Regularization of the differential inverse orientation problem of generic serial revolute joint manipulators*

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***Abstract*— Orienting manipulators in robotics are used to achieve the desired orientation of the end effector of the manipulator. These manipulators are composed of rotational joints, and thus inherently burdened with singularities. In singular configurations, the differential inverse orientation problem can not be applied since the Jacobian becomes singular. A regularization method is discussed that regularizes the Jacobian in singular configurations, by first transforming the angle-axis representation of angular velocities to infinitezimal rotation about an axis and infinitezimal translation perpendicular to that axis, then regularizing the infinitezimal translational motion in the new representation. It is shown that the Jacobian of generic manipulators composed of three rotational joints is always regularizable, so methods based on the Jacobian can be applied even in singular configurations. The method is generalized to redundant orienting manipulators as well, and its application is demonstrated on two examples: a 3R Euler wrist and a 4R Hamilton wrist.**

I. INTRODUCTION

Orienting manipulators are composed of revolute joints, thus their kinematic mappings are inherently burdened with singularities [1]. These singularities may cause undesired behavior in robot motion (e.g. discontinuous joint paths when movement is done in the singular direction), and make the application of the Jacobian of the kinematic map impossible.

There are several known methods to overcome the problems caused by singularities. Several authors examined the effect of adding extra degrees of freedom to the wrist that can be used to avoid singularities (e.g. [2], [3], [4]). However, regardless of the extra degrees of freedom, these manipulators still have singular configurations [1]. Other methods are numerical techniques used to calculate the pseudoinverse of the Jacobian, e.g. the Damped Least Squares (DLS) method (e.g. [5], [6], [7], [8]) or the Levenberg-Marquardt (LM) method [9]. The main problem with these methods is that the invese Jacobian mapping is not full rank, so they can not generate motion in singular direction [10], [11].

The aim of this work is to regularize the Jacobian, so the inverse mapping becomes full rank, and motion in singular direction can be generated using Jacobian-based techniques. Regularization of the inverse positioning problem has been discussed in [11], [10], and the regularization of the differential inverse orientation problem has already been discussed in [12], [10]. The previous work is extended here by giving a proof for the regularizability of generic manipulators in Section III, and giving an algorithm to solve the differential inverse orientation problem of general nR orienting manipulators in Section IV.

The regularization technique can not change the natural behavior of the manipulators around singularities, but can be used to generate motions in singular directions with the application of the Jacobian only. In Section IV, the algorithm is demonstrated on an Euler wrist [13] and a Hamilton wrist composed of four revolute joints [2]. It is shown that based on the parameters of the regularization, we can generate motion close to the analytical solution (that has discontinuous joint paths), and we can also generate damped motion with smaller joint velocities.

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II. PRELIMINARIES

Manipulators used for achieving a desired orientation are composed of rotational joints (since prismatic joints do not have effect on the orientation), so we will suppose here that the manipulator under consideration has rotational joints only. Our goal is to achieve arbitrary orientation in three dimensions, so the manipulator needs to have at least three joints, thus we consider nR manipulators (manipulators composed of n revolute joints) with $n \geq 3$.

The joint axes of the manipulators in the current configuration will be denoted by $\omega_1, \omega_2, \ldots, \omega_n$, that also represent angular velocities generated by the motion of the corresponding joints, so the task Jacobian of the manipulator in the current configuration is the matrix

$$J = (\omega_1, \omega_2, \ldots, \omega_n). \tag{1}$$

If the vectors $\omega_1, \omega_2, \ldots, \omega_n$ are the joint axes in the reference configuration (the configuration where the joint variables are $\theta_0 = (0, 0, \ldots, 0)^\top$), then the orientation of the end effector in a joint configuration $\theta = (\theta_1, \theta_2, \ldots, \theta_n)^\top$ is given by the formula (see e.g. [14], [15])

$$R(\theta) = \exp(\hat{\omega}_1\theta_1)\exp(\hat{\omega}_2\theta_2)\cdot \ldots \cdot \exp(\hat{\omega}_n\theta_n)R(0) \tag{2}$$

where $R(0)$ is the orientation of the end effector in the reference configuration given in the fixed spatial frame, and $\hat{\omega}$ is the skew symmetric cross product operator matrix of the vector ω given by

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \leftrightarrow \hat{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_x \\ \omega_z & 0 & -\omega_y \\ -\omega_x & \omega_y & 0 \end{pmatrix}. \tag{3}$$

If we want to reach an arbitrary orientation in three dimensions, then we need a manipulator composed of at least three revolute joints. In order to be able to achieve arbitrary

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orientation with a 3R manipulator, the adjacent joints of the manipulator must not be parallel. Suppose e.g. that $\omega_2 = \omega_3$ for a 3R manipulator, then the orientation of the manipulator (assuming $R(0) = I$ for simplicity) is given by

$$\begin{aligned} R(\theta) &= \exp(\hat{\omega}_1\theta_1)\exp(\hat{\omega}_2\theta_2)\exp(\hat{\omega}_3\theta_3) \\ &= \exp(\hat{\omega}_1\theta_1)\exp(\hat{\omega}_2\theta_{23}) \end{aligned} \quad (4)$$

with $\theta_{23} = \theta_2 + \theta_3$, thus it is equivalent to a 2R manipulator, and not suitable for reaching arbitrary orientation in three dimensions. So in what follows, we will suppose that the adjacent joints of 3R manipulators are not parallel, and at least the last three joints of nR manipulators satisfy that the adjacent joints are not parallel.

Pai and Leu [16, Theorem 5] showed that 3R orienting manipulators whose adjacent joints are not parallel are also generic, meaning that the kinematic mapping (2) of the manipulator is generic. Generic mappings gained interest in robotics literature because their singular points form smooth manifolds that can be traced out using local methods [17], [16], [18], [19], [20]. We will show that the Jacobian of generic 3R orienting manipulators (and nR manipulators whose last three joints form a generic 3R manipulator) can be regularized by a technique given in the next Section.

III. REGULARIZATION

The kinematic descriptions usually use the angle-axis representation to give the angular velocity of the end effector of robot manipulators, i.e. the angular velocity is described by a three-dimensional vector, and the coordinates of the vector are angular velocities around the corresponding axes of the fixed spatial frame (see Fig. 1, left). In this Section, we transform the angular velocities to describe the orientation as motion on the surface of a unit sphere, and rotation about the normal vector of the sphere (see the right of Fig. 1). We will call this representation the spherical representation [12], [10]. The main motivation behind this transformation is that it transforms the singular direction into an infinitezimal translation that can be regularized by a technique developed for the regularization of the inverse positioning problem of regional manipulators in [11], [10].

Let ω_r denote the normal vector of the unit sphere used in the spherical representation. Then an angular velocity vector ω given in the angle-axis representation is transformed to the spherical representation as

$$\omega_s = \omega_r\omega_r^\top\omega + \omega_r \times \omega. \quad (5)$$

The first term on the right-hand side of (5) is the component of the angular velocity parallel to the normal vector ω_r, representing infinitezimal rotation about the fictive axis ω_r. The second term is created from the component of the angular velocity that is perpendicular to the normal vector ω_r, and represents infinitezimal translation on the surface of the sphere. We will denote the one-dimensional subspace of infinitezimal rotations about ω_r by $\mathscr{S}_\Omega$, and the two-dimensional subspace of infinitezimal translations perpendicular to ω_r (spanned by v_1 and v_2 in Fig. 1) by $\mathscr{S}_V$.

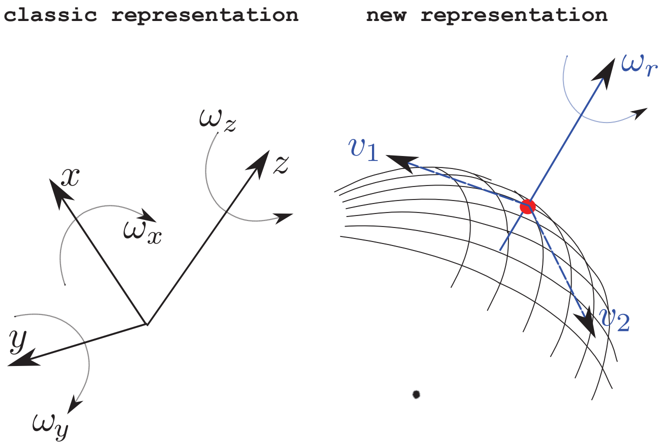

Fig. 1. The classical angle-axis representation of angular velocities (infinitezimal rotation about the x,y and z-axis by ω_x, ω_y, ω_z respectively, left) and the new spherical representation of the angular velocities (infinitezimal rotation about ω_r and infinitezimal translations on the tangent of the sphere spanned by v_1 and v_2, right)

The task Jacobian is transformed to the spherical representation as $S = PJ + J \times \omega_r$, where $P = \omega_r\omega_r^\top$ is the projector to the subspace $\mathscr{S}_\Omega$, and the cross product of a matrix and a vector means column-wise cross product, i.e.

$$J \times \omega_r := (\omega_1 \times \omega_r, \omega_2 \times \omega_2, \ldots, \omega_n \times \omega_r). \quad (6)$$

We will call the Jacobian transformed to the spherical representation the spherical Jacobian and denote it by S. In order to get an equivalent representation (such that the spherical Jacobian has the same rank as the original task Jacobian in every joint configuration), ω_r has to be chosen to be in the image space of the task Jacobian, thus it has to satisfy $\omega_r J \neq 0$, as it has been shown in [12], [10]. We will choose $\omega_r := \omega_n$ throughout the paper, so the ω_r vector is the axis of the last joint of the manipulator.

The advantage of the spherical representation is that if the spherical Jacobian is singular, the singular direction is in the subspace $\mathscr{S}_V$ (see Proposition 4 in [12]), i.e. it is an infinitezimal translation, and we can apply the regularization techniques developed for the positioning problem in [11], [10]. The regularization of the inverse positioning problem is done be removing the end effector point whose velocity is considered in the direction defined by the regularization vector (we will denote it by r and assume it is a unit vector) at a certain distance that will be denoted by γ. As a result, the partial linear velocities corresponding to the new end effector point will become linearly independent, resulting in a full rank Jacobian.

In this case, this technique should be applied for the components of the spherical Jacobian that are in the subspace $\mathscr{S}_V$, so we define the projector $P^\perp = I - P$ that projects velocities to $\mathscr{S}_V$ and give the regularized spherical Jacobian [12] as

$$S_{reg} = PJ + J \times \omega_r + \gamma P^\perp(\omega_1, \omega_2, \ldots, \omega_{n-1}, 0) \times r. \quad (7)$$

Note that the last column in the regularization term is zero, since the last joint axis of the manipulator is coincident with

ω_r, so it can not contribute to the motion in the subspace $\mathscr{S}_V$. In the following theorem, we show that for 3R generic orienting manipulators there always exist $r \in \mathbb{R}^3$ and $\gamma \in \mathbb{R}$ such that S_{reg} is full rank.

Theorem 1: The orientation task Jacobian of every 3R generic orienting manipulator is regularizable.

Proof: Let the task Jacobian of the 3R orienting manipulator be $J = (\omega_1, \omega_2, \omega_3)$. Choose $\omega_r := \omega_3$, let r be the regularization vector, $P = \omega_r \omega_r^\top$ the projector to the subspace $\mathscr{S}_\Omega$ and $P^\perp = I - P$ the projector to the subspace $\mathscr{S}_V$. Then the regularized spherical Jacobian is

$$S_{reg} = PJ + J \times \omega_r + \gamma P^\perp (\omega_1, \omega_2, 0) \times r. \quad (8)$$

The third column of S_{reg} is $\omega_r = \omega_3$, while the components orthogonal to ω_r in the first and second columns of S_{reg} are

$$v_1 =: P^\perp (\omega_1 \times \omega_r + \gamma \omega_1 \times r), \quad (9)$$
$$v_2 =: P^\perp (\omega_2 \times \omega_r + \gamma \omega_2 \times r) \quad (10)$$

respectively. Since these are orthogonal to ω_r (if they are not zero), it follows that S_{reg} is full rank if and only if v_1 and v_2 are linearly independent.

First, examine the independence of the vectors before the projection with $P^\perp$, so investigate if the vectors

$$v_1' =: \omega_1 \times \omega_r + \gamma \omega_1 \times r \quad (11)$$
$$v_2' =: \omega_2 \times \omega_r + \gamma \omega_2 \times r \quad (12)$$

are linearly independent. We will do this investigation by looking at the situations in which the vectors are dependent. The vectors v_1' and v_2' are dependent if and only if

$$v_1' \times v_2' = (\omega_1 \times (\omega_r + \gamma r)) \times (\omega_2 \times (\omega_r + \gamma r)) = 0 \quad (13)$$

that can be rearranged using the identity $(a \times b) \times (a \times c) = [a \cdot (b \times c)]a$ to get

$$[(\omega_r + \gamma r) \cdot (\omega_1 \times \omega_2)] (\omega_r + \gamma r) = 0. \quad (14)$$

Let the regularization vector be chosen as $r := \omega_1 \times \omega_2$. Then (14) becomes

$$[(\omega_r + \gamma \omega_1 \times \omega_2) \cdot (\omega_1 \times \omega_2)] (\omega_r + \gamma \omega_1 \times \omega_2) =$$
$$\left[\omega_r \cdot (\omega_1 \times \omega_2) + \gamma \|\omega_1 \times \omega_2\|^2\right] (\omega_r + \gamma \omega_1 \times \omega_2) = 0 \quad (15)$$

that holds if and only if at least one of the following equations are satisfied:

$$\omega_1 \times \omega_2 = 0 \quad (16)$$
$$\omega_r \cdot (\omega_1 \times \omega_2) + \gamma \|\omega_1 \times \omega_2\|^2 = 0. \quad (17)$$

Condition (16) can never hold, since the manipulator is generic, so the adjacent joint axes can not be parallel. Condition (17) holds if and only if γ is chosen as

$$\gamma = -\frac{\omega_r \cdot (\omega_1 \times \omega_2)}{\|\omega_1 \times \omega_2\|^2}. \quad (18)$$

So if the length of the regularization vector is chosen such that (18) does not hold, then the vectors v_1' and v_2' are linearly independent.

Now examine the vectors after the projection to the subspace $\mathscr{S}_V$ using the projector $P^\perp$. Suppose that γ is chosen such that $v_1' \times v_2' \neq 0$, so the vectors before projection are independent. The vectors after projection become dependent if and only if

$$v_1 \times v_2 = \left(P^\perp v_1'\right) \times \left(P^\perp v_2'\right) = 0 \quad (19)$$

holds that happens if and only if

$$\left(\omega_r \times v_1'\right) \times \left(\omega_r \times v_2'\right) = 0 \quad (20)$$

is satisfied. Condition (20) can be reformulated to get

$$\left[\omega_r \cdot \left(v_1' \times v_2'\right)\right] \omega_r = 0 \quad (21)$$

that holds if and only if ω_r is perpendicular to $v_1' \times v_2'$. Due to (14), the cross product $v_1' \times v_2'$ can be written as $\alpha(\omega_r + \gamma \omega_1 \times \omega_2)$ with $\alpha \neq 0$ being a real number, so the term in the square brackets in (21) becomes

$$\alpha \omega_r \cdot \omega_r + \alpha \gamma \omega_r \cdot (\omega_1 \times \omega_2) = 0. \quad (22)$$

If ω_r is perpendicular to $\omega_1 \times \omega_2$, then (22) never holds, and v_1 and v_2 are linearly independent. If ω_r is not perpendicular to $\omega_1 \times \omega_2$, then (22) is true if and only if

$$\gamma = -\frac{\|\omega_r\|^2}{\omega_r \cdot (\omega_1 \times \omega_2)} \quad (23)$$

that has a unique solution, so γ can be chosen such that this equation is not satisfied. So if we choose $r := \omega_1 \times \omega_2$, and choose γ such that (18) and (23) are not true, then the regularized Jacobian is full rank. ■

Note that the choice $r = \omega_1 \times \omega_2$ in the proof is not unique, the regularization vector can be chosen in other ways too. For example in [10], it was proved that the regularization vector can be chosen as the second column of the spherical Jacobian as well.

If the regularization vector is chosen to be $r = \omega_1 \times \omega_2$, and (18) is not zero, then there are two nonzero γ values defined by (18) and (23) for which the regularized spherical Jacobian is not full rank. However, both solutions for γ have the same sign, so choosing the opposite sign for γ ensures that the regularized spherical Jacobian is full rank. We have also assumed previously, that the regularization vector is a unit vector, however $\omega_1 \times \omega_2$ is not necessarily a unit vector. We will normalize this vector in the algorithm given in the following Section, however we omitted this normalization in the previous proof for the sake of simplicity.

The theorem can be generalized easily for nR orienting manipulators. Suppose that the last three joints of the nR manipulator form a generic manipulator, i.e. the adjacent joints are not parallel. Then the orienting task Jacobian can be regularized, by considering $\omega_r := \omega_n$ and $r = \omega_{n-2} \times \omega_{n-1}$. If we apply the regularization technique to this manipulator with these choices, then the last three columns of the regularized spherical Jacobian will be linearly independent, and since the regularized Jacobian has only three rows, it means that it is full rank. In the algorithm given in the following Section we used the choices for the regularization given in this paragraph.

Note that it is not necessary to use the last three joints of the manipulator for regularization, as it is not necessary either to use the choice $\omega_r := \omega_n$. However we use this convention, since in the practical designs usually the last three joints of the manipulators are suitable for reaching arbitrary orientation.

IV. APPLICATIONS

The regularization technique defined in the previous Section can be used in any algorithm that requires the task Jacobian. In this Section the technique is applied for the calculation of the joint velocities in a differential inverse kinematics algorithm. The algorithm is illustrated on two generic examples: a 3R Euler wrist [13] and a 4R Hamiltonian wrist [2]. The motion of the wrists are initiated in a singular configuration, and the motion is generated in the singular direction for a certain amount of time, then the wrists are moved back to the singular configuration by generating motion at the opposite direction for the same amount of time. The motions are generated with a relatively small and large regularization vector length; application of small regularization vector length yields results close to the analytical solution, while large regularization vector length yields damped results.

The task vector for both wrists will be $t[k] = \kappa(0.01, 0, 0)^\top$ rad/sec, $k = 0, 1, 2, \ldots, 200$, with $\kappa = 1$ if $k < 100$, and $\kappa = -1$ otherwise, i.e. rotation about the x-axis of the spatial frame is done in the positive direction in the first half, and in the negative direction in the second half of the simulation. The joint variables for the each time step are calculated as

$$\theta[k+1] = \theta[k] + T_s \Delta\theta[k] \tag{24}$$

with the sampling time being $T_s = 0.1$ sec in the simulations, and the joint difference is calculated using the following algorithm.

Algorithm 1 Algorithm for calculating the joint difference in each time step

Input: The current task Jacobian $J(\theta[k]) = (\omega_1[k], \omega_2[k], \ldots, \omega_n[k])$, the task vector $t[k]$.
Output: The current joint difference $\Delta\theta[k]$.

```
μ[k] = sqrt(det(J(θ[k])^T J(θ[k])))
if (μ[k] < μ_s) then
    γ[k] = γ_0 cos( (μ[k]/μ_s) (π/2) )
    ω_r[k] = ω_n[k]
    P[k] = ω_r[k] ω_r[k]^T
    P^⊥[k] = I − P[k]
    S[k] = P[k] J(θ[k]) + J(θ[k]) × ω_r[k]
    r[k] = (ω_{n−2}[k] × ω_{n−1}[k]) / ||ω_{n−2}[k] × ω_{n−1}[k]||
    S_reg[k] = S[k] + P^⊥[k] (ω_1[k], ω_2[k], ..., ω_{n−1}[k], 0) × r[k]
    t_s[k] = P t[k] + t[k] × ω_r
    Δθ[k] = S_reg[k]^# t_s[k]
else
    Δθ[k] = J(θ[k])^# t[k]
end if
```

First, the manipulability index [21], [22] $\mu[k]$ is calculated, and if it is smaller than the predefined limit μ_s (the value used in the simulations is 0.05), i.e. the manipulator is close to a singular configuration, then the joint difference is calculated using the regularization technique, otherwise it is calculated the conventional way.

The maximal length of the regularization vector is γ_0, and the actual length $\gamma[k]$ depends on the distance from the singular configuration measured by the manipulability index. The actual length is γ_0 in a singular configuration, and decreases as μ increases, and reaches zero if $\mu[k] = \mu_s$. This ensures a continuous transition between the regularization and the conventional technique. In the algorithm, a cosine function is used to describe the relationship between the manipulability index and the length of the regularization vector, however other functions possessing the above-mentioned properties could be used as well.

A. *Euler wrist*

First, a 3R Euler wrist is examined with the joint axes in the reference configuration being

$$\omega_1 = (0,0,1)^\top, \quad \omega_2 = (0,1,0)^\top, \quad \omega_3 = (0,0,1)^\top, \tag{25}$$

with the joint axes intersecting at the same point. The motion is initiated in the initial configuration $\theta[0] = (0,0,0)^\top$, where the task Jacobian is

$$J(\theta[0]) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \tag{26}$$

so rotation about the x-axis is indeed a singular direction.

The results of the differential inverse kinematics algorithm with $\gamma_0 = 1/100$ are shown in Figs. 2 and 3. The resulting joint paths in Fig. 2 are very close to the analytic solution: initially, the first joint is rotated by the amount of $\pi/2$ so as a result the second joint axis points in the direction of the x-axis, thus rotation about the x-axis becomes possible, while the third joint is rotated in the opposite direction by $\pi/2$ to ensure that the orientation of the end effector remains the same during the change of posture. This posture change is done in a short time resulting in high joint velocities. After the posture change, the rotation about the x-axis is simply done by moving the second joint. Finally, the wrist moves back to a singular configuration with same orientation as the initial orientation, however it is not the same configuration as the initial singular joint configuration, since the second joint axis is rotated by $\pi/2$. The angular velocities of the end effector are shown in Fig. 3, demonstrating that the rotation is done about the x-axis as described by the task vector, and rotation about other axes is zero as desired.

The results with $\gamma_0 = 1/10$ are shown in Figs. 4 and 5. The same posture change is done as in the previous example, however the joint velocities are smaller this time, resulting in a damped solution. Fig. 5 shows that the angular velocity of the end effector becomes the desired velocity only after a few seconds, while there are undesired, but small rotations around the y-axis too. In this case, the robot moves back to the initial joint configuration as it can be seen in Fig. 4.

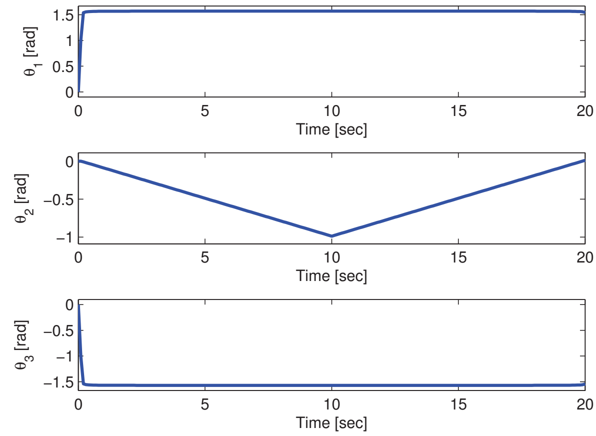

Fig. 2. The joints paths of the Euler wrist during the motion with $\gamma_0 = 1/100$.

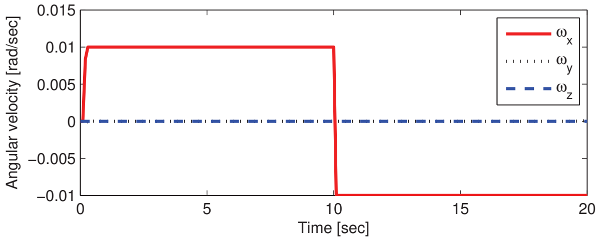

Fig. 3. The angular velocity of the end effector of the Euler wrist during the motion with $\gamma_0 = 1/100$, given by rotation about the axes of the spatial frame

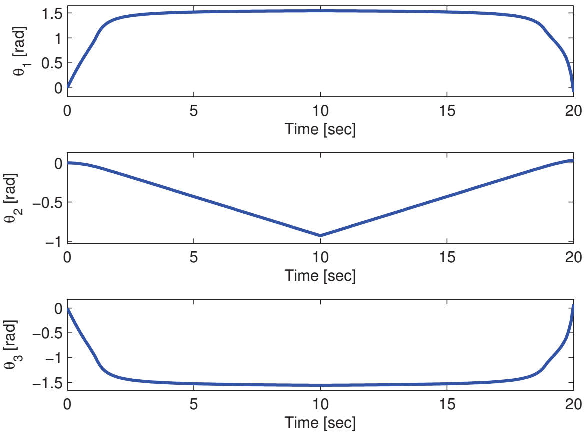

Fig. 4. The joints paths of the Euler wrist during the motion with $\gamma_0 = 1/10$.

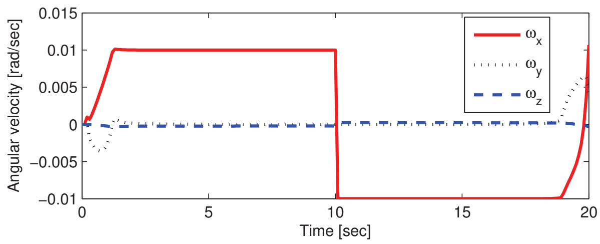

Fig. 5. The angular velocity of the end effector of the Euler wrist during the motion with $\gamma_0 = 1/10$, given by rotation about the axes of the spatial frame

B. *Hamilton wrist*

Second, a 4R Hamilton wrist is examined, whose joint axes in the reference configuration (as described in [2], Fig. 3) are

$$\begin{aligned} \omega_1 &= (0,0,1)^\top, \quad \omega_2 = (0,1,0)^\top, \\ \omega_3 &= (-1,0,0)^\top, \quad \omega_4 = (0,0,1)^\top, \end{aligned} \tag{27}$$

with the joint axes intersecting at the same point. The motion is initiated in the initial configuration $\theta[0] = (0,\pi/2,\pi/2,0)^\top$, where the task Jacobian is

$$J(\theta[0]) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \tag{28}$$

so rotation about the x-axis is indeed a singular direction.

The resulting motion with $\gamma_0 = 1/100$ are shown in Figs. 6 and 7, while the results with $\gamma_0 = 1/10$ are in Figs. 8 and 9. Similar to the results with the Euler wrist, the motion with $\gamma_0 = 1/100$ is close to the analytic solution, the posture change is done with high velocities, and the manipulator moves back to a different joint configuration, but same orientation as in the initial state. The motion with $\gamma_0 = 1/10$ is damped, the posture change is done with smaller joint velocities, and the manipulator moves back towards the original joint configuration. Note that rotation about undesired axes is much smaller for the damped motion of the 4R wrist compared to the damped motion of the 3R wrist.

Note that inverse kinematics algorithms using the DLS or LM methods fail to work in the situations examined in the examples, since the motion is initiated in a singular configuration, and the task vector points in a singular direction, and DLS and LM methods can not generate motion in singular directions.

V. CONCLUSION

A regularization technique was discussed here that can be applied to generate motion for orienting manipulators based on the Jacobian even in singular configurations. It was proved that the Jacobian of 3R generic orienting manipulators can be regularized, thus the Jacobian of all 3R manipulators that are suitable for solving the inverse orientation task can be regularized. The algorithm for the regularization was given for nR orienting manipulators, provided that the last three joints of the manipulator represent a generic orienting manipulator, and was demonstrated on an example where conventional methods like the DLS or LM method fail.

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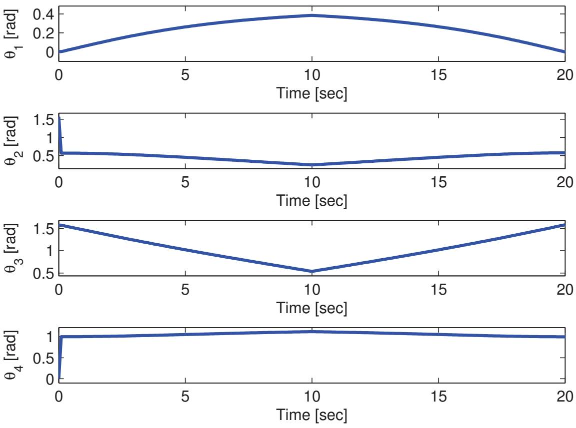

Fig. 6. The joints paths of the Hamilton wrist during the motion with $\gamma_0 = 1/100$.

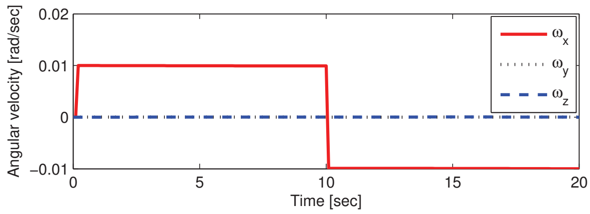

Fig. 7. The angular velocity of the end effector of the Hamilton wrist during the motion with $\gamma_0 = 1/100$, given by rotation about the axes of the spatial frame

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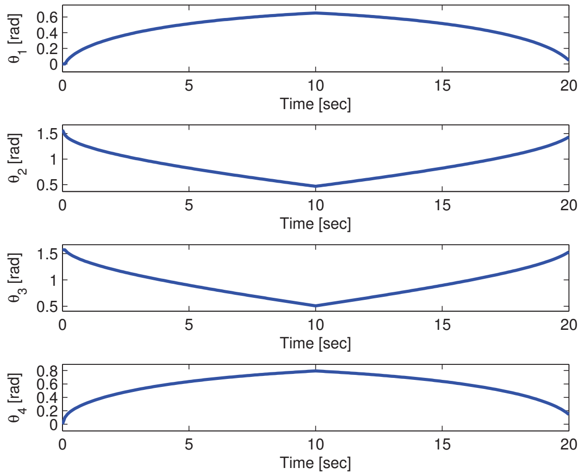

Fig. 8. The joints paths of the Hamilton wrist during the motion with $\gamma_0 = 1/10$.

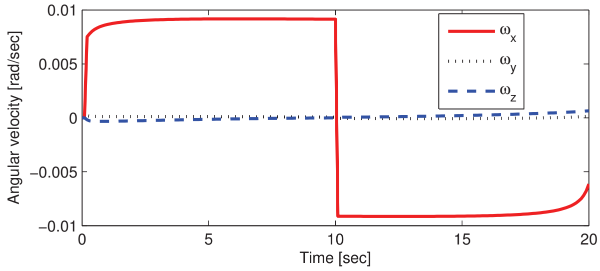

Fig. 9. The angular velocity of the end effector of the Hamilton wrist during the motion with $\gamma_0 = 1/10$, given by rotation about the axes of the spatial frame

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