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*Research Article*

# Observer-Based Finite-Time $H_\infty$ Control of Singular Markovian Jump Systems

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This paper addresses the problem of finite-time $H_\infty$ control via observer-based state feedback for a family of singular Markovian jump systems (SMJSs) with time-varying norm-bounded disturbance. Firstly, the concepts of singular stochastic finite-time boundedness and singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback are given. Then an observer-based state feedback controller is designed to ensure singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback of the resulting closed-loop error dynamic SMJS. Sufficient criteria are presented for the solvability of the problem, which can be reduced to a feasibility problem involving linear matrix inequalities with a fixed parameter. As an auxiliary result, we also discuss the problem of finite-time stabilization via observer-based state feedback of a class of SMJSs and give sufficient conditions of singular stochastic finite-time stabilization via observer-based state feedback for the class of SMJSs. Finally, illustrative examples are given to demonstrate the validity of the proposed techniques.

## 1. Introduction

In practice, there exist many concerned problems which described that system state does not exceed some bound during some time interval, for instance, large values of the state are not acceptable in the presence of saturations [1–3]. Therefore, we need to check the unacceptable values to see whether the system states remain within the prescribed bound in a fixed finite-time interval. Compared with classical Lyapunov asymptotical stability, in order to deal with these transient performances of control dynamic systems, finite-time stability or short-time stability was introduced in the literatures [4, 5]. Applying Lyapunov function approach, some appealing results were obtained to ensure finite-time stability, finite-time boundedness, and finite-time stabilization of various systems including linear systems, nonlinear systems, and stochastic systems. For instance, Amato et al. [6] investigated the output feedback finite-time stabilization for continuous linear systems. Zhang and An [7] considered finite-time control problems for linear stochastic systems. Recently, Meng and Shen [8] extended the definition of $H_\infty$ control to finite-time $H_\infty$ control for linear continuous systems, and a state feedback controller was designed to ensure finite-time boundedness of the resulting systems and the effect of the disturbance input on the controlled output satisfying a prescribed level. For more details of the literature related to finite-time stability, finite-time boundedness, and finite-time $H_\infty$ control, the reader is referred to [9–20] and the references therein.

On the other hand, singular systems referred to as descriptor systems, differential-algebraic systems, generalized state-space systems, or semistate systems have attracted many researchers since the class of systems have been extensively applied to deal with mechanical systems, electric circuits, chemical process, power systems, interconnected systems, and so on; see more practical examples in [21, 22] and the references therein. A great number of results based on the theory of regular systems or state-space systems have been extensively generalized to singular systems with or without time delay, such as stability [23], stabilization [24], $H_\infty$ control [25–29], and other issues. Meanwhile, Markovian jump systems are referred to as a special family of hybrid systems and stochastic systems, which are very appropriate to model plants whose parameters are subject to random abrupt changes [30]. Thus, many attracting results and a large variety of control problems have been studied, such as stochastic Lyapunov stability [31–33], sliding mode control [34, 35], robust control [36–40], $H_\infty$ filtering [41–45], dissipative control [46], passive control [47], guaranteed cost control [48], tracking control [49], and other issues, the readers are refered to [31] and the references therein. It is pointed out that the problem of state feedback stabilization, just as was mentioned above, requires to assume the complete access to the state vector. Practically this assumption is not realistic for many reasons like the nonexistence of the appropriate sensors to measure some of the states or the limitation in the control strategies. Thus, the observer-based control and output feedback control are probably well suited in such situation for feedback control, such as stability [31], $H_\infty$ control [50–54], passive control [55], and finite-time control [6, 20]. However, to date, the problems of observer-based finite-time stabilization of singular stochastic systems have not been investigated. The problems are important and challenging in many practice applications, which motivates the main purpose of our research.

In this paper, we consider the problem of finite-time $H_\infty$ control via observer-based state feedback of singular Markovian jump systems (SMJSs) with time-varying norm-bounded disturbance. The results of this paper are totally different from those previous results, although some studies on finite-time control for singular stochastic systems have been conduced, see [18, 56, 57]. The concepts of singular stochastic finite-time boundedness (SSFTB) and singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback of singular stochastic systems are given. The main contribution of the paper is to design an observer-based state feedback controller which ensures singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback of the resulting closed-loop error dynamic SMJS. Sufficient criterions are presented for the solvability of the problem, which can be reduced to a feasibility problem in terms of linear matrix inequalities with a fixed parameter. As an auxiliary result, we also investigate the problem of observer-based finite-time stabilization via state feedback of a class of SMJSs and give sufficient conditions of singular stochastic finite-time stabilization via observer-based state feedback for the class of SMJSs.

The rest of this paper is organized as follows. In Section 2, the problem formulation and some preliminaries are introduced. The results of singular stochastic finite-time $H_\infty$

stabilization via observer-based state feedback are given for a class of SMJSs in Section 3. Section 4 presents numerical examples to show the validity of the proposed methodology. Some conclusions are drawn in Section 5.

*Notations.* Throughout the paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the sets of $n$ component real vectors and $n \times m$ real matrices, respectively. The superscript $T$ stands for matrix transposition or vector. $\mathbb{E}\{\cdot\}$ denotes the expectation operator respective to some probability measure $\mathbb{P}$. In addition, the symbol $*$ denotes the transposed elements in the symmetric positions of a matrix, and diag$\{\cdots\}$ stands for a block-diagonal matrix. $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the smallest and the largest eigenvalues of matrix $P$, respectively. Notations sup and inf denote the supremum and infimum, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2. Problem Formulation

Let us consider the following continuous-time singular Markovian jump system (SMJS):

$$E(r_t)\dot{x}(t) = A(r_t)x(t) + B(r_t)u(t) + G(r_t)w(t), \tag{2.1a}$$

$$z(t) = C(r_t)x(t) + D_1(r_t)u(t) + D_2(r_t)w(t), \tag{2.1b}$$

$$y(t) = C_y(r_t)x(t), \tag{2.1c}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $z(t) \in \mathbb{R}^l$ is the controlled output, $y(t) \in \mathbb{R}^q$ is the measured output, $w(t) \in \mathbb{R}^p$ is the controlled input, $z(t) \in \mathbb{R}^p$ is the controlled output, $E(r_t)$ is a singular matrix with rank$(E(r_t)) = r_{r_t} < n$, $\{r_t,\ t \geq 0\}$ is continuous-time Markov stochastic process taking values in a finite space $\mathbb{M} := \{1, 2, \ldots, N\}$ with transition matrix $\Gamma = (\pi_{ij})_{N\times N}$, and the transition probabilities are described as follows:

$$\Pr\left(r_{t+\Delta t} = j \mid r_t = i\right) = \begin{cases} \pi_{ij}\Delta t + o(\Delta t) & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t) & \text{if } i = j, \end{cases} \tag{2.2}$$

where $\lim_{\Delta\to 0} o(\Delta t)/\Delta t = 0$, $\pi_{ij}$ satisfies $\pi_{ij} \geq 0$ $(i \neq j)$, and $\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}$ for all $i, j \in \mathbb{M}$. Moreover, the disturbance $w(t) \in \mathbb{R}^p$ satisfies the following constraint condition:

$$\int_0^T w^T(t)w(t)dt \leq d^2, \quad d \geq 0, \tag{2.3}$$

and the matrices $A(r_t), B(r_t), G(r_t), C(r_t), D_1(r_t)$, and $D_2(r_t)$ are coefficient matrices and of appropriate dimension for all $r_t \in \mathbb{M}$.

For notational simplicity, in the sequel, for each possible $r_t = i,\ i \in \mathbb{M}$, a matrix $K(r_t)$ will be denoted by $K_i$; for instance, $A(r_t)$ will be denoted by $A_i$, $B(r_t)$ by $B_i$, and so on.

In this paper, we construct the following state observer and feedback controller:

$$E_i\dot{\tilde{x}}(t) = A_i\tilde{x}(t) + B_iu(t) + H_i\big(y(t) - \tilde{y}(t)\big), \tag{2.4a}$$

$$\tilde{y}(t) = C_{yi}\tilde{x}(t), \tag{2.4b}$$

$$u(t) = K_i\tilde{x}(t), \tag{2.4c}$$

where $\tilde{x}(t)$ and $\tilde{y}(t)$ are the estimated state and output, $\tilde{x}(0)$ is an estimated initial state, $K_i$ is to be a designed state feedback gain, and $H_i$ is an observer gain to be designed. Define the state estimated error $e(t) = x(t) - \tilde{x}(t)$ and $\overline{x}(t) = [x^T(t) \;\; e^T(t)]^T$. Then the resulting closed-loop error dynamic SMJS can be written in the form as follows:

$$\overline{E}_i\dot{\overline{x}}(t) = \overline{A}_i\overline{x}(t) + \overline{G}_iw(t), \tag{2.5a}$$

$$z(t) = \overline{C}_i\overline{x}(t) + D_{2i}w(t), \tag{2.5b}$$

where

$$\begin{aligned} \overline{E}_i &= \begin{bmatrix} E_i & 0 \\ * & E_i \end{bmatrix}, \qquad \overline{A}_i = \begin{bmatrix} A_i + B_iK_i & -B_iK_i \\ 0 & A_i - H_iC_{yi} \end{bmatrix}, \\ \overline{G}_i &= \begin{bmatrix} G_i \\ G_i \end{bmatrix}, \qquad \overline{C}_i = \begin{bmatrix} C_i + D_{1i}K_i & -D_{1i}K_i \end{bmatrix}. \end{aligned} \tag{2.6}$$

*Definition 2.1* (regular and impulse-free, see [21, 22]). The SMJS (2.1a) with $u(t) = 0$ is said to be regular in time interval $[0, T]$ if the characteristic polynomial $\det(sE_i - A_i)$ is not identically zero for all $t \in [0, T]$. The SMJS (2.1a) with $u(t) = 0$ is said to be impulse-free in time interval $[0, T]$ if $\deg(\det(sE_i - A_i)) = \operatorname{rank}(E_i)$ for all $t \in [0, T]$.

*Definition 2.2* (singular stochastic finite-time stability (SSFTS)). The SMJS (2.1a) with $w(t) = 0$ is said to be SSFTS with respect to $(c_1, c_2, T, R_i)$, with $c_1 < c_2$ and $R_i > 0$, if the stochastic system is regular and impulse-free in time interval $[0, T]$ and satisfies

$$\mathbb{E}\left\{x^T(0)E_i^TR_iE_ix(0)\right\} \le c_1^2 \Longrightarrow \mathbb{E}\left\{x^T(t)E_i^TR_iE_ix(t)\right\} < c_2^2, \quad \forall t \in [0, T]. \tag{2.7}$$

*Definition 2.3* (singular stochastic finite-time boundedness (SSFTB)). The SMJS (2.1a) which satisfies (2.3) is said to be SSFTB with respect to $(c_1, c_2, T, R_i, d)$, with $c_1 < c_2$ and $R_i > 0$, if the stochastic system is regular and impulse-free in time interval $[0, T]$, and condition (2.7) holds.

*Remark 2.4.* The definition of SSFTB is the generalization of finite-time boundedness [1]. SSFTB implies that the whole mode of the singular stochastic system is finite-time bounded since the static mode is regular and impulse-free.

*Definition 2.5* (singular stochastic finite-time stabilization via observer-based state feedback). The error dynamic SMJS (2.5a) and (2.5b) is said to be singular stochastic finite-time stabilization via observer-based state feedback with respect to $(c_1, c_2, T, \overline{R}_i, d)$, with $c_1 < c_2$ and $\overline{R}_i > 0$, if there exists a state feedback control law and a state observer in the form (2.4a)–(2.4c), such that the error dynamic SMJS (2.5a) and (2.5b) is regular and impulse-free in time interval $[0, T]$ and satisfies the following constraint relation:

$$\mathbb{E}\left\{\overline{x}^T(0)\overline{E}_i^T\overline{R}_i\overline{E}_i\overline{x}(0)\right\} \le c_1^2 \Longrightarrow \mathbb{E}\left\{\overline{x}^T(t)\overline{E}_i^T\overline{R}_i\overline{E}_i\overline{x}(t)\right\} < c_2^2, \quad \forall t \in [0, T]. \tag{2.8}$$

*Definition 2.6* (see [30, 33]). Let $V(x(t),\ r_t = i,\ t \ge 0)$ be the stochastic function, and define its weak infinitesimal operator $\mathbf{J}$ of stochastic process $\{(x(t),\ r_t = i),\ t \ge 0\}$ by

$$\begin{aligned} \mathbf{J}V(x(t), r_t = i, t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\{\mathbb{E}\{V(x(t+\Delta t), r_{t+\Delta t}, t+\Delta t) \mid x(t) = x, r_t = i\} - V(x(t), i, t)\} \\ &= V_t(x(t), i, t) + V_x(x(t), i, t)\dot{x}(t, i) + \sum_{j=1}^{N}\pi_{ij}V(x(t), j, t). \end{aligned} \tag{2.9}$$

*Definition 2.7* (singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback). The closed-loop error dynamic SMJS (2.5a) and (2.5b) is said to be singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback with respect to $(c_1, c_2, T, \overline{R}_i, \gamma, d)$, with $c_1 < c_2$ and $\overline{R}_i > 0$, if there exists a state observer and feedback controller in the form (2.4a)–(2.4c), such that the error dynamic SMJS (2.5a) and (2.5b) is SSFTB with respect to $(c_1, c_2, T, \overline{R}_i, d)$, and under the zero-initial condition, the controlled output $z$ satisfies

$$\mathbb{E}\left\{\int_0^T z^T(t)z(t)dt\right\} < \gamma^2\mathbb{E}\left\{\int_0^T w^T(t)w(t)dt\right\}, \tag{2.10}$$

for any nonzero $w(t)$ which satisfies (2.3), where $\gamma$ is a prescribed positive scalar.

The main objective of this paper being to concentrate on designing a state observer and feedback controller of the form (2.4a)–(2.4c) that ensures singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback of the error dynamic SMJS (2.5a) and (2.5b), we require the following lemmas.

**Lemma 2.8** (Schur complement lemma, see [57, 58]). *The linear matrix inequality* $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} < 0$ *is equivalent to* $S_{22} < 0$ *and* $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$ *with* $S_{11} = S_{11}^T$ *and* $S_{22} = S_{22}^T$.

**Lemma 2.9** (see [57]). *The following items are true.*

(i) *Assume that* $\operatorname{rank}(E) = r$, *then there exist two orthogonal matrices* $U$ *and* $V$ *such that* $E$ *has the decomposition as*

$$E = U\begin{bmatrix} \Sigma_r & 0 \\ * & 0 \end{bmatrix}V^T = U\begin{bmatrix} I_r & 0 \\ * & 0 \end{bmatrix}\mathcal{V}^T, \tag{2.11}$$

*where* $\Sigma_r = \operatorname{diag}\{\delta_1, \delta_1, \dots, \delta_r\}$ *with* $\delta_k > 0$ *for all* $k = 1, 2, \dots, r$. *Partition* $U = [U_1\ \ U_2]$, $V = [V_1\ \ V_2]$, *and* $\mathcal{V} = [V_1\Sigma_r\ \ V_2]$ *with* $EV_2 = 0$ *and* $U_2^TE = 0$.

(ii) *If $P$ satisfies*

$$EP^T = PE^T \geq 0, \tag{2.12}$$

*then* $\widetilde{P} = \mathcal{U}^T P \mathcal{V}$ *with* $\mathcal{U}$ *and* $\mathcal{V}$ *satisfying* (2.11) *if and only if*

$$\widetilde{P} = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}, \tag{2.13}$$

*with* $P_{11} \geq 0 \in \mathbb{R}^{r\times r}$. *In addition, when $P$ is nonsingular, one has* $P_{11} > 0$ *and* $\det(P_{22}) \neq 0$. *Furthermore, $P$ satisfying* (2.12) *can be parameterized as*

$$P = E\mathcal{V}^{-T} X \mathcal{V}^{-1} + \mathcal{U} Z V_2^T, \tag{2.14}$$

*where* $X = \operatorname{diag}\{P_{11}, \Psi\}$, $Z = [P_{12}^T \;\; P_{22}^T]^T$, *and* $\Psi \in \mathbb{R}^{(n-r)\times(n-r)}$ *is an arbitrary parameter matrix.*

(iii) *If $P$ is a nonsingular matrix, $R$ and $\Psi$ are two symmetric positive definite matrices, $P$ and $E$ satisfy* (2.12), *$X$ is a diagonal matrix from* (2.14), *and the following equality holds:*

$$P^{-1}E = E^T R^{1/2} S R^{1/2} E. \tag{2.15}$$

*Then the symmetric positive definite matrix* $S = R^{-1/2} \mathcal{U} X^{-1} \mathcal{U}^T R^{-1/2}$ *is a solution of* (2.15).

## 3. Main Results

In this section, LMI conditions are established to design a state observer and feedback controller that guarantees the error dynamic SMJS of the class we are considering is singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback.

**Theorem 3.1.** *The error dynamic SMJS* (2.5a) *and* (2.5b) *is singular stochastic finite-time stabilization via observer-based state feedback with respect to* $(c_1, c_2, T, \overline{R}_i, d)$ *if there exist positive scalars* $\alpha, c_2$, *a set of mode-dependent nonsingular matrices* $\{\overline{P}_i,\ i \in \mathbb{M}\}$, *and sets of mode-dependent symmetric positive-definite matrices* $\{\overline{S}_i,\ i \in \mathbb{M}\}$, $\{\Theta_i,\ i \in \mathbb{M}\}$, *and for all* $i \in \mathbb{M}$, *such that the following inequalities hold:*

$$\overline{P}_i \overline{E}_i^T = \overline{E}_i \overline{P}_i^T \geq 0, \tag{3.1a}$$

$$\begin{bmatrix} \overline{A}_i \overline{P}_i^T + \overline{P}_i \overline{A}_i^T + \sum_{j=1}^{N} \pi_{ij} \overline{P}_i \overline{P}_j^{-1} \overline{E}_j \overline{P}_i^T - \alpha \overline{E}_i \overline{P}_i^T & \overline{G}_i \\ * & -\Theta_i \end{bmatrix} < \mathbf{0}, \tag{3.1b}$$

$$\overline{P}_i^{-1} \overline{E}_i = \overline{E}_i^T \overline{R}_i^{1/2} \overline{S}_i \overline{R}_i^{1/2} \overline{E}_i, \tag{3.1c}$$

$$\overline{\lambda}_1 c_1^2 + \overline{\lambda}_2 d^2 < c_2^2 \underline{\lambda}_1 e^{-\alpha T}, \tag{3.1d}$$

*where* $\overline{\lambda}_1 = \sup_{i\in M}\{\lambda_{\max}(\overline{S}_i)\}$, $\underline{\lambda}_1 = \inf_{i\in M}\{\lambda_{\min}(\overline{S}_i)\}$, *and* $\overline{\lambda}_2 = \sup_{i\in \mathbb{M}}\{\lambda_{\max}(\Theta_i)\}$.

*Proof.* Firstly, we prove that the error dynamic SMJS (2.5a) and (2.5b) is regular and impulse-free in time interval $[0, T]$. Applying Lemma 2.8, condition (3.1b) implies

$$\overline{A}_i\overline{P}_i^T + \overline{P}_i\overline{A}_i^T + (\pi_{ii}-\alpha)\overline{E}_i\overline{P}_i^T < -\sum_{j=1, j\neq i}^{N}\pi_{ij}\overline{P}_i\overline{P}_j^{-1}\overline{E}_j\overline{P}_i^T \leq 0. \tag{3.2}$$

Now, there exist two orthogonal matrices $\overline{U}_i$ and $\overline{V}_i$ such that $\overline{E}_i$ has the decomposition as

$$\overline{E}_i = \overline{U}_i\begin{bmatrix}\overline{\Sigma}_{\overline{r}_i} & 0\\ * & 0\end{bmatrix}\overline{V}_i^T = \overline{U}_i\begin{bmatrix}I_{\overline{r}_i} & 0\\ * & 0\end{bmatrix}\overline{\mho}_i^T, \tag{3.3}$$

where $\overline{\Sigma}_{\overline{r}_i} = \operatorname{diag}\{\overline{\delta}_{i1}, \overline{\delta}_{i2}, \ldots, \overline{\delta}_{i\overline{r}_i}\}$ with $\overline{\delta}_k > 0$ for all $k = 1, 2, \ldots, \overline{r}_i$. Partition $\overline{U}_i = [\overline{U}_{i1} \;\; \overline{U}_{i2}]$, $\overline{V}_i = [\overline{V}_{i1} \;\; \overline{V}_{i2}]$, and $\overline{\mho}_i = [\overline{V}_{i1}\overline{\Sigma}_{\overline{r}} \;\; \overline{V}_{i2}]$ with $\overline{E}_i\overline{V}_{i2} = 0$ and $\overline{U}_{i2}^T\overline{E}_i = 0$. Denote

$$\overline{U}_i^T\overline{A}_i\overline{\mho}_i^{-T} = \begin{bmatrix}\overline{A}_{11i} & \overline{A}_{12i}\\ \overline{A}_{21i} & \overline{A}_{22i}\end{bmatrix}, \qquad \overline{U}_i^T\overline{P}_i\overline{\mho}_i = \begin{bmatrix}\overline{P}_{11i} & \overline{P}_{12i}\\ \overline{P}_{21i} & \overline{P}_{22i}\end{bmatrix}. \tag{3.4}$$

Noting that condition (3.1a) and $\overline{P}_i$ is a nonsingular matrix, by Lemma 2.9, we have $\overline{P}_{21i} = 0$ and $\det(\overline{P}_{22i}) \neq 0$. Before and after multiplying (3.2) by $\overline{U}_i^T$ and $\overline{U}_i$, respectively, this results in that the following matrix inequality holds:

$$\begin{bmatrix}\star & \star\\ \star & \overline{A}_{22i}\overline{P}_{22i}^T + \overline{P}_{22i}\overline{A}_{22i}^T\end{bmatrix} < 0, \tag{3.5}$$

where the star$^\star$ will not be used in the following discussion. By Lemma 2.8, we have $\overline{A}_{22i}\overline{P}_{22i}^T + \overline{P}_{22i}\overline{A}_{22i}^T < 0$. Therefore, $\overline{A}_{22i}$ is nonsingular, which implies that the error dynamic SMJS (2.5a) and (2.5b) is regular and impulse-free in time interval $[0, T]$.

For the given mode-dependent nonsingular matrix $\overline{P}_i$, let us consider the following quadratic function as:

$$V(\overline{x}(t), i) = \overline{x}^T(t)\overline{P}_i^{-1}\overline{E}_i\overline{x}(t). \tag{3.6}$$

Computing the weak infinitesimal operator $\mathbf{J}$ emanating from the point $(\overline{x}, i)$ at time $t$ along the solution of error dynamic SMJS (2.5a) and (2.5b) and noting the condition (3.1a), we obtain

$$\mathbf{J}V(\overline{x}(t), i) = \xi^T(t)\begin{bmatrix}\overline{P}_i^{-1}\overline{A}_i + \overline{A}_i^T\overline{P}_i^{-T} + \sum_{j=1}^{N}\pi_{ij}\overline{P}_j^{-1}\overline{E}_j & \overline{P}_i^{-1}\overline{G}_i\\ * & 0\end{bmatrix}\xi(t), \tag{3.7}$$

where $\xi(t) = [\overline{x}^T(t), w^T(t)]^T$. Before and after multiplying (3.1b) by $\mathrm{diag}\{\overline{P}_i^{-1}, I\}$ and $\mathrm{diag}\{\overline{P}_i^{-T}, I\}$, respectively, this results in the following matrix inequality

$$\begin{bmatrix} \overline{P}_i^{-1}\overline{A}_i + \overline{A}_i^T\overline{P}_i^{-T} + \sum_{j=1}^{N}\pi_{ij}\overline{P}_j^{-1}\overline{E}_j - \alpha\overline{P}_i^{-1}\overline{E}_i & \overline{P}_i^{-1}\overline{G}_i \\ * & -\Theta_i \end{bmatrix} < 0. \tag{3.8}$$

From (3.7) and (3.8), we can obtain

$$\mathbf{J}V(\overline{x}(t), i) < \alpha V(\overline{x}(t), i) + w^T(t)\Theta_i w(t). \tag{3.9}$$

Further, (3.9) can be rewritten as

$$\mathbb{E}\{\mathbf{J}[e^{-\alpha t}V(\overline{x}(t), i)]\} < \mathbb{E}\left\{e^{-\alpha t}w^T(t)\Theta_i w(t)\right\}. \tag{3.10}$$

Integrating (3.10) from 0 to $t$, with $t \in [0, T]$, we obtain

$$e^{-\alpha t}\mathbb{E}\{V(\overline{x}(t), i)\} < \mathbb{E}\{V(\overline{x}(0), i = r_0)\} + \int_0^t \mathbb{E}\left\{e^{-\alpha\tau}w^T(\tau)\Theta_i w(\tau)\right\}d\tau. \tag{3.11}$$

Noting $\alpha \geq 0$, $t \in [0, T]$, and condition (3.1c), we have

$$\begin{aligned} \mathbb{E}\left\{\overline{x}^T(t)\overline{P}_i^{-1}\overline{E}_i\overline{x}(t)\right\} &= \mathbb{E}\{V(\overline{x}(t), i)\} \\ &< e^{\alpha t}\left\{\mathbb{E}\{V(\overline{x}(0), i = r_0)\} + \int_0^t \mathbb{E}\left\{e^{-\alpha\tau}w^T(\tau)\Theta_i w(\tau)\right\}d\tau\right\} \\ &\leq e^{\alpha t}\left(\overline{\lambda}_1 c_1^2 + \overline{\lambda}_2 d^2\right). \end{aligned} \tag{3.12}$$

Taking into account that

$$\begin{aligned} \mathbb{E}\left\{\overline{x}^T(t)\overline{P}_i^{-1}\overline{E}_i x(t)\right\} &= \mathbb{E}\left\{\overline{x}^T(t)\overline{E}_i^T\overline{R}_i^{1/2}\overline{S}_i\overline{R}_i^{1/2}\overline{E}_i\overline{x}(t)\right\} \\ &\geq \underline{\lambda}_1\mathbb{E}\left\{\overline{x}^T(t)\overline{E}_i^T\overline{R}_i\overline{E}_i\overline{x}(t)\right\}, \end{aligned} \tag{3.13}$$

we obtain

$$\mathbb{E}\left\{\overline{x}^T(t)\overline{E}_i^T\overline{R}_i\overline{E}_i\overline{x}(t)\right\} < e^{\alpha T}\frac{\overline{\lambda}_1 c_1^2 + \overline{\lambda}_2 d^2}{\underline{\lambda}_1}. \tag{3.14}$$

Therefore, it follows that condition (3.1d) implies $\mathbb{E}\{\overline{x}^T(t)\overline{E}_i^T\overline{R}_i\overline{E}_i\overline{x}(t)\} < c_2^2$ for all $t \in [0, T]$. This completes the proof of the theorem. □

**Theorem 3.2.** *The error dynamic SMJS* (2.5a) *and* (2.5b) *is singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback with respect to $(c_1, c_2, T, \overline{R}_i, \gamma, d)$ if there exist positive scalars $\alpha, c_2, \gamma$, a set of mode-dependent nonsingular matrices $\{\overline{P}_i,\ i \in \mathbb{M}\}$, and a set of mode-dependent symmetric positive-definite matrices $\{\overline{S}_i,\ i \in \mathbb{M}\}$, and for all $i \in \mathbb{M}$, such that* (3.1a), (3.1c), *and the following inequalities:*

$$\begin{bmatrix} \overline{\Xi}_i + \overline{P}_i \overline{C}_i^T \overline{C}_i \overline{P}_i^T & \overline{P}_i \overline{C}_i^T D_{2i} + \overline{G}_i \\ * & -\gamma^2 e^{-\alpha T} I + D_{2i}^T D_{2i} \end{bmatrix} < 0, \tag{3.15a}$$

$$\overline{\lambda}_1 c_1^2 e^{\alpha T} + \gamma^2 d^2 < c_2^2 \underline{\lambda}_1 \tag{3.15b}$$

*hold, where* $\overline{\Xi}_i = \overline{A}_i \overline{P}_i^T + \overline{P}_i \overline{A}_i^T + \sum_{j=1}^{N} \pi_{ij} \overline{P}_i \overline{P}_j^{-1} \overline{E}_j \overline{P}_i^T - \alpha \overline{E}_i \overline{P}_i^T$.

*Proof.* Note that

$$\begin{bmatrix} \overline{P}_i \overline{C}_i^T \overline{C}_i \overline{P}_i^T & \overline{P}_i \overline{C}_i^T D_{2i} \\ * & D_{2i}^T D_{2i} \end{bmatrix} = \begin{bmatrix} \overline{P}_i \overline{C}_i^T \\ D_{2i}^T \end{bmatrix} \begin{bmatrix} \overline{C}_i \overline{P}_i^T & D_{2i} \end{bmatrix} \geq 0. \tag{3.16}$$

Thus, condition (3.15a) implies that

$$\begin{bmatrix} \overline{A}_i \overline{P}_i^T + \overline{P}_i \overline{A}_i^T + \displaystyle\sum_{j=1}^{N} \pi_{ij} \overline{P}_i \overline{P}_j^{-1} \overline{E}_j \overline{P}_i^T - \alpha \overline{E}_i \overline{P}_i^T & \overline{G}_i \\ * & -\gamma^2 e^{-\alpha T} I \end{bmatrix} < 0. \tag{3.17}$$

Let $\Theta_i = -\gamma^2 e^{-\alpha T} I$ for all $i \in \mathbb{M}$, by Theorem 3.1, conditions (3.1a), (3.1c), (3.15b), and (3.17) can guarantee that the error dynamic SMJS (2.5a) and (2.5b) is singular stochastic finite-time stabilization via observer-based state feedback with respect to $(c_1, c_2, T, \overline{R}_i, d)$. Therefore, we only need to prove that the constraint relation (2.10) holds. Let us choose the Lyapunov-Krasovskii function $V(\overline{x}(t), i)$ in the form (3.6) in Theorem 3.1 and noting (3.7) and (3.15a), we obtain

$$\mathbf{J}V(\overline{x}(t), i) < \alpha V(\overline{x}(t), i) + \gamma^2 e^{-\alpha T} w^T(t) w(t) - z^T(t) z(t). \tag{3.18}$$

Further, (3.18) can be represented as

$$\mathbf{J}\left[e^{-\alpha t} V(\overline{x}(t), i)\right] < e^{-\alpha t} \left[\gamma^2 e^{-\alpha T} w^T(t) w(t) - z^T(t) z(t)\right]. \tag{3.19}$$

Integrating (3.19) from 0 to $T$ and noting that under-zero initial condition, we have

$$\int_0^T e^{-\alpha t} \left[z^T(t) z(t) - \gamma^2 e^{-\alpha T} w^T(t) w(t)\right] dt < -\int_0^T \mathbf{J}\left[e^{-\alpha t} V(\overline{x}(t), i)\right] dt \leq V(\overline{x}(0), r_0) = 0. \tag{3.20}$$

Using the Dynkin formula, it results that

$$\mathbb{E}\left\{\int_0^T e^{-\alpha t}\left[z^T(t)z(t)-\gamma^2 e^{-\alpha T}w^T(t)w(t)\right]wt\right\}<0. \tag{3.21}$$

Thus, for all $t \in [0,T]$ and under-zero initial condition, we have

$$\begin{aligned}\mathbb{E}\left\{\int_0^T z^T(t)z(t)dt\right\} &\leq e^{\alpha T}\mathbb{E}\left\{\int_0^T e^{-\alpha t}z^T(t)z(t)dt\right\}\\ &< e^{\alpha T}\mathbb{E}\left\{\int_0^T \gamma^2 e^{-\alpha(t+T)}w^T(t)w(t)dt\right\}\\ &\leq \gamma^2\mathbb{E}\left\{\int_0^T w^T(t)w(t)dt\right\}.\end{aligned} \tag{3.22}$$

This completes the proof of the theorem. □

Let $\overline{P}_i = \operatorname{diag}\{P_i, P_i\}$, $\overline{S}_i = \operatorname{diag}\{S_i, S_i\}$, and $\overline{R}_i = \operatorname{diag}\{R_i, R_i\}$, then the following theorem gives LMI conditions to ensure singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback of the error dynamic SMJS (2.5a) and (2.5b).

**Theorem 3.3.** *There exist a state feedback controller $u(t) = K_i\tilde{x}(t)$ with $K_i = Y_iP_i^{-T}$ and a state observer $H_i = -P_iC_{yi}^T$ such that the error dynamic SMJS* (2.5a) *and* (2.5b) *is singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback with respect to $(c_1, c_2, T, \overline{R}_i, \gamma, d)$ if there exist positive scalars $\alpha, c_2, \gamma, \sigma_1$, and sets of mode-dependent symmetric positive-definite matrices $\{X_i,\ i \in \mathbb{M}\}$, $\{\Phi_i,\ i \in \mathbb{M}\}$, sets of mode-dependent matrices $\{Y_i,\ i \in \mathbb{M}\}$, $\{Z_i,\ i \in \mathbb{M}\}$, and for all $r_t = i \in \mathbb{M}$, such that the following inequalities hold:*

$$0 \leq P_iE_i^T = E_iP_i^T = E_i\mathcal{U}_i^{-T}X_i\mathcal{U}_i^{-1}E_i^T \leq \Phi_i, \tag{3.23a}$$

$$\begin{bmatrix}\Pi_{11i} & \Pi_{12i}\\ * & \Pi_{22i}\end{bmatrix}<0, \tag{3.23b}$$

$$\sigma_1 R_i^{-1} < U_iX_iU_i^T < R_i^{-1}, \tag{3.23c}$$

$$\begin{bmatrix}e^{-\alpha T}\left(-c_2^2+\gamma^2d^2\right) & c_1\\ * & -\sigma_1\end{bmatrix}<0, \tag{3.23d}$$

*where*

$$
\begin{aligned}
\Pi_{11i} &= \begin{bmatrix} L_1(P_i, Y_i) & -B_i Y_i & G_i \\ * & L_2(P_i, Y_i) & G_i \\ * & * & -\gamma^2 e^{-\alpha T} I \end{bmatrix}, \\
\Pi_{12i} &= \begin{bmatrix} P_i C_i^T + Y_i^T D_{1i}^T & 0 & \Upsilon_i & 0 \\ -Y_i^T D_{1i}^T & P_i C_{yi}^T & 0 & \Upsilon_i \\ D_{2i}^T & 0 & 0 & 0 \end{bmatrix}, \\
\Pi_{22i} &= -\operatorname{diag}\left\{ I, \frac{I}{2}, W_i, W_i \right\}, \\
L_1(P_i, Y_i) &= P_i A_i^T + A_i P_i^T + B_i Y_i + Y_i^T B_i^T + (\pi_{ii} - \alpha) E_i P_i^T, \\
L_2(P_i, Y_i) &= P_i A_i^T + A_i P_i^T + (\pi_{ii} - \alpha) E_i P_i^T, \\
\Upsilon_i &= \left[ \sqrt{\pi_{i1}} P_i, \dots, \sqrt{\pi_{i(i-1)}} P_i, \sqrt{\pi_{i(i+1)}} P_i, \dots, \sqrt{\pi_{iN}} P_i \right], \\
W_i &= \operatorname{diag}\left\{ P_1^T + P_1 - \Phi_1, \dots, P_{i-1}^T + P_{i-1} - \Phi_{i-1}, P_{i+1}^T + P_{i+1} - \Phi_{i+1}, \dots, P_N^T + P_N - \Phi_N \right\}.
\end{aligned}
\tag{3.24}
$$

*In addition, the form of* $P_i = E_i \mathcal{V}_i^{-T} X_i \mathcal{V}_i^{-1} + U_i Z_i V_{i2}^T$ *is from* (3.36).

*Proof.* We firstly prove that condition (3.23b) implies condition (3.15a). Let $\overline{P}_i = \operatorname{diag}\{P_i, P_i\}$, $\overline{S}_i = \operatorname{diag}\{S_i, S_i\}$, and $\overline{R}_i = \operatorname{diag}\{R_i, R_i\}$, then conditions (3.1a), (3.1c), and (3.1d) are equivalent to

$$P_i E_i^T = E_i P_i^T \geq 0, \tag{3.25a}$$

$$P_i^{-1} E_i = E_i^T R_i^{1/2} S_i R_i^{1/2} E_i, \tag{3.25b}$$

$$\overline{\kappa}_1 c_1^2 e^{\alpha T} + \gamma^2 d^2 < c_2^2 \underline{\kappa}_1, \tag{3.25c}$$

where $\overline{\kappa}_1 = \sup_{i \in M}\{\lambda_{\max}(S_i)\}$ and $\underline{\kappa}_1 = \inf_{i \in M}\{\lambda_{\min}(S_i)\}$. By condition (3.23a), we have

$$P_j^{-1} E_j \leq P_j^{-1} \Phi_j P_j^{-T}, \quad \forall j \in \mathbb{M}. \tag{3.26}$$

Thus, the inequality

$$
\begin{aligned}
\sum_{j=1, j \neq i}^{N} \pi_{ij} P_i P_j^{-1} E_j P_i^T &\leq \sum_{j=1, j \neq i}^{N} \pi_{ij} P_i P_j^{-1} \Phi_j P_j^{-T} P_i^T \\
&\leq \Upsilon_i V_i^{-1} \Upsilon_i^T
\end{aligned}
\tag{3.27}
$$

holds, where

$$\Upsilon_i = [\sqrt{\pi_{i,1}}P_i, \ldots, \sqrt{\pi_{i,i-1}}P_i, \sqrt{\pi_{i,i+1}}P_i, \ldots, \sqrt{\pi_{i,N}}P_i],$$
$$V_i = \operatorname{diag}\left\{P_1^T\Phi_1^{-1}P_1, \ldots, P_{i-1}^T\Phi_{i-1}^{-1}P_{i-1}, P_{i+1}^T\Phi_{i+1}^{-1}P_{i+1}, \ldots, P_N^T\Phi_N^{-1}P_N\right\}. \tag{3.28}$$

Noting that the inequality

$$P_j^T\Phi_j^{-1}P_j \geq P_j^T + P_j - \Phi_j \tag{3.29}$$

holds for each $j \in \mathbb{M}$, thus,

$$\sum_{j=1, j\neq i}^{N} \pi_{ij}P_iP_j^{-1}E_jP_i^T \leq \Upsilon_i W_i^{-1}\Upsilon_i^T, \tag{3.30}$$

where $W_i = \operatorname{diag}\{P_1^T + P_1 - \Phi_1, \ldots, P_{i-1}^T + P_{i-1} - \Phi_{i-1}, P_{i+1}^T + P_{i+1} - \Phi_{i+1}, \ldots, P_N^T + P_N - \Phi_N\}$. Let $\overline{\Upsilon}_i = \operatorname{diag}\{\Upsilon_i, \Upsilon_i\}$ and $\overline{W}_i = \operatorname{diag}\{W_i, W_i\}$, then a sufficient condition for (3.15a) to guarantee is that

$$\begin{bmatrix} \overline{\Theta}_i + \overline{\Upsilon}_i\overline{W}_i^{-1}\overline{\Upsilon}_i^T + \overline{P}_i\overline{C}_i^T\overline{C}_i\overline{P}_i^T & \overline{P}_i\overline{C}_i^T D_{2i} + \overline{G}_i \\ * & -\gamma^2 e^{-\alpha T} I + D_{2i}^T D_{2i} \end{bmatrix} < 0, \tag{3.31}$$

where $\overline{\Theta}_i = \overline{A}_i\overline{P}_i^T + \overline{P}_i\overline{A}_i^T + (\pi_{ii} - \alpha)\overline{E}_i\overline{P}_i^T$. Noting the forms of $\overline{A}_i, \overline{B}_i, \overline{C}_i$, and $\overline{G}_i$, then the inequality (3.31) is equivalent to the following:

$$\begin{bmatrix} \Lambda_{11i} & -B_iK_iP_i^T & G_i & P_iC_i^T + P_iK_i^TD_{1i}^T \\ * & \Lambda_{22i} & G_i & -P_iK_i^TD_{1i}^T \\ * & * & -\gamma^2 e^{-\alpha T} I & D_{2i}^T \\ * & * & * & -I \end{bmatrix} < 0, \tag{3.32}$$

where

$$\Lambda_{11i} = P_iA_i^T + A_iP_i^T + B_iK_iP_i^T + P_iK_i^TB_i^T + \Upsilon_iW_i^{-1}\Upsilon_i^T + (\pi_{ii} - \alpha)E_iP_i^T,$$
$$\Lambda_{22i} = P_iA_i^T + A_iP_i^T - H_iC_{yi}P_i^T - P_iC_{yi}^TH_i^T + \Upsilon_iW_i^{-1}\Upsilon_i^T + (\pi_{ii} - \alpha)E_iP_i^T. \tag{3.33}$$

Let $H_i = -P_iC_{yi}^T$, we obtain

$$\Lambda_{22i} = P_iA_i^T + A_iP_i^T + 2P_iC_{yi}^TC_{yi}P_i^T + \Upsilon_iW_i^{-1}\Upsilon_i^T + (\pi_{ii} - \alpha)E_iP_i^T. \tag{3.34}$$

Letting $Y_i = K_iP_i^T$ and applying Lemma 2.8, it follows that (3.32) is equivalent to (3.23b).

Noting that $P_i$ is nonsingular matrix, by Lemma 2.9, there exist two orthogonal matrices $U_i$ and $V_i$, such that $E_i$ has the decomposition as

$$E_i = U_i \begin{bmatrix} \Sigma_{r_i} & 0 \\ * & 0 \end{bmatrix} V_i^T = U_i \begin{bmatrix} I_{r_i} & 0 \\ * & 0 \end{bmatrix} \mathcal{V}_i^T, \tag{3.35}$$

where $\Sigma_{r_i} = \text{diag}\{\delta_{i1}, \delta_{i1}, \ldots, \delta_{ir_i}\}$ with $\delta_{ik} > 0$ for all $k = 1, 2, \ldots, r_i$. Partition $U_i = [U_{i1} \;\; U_{i2}]$, $V_i = [V_{i1} \;\; V_{i2}]$, and $\mathcal{V}_i = [V_{i1}\Sigma_{r_i} \;\; V_{i2}]$ with $E_i V_{i2} = 0$ and $U_{i2}^T E_i = 0$. Let $\widetilde{P}_i = U_i^T P_i \mathcal{V}_i$, from (3.23a), $\widetilde{P}_i$ is of the following form $\left[\begin{smallmatrix} P_{11i} & P_{12i} \\ 0 & P_{22i} \end{smallmatrix}\right]$, and $P_i$ can be expressed as

$$P_i = E_i \mathcal{V}_i^{-T} X_i \mathcal{V}_i^{-1} + U_i Z_i V_{i2}^T, \tag{3.36}$$

where $Z_i = [P_{12i}^T \;\; P_{22i}^T]^T$ and $X_i = \text{diag}\{P_{11i}, \Psi_i\}$ with a parameter matrix $\Psi_i$. If we choose $\Psi_i$ as a symmetric positive definite matrix, then $X_i$ is a symmetric positive definite matrix. Thus, $S_i = R_i^{-1/2} U_i X_i^{-1} U_i^T R_i^{-1/2}$ is a solution of (3.25b), and $P_i$ satisfies $P_i E_i^T = E_i P_i^T = E_i \mathcal{V}_i^{-T} X_i \mathcal{V}_i^{-1} E_i^T$.

Let $I < S_i < \sigma_1^{-1} I$, then it is easy to check that conditions (3.23c) and (3.23d) can guarantee that conditions (3.25c) hold. This completes the proof of the theorem. □

**Corollary 3.4.** *There exist a state feedback controller $u(t) = K_i \tilde{x}(t)$ with $K_i = Y_i P_i^{-T}$ and a state observer $H_i = -P_i C_{yi}^T$ such that the error dynamic SMJS (2.5a) is singular stochastic finite-time stabilization via observer-based state feedback with respect to $(c_1, c_2, T, \overline{R}_i, d)$ if there exist positive scalars $\alpha, c_2, \sigma_1, \sigma_2$, and sets of mode-dependent symmetric positive-definite matrices $\{X_i,\ i \in \mathbb{M}\}$, $\{\Phi_i,\ i \in \mathbb{M}\}$, $\{\Theta_i,\ i \in \mathbb{M}\}$, sets of mode-dependent matrices $\{Y_i,\ i \in \mathbb{M}\}$, $\{Z_i,\ i \in \mathbb{M}\}$, and for all $r_t = i \in \mathbb{M}$, such that the following inequalities hold:*

$$0 \le P_i E_i^T = E_i P_i^T = E_i \mathcal{V}_i^{-T} X_i \mathcal{V}_i^{-1} E_i^T \le \Phi_i, \tag{3.37a}$$

$$\begin{bmatrix} L_1(P_i, Y_i) & -B_i Y_i & G_i & 0 & \Upsilon_i & 0 \\ * & L_2(P_i, Y_i) & G_i & P_i C_{yi}^T & 0 & \Upsilon_i \\ * & * & -\Theta_i & 0 & 0 & 0 \\ * & * & * & -\dfrac{I}{2} & 0 & 0 \\ * & * & * & * & -W_i & 0 \\ * & * & * & * & * & -W_i \end{bmatrix} < 0, \tag{3.37b}$$

$$\sigma_1 R_i^{-1} < U_i X_i U_i^T < R_i^{-1}, \quad 0 < \Theta_i < \sigma_2 I, \tag{3.37c}$$

$$\begin{bmatrix} -e^{-\alpha T} c_2^2 + \sigma_2 d^2 & c_1 \\ * & -\sigma_1 \end{bmatrix} < 0, \tag{3.37d}$$

*where $L_1(P_i, Y_i), L_2(P_i, Y_i), \Upsilon_i, W_i$, and $P_i$ are the same as Theorem 3.3.*

*Remark 3.5.* The feasibility of conditions stated in Theorem 3.3 and Corollary 3.4 can be turned into the following LMIs-based feasibility problem with a parameter $\alpha$, respectively:

$$\begin{aligned}
&\min \quad (\beta+\rho)\\
&X_i, Y_i, Z_i, \Phi_i, \sigma_1\\
&\text{s.t. } (3.23\text{a})–(3.23\text{d}) \quad \text{with } \beta = c_2^2, \qquad \rho = \gamma^2,
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
&\min \quad \beta\\
&X_i, Y_i, Z_i, \Phi_i, \Theta_i, \sigma_1, \sigma_2\\
&\text{s.t. } (3.37\text{a})–(3.37\text{d}) \quad \text{with } \beta = c_2^2.
\end{aligned} \tag{3.39}$$

Furthermore, we can also find the parameter $\alpha$ by an unconstrained nonlinear optimization approach, in which a locally convergent solution can be obtained by using the program *fminsearch* in the optimization toolbox of Matlab.

*Remark 3.6.* If we can find feasible solution with parameter $\alpha = 0$, by the above discussion, we can obtain the designed state observer, and state feedback controller cannot only ensure SSFTB and stochastic stabilization of the error dynamic SMJS (2.5a) and (2.5b) but also the effect of the disturbance input of the disturbance input on the controlled output satisfying $\|T_{wz}\| < \gamma$ for the error dynamic SMJS.

## 4. Numerical Examples

In this section, we present numerical examples to illustrate the proposed methods.

*Example 4.1.* To show the results of singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback of the error dynamic SMJS (2.5a) and (2.5b), consider a two-mode SMJS (2.1a)–(2.1c) with parameters as follows:

(i) mode no. 1:

$$\begin{gathered}
E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 1.5 \\ 2.2 & -3 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -1 & 0.2 \\ 0.5 & -0.1 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\\
C_1 = \begin{bmatrix} 1 & -0.3 \end{bmatrix}, \qquad D_{11} = \begin{bmatrix} 0.5 & -0.6 \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} -0.3 \end{bmatrix}, \qquad C_{y1} = \begin{bmatrix} 0.6 & -0.6 \end{bmatrix},
\end{gathered} \tag{4.1}$$

mode no. 2:

$$\begin{gathered}
E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.5 & 1.2 \\ 1.6 & -1.5 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -1 & 2 \\ 0.5 & -1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix},\\
C_2 = \begin{bmatrix} 1 & -0.2 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} -0.8 & 0.6 \end{bmatrix}, \qquad D_{22} = \begin{bmatrix} -0.4 \end{bmatrix}, \qquad C_{y2} = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}.
\end{gathered} \tag{4.2}$$

In addition, the transition rate matrix is described by $\Gamma = \left[\begin{smallmatrix} -0.5 & 0.5 \\ 0.6 & -0.6 \end{smallmatrix}\right]$.

Then, we choose $R_1 = R_2 = I_2$, $T = 2$, $d = 2$, $c_1 = 1$, and $\alpha = 2$, and Theorem 3.3 yields $\gamma = 3.0073$, $c_2 = 10.7667$, and

$$X_1 = \begin{bmatrix} 0.9982 & 0.0000 \\ 0.0000 & 0.8569 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 0.2129 & -0.1927 \\ 0.6038 & -0.2995 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 0.9166 \\ 1.0377 \end{bmatrix},$$
$$X_2 = \begin{bmatrix} 0.6857 & -0.0000 \\ -0.0000 & 0.8569 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.5051 & 0.0344 \\ 0.1119 & 0.2116 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} -0.1895 \\ 0.8802 \end{bmatrix}, \tag{4.3}$$
$$\Phi_1 = \begin{bmatrix} 1.0041 & -0.0333 \\ -0.0333 & 0.2897 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0.8400 & -0.2934 \\ -0.2934 & 0.6147 \end{bmatrix}, \quad \sigma_1 = 0.6852.$$

This also gives the following gains:

$$K_1 = \begin{bmatrix} 0.3838 & -0.1857 \\ 0.8700 & -0.2886 \end{bmatrix}, \quad H_1 = \begin{bmatrix} -0.0490 \\ 0.6226 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 0.7366 & 0.0401 \\ 0.1632 & 0.2470 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.5663 \\ -0.4151 \end{bmatrix}. \tag{4.4}$$

Moreover, by Theorem 3.3, the optimal bound with minimum value of $c_2^2 + \gamma^2$ relies on parameter $\alpha$. Let $R_1 = R_2 = I_2, T = 2, d = 2$, and $c_1 = 1$, then we can find feasible solution when $1.27 \le \alpha \le 10.11$. Figure 1 shows the optimal value with a different value of $\alpha$. Furthermore, by using the program *fminsearch* in the optimization toolbox of Matlab starting at $\alpha = 2$, the locally convergent solution can be derived as

$$K_1 = \begin{bmatrix} 0.3251 & -0.1199 \\ 0.5985 & -0.1072 \end{bmatrix}, \quad H_1 = \begin{bmatrix} -0.1564 \\ 0.4356 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 1.1810 & -0.1137 \\ 0.6294 & -0.0173 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.4956 \\ -0.2904 \end{bmatrix}, \tag{4.5}$$

with $\alpha = 1.7043$ and the optimal values $\gamma = 2.5572$ and $c_2 = 9.7515$.

*Remark 4.2.* To show the results of singular stochastic finite-time stabilization via observer-based state feedback of the error dynamic SMJS (2.5a), consider a two-mode SMJS (2.1a) and (2.1c) with parameters that the matrical variables and the transition rate matrix are the same as the above example. Let $R(1) = R(2) = I_2$, $T = 2$, $d = 2$, and $c_1 = 1$, by Corollary 3.4, the optimal bound with minimum value of $c_2^2$ relies on the parameter $\alpha$. We can find feasible solution when $1.26 \le \alpha \le 10.35$. Figure 2 shows the optimal value with different value of $\alpha$. Then using the program *fminsearch* in the optimization toolbox of Matlab starting at $\alpha = 2$, we can obtain the locally convergent solution $c_2 = 9.7037$ with $\alpha = 1.7001$.

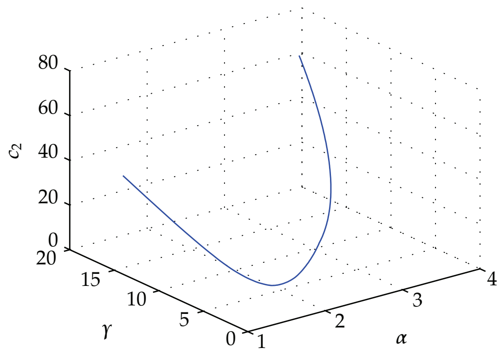


**Figure 1:** The local optimal bound of $\gamma$ and $c_2$.

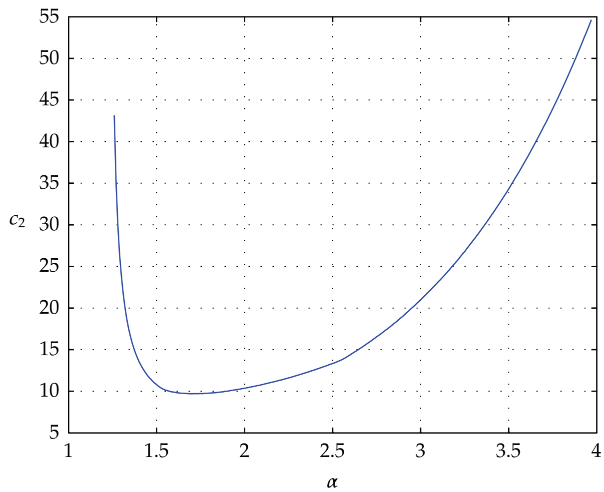


**Figure 2:** The local optimal bound of $c_2$.

*Example 4.3.* To show SSFTB and stochastic stabilization of the error dynamic SMJS (2.5a) and (2.5b) and the effect of the disturbance input of the disturbance input on the controlled output satisfying $\|T_{wz}\| < \gamma$ for the error dynamic SMJS, let

$$A_1 = \begin{bmatrix} -4 & 1.45 \\ -2.5 & -3.2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2.2 & -1.45 \\ -1 & -1.5 \end{bmatrix}, \tag{4.6}$$

and other matrical variables and the transition rate matrix are the same as Example 4.1.

Then, let $R_1 = R_2 = I_2$, $d = 2$, and $c_1 = 1$. By Theorem 3.3, we can find the feasible solution when $\alpha = 0$. Noting that when $\alpha = 0$, Theorem 3.3 yields the optimal value $\gamma = 0.4001$ and $c_2 = 1.2817$. Thus, the above error dynamic SMJS is stochastically stabilizable, and the effect of the disturbance input of the disturbance input on the controlled output satisfies $\|T_{wz}\| < 0.4001$.

## 5. Conclusions

This paper investigates the problem of finite-time $H_\infty$ control via observer-based state feedback for a family of SMJSs with time-varying norm-bounded disturbance. An observer-based state feedback controller is designed which ensures singular stochastic finite-time $H_\infty$ stabilization via observer-based state feedback of the resulting closed-loop error dynamic SMJS. Sufficient criterions are presented for the solvability of the problem, which can be reduced to a feasibility problem in the form of linear matrix inequalities with a fixed parameter. In addition, we also give the problem of finite-time stabilization via observer-based state feedback of a class of SMJSs and present sufficient conditions of singular stochastic finite-time stabilization via observer-based state feedback for the class of SMJSs. Numerical examples are also given to show the validity of the proposed methodology.

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