implies $1 = 2 \cdot \frac{1}{2}$ proving that 2 is the inverse of ½. Conversely, we begin by using Proposition 4.1 to obtain, from the mobi, an IMM structure $(A, \overline{()}, \oplus, \cdot, 1)$ in which $\frac{1}{2} = \overline{1} \oplus 1$, $a \cdot b = p(0, a, b)$ and $a \oplus b = p(a, \frac{1}{2}, b)$. Within this structure, if 2 is the inverse of ½, we have, by Theorem 5.2, that $a \oplus b = \frac{1}{2} \cdot (a + b)$ which is equivalent to $a + b = 2 \cdot p(a, \frac{1}{2}, b)$. □

Theorem 7.2. *Let $(A, +, \cdot, 0, 1)$ be a unitary ring with a distinguished element $\frac{1}{2} \in A$.*

The structure $(A, p, 0, ½, 1)$*, with* $p(a, b, c) = a + bc - ba$*, is a mobi algebra if and only if* ½ *is the inverse of* $1 + 1$*.*

Proof. If $(A, p, 0, ½, 1)$ is a mobi algebra, with $p(a, b, c) = a + bc - ba$, then axiom **(A1)**, $p(1, ½, 0) = ½$, reads $1 - ½ = ½$, i.e. $(1+1) \cdot ½ = 1$. On the other hand, if ½ is the inverse of $1+1$, Theorem 5.3 gives us an IMM algebra $(A, \overline{()}, \oplus, \cdot, 1)$ in which $\overline{b} = 1-b$ and $a+b = (1+1)\cdot(a \oplus b)$. This implies, in particular, that $1 = (1 + 1) \cdot (\overline{1} \oplus 1)$. Hence, through Corollary 6.3, we conclude that $(A, p, 0, ½, 1)$ is a mobi algebra, with $p(a, b, c) = (\overline{1} \oplus 1)^{-1} \cdot (((1 - b) \cdot a) \oplus (b \cdot c)) = (a - ba) + bc$. □

Theorem 7.3. *There is a bijective correspondence between unitary rings containing the element* 2^{-1} *and mobi algebras containing* 2*.*

Proof. Let $(A, +, \cdot, 0, 1)$ be a unitary ring such that $2^{-1} \in A$, with $2 = 1 + 1$. Then, by Theorem 7.2, the system $(A, p, 0, 2^{-1}, 1)$ where $p(a, b, c) = a + bc - ba$ is a mobi algebra. This mobi algebra contains 2 (the inverse of 2^{-1}) and, consequently, by Theorem 7.1, it determines a unitary ring $(A, +', \cdot', 0, 1)$. This ring is identical to the initial ring $(A, +, \cdot, 0, 1)$ because:

$$\begin{aligned} a \cdot' b &= p(0, a, b) = 0 + a \cdot b - a \cdot 0 = a \cdot b \\ a +' b &= 2 \cdot' p(a, \frac{1}{2}, b) = 2 \cdot (a + \frac{1}{2} \cdot b - \frac{1}{2} \cdot a) = a + b. \end{aligned}$$

Conversely, let $(A, p, 0, ½, 1)$ be a mobi algebra such that $p(0, ½, 2) = 1$ with $2 \in A$. Then, by Theorem 7.1, $(A, +, \cdot, 0, 1)$ with $a \cdot b = p(0, a, b)$ and $a+b = 2\cdot p(a, ½, b)$ is a unitary ring. This ring contains ½ and, consequently, by Theorem 7.2, it determines a mobi algebra $(A, p', 0, ½, 1)$. This mobi algebra is identical to the initial one. Indeed, by definition of p', we have

$$p'(a, b, c) = (1 - b) \cdot a + b \cdot c = 2 \cdot p((1 - b) \cdot a, ½, b \cdot c).$$

Then, as shown in the proof of Proposition 4.1, we get $p(\overline{b}, ½, b) = ½$. When 2 exists, this equality may be written as $\overline{b} + b = 1$, i.e., $1 - b = \overline{b}$. Therefore

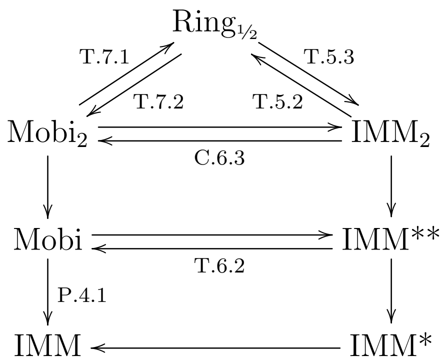
$$p'(a, b, c) = 2 \cdot p(\overline{b} \cdot a, ½, b \cdot c)$$

which, using property (23), implies that $p'(a, b, c) = p(a, b, c)$. □

The finite case is of particular interest because every finite mobi is such that its element ½ is invertible, and hence it is uniquely determined by a unitary ring structure in which 2 is invertible (see Example 15 in Section 3).

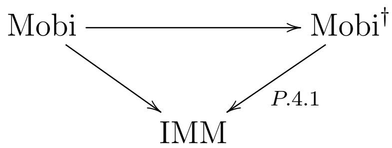
8. Conclusion

We conclude with a schematic diagram relating the algebraic structures considered here and the results that relate them. We use an arrow labelled with the number of the Theorem, Proposition or Corollary where the result is proved on the direction indicated by the arrow. For example, the arrow labelled P.4.1, with source Mobi and target IMM, simply means that the Proposition 4.1 establishes an effective passage from the algebraic structure of a mobi algebra to the algebraic structure of an IMM algebra. Moreover, we use the name IMM** to designate an IMM* algebra in which condition (i) of Theorem 6.2 holds. This structure, by Corollary 6.3, is clearly in between IMM algebras, in which ½ is invertible (that we are denoting by IMM_2), and IMM* algebras. Following the same line, we denote by $\mathrm{Ring}_{1/2}$ the rings in which 2 is invertible and by Mobi_2 the mobi algebras in which ½ is invertible.



(41)

We observe that in the proof of property P.4.1, the axiom **(A6)** is not used which suggests that we could also consider a mobi without this axiom. Representing such a structure by Mobi†, we get the following additional diagram.



(42)

Furthermore, the inclusion Mobi $\subset$ Mobi† is strict. Indeed, the example IMM 2 (see Appendix) is an IMM algebra which is obtained from a Mobi† that is not a Mobi (axiom **(A6)** is not satisfied). Moreover, there are IMM algebras which are not obtained from Mobi† as it is shown by example IMM 3 in the Appendix.

Our last comment is that the connection between IMM algebras and rings can be lifted to the more general case of semi-rings. Indeed, if in the definition of IMM algebra we remove the unary operation $\overline{()}$ while keeping the existence of the element $\overline{1}$ such that $\overline{1} \cdot x = \overline{1}$ then, in Theorem 5.2, we can replace rings by semi-rings.

Appendix

In this appendix, we present some examples of finite IMM algebras with 5 elements. Let $A = \{\alpha, 0, ½, 1, \beta\}$ be a set with 5 elements. In the three examples below, the unary operation $\overline{()}$ is defined by $\overline{\alpha} = \beta$, $\overline{0} = 1$, $\overline{1} = 0$, $\overline{½} = ½$, $\overline{\beta} = \alpha$.

IMM 1: The system $(A, \overline{()}, \oplus, \cdot, 1)$, with $\oplus$ and $\cdot$ defined as follows, is an IMM algebra.

$\oplus$	α	0	½	1	β
α	α	β	1	0	½
0	β	0	α	½	1
½	1	α	½	β	0
1	0	½	β	1	α
β	½	1	0	α	β

$\cdot$	α	0	½	1	β
α	1	0	β	α	½
0	0	0	0	0	0
½	β	0	α	½	1
1	α	0	½	1	β
β	½	0	1	β	α

Defining $p(a, b, c) = \beta \cdot ((\overline{b} \cdot a) \oplus (b \cdot c))$, it can be checked that $(A, p, 0, ½, 1)$ is a mobi algebra.

IMM 2: The system $(A, \overline{()}, \oplus, \cdot, 1)$, with $\oplus$ and $\cdot$ defined as follows, is an IMM algebra.

$\oplus$	α	0	½	1	β
α	α	½	½	α	½
0	½	0	½	½	β
½	½	½	½	½	½
1	α	½	½	1	½
β	½	β	½	½	β

$\cdot$	α	0	½	1	β
α	α	0	½	α	β
0	0	0	0	0	0
½	½	0	½	½	β
1	α	0	½	1	β
β	β	0	β	β	0

It is obvious that the operation $\oplus$ is not cancellative. Nevertheless, the equation $½ \cdot p = (\overline{b} \cdot a) \oplus (b \cdot c)$ can be solved for all $a, b, c \in A$. It can be shown that there are solutions p of that equation for which $(A, p, 0, ½, 1)$ is a Mobi† (a mobi algebra without axiom **(A6)**).

IMM 3: The system $(A, \overline{()}, \oplus, \cdot, 1)$, with $\oplus$ and $\cdot$ defined as follows, is an IMM algebra.

$\oplus$	α	0	½	1	β
α	α	½	½	½	½
0	½	0	½	½	½
½	½	½	½	½	½
1	½	½	½	1	½
β	½	½	½	½	β

$\cdot$	α	0	½	1	β
α	β	0	½	α	1
0	0	0	0	0	0
½	½	0	½	½	½
1	α	0	½	1	β
β	1	0	½	β	α

This IMM cannot be induced by a mobi algebra because the equation $½ \cdot p = (\overline{b} \cdot a) \oplus (b \cdot c)$ does not have a solution p for all $a, b, c \in A$, in contradiction with (23). Indeed, for example, the equation $½ \cdot p = (\overline{\beta} \cdot \beta) \oplus (\beta \cdot \alpha)$ is equivalent to $½ \cdot p = 1$ which does not have a solution.

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