Analyzing Controllability of Neural Networks

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Abstract: In recent years, due to the relation between cognitive control and mathematical concept of control dynamical systems, there has been growing interest in the descriptive analysis of complex networks with linear dynamics, permeating many aspects from everyday life, obtaining considerable advances in the description of their structural and dynamical properties. Nevertheless, much less effort has been devoted to studying the controllability of the dynamics taking place on them. Concretely, for complex systems is of interest to study the exact controllability, this measure is defined as the minimum set of controls that are needed to steer the whole system toward any desired state. In this paper, a revision of controllability concepts is presented and provides conditions for exact controllability for the multiagent systems.

Key–Words: Neural network, controllability, exact controllability, eigenvalues, eigenvectors, linear systems

1 Introduction

The brain structure is a deep recurrent complex neuronal network.

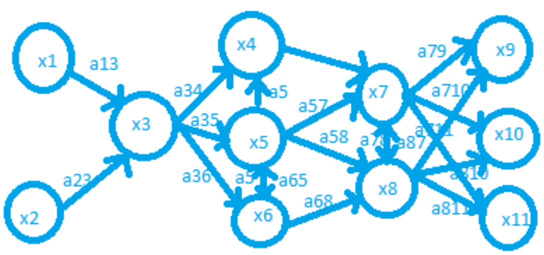

Figure 1: Deep Recurrent Neural Network.

The term neural network refers to a particular model for understanding brain function, in which neurons are the basic computational units, and computation is interpreted in terms of network interactions.

As Kriegeskorte argues in [13], neural network models mark the beginning of a new era of computational neuroscience, in which participants share in real-world tasks that require extensive knowledge and complex computations.

Neural systems allow humans to perform the multiple complex cognitive functions necessary for daily life and these can alter their dynamics to meet the demands of tasks. These capabilities are known as control. The concept of cognitive control is analogous to the mathematical concept of control of dynamic systems used in engineering, where the state of a complex system can be modulated by the energy input. Neural network systems, such as the brain, are very attractive systems for the study of control due to their structure that predisposes certain components to specific control actions. The neuronal sets of the brain can be interpreted as the nodes of a complex system and the anatomical cables of interconnection as the axes, this system exerts an impact on the neural function. It is therefore plausible that the brain regulates cognitive function through a process of transient network level control similar to technological systems modelled mathematically as complex systems. Although the complete understanding of the relationship between mathematical control measures and the notions of cognitive control of neuroscience are difficult to achieve, small advances in the study can favour the study and action against learning difficulties such as dyscalculia or other disturbances like the phenomena of forgetting, ([9, 8]).

In these recent years, the study of the control of complex networks with linear dynamics has gained importance in both science and engineering. Controllability of a dynamical system has being largely studied by several authors and under many different points of view, (see [1], [2], [3], [11], [14], [7], [18] and [5], for example). Between different aspects in which we can study the controllability we have the notion of structural controllability that has been proposed by Lin [16] as a framework for studying the controllability properties of directed complex networks where the dynamics of the system is governed by a linear system: $\dot{x}(t) = Ax(t) + Bu(t)$ usually the matrix A of the system is linked to the adjacency matrix of the network, $x(t)$ is a time dependent vector of the state variables of the nodes, $u(t)$ is the vector of input signals, and B which defines how the input signals are connected to the nodes of the network and it is the called input matrix. Structurally controllable means that there exists a matrix $\bar{A}$ in which is not allowed to contain a non-zero entry when the corresponding entry in A is zero such that the network can be driven from any initial state to any final state by appropriately choosing the input signals $u(t)$. Recent studies over the structural controllability can be found on [17].

Another important aspect of control is the notion of output controllability that describes the ability of an external data to move the output from any initial condition to any final in a finite time. Some results about can be found in [7].

In this paper we revise system theoretic properties of neural networks, and we analyze the exact controllability concept for multiagent neural network following definition given in [21, 6]. This concept is based on the maximum multiplicity to identify the minimum set of driver nodes required to achieve full control of networks with arbitrary structures and link-weight distributions.

2 Preliminaries

2.1 Algebraic Graph theory

We consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order N with the set of vertices $\mathcal{V} = \{1, \ldots, N\}$ and the set of edges $\mathcal{E} = \{(i,j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

Given an edge (i,j) i is called the parent node and j is called the child node and j is in the neighbor of i, concretely we define the neighbor of i and we denote it by $\mathcal{N}_i$ to the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}$.

The graph is called undirected if verifies that $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$. The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph we consider a matrix $G = (g_{ij})$ called (unweighted) adjacency matrix defined as follows $a_{ii} = 0$, $g_{ij} = 1$ if $(i,j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise.

(In a more general case we can consider a weighted adjacency matrix is $A = (a_{ij})$ with $a_{ii} = 0$, $a_{ij} > 0$ if $(i,j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise).

The adjacency matrix corresponding to the graph gven in figure 2 is as follows

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{13} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{34} & 0 & a_{54} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{35} & 0 & 0 & a_{65} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{36} & 0 & a_{56} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{47} & a_{57} & 0 & 0 & a_{87} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{58} & a_{68} & a_{78} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{79} & a_{89} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{710} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{711} & a_{811} & 0 & 0 & \end{pmatrix}$$

The Laplacian matrix of the graph is

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

For more details about graph theory see [20].

A possible manner to study the control of the neural networks can be associating a dynamical system to graph:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where

$x = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}^t$ stands for the states nodes, $A = (a_{ij})$ is the adjacency matrix to the graph where a_{ij} represents the weight of a directed link from node i to j, u is the vector of m controllers: $u = \begin{pmatrix} u_1, \ldots, u_m \end{pmatrix}^t$ and B is the $n \times m$ control matrix.

2.2 Controllability

Controllability is one of the most important properties of dynamical systems. A system is controllable if we can drive the state variables from any initial to any desired values within a finite period of time with properly selected inputs, more concretely:

Definition 1 *The system 1 is called controllable if, for any $t_1 > 0$, $x(0) \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$, there exists a control input $u(t)$ sufficiently smooth such that $x(t_1) = w$.*

The controllability character can be computed by means of the well-known Kalman's rank condition

The system 1 is controllable if and only if:

Proposition 2 ([12])

$$\text{rank} \begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix} = n \tag{2}$$

or by means the Hautus Test for controllability of linear dynamical systems.

Proposition 3 ([10])

$$\text{rank} \begin{pmatrix} sI - A & B \end{pmatrix} = n, \ \forall s \in \mathbb{C} \ \Big\}. \tag{3}$$

To ensure controllability with a minimal number of inputs the brute force approach should generate $2^N - 1$ configurations of the B matrix [15]. To solve this challenging task, Y. Y. Liu et al. proposed the maximum matching algorithm based on the network representation of the A matrix to select the control1 and observer2 nodes that ensure controllable and observable systems.

2.3 Exact controllability

Given a state space representation of a linear dynamical system as in equation 1
that for simplicity, from now on we will write as the pair of matrices (A, B). It is well known that There are many possible control matrices B in the system 1 that satisfy the controllability condition.

The goal is to find the set of all possible matrices B, havin g the minimum number of columns corresponding to the minimum number $n_D(A)$ of independent controllers required to control the whole network.

Definition 4 *Let A be a matrix. The exact controllability $n_D(A)$ is the minimum of the rank of all possible matrices B making the system 1 controllable.*

$$n_D(A) = \min\{\text{rank}\, B, \forall B \in M_{n\times i}\; 1 \leq i \leq n\; (A, B) \text{ controllable}\}.$$

If confusion is not possible we will write simply n_D.

It is straightforward that n_D is invariant under similarity, that is to say: for any invertible matrix S we have $n_D(A) = n_D(S^{-1}AS)$. As a consequence, if necessary we can consider A in its canonical Jordan form.

Example 5 *1) If $A = 0$, $n_D = n$*

2) If $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \neq \lambda_j$ for all $i \neq j$, then $n_D = 1$, (it suffices to take $B = (1 \ldots 1)^t$).

3) Not every matrix B having n_D columns is valid to make the system controllable. For example if $A = \text{diag}(1, 2, 3)$ and $B = (1, 0, 0)^t$, the system (A, B) is not controllable, (rank $\begin{pmatrix} B & AB & A^2B \end{pmatrix} = 1 < 3$, or equivalently rank $\begin{pmatrix} A - \lambda I & B \end{pmatrix} = 2$ for $\lambda = 2, 3$. Observe that, in this case, the matrix B corresponds to an eigenvector of the operator A.

Proposition 6 ([21])

$$n_D = \max_i \{\mu(\lambda_i)\}$$

where $\mu(\lambda_i) = \dim \text{Ker}\,(A - \lambda_i I)$ is the geometric multiplicity of the eigenvalue λ_i.

2.4 Structural controllability

We recall now the concept of structural controllability[16]. Structural controllability is a generalization of the controllability concept. It is of great interest because many times we know the entries of the matrices only approximately. Roughly speaking, a linear system is said to be structurally controllable if one can find a set of values for the parameters in the matrices such that the corresponding system is controllable. More concretely, the definition is as follows.

Definition 7 *The linear system 1 is structurally controllable if and only if $\forall \varepsilon > 0$, there exists a completely controllable linear system $\dot{x}(t) = \overline{A}x(t) + \overline{B}u(t)$, of the same structure as $\dot{x}(t) = Ax(t)+Bu(t)$ such that $\|\overline{A} - A\| < \varepsilon$ and $\|\overline{B} - B\| < \varepsilon$.*

Recall that, a linear dynamic system $\dot{x}(t) = Ax(t)+Bu(t)$ has the same structure as another linear dynamical system $\dot{x}(t) = \overline{A}x(t)+\overline{B}u(t)$, of the same dimensions, if for every fixed zero entry of the pair of matrices (A, B), the corresponding entry of the pair of matrices $(\overline{A}, \overline{B})$ is fixed zero and vice versa.

3 Controllability of multiagent neural networks

The complexity of the brain drives that in order to study control problems, the global model is divided into several local submodels, each with its complex and interrelated network structure. Structuring, in this way, the brain as a neuronal multi-network with a common goal.

Let us consider a group of k identical agents. The dynamic of each agent is given by the following linear dynamical systems

$$\begin{array}{rl} \dot{x}^1(t) & = A_1x^1(t) + B_1u^1(t) \\ \vdots & \\ \dot{x}^k(t) & = A_kx^k(t) + B_ku^k(t) \end{array} \tag{4}$$

$x^i(t) \in \mathbb{R}^n$, $u^i(t) \in \mathbb{R}^m$, $1 \leq i \leq k$.

We consider the undirected graph $\mathcal{G}$ with

i) Vertex set: $V = \{1, \ldots, k\}$

ii) Edge set: $\mathcal{E} = \{(i, j) \mid i, j \in V\} \subset V \times V$

defining the communication topology among agents.

Writing

$$\mathcal{X}(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^k(t) \end{pmatrix}, \quad \dot{\mathcal{X}}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \vdots \\ \dot{x}^k(t) \end{pmatrix},$$

$$\mathcal{U}(t) = \begin{pmatrix} u^1(t) \\ \vdots \\ u^k(t) \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix},$$

Following this notation we can describe the multisystem as a system:

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}\mathcal{U}(t).$$

and we are interested in take the output of the system to a reference value and keep it there, we can ensure that if the system is controllable. Clearly, this system is controllable if and only if each subsystem is controllable, and, in this case, there exist a feedback in which we obtain the desired solution.

But, in our case, not all possible feedbacks are available due the restriction of interconnection of agents. So we are interested in a feedback K in such a way that the with control

$$u^i(t) = K \sum_{j \in \mathcal{N}_i} (x^i(t) - x^j(t)), \ 1 \leq i \leq k$$

the system has prescribed eigenvalues in order to take a desired output of the system.

In our particular setup, we are interested in a solution such that

$$\lim_{t\to\infty} \|x^i - x^j\| = 0, \ 1 \leq i, j \leq k.$$

That is to say, founding solutions of each subsystem arriving all, to the same point.

Proposition 8 *Taking the control* $u^i(t) = K\sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t)), \ 1 \leq i \leq k$ *the closed-loop system can be described as*

$$\dot{\mathcal{X}}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K}(\mathcal{L} \otimes I_n))\mathcal{X}(t).$$

where $\mathcal{K} = \begin{pmatrix} K & & \\ & \ddots & \\ & & K \end{pmatrix}$.

Computing the matrix $\mathcal{A}+\mathcal{B}\mathcal{K}(\mathcal{L}\otimes I_n)$ we obtain

$$\begin{pmatrix} A_1 + l_{11}B_1K & l_{12}B_1K & \dots & l_{1k}B_1K \\ l_{21}B_2K & A_2 + l_{22}B_2K & \dots & l_{2k}B_2K \\ \vdots & \vdots & \ddots & \vdots \\ l_{k1}B_kK & l_{k2}B_kK & \dots & l_{kk}B_kK \end{pmatrix}$$

Example 9 *We consider 3 agents with the following dynamics of each agent*

$$\begin{array}{ll} \dot{x}^1 & = A_1x^1 + Bu^1 \\ \dot{x}^2 & = A_2x^2 + Bu^2 \\ \dot{x}^3 & = A_3x^3 + Bu^3 \end{array} \qquad (5)$$

with $A_1 = A_2 = A_3 = \begin{pmatrix} 0 & 1 \\ -0.1 & -0.5 \end{pmatrix}$, *0 , and* $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The communication topology is defined by the graph $(\mathcal{V}, \mathcal{E})$*:*

$V = \{1, 2, 3\}$

$\mathcal{E} = \{(i, j) \mid i, j \in V\} = \{(1, 2), (1, 3)\} \subset V \times V$

The neighbors of the parent nodes are $\mathcal{N}_1 = \{2, 3\}$, $\mathcal{N}_2 = \{1\}$, $\mathcal{N}_3 = \{1\}$.

The Laplacian matrix of the graph is

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Taking $K = \begin{pmatrix} k & \ell \end{pmatrix}$

The matrix of the system is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2k - \frac{1}{10} & 2\ell - \frac{1}{2} & -k & -\ell & -k & -\ell \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k & -\ell & k - \frac{1}{10} & \ell - \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k & -\ell & 0 & 0 & k - \frac{1}{10} & \ell - \frac{1}{2} \end{pmatrix}$$

Taking $K = \begin{pmatrix} -0.5 & -0.2 \end{pmatrix}$ *the eigenvalues are* $-0.5500 + 1.1391i$, $-0.5500 - 1.1391i$, $-0.2500 + 0.1936i$, $-0.2500 - 0.1936i$, $-0.3500 + 0.6910i$, $-0.3500 - 0.6910i$, *then the system has a stable solution and the the three trajectories arrive at a common point as we can see in the graphic.*

As we can see in the example, all agents on the multi-agent system, have an identical linear dynamic mode. In this particular case proposition 8 can be rewritten in the following manner (see [4, 19])

Proposition 10 *Taking the control* $u^i(t) = K\sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t)), \ 1 \leq i \leq k$ *the closed-loop system for a multiagents having identical linear dynamical mode, can be described as*

$$\dot{\mathcal{X}} = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L} \otimes I_n))\mathcal{X}.$$

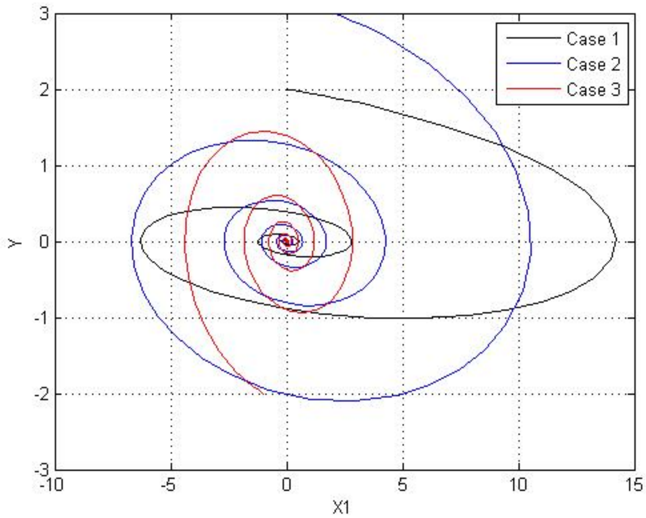

Figure 2: Trajectories of each system.

3.1 Exact controllability

Let us consider a group of k agents. The dynamic of each agent is given by the following homogeneous linear dynamical systems

$$\begin{aligned} \dot{x}^1(t) &= A_1 x^1(t) \\ &\vdots \\ \dot{x}^k(t) &= A_k x^k(t) \end{aligned} \tag{6}$$

$x^i(t) \in \mathbb{R}^n, 1 \leq i \leq k$.

We ask for minimum number of columns that a matrix B must have for the system

$$\begin{aligned} \dot{x}^1(t) &= A_1 x^1(t) + Bu^1(t) \\ &\vdots \\ \dot{x}^k(t) &= A_k x^k(t) + Bu^k(t) \end{aligned} \tag{7}$$

to be controllable (the matrix B the same for aech agent).

Let $\lambda_{i_1}, \dots, \lambda_{i_{r_i}}$ the eigenvalues of the matrix A_i with geometrical multiplicities $\mu(\lambda_{i_1}), \dots, \mu(\lambda_{i_{r_i}})$. Then

Proposition 11

$n_D(\mathcal{A}) =$ *max*$(\mu(\lambda_{1_1}), \dots, \mu(\lambda_{1_{r_1}}), \dots, \mu(\lambda_{k_1}), \dots, \mu(\lambda_{k_{r_k}})$

Corollary 12

$$n_D(\mathcal{A}) = n_D(A_i)$$

for some $i = 1, \dots, k$.

Remark 13
- *Not all matrices B having $n_D(A_i)$ columns and making the system $\dot{x}^i = A_i x^i + Bu^i$ controllable are available for the multiagent system.*
- *Taking all possible matrices B making the system controllable we can consider the existence of the matrix K.*

Example 14 *Suppose* $A_1 = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.

The matrix $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *make the system* $\dot{x}(t) = A_1 x(t) + Bu(t)$ *controllable but not* $\dot{x}(t) = A_2 x(t) + Bu(t)$.

The matrix $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *make the system* $\dot{x}(t) = A_2 x(t) + Bu(t)$ *controllable but not* $\dot{x}(t) = A_1 x(t) + Bu(t)$.

Nevertheless, the matrix $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *make both systems controllable.*

4 Conclusions

The exact controllability for multi-agent systems where all agents have an identical linear dynamic mode are analyzed with an important objective: To help people with dyscalculia.

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