Numerical analysis of a thermoelastic problem with dual-phase-lag heat conduction

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Abstract

In this paper we study, from the numerical point of view, a thermoelastic problem with dual-phase-lag heat conduction. The variational formulation is written as a coupled system of hyperbolic linear variational equations. An existence and uniqueness result is recalled. Then, fully discrete approximations are introduced by using the finite element method and the implicit Euler scheme. A discrete stability result is proved and a priori error estimates are obtained, from which the linear convergence of the algorithm is deduced under suitable additional regularity conditions. Finally, some two-dimensional numerical simulations are presented to demonstrate the accuracy of the approximation and the behaviour of the solution.

Keywords. Thermoelasticity, Dual-phase-lag, finite elements, discrete stability, a priori estimates, numerical results.

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1 Introduction

The heat conduction theory based on the Fourier law is compatible with the fact that the thermal perturbations at some point will be felt instantly anywhere. This is a drawback of the model and for this reason several authors have tried to overcome this difficulty proposing alternative theories. Most celebrated one is the Cattaneo-Maxwell law [1]. There exist two thermoelastic theories based in this law. Green and Lindsay [2] proposed the first one and the second one is due to Lord and Shulman [3]. Both cases introduce thermoelastic theories described by means of hyperbolic equations. In the decade of the 90s Green and Naghdi [4–6] suggested three alternative theories which were based on an entropy balance which represents an alternative approach to the usual one based on the entropy inequality. The proposition was based on the axioms of thermomechanics and it was developed from a rational point of view. The main difference between these three theories came from the choice of the independent variables.

In 1995, Tzou [7,8] proposed a modification of the Fourier law where the delay parameters were present. That is, the constitutive equation takes the form

$$\boldsymbol{q}(\boldsymbol{x}, t+\tau_q) = -\kappa \nabla \theta(\boldsymbol{x}, t+\tau_\theta),$$

where θ is the temperature, $\boldsymbol{q}$ is the heat flux vector and τ_θ and τ_q represent the phase-lag of the temperature and the heat-flux. One thinks that the time delay τ_θ is caused by microstructural interactions such as phonon scattering or phonon-electron interactions, meanwhile τ_q can be seen as the relaxation time due to the fast transient effects of thermal inertia. However, if we adjoin our equation with the heat equation

$$\theta_t + \operatorname{div} \boldsymbol{q} = 0$$

the problem becomes ill-posed in the sense of Hadamard (see [9] for details). At the same time, as it has been pointed out in [10], this model is not in agreement with the second law of thermodynamics. The solutions have a very explosive behaviour and we may conclude that the problem cannot be a good candidate to describe the heat conduction nor from the mathematical point of view neither the thermomechanical one. Nevertheless, many people have been attracted by the theories obtained when we substitute the proposed constitutive equation by the Taylor approximations with respect to the delay

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parameters. Many papers have been published dealing with mathematical and numerical issues as existence, uniqueness, energy decay, spatial behaviour, numerical resolution and so on (see, for instance, [10–24]).

In this paper, we are going to consider the thermoelastic theory associated to the constitutive equation

$$\boldsymbol{q} + \tau_q \boldsymbol{q}_t + \frac{\tau_q^2}{2}\boldsymbol{q}_{tt} = -\kappa \nabla\theta - \kappa\tau_\theta \nabla\theta_t.$$

It is worth noting that this model is in agreement with the second law of thermodynamics under suitable conditions for the delay parameters (see [10, 15]). As it has been pointed out in [24] the involvement of high-order terms in the delay parameters are the natural consequence of the handling of systems in which multiple energy carriers are involved (see also [25, p. 376]).

In this work we revisit the thermoelastic model based on the previous constitutive equation which can be obtained following the arguments of Chandrasekharaiah [26]. The system of equations was studied in [20], where the existence of a unique solution was proved as well as the spatial behaviour of the solutions. The exponential stability for the one-dimensional case was also obtained. We here continue the research of this problem, providing the numerical analysis of the variational problem, including a discrete stability property and a priori error estimates, and performing two-dimensional numerical simulations which demonstrate the accuracy of the approximation and the behaviour of the solution.

The paper is structured as follows. The mechanical and variational models are presented in Section 2 following [20], and an existence and uniqueness result is recalled. Then, in Section 3 a fully discrete approximation is introduced, based on the finite element method to approximate the spatial domain and the backward Euler scheme to discretize the time derivatives. A discrete stability property and a priori error estimates are proved, from which, under suitable additional regularity conditions, the linear convergence of the algorithm is deduced. Finally, some two-dimensional numerical simulations are presented in Section 4, and some conclusions are shown in Section 5.

2 The mechanical and variational problems: existence and uniqueness

In this section, we present a brief description of the model and we obtain its mechanical and variational formulations (details can be found in [20]). We also recall an existence and uniqueness result.

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the domain and denote by $[0, T]$, $T > 0$, the time interval of interest. The boundary of the body $\Gamma = \partial\Omega$ is assumed to be Lipschitz, with outward unit normal vector $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$. Moreover, let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x} = (x_j)_{j=1}^d$ and t, and a subscript after a comma under a variable represents its spatial derivative with respect to the prescribed variable, i.e. $f_{i,j} = \dfrac{\partial f_i}{\partial x_j}$. The time derivatives are represented as a point for the first order, two points for the second order and three points for the third order, over each variable. Finally, as usual the repeated index notation is used for the summation.

We denote by $\boldsymbol{u}$ and θ the "displacement" field and the temperature, respectively. We note that in [20] the displacements are represented as $\tilde{\boldsymbol{u}}$ because it is really a transformation of the actual displacement field. However, for the sake of simplicity in the writing we have decided to remove that notation in this work.

Assuming that the material is isotropic and homogeneous and following the work by Quintanilla and Racke [20], the model is written as follows, for $i, j = 1, \ldots, d$,

$$\begin{aligned} &\rho\ddot{u}_i - \mu u_{i,jj} - (\lambda + \mu)u_{j,ji} - m(\theta_{,i} + \tau_q\dot{\theta}_{,i} + \frac{\tau_q^2}{2}\ddot{\theta}_{,i}) = H_i, \\ &\frac{\tau_q^2}{2}\dddot{\theta} + \tau_q\ddot{\theta} + \dot{\theta} - \kappa(\theta_{,ii} + \tau_\theta\dot{\theta}_{,ii}) - m\theta^*\dot{u}_{i,i} = P. \end{aligned} \tag{1}$$

In the above equations, $\boldsymbol{H} = (H_i)_{i=1}^d$ are an external body forces and P represents a heat supply. Constants ρ and κ denote the mass density and the thermal diffusion coefficient, respectively, and λ and μ represent the Lamé's coefficients. Moreover, m is a thermal expansion coefficient, and θ^* denotes a reference temperature.

As boundary conditions, we assume that

$$u_i(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) = 0 \quad \text{for} \quad i = 1, \ldots, d \text{ and } (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{2}$$

We point out that other boundary conditions could be used but we restrict ourselves to this case for the sake of simplicity.

In order to complete the definition of the mechanical problem we impose the following initial conditions:

$$\begin{aligned} &u_i(\boldsymbol{x}, 0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_i^0(\boldsymbol{x}), \quad \theta(\boldsymbol{x}, 0) = \theta^0(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \Omega, \\ &\dot{\theta}(\boldsymbol{x}, 0) = e^0(\boldsymbol{x}), \quad \ddot{\theta}(\boldsymbol{x}, 0) = \xi^0(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \Omega, \end{aligned} \tag{3}$$

where $\boldsymbol{u}^0 = (u_i^0)_{i=1}^d$, $\boldsymbol{v}^0 = (v_i^0)_{i=1}^d$, θ^0, e^0 and ξ^0 are prescribed functions.

Therefore, the thermo-mechanical problem modelling the deformation of a thermoelastic body with dual-phase-lag heat conduction is the following (see [20] for details).

Problem P. *Find the displacement field* $\boldsymbol{u} = (u_i)_{i=1}^d : \overline{\Omega} \times [0,T] \to \mathbb{R}^d$ *and the temperature* $\theta : \overline{\Omega} \times [0,T] \to \mathbb{R}$ *such that equations (1), boundary conditions (2) and initial conditions (3) are fulfilled.*

Remark 1 *We note that the analysis presented in this work could be extended to another dual-phase-lag model (see [27]), where a term of the form* $\frac{\tau_\theta^2}{2}\ddot{\theta}_{,ii}$ *is added. In this case, the numerical analysis is even simpler and it can be done following the arguments presented in the next section. Hence, we skip the details.*

Now, in order to obtain the variational formulation of Problem P, let $Y = L^2(\Omega)$, $H = [L^2(\Omega)]^d$ and $Q = [L^2(\Omega)]^{d\times d}$ and denote by $(\cdot,\cdot)_Y$, $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$, $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational spaces V and E as follows,

$$V = \{\boldsymbol{z} \in [H^1(\Omega)]^d \,;\, \boldsymbol{z} = \boldsymbol{0} \quad \text{on} \quad \Gamma\},$$
$$E = \{r \in H^1(\Omega) \,;\, r = 0 \quad \text{on} \quad \Gamma\},$$

with respective scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_E$, and norms $\|\cdot\|_V$ and $\|\cdot\|_E$.

By using Green's formula and boundary conditions (2), we write the variational formulation of Problem P in terms of the velocity field $\boldsymbol{v} = \dot{\boldsymbol{u}}$ and the thermal acceleration $\xi = \ddot{\theta}$.

Problem VP. *Find the velocity field* $\boldsymbol{v} : [0,T] \to V$ *and the thermal acceleration* $\xi : [0,T] \to E$ *such that* $\boldsymbol{v}(0) = \boldsymbol{v}^0$, $\xi(0) = \xi^0$, *and, for a.e.* $t \in (0,T)$ *and for all* $\boldsymbol{w} \in V$, $r \in E$,

$$\begin{aligned}\rho(\dot{\boldsymbol{v}}(t),\boldsymbol{w})_H &+ (\lambda+\mu)(\operatorname{div}\boldsymbol{u}(t),\operatorname{div}\boldsymbol{w})_Y + \mu(\nabla\boldsymbol{u}(t),\nabla\boldsymbol{w})_Q \\ &-m\left(\frac{\tau_q^2}{2}\nabla\xi(t) + \tau_q\nabla e(t) + \nabla\theta(t),\boldsymbol{w}\right)_H = (\boldsymbol{H}(t),\boldsymbol{w})_H, \end{aligned} \tag{4}$$

$$\begin{aligned}\left(\frac{\tau_q^2}{2}\dot{\xi}(t) + \tau_q\xi(t) + e(t), r\right)_Y &+ \kappa(\tau_\theta\nabla e(t) + \nabla\theta(t),\nabla r)_H \\ &-m\theta^*(\operatorname{div}\boldsymbol{v}(t), r)_Y = (P(t), r)_Y, \end{aligned} \tag{5}$$

where the displacement, thermal velocity and temperature fields are then recovered from the relations

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\, ds + \boldsymbol{u}^0, \quad e(t) = \int_0^t \xi(s)\, ds + e^0, \quad \theta(t) = \int_0^t e(s)\, ds + \theta^0, \tag{6}$$

and we note that div represents the classical divergence operator.

Theorem 2 *Let the following conditions on the constitutive coefficients hold:*

$$\rho > 0, \quad \mu > 0, \quad \lambda > 0, \quad \kappa > 0, \quad \tau_q > 0, \quad \tau_\theta > 0.$$

If the initial conditions, the body forces and the heat supply satisfy:

$$\begin{aligned} &\boldsymbol{u}^0, \boldsymbol{v}^0 \in V, \quad \xi^0, e^0, \theta^0 \in E, \\ &\boldsymbol{H} \in C^1([0,T]; H), \quad P \in C^1([0,T]; Y), \end{aligned}$$

then there exists a unique solution to Problem VP with the following regularity:

$$\boldsymbol{u} \in C^1([0,T]; V) \cap C^2([0,T]; H), \quad \theta \in C^2([0,T]; E) \cap C^3([0,T]; Y).$$

Remark 3 *We note that we have imposed the simpler condition $\lambda > 0$ instead of a more complicated one (as it was done in [27]), for the sake of simplicity. We could adapt the analysis presented in the following section to such new condition.*

3 Fully discrete approximations: an a priori error analysis

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the domain $\overline{\Omega}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of [28]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$V^h = \{\boldsymbol{z}^h \in [C(\overline{\Omega})]^d \,;\, \boldsymbol{z}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \quad \boldsymbol{z}^h = \boldsymbol{0} \quad \text{on} \quad \Gamma\}, \tag{7}$$

$$E^h = \{r^h \in C(\overline{\Omega}) \,;\, r^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad r^h = 0 \quad \text{on} \quad \Gamma\}, \tag{8}$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr, i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, θ^{0h}, e^{0h} and ξ^{0h}, are given by

$$\begin{aligned} &\boldsymbol{u}^{0h} = \mathcal{P}_1^h \boldsymbol{u}^0, \quad \boldsymbol{v}^{0h} = \mathcal{P}_1^h \boldsymbol{v}^0, \quad \theta^{0h} = \mathcal{P}_2^h \theta^0, \quad e^{0h} = \mathcal{P}_2^h e^0, \\ &\xi^{0h} = \mathcal{P}_2^h \xi^0, \end{aligned} \tag{9}$$

where $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ are the classical finite element interpolation operators over V^h and E^h, respectively (see, e.g., [28]).

Secondly, we consider a partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. *Find the discrete velocity field $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete thermal acceleration $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset E^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}^{0h}$, $\xi_0^{hk} = \xi^{0h}$, and, for $n = 1, \ldots, N$ and for all $\boldsymbol{w}^h \in V^h$, $r^h \in E^h$,*

$$\rho(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + (\lambda + \mu)(\operatorname{div} \boldsymbol{u}_n^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y + \mu(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{w}^h)_Q \\ -m\left(\frac{\tau_q^2}{2}\nabla \xi_n^{hk} + \tau_q \nabla e_n^{hk} + \nabla \theta_n^{hk}, \boldsymbol{w}^h\right)_H = (\boldsymbol{H}_n, \boldsymbol{w}^h)_H, \tag{10}$$

$$\left(\frac{\tau_q^2}{2}\delta \xi_n^{hk} + \tau_q \xi_n^{hk} + e_n^{hk}, r^h\right)_Y + \kappa(\tau_\theta \nabla e_n^{hk} + \nabla \theta_n^{hk}, \nabla r^h)_H \\ -m\theta^*(\operatorname{div} \boldsymbol{v}_n^{hk}, r^h)_Y = (P_n, r^h)_Y, \tag{11}$$

where the discrete displacement, thermal velocity and temperature fields are then recovered from the relations

$$\boldsymbol{u}_n^{hk} = k\sum_{j=1}^n \boldsymbol{v}_j^{hk} + \boldsymbol{u}^{0h}, \quad e_n^{hk} = k\sum_{j=1}^n \xi_j^{hk} + e^{0h}, \quad \theta_n^{hk} = k\sum_{j=1}^n e_j^{hk} + \theta^{0h}. \tag{12}$$

We note that the existence of a unique discrete solution to Problem VP^{hk} is obtained in a straightforward way using the classical Lax-Milgram lemma.

Remark 4 *We note that we have chosen continuous and piecewise affine functions for the spatial discretization and the backward Euler scheme to discretize the time derivatives for the sake of simplicity in the calculations presented in this section, and to obtain the discrete stability property. Anyway, the a priori error estimates proved below could be easily extended to higher order schemes assuming additional regularity on the continuous solution.*

We have the following stability result.

Lemma 5 *Under the assumptions of Theorem 2, it follows that the sequences $\{\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \theta^{hk}, e^{hk}, \xi^{hk}\}$ generated by Problem VP^{hk} satisfy the stability esti-*

mate:

$$\|\boldsymbol{v}_n^{hk}\|_H^2 + \|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2 + \|\nabla\boldsymbol{u}_n^{hk}\|_Q^2 + \|\xi_n^{hk}\|_Y^2 + \|e_n^{hk}\|_E^2 + \|\theta_n^{hk}\|_E^2 \le C,$$

where C is a positive constant which is independent of the discretization parameters h and k.

PROOF. In order to simplify the writing, in this proof we remove the superscripts h and k in all the variables, and we assume that $\theta^* = 1$.

We point out that the estimates involving the dual-phase-lag terms are obtained using some of the arguments already employed in [29]. However, for the sake of the reading we detail it below.

Taking $r^h = \xi_n$ as a test function in discrete variational equation (11) we have

$$\frac{\tau_q^2}{2}(\delta\xi_n, \xi_n)_Y + \tau_q(\xi_n, \xi_n)_Y + (e_n, \xi_n)_Y + \kappa(\nabla\theta_n, \nabla\xi_n)_H + \kappa\tau_\theta(\nabla e_n, \nabla\xi_n)_H$$
$$-m(\mathrm{div}\,\boldsymbol{v}_n, \xi_n)_Y = (P_n, \xi_n)_Y.$$

Thus, keeping in mind that

$$(\delta\xi_n, \xi_n)_Y \ge \frac{1}{2k}\left\{\|\xi_n\|_Y^2 - \|\xi_{n-1}\|_Y^2\right\},$$
$$(e_n, \xi_n)_Y \ge \frac{1}{2k}\left\{\|e_n\|_Y^2 - \|e_{n-1}\|_Y^2\right\},$$
$$(\nabla e_n, \nabla\xi_n)_H \ge \frac{1}{2k}\left\{\|\nabla e_n\|_H^2 - \|\nabla e_{n-1}\|_H^2\right\},$$
$$(\nabla\theta_n, \nabla\xi_n)_H = \frac{1}{k}\left\{(\nabla\theta_n, \nabla e_n)_H - (\nabla\theta_{n-1}, \nabla e_{n-1})_H\right\} - (\nabla e_n, \nabla e_{n-1})_H,$$

using Cauchy-Schwarz inequality and the following Cauchy's inequality

$$ab \le \eta a^2 + \frac{1}{4\eta}b^2 \quad \forall a, b \in \mathbb{R},\ \eta > 0, \tag{13}$$

we find that

$$\frac{\tau_q^2}{4k}\left\{\|\xi_n\|_Y^2 - \|\xi_{n-1}\|_Y^2\right\} + \frac{1}{2k}\left\{\|e_n\|_Y^2 - \|e_{n-1}\|_Y^2\right\} + \frac{\kappa\tau_\theta}{2k}\left\{\|\nabla e_n\|_H^2 - \|\nabla e_{n-1}\|_H^2\right\}$$
$$+\frac{\kappa}{k}\left\{(\nabla\theta_n, \nabla e_n)_H - (\nabla\theta_{n-1}, \nabla e_{n-1})_H\right\} - m(\mathrm{div}\,\boldsymbol{v}_n, \xi_n)_Y$$
$$\le C\left(\|\nabla e_n\|_H^2 + \|\nabla e_{n-1}\|_H^2\right) + C\|\xi_n\|_Y.$$

Next, we obtain the estimates for the velocity field. Taking $\boldsymbol{w}^h = \boldsymbol{v}_n$ as a test function in discrete variational equation (10) it follows that

$$\rho(\delta \boldsymbol{v}_n, \boldsymbol{v}_n)_H + (\lambda+\mu)(\operatorname{div}\boldsymbol{u}_n, \operatorname{div}\boldsymbol{v}_n)_Y + \mu(\nabla\boldsymbol{u}_n, \nabla\boldsymbol{v}_n)_Q \\ -m\left(\frac{\tau_q^2}{2}\nabla\xi_n + \tau_q\nabla e_n + \nabla\theta_n, \boldsymbol{v}_n\right)_H = (\boldsymbol{H}_n, \boldsymbol{v}_n)_H,$$

and, taking into account that

$$\begin{aligned}
&(\delta\boldsymbol{v}_n, \boldsymbol{v}_n)_H \geq \frac{1}{2k}\left\{\|\boldsymbol{v}_n\|_H^2 - \|\boldsymbol{v}_{n-1}\|_H^2\right\},\\
&(\operatorname{div}\boldsymbol{u}_n, \operatorname{div}\boldsymbol{v}_n)_Y \geq \frac{1}{2k}\left\{\|\operatorname{div}\boldsymbol{u}_n\|_Y^2 - \|\operatorname{div}\boldsymbol{u}_{n-1}\|_Y^2\right\},\\
&(\nabla\boldsymbol{u}_n, \nabla\boldsymbol{v}_n)_Q \geq \frac{1}{2k}\left\{\|\nabla\boldsymbol{u}_n\|_Q^2 - \|\nabla\boldsymbol{u}_{n-1}\|_Q^2\right\},\\
&|(\nabla(e_n+\theta_n), \boldsymbol{v}_n)_H| \leq C\left(\|\nabla e_n\|_H^2 + \|\nabla\theta_n\|_H^2 + \|\boldsymbol{v}_n\|_H^2\right),\\
&-m(\nabla\xi_n, \boldsymbol{v}_n)_H = m(\xi_n, \operatorname{div}\boldsymbol{v}_n)_Y,
\end{aligned}$$

using again Cauchy-Schwarz and the previous Young inequalities we find that

$$\begin{aligned}
&\frac{\rho}{\tau_q^2 k}\left\{\|\boldsymbol{v}_n\|_H^2 - \|\boldsymbol{v}_{n-1}\|_H^2\right\} + \frac{\lambda+\mu}{\tau_q^2 k}\left\{\|\operatorname{div}\boldsymbol{u}_n\|_Y^2 - \|\operatorname{div}\boldsymbol{u}_{n-1}\|_Y^2\right\} + m(\operatorname{div}\boldsymbol{v}_n, \xi_n)_Y\\
&\qquad + \frac{\mu}{\tau_q^2 k}\left\{\|\nabla\boldsymbol{u}_n\|_Q^2 - \|\nabla\boldsymbol{u}_{n-1}\|_Q^2\right\}\\
&\quad \leq C\left(\|\nabla e_n\|_H^2 + \|\nabla\theta_n\|_H^2 + \|\boldsymbol{v}_n\|_H^2\right) + C\|\boldsymbol{v}_n\|_H.
\end{aligned}$$

Combining the previous estimates, we have

$$\begin{aligned}
&\frac{1}{k}\left\{\|\boldsymbol{v}_n\|_H^2 - \|\boldsymbol{v}_{n-1}\|_H^2\right\} + \frac{1}{k}\left\{\|\operatorname{div}\boldsymbol{u}_n\|_Y^2 - \|\operatorname{div}\boldsymbol{u}_{n-1}\|_Y^2\right\}\\
&\quad + \frac{1}{k}\left\{\|\nabla\boldsymbol{u}_n\|_Q^2 - \|\nabla\boldsymbol{u}_{n-1}\|_Q^2\right\} + \frac{1}{k}\left\{\|\xi_n\|_Y^2 - \|\xi_{n-1}\|_Y^2\right\} + \frac{1}{k}\left\{\|e_n\|_Y^2 - \|e_{n-1}\|_Y^2\right\}\\
&\quad + \frac{1}{k}\left\{(\nabla\theta_n, \nabla e_n)_H - (\nabla\theta_{n-1}, \nabla e_{n-1})_H\right\} + \frac{1}{k}\left\{\|\nabla e_n\|_H^2 - \|\nabla e_{n-1}\|_H^2\right\}\\
&\quad \leq C\left(\|\nabla e_n\|_H^2 + \|\nabla e_{n-1}\|_H^2 + \|\nabla\theta_n\|_H^2 + \|\boldsymbol{v}_n\|_H^2\right) + C\|\boldsymbol{v}_n\|_H + C\|\xi_n\|_Y.
\end{aligned}$$

Multiplying the previous estimates by k and summing up the resulting equation it follows that

$$\begin{aligned}
&\|\boldsymbol{v}_n\|_H^2 + \|\operatorname{div}\boldsymbol{u}_n\|_Y^2 + \|\nabla\boldsymbol{u}_n\|_Q^2 + \|\xi_n\|_Y^2 + \|e_n\|_Y^2 + \|\nabla e_n\|_H^2 + (\nabla\theta_n, \nabla e_n)_H\\
&\quad \leq Ck\sum_{j=1}^{n}\left(\|\nabla e_j\|_H^2 + \|\nabla\theta_j\|_H^2 + \|\boldsymbol{v}_j\|_H^2 + \|\xi_j\|_Y^2\right) + C\Big(1 + \|\boldsymbol{v}^0\|_H^2\\
&\qquad + \|\operatorname{div}\boldsymbol{u}^0\|_Y^2 + \|\nabla\boldsymbol{u}^0\|_Q^2 + \|\xi^0\|_Y^2 + \|e^0\|_Y^2 + \|\nabla e^0\|_H^2 + \|\nabla\theta^0\|_H^2\Big),
\end{aligned}$$

and so, taking into account that

$$
\begin{aligned}
&|(\nabla\theta_n, \nabla e_n)_H| \leq C\|\nabla\theta_n\|_H^2 + \epsilon\|\nabla e_n\|_H^2,\\
&\|\theta_n\|_Y^2 \leq Ck\sum_{j=1}^n \|e_j\|_Y^2 + C\|\theta^0\|_Y^2,\\
&\|\nabla\theta_n\|_H^2 \leq Ck\sum_{j=1}^n \|\nabla e_j\|_H^2 + C\|\nabla\theta^0\|_H^2,
\end{aligned}
$$

where $\epsilon > 0$ is assumed small enough, we find that

$$
\begin{aligned}
&\|\boldsymbol{v}_n\|_H^2 + \|\mathrm{div}\,\boldsymbol{u}_n\|_Y^2 + \|\nabla\boldsymbol{u}_n\|_Q^2 + \|\xi_n\|_Y^2 + \|e_n\|_E^2 + \|\theta_n\|_E^2\\
&\quad\leq Ck\sum_{j=1}^n \Big(\|e_j\|_E^2 + \|\theta_j\|_E^2 + \|\boldsymbol{v}_j\|_H^2 + \|\xi_j\|_Y^2\Big) + C\Big(1 + \|\boldsymbol{v}^0\|_H^2 + \|\mathrm{div}\,\boldsymbol{u}^0\|_Y^2\\
&\qquad\quad + \|\nabla\boldsymbol{u}^0\|_Q^2 + \|\xi^0\|_Y^2 + \|e^0\|_Y^2 + \|\nabla e^0\|_H^2 + \|\nabla\theta^0\|_H^2 + \|\theta^0\|_Y^2\Big).
\end{aligned}
$$

Finally, the desired stability estimates are a straightforward consequence of the application of a discrete version of Gronwall's inequality (see, e.g., [30]), the properties of the interpolation operators P_1^h and P_2^h (see [28]) and the regularities on the initial conditions.

Now, we will obtain some a priori error estimates on the numerical errors $\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}$ and $\xi_n - \xi_n^{hk}$. We have the following.

Theorem 6 *Under the assumptions of Theorem 2, if we denote by* $(\boldsymbol{u}, \boldsymbol{v}, \theta, e, \xi)$ *the solution to Problem* VP *and by* $(\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \theta^{hk}, e^{hk}, \xi^{hk})$ *the solution to Problem* VP^{hk}*, then we have the following a priori error estimates, for all* $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$ *and* $r^h = \{r_j^h\}_{j=0}^N \subset E^h$*,*

$$
\begin{aligned}
&\max_{0\leq n\leq N}\Big\{\|\xi_n - \xi_n^{hk}\|_Y^2 + \|e_n - e_n^{hk}\|_Y^2 + \|\nabla(e_n - e_n^{hk})\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\theta_n - \theta_n^{hk}\|_Y^2\\
&\qquad + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2\Big\}\\
&\quad\leq Ck\sum_{j=1}^N \Big(\|\dot\xi_j - \delta\xi_j\|_Y^2 + \|\xi_j - r_j^h\|_E^2 + J_j^2 + I_j^2 + \|\nabla(\dot e_j - \delta e_j)\|_H^2\\
&\qquad + \|\nabla(\dot\theta_j - \delta\theta_j)\|_H^2 + \|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\dot{\boldsymbol{u}}_j - \delta\boldsymbol{u}_j\|_V^2\Big)\\
&\qquad + \frac{C}{k}\sum_{j=1}^{N-1}\Big\{\|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2 + \|\xi_j - r_j^h - (\xi_{j+1} - r_{j+1}^h)\|_Y^2\Big\}\\
&\qquad + C\Big(\|\xi^0 - \xi^{0h}\|_Y^2 + \|e^0 - e^{0h}\|_E^2 + \|\boldsymbol{v}^0 - \boldsymbol{v}^{0h}\|_H^2 + \|\theta^0 - \theta^{0h}\|_E^2\\
&\qquad + \|\mathrm{div}\,(\boldsymbol{u}^0 - \boldsymbol{u}^{0h})\|_Y^2 + \|\nabla(\boldsymbol{u}^0 - \boldsymbol{u}^{0h})\|_Q^2\Big), \qquad (14)
\end{aligned}
$$

where $C>0$ is a positive constant which is be independent of the discretization parameters h and k, but depending on the continuous solution, and $\delta\xi_j = (\xi_j-\xi_{j-1})/k$, $\delta e_j=(e_j-e_{j-1})/k$, $\delta \boldsymbol{v}_j=(\boldsymbol{v}_j-\boldsymbol{v}_{j-1})/k$, $\delta\boldsymbol{u}_j=(\boldsymbol{u}_j-\boldsymbol{u}_{j-1})/k$ and $\delta\theta_j=(\theta_j-\theta_{j-1})/k$, and I_j and J_j are the integration errors given by

$$I_j=\left\|\int_0^{t_j} e(s)\,ds - k\sum_{l=1}^{l} e_l\right\|_Y,\quad J_j=\left\|\int_0^{t_j}\nabla e(s)\,ds-k\sum_{l=1}^{j}\nabla e_l\right\|_H.$$

PROOF. First, we obtain some estimates for the velocity field. Then, we subtract variational equation (4) at time $t=t_n$ for a test function $\boldsymbol{w}=\boldsymbol{w}^h\in V^h\subset V$ and discrete variational equation (10) to obtain, for all $\boldsymbol{w}^h\in V^h$,

$$\begin{aligned}&\frac{2\rho}{\tau_q^2}(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)_H+\frac{2(\lambda+\mu)}{\tau_q^2}(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}\boldsymbol{w}^h)_Y+\frac{2\mu}{\tau_q}^2(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla\boldsymbol{w}^h)_Q\\&\qquad-m(\nabla(\xi_n-\xi_n^{hk})+\frac{2}{\tau_q}\nabla(e_n-e_n^{hk})+\frac{2}{\tau_q^2}\nabla(\theta_n-\theta_n^{hk}),\boldsymbol{w}^h)_H=0,\end{aligned}$$

and so, we have, for all $\boldsymbol{w}^h\in V^h$,

$$\begin{aligned}&\frac{2\rho}{\tau_q^2}(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H+\frac{2(\lambda+\mu)}{\tau_q^2}(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\\&\quad-m(\nabla(\xi_n-\xi_n^{hk})+\frac{2}{\tau_q}\nabla(e_n-e_n^{hk})+\frac{2}{\tau_q^2}\nabla(\theta_n-\theta_n^{hk}),\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H\\&\quad+\frac{2\mu}{\tau_q^2}(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q\\&=\frac{2\rho}{\tau_q^2}(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H+\frac{2(\lambda+\mu)}{\tau_q^2}(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{w}^h))_Y\\&\quad-m(\nabla(\xi_n-\xi_n^{hk})+\frac{2}{\tau_q}\nabla(e_n-e_n^{hk})+\frac{2}{\tau_q^2}\nabla(\theta_n-\theta_n^{hk}),\boldsymbol{v}_n-\boldsymbol{w}^h)_H\\&\quad+\frac{2\mu}{\tau_q^2}(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{w}^h))_Q.\end{aligned}$$

Taking into account that

$$\begin{aligned}&(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H\geq(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n,\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H\\&\qquad+\frac{1}{2k}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\},\\&(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\geq(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n))_Y\\&\qquad+\frac{1}{2k}\left\{\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\operatorname{div}(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\},\\&(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q\geq(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n))_Q\end{aligned}$$

$$+\frac{1}{2k}\left\{\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\},$$
$$-m(\nabla(\xi_n-\xi_n^{hk}),\boldsymbol{v}_n-\boldsymbol{w}^h)_H=m(\xi_n-\xi_n^{hk},\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{w}^h))_Y,$$

using again Cauchy-Schwarz inequality and Young's inequality (13) it follows that, for all $\boldsymbol{w}^h\in V^h$,

$$\begin{aligned}
&\frac{\rho}{\tau_q^2 k}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}-m(\nabla(\xi_n-\xi_n^{hk}),\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H\\
&\quad+\frac{\lambda+\mu}{\tau_q^2 k}\left\{\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\mathrm{div}\,(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\}\\
&\quad+\frac{\mu}{\tau_q^2 k}\left\{\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\}\\
&\le C\Big(\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_V^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n\|_V^2\\
&\quad+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\nabla(e_n-e_n^{hk})\|_H^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2\\
&\quad+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H\Big).
\end{aligned}\tag{15}$$

Now, we obtain the error estimates on the thermal acceleration. Then, we subtract variational equation (5) at time $t=t_n$ for a test function $r=r^h\in E^h\subset E$ and discrete variational equation (11) to obtain, for all $r^h\in E^h$,

$$\begin{aligned}
&(\frac{\tau_q^2}{2}(\dot{\xi}_n-\delta\xi_n^{hk})+\tau_q(\xi_n-\xi_n^{hk})+e_n-e_n^{hk},r^h)_Y-m(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),r^h)_Y\\
&\quad+\kappa(\tau_\theta\nabla(e_n-e_n^{hk})+\nabla(\theta_n-\theta_n^{hk}),\nabla r^h)_H=0,
\end{aligned}$$

and so we have, for all $r^h\in E^h$,

$$\begin{aligned}
&(\frac{\tau_q^2}{2}(\dot{\xi}_n-\delta\xi_n^{hk})+\tau_q(\xi_n-\xi_n^{hk})+e_n-e_n^{hk},\xi_n-\xi_n^{hk})_Y-m(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\xi_n-\xi_n^{hk})_Y\\
&\quad+\kappa(\tau_\theta\nabla(e_n-e_n^{hk})+\nabla(\theta_n-\theta_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\\
&=(\frac{\tau_q^2}{2}(\dot{\xi}_n-\delta\xi_n^{hk})+\tau_q(\xi_n-\xi_n^{hk})+e_n-e_n^{hk},\xi_n-r^h)_Y-m(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\xi_n-r^h)_Y\\
&\quad+\kappa(\tau_\theta\nabla(e_n-e_n^{hk})+\nabla(\theta_n-\theta_n^{hk}),\nabla(\xi_n-r^h))_H.
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
(\dot{\xi}_n-\delta\xi_n^{hk},\xi_n-\xi_n^{hk})_Y&\ge(\dot{\xi}_n-\delta\xi_n,\xi_n-\xi_n^{hk})_Y\\
&\quad+\frac{1}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\},
\end{aligned}$$

$$
\begin{aligned}
&(e_n - e_n^{hk}, \xi_n - \xi_n^{hk})_Y \geq (e_n - e_n^{hk}, \dot{e}_n - \delta e_n)_Y \\
&\qquad + \frac{1}{2k}\left\{ \|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2 \right\}, \\
&(\nabla(e_n - e_n^{hk}), \nabla(\xi_n - \xi_n^{hk}))_Y \geq (\nabla(e_n - e_n^{hk}), \nabla(\dot{e}_n - \delta e_n))_H \\
&\qquad + \frac{1}{2k}\left\{ \|\nabla(e_n - e_n^{hk})\|_H^2 - \|\nabla(e_{n-1} - e_{n-1}^{hk})\|_H^2 \right\}, \\
&(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \xi_n - \xi_n^{hk})_Y = -(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\xi_n - \xi_n^{hk}))_H, \\
&(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \xi_n - r^h)_Y = -(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\xi_n - r^h))_H, \\
&\frac{1}{k}\Big\{ (\nabla(\theta_n - \theta_n^{hk}), \nabla(e_n - e_n^{hk}))_H - (\nabla(\theta_{n-1} - \theta_{n-1}^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H \Big\} \\
&\quad = (\nabla(\theta_n - \theta_n^{hk}), \nabla(\delta e_n - \dot{e}_n))_H + (\nabla(\theta_n - \theta_n^{hk}), \nabla(\xi_n - \xi_n^{hk}))_H \\
&\qquad + (\nabla(\delta\theta_n - \dot{\theta}_n), \nabla(e_{n-1} - e_{n-1}^{hk}))_H + (\nabla(e_n - e_n^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H, \\
&\|\theta_n - \theta_n^{hk}\|_Y^2 \leq C\Big(I_n^2 + \sum_{j=1}^n k\|e_j - e_j^{hk}\|_Y^2 + \|\theta_0 - \theta_0^h\|_Y^2 \Big), \\
&\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 \leq C\Big(J_n^2 + \sum_{j=1}^n k\|\nabla(e_j - e_j^{hk})\|_H^2 + \|\nabla(\theta_0 - \theta_0^h)\|_H^2 \Big),
\end{aligned}
$$

where we recall that the integration errors I_n and J_n are given by

$$
I_n = \left\| \int_0^{t_n} e(s)\,ds - k\sum_{j=1}^n e_j \right\|_Y, \quad J_n = \left\| \int_0^{t_n} \nabla e(s)\,ds - k\sum_{j=1}^n \nabla e_j \right\|_H,
$$

using several times Cauchy-Schwarz and Young inequalities we find that, for all $r^h \in E^h$,

$$
\begin{aligned}
&\frac{\tau_q^2}{2k}\left\{ \|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k}\left\{ \|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2 \right\} \\
&\qquad + \frac{\tau_\theta}{2k}\left\{ \|\nabla(e_n - e_n^{hk})\|_H^2 - \|\nabla(e_{n-1} - e_{n-1}^{hk})\|_H^2 \right\} + m(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\xi_n - \xi_n^{hk}))_H \\
&\quad \leq C\Big(\|\dot{\xi}_n - \delta\xi_n\|_Y^2 + \|\xi_n - r^h\|_E^2 + \|e_n - e_n^{hk}\|_Y^2 + \|\nabla(e_n - e_n^{hk})\|_H^2 \\
&\qquad + (\delta\xi_n - \delta\xi_n^{hk}, \xi_n - r^h)_Y + J_n^2 + \|\nabla(\dot{e}_n - \delta e_n)\|_H^2 + \|\nabla(\dot{\theta}_n - \delta\theta_n)\|_H^2 \\
&\qquad + \|\nabla(e_{n-1} - e_{n-1}^{hk})\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla(\theta_0 - \theta_0^h)\|_H^2 \\
&\qquad + \sum_{j=1}^n k\|\nabla(e_j - e_j^{hk})\|_H^2 \Big).
\end{aligned} \tag{16}
$$

Combining now (15) and (16) we find that

$$
\frac{1}{2k}\left\{ \|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k}\left\{ \|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2 \right\}
$$

$$
\begin{aligned}
&+\frac{1}{2k}\left\{\|\nabla(e_n-e_n^{hk})\|_H^2-\|\nabla(e_{n-1}-e_{n-1}^{hk})\|_H^2\right\}\\
&+\frac{1}{k}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}\\
&+\frac{1}{k}\left\{\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\mathrm{div}\,(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\}\\
&+\frac{1}{k}\left\{\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\}\\
\le{}& C\Big(\|\dot\xi_n-\delta\xi_n\|_Y^2+\|\xi_n-r^h\|_E^2+\|e_n-e_n^{hk}\|_Y^2+\|\nabla(e_n-e_n^{hk})\|_H^2\\
&+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-r^h)_Y+J_n^2+\|\nabla(\dot e_n-\delta e_n)\|_H^2+\|\nabla(\dot\theta_n-\delta\theta_n)\|_H^2\\
&+\|\nabla(e_{n-1}-e_{n-1}^{hk})\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla(\theta^0-\theta^{0h})\|_H^2\\
&+\sum_{j=1}^n k\|\nabla(e_j-e_j^{hk})\|_H^2+\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_V^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2\\
&+\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n\|_V^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H\Big).
\end{aligned}
$$

Multiplying the previous estimates by k and summing up the resulting equation, using the estimates on the temperature fields given above we have

$$
\begin{aligned}
&\|\xi_n-\xi_n^{hk}\|_Y^2+\|e_n-e_n^{hk}\|_Y^2+\|\nabla(e_n-e_n^{hk})\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\theta_n-\theta_n^{hk}\|_Y^2\\
&\quad+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2\\
&\le Ck\sum_{j=1}^n\Big(\|\dot\xi_j-\delta\xi_j\|_Y^2+\|\xi_j-r_j^h\|_E^2+\|e_j-e_j^{hk}\|_Y^2+\|\nabla(e_j-e_j^{hk})\|_H^2\\
&\quad+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-r_j^h)_Y+I_j^2+J_j^2+\|\nabla(\dot e_j-\delta e_j)\|_H^2+\|\nabla(\dot\theta_j-\delta\theta_j)\|_H^2\\
&\quad+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2\\
&\quad+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\mathrm{div}\,(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\Big)\\
&\quad+C\Big(\|\xi^0-\xi^{0h}\|_Y^2+\|e^0-e^{0h}\|_Y^2+\|\nabla(e^0-e^{0h})\|_H^2+\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2\\
&\quad+\|\mathrm{div}\,(\boldsymbol{u}^0-\boldsymbol{u}^{0h})\|_Y^2+\|\nabla(\boldsymbol{u}^0-\boldsymbol{u}^{0h})\|_Q^2+\|\theta^0-\theta^{0h}\|_Y^2+\|\nabla(\theta^0-\theta^{0h})\|_H^2\Big).
\end{aligned}
$$

Finally, taking into account that

$$
\begin{aligned}
&k\sum_{j=1}^n(\delta\xi_j-\delta\xi_j^{hk},\xi_j-r_j^h)_Y=\sum_{j=1}^n(\xi_j-\xi_j^{hk}-(\xi_{j-1}-\xi_{j-1}^{hk}),\xi_j-r_j^h)_Y\\
&\quad=(\xi_n-\xi_n^{hk},\xi_n-r_n^h)_Y+(\xi^{0h}-\xi^0,\xi_1-r_1^h)_Y\\
&\qquad+\sum_{j=1}^{n-1}(\xi_j-\xi_j^{hk},\xi_j-r_j^h-(\xi_{j+1}-r_{j+1}^h))_Y,
\end{aligned}
$$

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta \boldsymbol{v}_j-\delta \boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H &= \sum_{j=1}^{n}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\\
&= (\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\boldsymbol{v}^{0h}-\boldsymbol{v}^0,\boldsymbol{v}_1-\boldsymbol{w}_1^h)_H\\
&\quad +\sum_{j=1}^{n-1}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h))_H,
\end{aligned}
$$

using the above estimates and a discrete version of Gronwall's inequality (see again [30]) we conclude the proof.

We note that error estimates (14) are the basis to get the convergence order of the approximations given by Problem VP^{hk}. Therefore, as an example, if we assume the following additional regularity:

$$
\begin{aligned}
&\theta \in H^4(0,T;Y)\cap W^{2,\infty}(0,T;H^2(\Omega))\cap H^3(0,T;E),\\
&\boldsymbol{u} \in H^3(0,T;H)\cap W^{1,\infty}(0,T;[H^2(\Omega)]^d)\cap H^2(0,T;V),
\end{aligned}
\tag{17}
$$

using the classical results on the approximation by finite elements and the regularities of the initial conditions (see, for instance, [28]), we have the following.

Corollary 7 *Let the assumptions of Theorem 2 hold. Under the additional regularity (17) it follows that the approximations obtained by Problem VP^{hk} are linearly convergent; that is, there exists a positive constant C, independent of the discretization parameters h and k, such that*

$$
\begin{aligned}
\max_{0\le n\le N}\Big\{&\|\xi_n-\xi_n^{hk}\|_Y+\|e_n-e_n^{hk}\|_E+\|\theta_n-\theta_n^{hk}\|_E+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H\\
&+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q\Big\}\le C(h+k).
\end{aligned}
$$

4 Numerical results

In this final section, we describe the numerical scheme implemented in the well-known finite element code FreeFem++ for solving Problem VP^{hk}, and we show some numerical examples to demonstrate the accuracy of the approximations and the behaviour of the solution.

4.1 *Numerical scheme*

As a first step, given the solution $\boldsymbol{u}_{n-1}^{hk}$, $\boldsymbol{v}_{n-1}^{hk}$, ξ_{n-1}^{hk}, e_{n-1}^{hk} and θ_{n-1}^{hk} at time t_{n-1}, the velocity and the thermal acceleration are obtained by solving the discrete linear system, for all $\boldsymbol{w}^h \in V^h$ and $r^h \in E^h$,

$$
\begin{aligned}
&\rho(\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + (\lambda+\mu)k^2(\operatorname{div} \boldsymbol{v}_n^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y + \mu k^2(\nabla \boldsymbol{v}_n^{hk}, \nabla \boldsymbol{w}^h)_Q \\
&\qquad -mk\left(\frac{\tau_q^2}{2}\nabla\xi_n^{hk} + \tau_q k\nabla\xi_n^{hk} + k^2\nabla\xi_n^{hk}, \boldsymbol{w}^h\right)_H \\
&\quad = k(\boldsymbol{H}_n, \boldsymbol{w}^h)_H + \rho(\boldsymbol{v}_{n-1}^{hk}, \boldsymbol{w}^h)_H - (\lambda+\mu)k(\operatorname{div} \boldsymbol{u}_{n-1}^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y \\
&\qquad -\mu k(\nabla \boldsymbol{u}_{n-1}^{hk}, \nabla \boldsymbol{w}^h)_Q - mk(\tau_q \nabla e_{n-1}^{hk} - k\nabla e_{n-1}^{hk} - \nabla\theta_{n-1}^{hk}, \boldsymbol{w}^h)_H, \\
&\left(\frac{\tau_q^2}{2}\xi_n^{hk} + k\tau_q\xi_n^{hk} + k^2\xi_n^{hk}, r^h\right)_Y + \kappa k(k\tau_\theta\nabla\xi_n^{hk} + k^2\nabla\xi_n^{hk}, \nabla r^h)_H - m\theta^* k(\operatorname{div} \boldsymbol{v}_n^{hk}, r^h)_Y \\
&\quad = k(P_n, r^h)_Y + \left(\frac{\tau_q^2}{2}\xi_{n-1}^{hk} - ke_{n-1}^{hk}, r^h\right)_Y - \kappa k(\tau_\theta\nabla e_{n-1}^{hk} + k\nabla e_{n-1}^{hk} + \nabla\theta_{n-1}^{hk}, \nabla r^h)_H,
\end{aligned}
$$

where the discrete displacements, the discrete thermal velocity and the discrete temperature are then recovered from the relations:

$$
\boldsymbol{u}_n^{hk} = k\boldsymbol{v}_n^{hk} + \boldsymbol{u}_{n-1}^{hk}, \quad e_n^{hk} = k\xi_n^{hk} + e_{n-1}^{hk}, \quad \theta_n^{hk} = ke_n^{hk} + \theta_{n-1}^{hk}.
$$

This numerical scheme was implemented on a 3.2 Ghz PC using FreeFem++ (see [31] for details) and a typical run ($h = k = 0.01$) took about 300 seconds of CPU time.

4.2 *First example: numerical convergence*

We will consider the following academic problem:

Problem P^{ex}. *Find the displacements* $\boldsymbol{u} : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^2$ *and the temperature* $\theta : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ *such that*

$$
\begin{aligned}
&\ddot{u}_i - u_{i,jj} - 2u_{j,ji} - (\theta_{,i} + \tau_q\dot{\theta}_{,i} + \frac{1}{2}\ddot{\theta}_{,i}) = H_i \ \textit{in}\ [0,1]\times[0,1]\times[0,1],\\
&\frac{1}{2}\dddot{\theta} + \ddot{\theta} + \dot{\theta} - (\theta_{,ii} + \dot{\theta}_{,ii}) - \dot{u}_{i,i} = P \ \textit{in}\ [0,1]\times[0,1]\times[0,1],\\
&u_i(x,y,t) = \theta(x,y,t) = 0 \quad \textit{for} \quad i=1,2 \ \textit{and}\ (x,y,t)\in\partial([0,1]\times[0,1])\times(0,1),\\
&u_i(x,y,0) = (xy(1-x)(1-y), xy(1-x)(1-y)) \quad \textit{for} \quad (x,y)\in[0,1]\times[0,1],\\
&\dot{u}_i(x,y,0) = (xy(1-x)(1-y), xy(1-x)(1-y)) \quad \textit{for} \quad (x,y)\in[0,1]\times[0,1],\\
&\theta(x,y,0) = xy(1-x)(1-y) \quad \textit{for} \quad (x,y)\in[0,1]\times[0,1],\\
&\dot{\theta}(x,y,0) = xy(1-x)(1-y) \quad \textit{for} \quad (x,y)\in[0,1]\times[0,1],\\
&\ddot{\theta}(x,y,0) = xy(1-x)(1-y) \quad \textit{for} \quad (x,y)\in[0,1]\times[0,1],
\end{aligned}
$$

where the body forces $\boldsymbol{H}$ and the heat supply P are given by

$$
\begin{aligned}
\boldsymbol{H}(x,y,t) &= e^t(x^2y^2 - x^2y - 2x^2 + 3xy^2 - 11xy + 6x - 8y^2 + 12y - 2,\\
&\qquad x^2y^2 + 3x^2y - 8x^2 - xy^2 - 11xy + 12x - 2y^2 + 6y - 2),\\
P(x,y,t) &= \frac{e^t}{2}\left(5x^2y^2 - x^2y - 12x^2 - xy^2 - 3xy + 12x - 12y^2 + 12y\right).
\end{aligned}
$$

We note that Problem P^{ex} corresponds to Problem P with the following data:

$$
\Omega = (0,1)\times(0,1), \quad T=1, \quad \kappa=1, \quad \tau_\theta=\tau_q=1, \quad \lambda=1, \quad \mu=1, \quad \theta^*=1,
$$

and the initial conditions, for all $(x,y)\in(0,1)\times(0,1)$,

$$
\begin{aligned}
&\boldsymbol{u}^0(x,y) = \boldsymbol{v}^0(x,y) = (xy(1-x)(1-y), xy(1-x)(1-y)),\\
&\theta^0(x,y) = e^0(x,y) = \xi^0(x,y) = xy(1-x)(1-y).
\end{aligned}
$$

It is worth recalling that $\tau_q < 2\tau_\theta$ is the condition required to guarantee the stability of the thermomechanical system (see [19, 32]), as well as the compatibility with the second law of thermodynamics [15].

Several uniform partitions for the domain have been performed dividing $\Omega = [0,1]\times[0,1]$ into $2(nd)^2$ triangles (that is, the spatial discretization parameter h equals $\frac{\sqrt{2}}{nd}$). Thus, the approximation errors estimated by

$$
\begin{aligned}
\max_{0\le n\le N} \Big\{ &\|\xi_n - \xi_n^{hk}\|_Y + \|e_n - e_n^{hk}\|_E + \|\theta_n - \theta_n^{hk}\|_E + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2\\
&+\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2\Big\}
\end{aligned}
$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly

$nd \downarrow k \rightarrow$	0.02	0.01	0.005	0.002	0.001
16	0.298784938	0.228314240	0.196912069	0.182066427	0.178721044
32	0.218118899	0.114676294	0.068450490	0.050788484	0.045673992
64	0.204728848	0.101276767	0.051792991	0.022819656	0.015199505
128	0.206267054	0.100131247	0.049121205	0.019904785	0.010370306
256	0.206787765	0.100639569	0.049477325	0.019502656	0.009715969

Table 1
Example 1: Numerical errors ($\times 100$) for some nd and k.

observed, and the linear convergence, stated in Corollary 7, does not seem to be achieved.

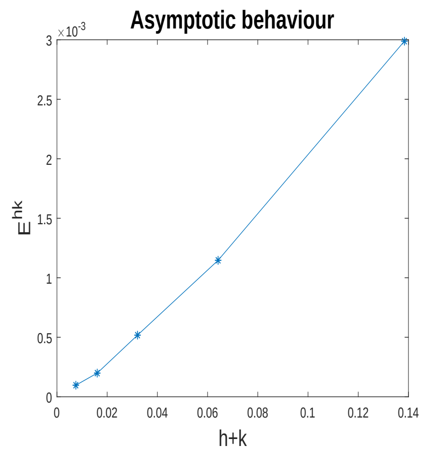

Fig. 1. Example 1: Asymptotic behaviour of the numerical scheme.

If we assume now that there are not volume forces, and we use the final time $T = 0.3$, the same data than in the previous example unless

$$\lambda = \mu = 10^4, \quad \tau_\theta = \tau_q = 0.001,$$

taking the discretization parameters $h = 0.01$ and $k = 0.001$, the evolution in time of the discrete energy E_n^{hk}, defined by

$$E_n^{hk} = \frac{1}{2}\Big\{\rho\|\boldsymbol{v}_n^{hk}\|_H^2 + \mu\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2 + (\lambda+\mu)\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2 + \|\theta_n^{hk} + \tau_q e_n^{hk} + \frac{\tau_q^2}{2}\xi_n^{hk}\|_Y^2$$
$$+\kappa\left((\tau_q+\tau_\theta)\|\nabla\theta_n^{hk}\|_H^2 + \frac{\kappa}{2}\tau_q^2\tau_\theta\|\nabla e_n^{hk}\|_H^2 + \kappa\tau_q^2(\nabla\theta_n^{hk}, \nabla e_n^{hk})_H\right)\Big\},$$

is plotted in Fig. 2 in both natural (left) and semi-log (right) scales. As can be seen, it converges to zero and an exponential decay seems to be achieved.

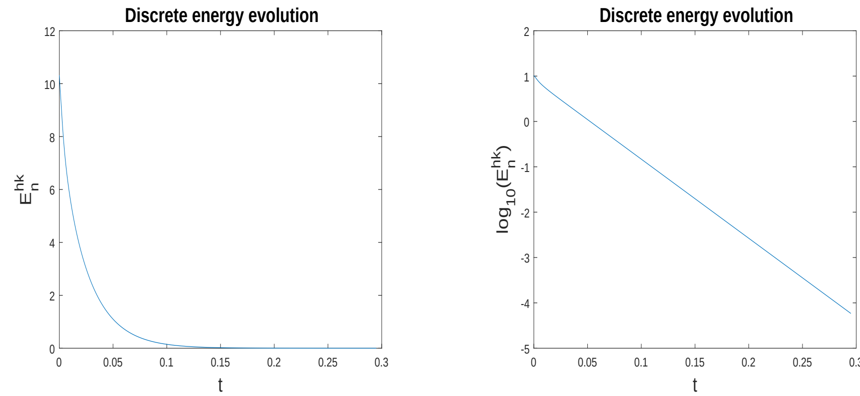

Fig. 2. Example 1: Discrete energy evolution in natural and semi-log scales.

4.3 Second example: effect of a heat supply

For this second example a rectangular domain $[0,4] \times [0,1]$ is considered, assumed to be clamped on its left part $\{0\} \times [0,1]$. An oscillating heat supply given by

$$P(x,y,t) = \begin{cases} xy(1-x)(1-y)\, \max\{\sin(10t\pi), 0\} & \text{if } x \leq 1, \\ 0 & \text{if } x > 1, \end{cases}$$

is applied for $T = 0.9$ seconds. The material contants used in this example are the same than those employed in the exponential decay simulation of the previous example. Moreover, we note that a decomposition of the boundary into Γ_D and Γ_F is used for the displacement field (on Γ_F we assume that it is traction-free). The analysis of this modified problem is done straightforwardly.

Using the time discretization parameter $k = 0.001$, in the left-hand side of Fig. 3 the evolution in time of the temperature field is plotted for three different points: $(0.5, 0.5), (2, 0.5)$ and $(3, 0.5)$; as it can be observed, an attenuation of the thermal waves is produced when the distance from the source increases. In the right-hand side, a zoom over the beginning of the simulation shows the delay on the temperature raising on points $(2, 0.5)$ and $(3, 0.5)$. The temperature field at final time is shown in Fig. 4.

4.4 Third example: real displacements recovery

As a final example, we recall that the solution calculated by the numerical scheme is a transformation of the actual displacement field. Here, we denote by $\tilde{\boldsymbol{u}}$ the solution obtained from Problem P and by $\boldsymbol{u}$ the actual displacement field.

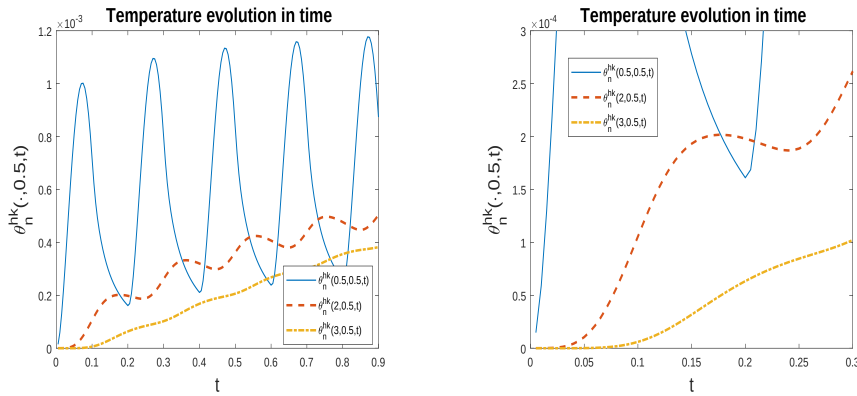

Fig. 3. Example 2: Temperature evolution at different points.

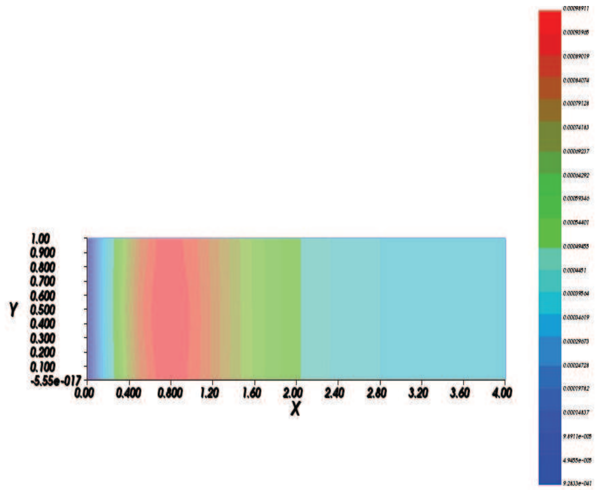

Fig. 4. Example 2: Temperature field at final time $T = 0.9$.

When the solution is calculated at each step time, the following ordinary differential equation must be solved for every node in order to recover the "real displacement" $\boldsymbol{u}$ (see [20]):

$$\frac{\tau_q^2}{2}\ddot{\boldsymbol{u}}(t) + \tau_q\dot{\boldsymbol{u}}(t) + \boldsymbol{u}(t) = \tilde{\boldsymbol{u}}(t),$$

which is computed through the equivalent system of first-order differential equations and discretized using a fourth-order Runge-Kutta method.

In this case, a circular domain is considered, with a round hole within its interior (see Fig. 5). An oscillating counterclockwise rotating body forces are applied for $T = 1.7$ seconds. The material constants used in this example are the same than in the previous one unless Lame's coefficient μ, which is assumed varying between 10000 and 40000. Again, a decomposition of the boundary into Γ_D and Γ_F is used for the displacement field (on Γ_F we assume that it is traction-free).

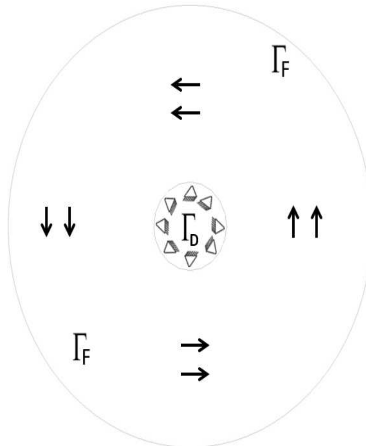

Fig. 5. Example 3: Physical setting

Using the time discretization parameter $k = 0.001$, both the evolution in time of the actual horizontal displacements and the temperature at point $(0, 1)$ are depicted in Fig. 6 for different values of coefficient μ, which allows to note its importance on the system. As expected, the displacement, and the oscillations of the temperature, increase when μ decreases.

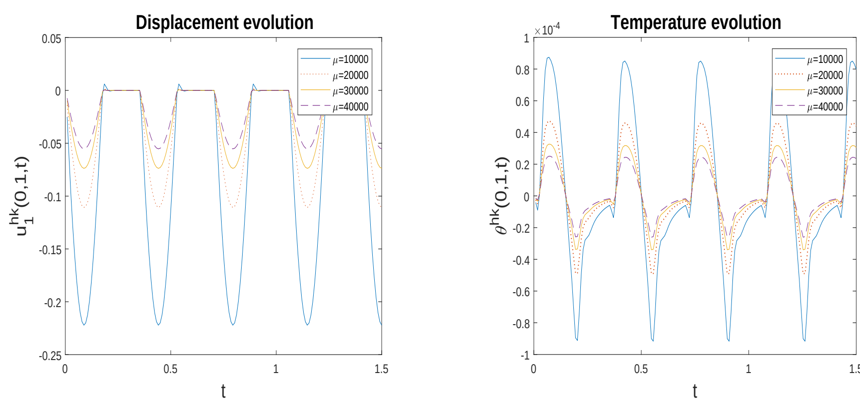

Fig. 6. Example 3: Horizontal displacement and temperature evolution at point $(0, 1)$.

5 Conclusions

In this paper we analyzed, from the numerical point of view, a dynamic problem involving a thermoelastic body. The dual-phase-lag heat conduction theory was used to model the thermal effects. The variational formulation was written as a hyperbolic system of coupled linear variational equations in terms of the thermal acceleration and the velocity field. Then, we introduced a fully discrete scheme using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. We provided a discrete stability result and we obtained some a priori error estimates. Finally, we presented some two-dimensional numerical simulations to demonstrate the convergence of the numerical scheme and the decay of the discrete energy (Example 1), the effect of the application of a heat supply (Example 2) and the dependence on a Lame's coefficient for the actual displacement field (Example 3).

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