

THE PARTER–WIENER THEOREM: REFINEMENT AND GENERALIZATION*

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Abstract. An important theorem about the existence of principal submatrices of a Hermitian matrix whose graph is a tree, in which the multiplicity of an eigenvalue increases, was largely developed in separate papers by Parter and Wiener. Here, the prior work is fully stated, then generalized with a self-contained proof. The more complete result is then used to better understand the eigenvalue possibilities of reducible principal submatrices of Hermitian tridiagonal matrices. Sets of vertices, for which the multiplicity increases, are also studied.

Key words. eigenvalues, Hermitian matrix, multiplicity, parter vertices, tree, vertex degrees

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Let T be a tree on n vertices $1, 2, \dots, n$, and suppose that $\mathcal{S}(T)$ is the set of all $n \times n$ complex Hermitian matrices whose graph is T ; the diagonal of $A \in \mathcal{S}(T)$ is not restricted by T . (All results also apply to $n \times n$ matrices $A = (a_{ij})$ for which $a_{ij}a_{ji} > 0$ when $\{i, j\}$ is an edge of T and the diagonal of A is real. Such matrices are diagonally similar to Hermitian matrices with the same graph.) For a complex Hermitian $n \times n$ matrix A , we denote the multiplicity of λ as an eigenvalue of A by $m_A(\lambda)$, and if $\alpha \subseteq N = \{1, \dots, n\}$ is an index set, we denote the principal submatrix of A resulting from deletion (retention) of the rows and columns α by $A(\alpha)$ ($A[\alpha]$). Often, α will consist of a single index i , and we abbreviate $A(\{i\})$ by $A(i)$. If $A = (a_{ij})$, identify $A[\{i\}]$ with a_{ii} . Note that when $A \in \mathcal{S}(T)$, the subgraph of T induced by deletion of vertex v , $T - v$, corresponds, in a natural way, to $A(v)$. In particular, $A(v)$ is a direct sum whose summands correspond to components of $T - v$ (which we call *branches* of T at v), the number of summands or components being the degree of v ($\deg v$) in T . We will often identify (such parts of) T with (such parts of) A for convenience. Throughout, for $\deg v = k + 1$, we identify the neighbors of a vertex v in T as u_0, u_1, \dots, u_k , and we denote the branch of T resulting from deletion of v and containing u_i by T_i , $i = 0, 1, \dots, k$.

According to the interlacing theorem for Hermitian eigenvalues [2], there is a simple relationship between $m_{A(i)}(\lambda)$ and $m_A(\lambda)$ when A is Hermitian:

$$m_{A(i)}(\lambda) = m_A(\lambda) - 1 \text{ or } m_{A(i)}(\lambda) = m_A(\lambda) \text{ or } m_{A(i)}(\lambda) = m_A(\lambda) + 1.$$

It is natural to imagine that the first possibility is generic, and, for sufficiently full Hermitian matrices A , it probably is. However, in [8] a very surprising observation

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was made: If T is a tree and $A \in \mathcal{S}(T)$ and $m_A(\lambda) \geq 2$, then there is a vertex i such that $m_{A(i)}(\lambda) \geq 3$ and λ is an eigenvalue of at least three components (branches) of $A(i)$. In particular, if $m_A(\lambda) = 2$, $m_{A(i)}(\lambda) = m_A(\lambda) + 1$. In [10] it was further shown that if $m_A(\lambda) \geq 2$, then there is a vertex i such that $m_{A(i)}(\lambda) = m_A(\lambda) + 1$.

We note that the principal results of neither [8] nor [10] apply when T is a path, as, then, $m_A(\lambda) > 1$ cannot occur. For self-containment, we give a simple proof of this known fact later, and our generalization of [8] and [10] will apply to this case.

It is curious that Parter did not identify the multiplicity increase for all values of $m_A(\lambda) \geq 2$ and Wiener did not explicitly identify the distribution of the eigenvalue among at least three branches, both of which are important, though it appears that each author might have, given the machinery they developed. When just one vertex is removed, we note that the “three branches” cannot generally be improved upon, as there are trees with maximum degree 3 and arbitrarily high possible multiplicities [1], [3]. However, as we shall see in Theorem 14, the “three branches” may be improved by removing more vertices.

These results have been important to us in our recent works on possible multiplicities of eigenvalues among matrices in $\mathcal{S}(T)$ [3], [4], [5], [6], [7]. Although not explicitly stated by either, we feel it appropriate to attribute the following theorem to Parter and Wiener.

THEOREM 1 (PW-theorem). *Let T be a tree on n vertices and suppose that $A \in \mathcal{S}(T)$ and that $\lambda \in \mathbb{R}$ is such that $m_A(\lambda) \geq 2$. Then, there is a vertex i of T such that $m_{A(i)}(\lambda) = m_A(\lambda) + 1$ and λ occurs as an eigenvalue in direct summands of A that correspond to at least three branches of T at i .*

Besides focusing attention on the complete statement of Theorem 1, our purpose here is to give a generalization of it (the PW-theorem will be a special case) and to apply the generalization in a few ways. We give new and rather complete information about the relationship between the eigenvalues of a tridiagonal Hermitian matrix and those of a principal submatrix of size one smaller. Our approach also gives a clear identification of the elements necessary in a proof of the original observations.

We call a vertex i in T a (weak) Parter vertex for $\lambda \in \mathbb{R}$ and $A \in \mathcal{S}(T)$ when $m_{A(i)}(\lambda) = m_A(\lambda) + 1$ and call a collection $\alpha \subseteq N$ a Parter set when $m_{A(\alpha)}(\lambda) = m_A(\lambda) + |\alpha|$. We also examine when a collection of Parter vertices is a Parter set, and related issues. We also have used the term (strong) Parter vertex for one satisfying the conclusion of Theorem 1 elsewhere, but this will not be needed here. That a collection of Parter vertices need not be a Parter set is noted by example in [9].

Our generalization of the PW-theorem follows.

THEOREM 2. *Let A be a Hermitian matrix whose graph is a tree T , and suppose that there exists a vertex v of T and a real number λ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. Then*

- (a) *there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$;*
- (b) *if $m_A(\lambda) \geq 2$, then v' may be chosen so that $\deg v' \geq 3$ and so that there are at least three components T_1, T_2 , and T_3 of $T - v'$ such that $m_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$;*
- (c) *if $m_A(\lambda) = 1$, then v' may be chosen so that $\deg v' \geq 2$ and so that there are two components T_1 and T_2 of $T - v'$ such that $m_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.*

Before continuing, we note that, even when $m_A(\lambda) \geq 2$, it can happen that $\deg v' = 1$ or $\deg v' = 2$ or λ appears in only one or two components of $T - v'$ even when $\deg v' \geq 3$. Of course, it also can happen that v does not qualify as a v' (v need

not increase the multiplicity of λ). Examples are easily constructed and some appear in [9].

Naturally, in the PW-theorem case ($m_A(\lambda) \geq 2$), $m_{A(v)}(\lambda) \geq 1$, so that our hypothesis is automatically satisfied for any v . Thus, Theorem 1 is a special case of Theorem 2.

The proof of Theorem 2 rests, in part, on two key lemmas, but first we record (and prove, for completeness) a well-known fact that we shall use.

LEMMA 3. *If A is a Hermitian matrix whose graph is a path on n vertices, then for any $\lambda \in \sigma(A)$, $m_A(\lambda) = 1$.*

Proof. Up to permutation similarity, A , and thus $A - \lambda I$, is tridiagonal. Since A is irreducible, the result of deletion of the first column and last row of $A - \lambda I$ is an $(n - 1) \times (n - 1)$ lower triangular matrix with nonzero diagonal, which is, therefore, nonsingular and rank $n - 1$. Since rank cannot increase by extracting a submatrix, $\text{rank}(A - \lambda I) = n - 1$, and, as A is Hermitian, $m_A(\lambda) = 1$. \square

LEMMA 4. *Let A be a Hermitian matrix whose graph is a tree T . If there is a vertex v of T and a real number λ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$, then there are adjacent vertices v' and u of T such that the component T_0 of $T - v'$ containing u satisfies $m_{A[T_0]}(\lambda) = m_{A[T_0-u]}(\lambda) + 1$.*

Proof. We argue by induction on the number n of vertices of T . For convenience, we actually prove a slightly stronger statement by adding to the induction hypothesis the statement that v is not a vertex of T_0 . If $n = 1$ or $n = 2$, the claimed implication is correct because it is not possible for the hypothesis to be satisfied, as may be easily checked. If $n = 3$, then T is a path and it can be easily checked that the hypothesis is satisfied only if A is a tridiagonal matrix whose first and last diagonal entries are both λ and v is the middle vertex. Then, taking v' to be the middle vertex v and u to be either the first or last vertex shows that the conclusion is satisfied (as the empty matrix cannot have λ as an eigenvalue).

Now, suppose that the claim is valid for all trees on fewer than n vertices, $n > 3$, and consider a tree on n vertices and a Hermitian matrix A such that there is a vertex v such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. First, try letting v' be the vertex v . If there is a neighbor u_j of v such that $m_{A[T_j]}(\lambda) \geq 1$ and $m_{A[T_j-u_j]}(\lambda) = m_{A[T_j]}(\lambda) - 1$, we are done. If not, there are, by the hypothesis, neighbors u_j such that $m_{A[T_j]}(\lambda) \geq 1$, and, by replacing v with u_j and applying induction, the claim follows. \square

The second lemma may be proven in two different ways, each giving different insights. We give one proof here, and another may be found in [9]. Both proofs rely on an expansion of the characteristic polynomial for Hermitian matrices whose graphs are trees. First, focus on a particular vertex v of T with neighbors u_0, \dots, u_k and expand $p_A(t)$ along the corresponding row of $A = (a_{ij})$ to obtain

$$(1) \quad p_A(t) = (t - a_{vv}) \prod_{j=0}^k p_{A[T_j]}(t) - \sum_{j=0}^k |a_{vu_j}|^2 p_{A[T_j-u_j]}(t) \prod_{\substack{l=0 \\ l \neq j}}^k p_{A[T_l]}(t),$$

and also

$$(2) \quad p_{A(T_i)}(t) = (t - a_{vv}) \prod_{\substack{j=0 \\ j \neq i}}^k p_{A[T_j]}(t) - \sum_{\substack{j=0 \\ j \neq i}}^k |a_{vu_j}|^2 p_{A[T_j-u_j]}(t) \prod_{\substack{l=0 \\ l \neq i, j}}^k p_{A[T_l]}(t).$$

(Here, we observe the standard convention that the characteristic polynomial of the empty matrix is identically 1.)

It will be convenient to focus on the identified neighbor u_0 of v and rewrite (1) and (2) by letting

$$f(t) = \sum_{j=1}^k |a_{vu_j}|^2 p_{A[T_j - u_j]}(t) \prod_{\substack{l=1 \\ l \neq j}}^k p_{A[T_l]}(t)$$

and

$$g(t) = \prod_{j=1}^k p_{A[T_j]}(t)$$

to obtain

$$(3) \quad p_A(t) = (t - a_{vv})p_{A[T_0]}(t)g(t) - |a_{vu_0}|^2 g(t)p_{A[T_0 - u_0]}(t) - f(t)p_{A[T_0]}(t)$$

and

$$(4) \quad p_{A(T_0)}(t) = (t - a_{vv})g(t) - f(t).$$

We also have a useful form for $p_A(t)$ when we focus on the edge connecting v and u_0 . Denote by T_v the tree $T - T_0$. We have $A(T_0) = A[T_v]$ and $g(t) = p_{A[T_v - v]}(t)$. From (3),

$$p_A(t) = [(t - a_{vv})g(t) - f(t)]p_{A[T_0]}(t) - |a_{vu_0}|^2 g(t)p_{A[T_0 - u_0]}(t),$$

and with (4) we obtain

$$(5) \quad p_A(t) = p_{A[T_v]}(t)p_{A[T_0]}(t) - |a_{vu_0}|^2 p_{A[T_v - v]}(t)p_{A[T_0 - u_0]}(t).$$

Using these expansions we now prove the following lemma.

LEMMA 5. *Let A be a Hermitian matrix, whose graph is a tree T . If there is a vertex v of T and a real number λ for which*

$$\lambda \in \sigma(A) \cap \sigma(A(v)),$$

and there is a branch T_0 of T at v such that

$$m_{A[T_0 - u_0]}(\lambda) = m_{A[T_0]}(\lambda) - 1,$$

in which u_0 is the neighbor of v in T_0 , then

$$m_{A(v)}(\lambda) = m_A(\lambda) + 1.$$

Proof. We employ (3) and (4) above, with v and u_0 corresponding to the hypothesis of the lemma. First, note that $p_{A(v)}(t) = p_{A[T_0]}(t)g(t)$. Let $m = m_A(\lambda)$ and $m_0 = m_{A[T_0]}(\lambda)$, so that $m_{A[T_0 - u_0]}(\lambda) = m_0 - 1$. (We note that if it happens that $m_0 = m + 1$, the conclusion is immediate. Although the proof is technically correct in any event, it may be convenient to assume $m_0 \leq m$.) Also, let m_f and m_g be the multiplicities of λ as a root of f and g , respectively. Since removal of u_0 from T leaves $A(T_0) \oplus A[T_0 - u_0]$, by the interlacing inequalities and the assumption that $m_{A[T_0 - u_0]}(\lambda) = m_0 - 1$, λ is a root of $p_{A(T_0)}$ at least $m - m_0$ times. Also by interlacing, $m - m_0 - 1 \leq m_g \leq m - m_0 + 1$. If $m_g = m - m_0 + 1$, we would be done;

so consider the other two possibilities. In either event, $m_f \geq m_g$ by a divisibility argument applied to (4). Returning to (3), we find that if $m_g = m - m_0$, a divisibility argument would contradict our hypothesis, as all terms except $|a_{vu_0}|^2 p_{A[T_0-u_0]}(t)g(t)$ would be divisible by $(t - \lambda)^m$. Suppose $m_g = m - m_0 - 1$. Then $m_f = m_g$, or else a divisibility argument would contradict the fact that λ is a root of $p_{A(T_0)}(t)$ at least $m - m_0$ times. However, then a divisibility argument applied to (3) leads to a contradiction, as λ is a root of the left-hand side and the first and third terms on the right at least $m - 1$ times each, but only $m - 2$ times in the second term on the right. \square

Although we have made the statement in the form we wish to apply it, we note that the statement of Lemma 5 remains correct (trivially) if the hypothesis “ $\lambda \in \sigma(A) \cap \sigma(A(v))$ ” is replaced by the weaker “ $\lambda \in \sigma(A(v))$.”

Another proof of this key lemma is given in [9]. This proof uses (5) and focuses primarily on the nature of the neighbor u_0 . See also [10] for a variant of Lemma 5 and a different approach.

We next turn to a proof of Theorem 2.

Proof of Theorem 2. If $m_A(\lambda) \geq 2$, the first part of the hypothesis of Lemma 5 is satisfied for any vertex of T , in particular the vertex v' guaranteed by Lemma 4. In this event, the entire hypothesis of Lemma 5 holds, verifying part (a) of the theorem. If $m_A(\lambda) = 1$ (and $m_{A(v)}(\lambda) \geq 1$), we still have from Lemma 4 the existence of v' , and since $m_{A(v')}(\lambda) \geq m_{A[T_0]}(\lambda) = m_{A[T_0-u]}(\lambda) + 1 \geq 1$, we have $m_A(\lambda) \geq 1$ and $m_{A(v')}(\lambda) \geq 1$. Thus, v' in place of v satisfies the hypothesis of Lemma 5 and part (a) of the theorem still holds.

For part (b) we argue by induction on the number n of vertices of T . If $n \leq 3$, the claimed implication is correct because it is not possible that the hypothesis is satisfied, as may be easily checked, or simply apply Lemma 3, as any tree on $n \leq 3$ vertices is a path.

If $n = 4$, the only tree on four vertices that is not a path is a star (one vertex of degree 3 and three pendant vertices). Since $m_A(\lambda) = m \geq 2$, there is a vertex v in T such that $m_{A(v)}(\lambda) = m + 1$. In that case, v must be the central vertex (the vertex of degree 3), since for any other vertex u , $T - u$ is a path. Thus, $T - v$ is a graph consisting of three isolated vertices with $m_{A(v)}(\lambda) \leq 3$. Therefore, $m = 2$ and $m_{A(v)}(\lambda) = 3$; i.e., λ is an eigenvalue of three components of $T - v$.

Now, suppose that the claimed result is valid for all trees on fewer than n vertices, $n > 4$, and consider a tree T on n vertices and a Hermitian matrix $A \in \mathcal{S}(T)$ such that λ is an eigenvalue of A with $m_A(\lambda) = m \geq 2$. By part (a) of Theorem 2, there is a vertex v in T such that $m_{A(v)}(\lambda) = m + 1$. If λ is an eigenvalue of at least three components of $T - v$, we are done. If not, there are two possible situations: λ is an eigenvalue of two components of $T - v$ (case 1) or λ is an eigenvalue of one component of $T - v$ (case 2).

In case 1, there is a component T' of $T - v$ with λ as an eigenvalue of $A[T']$ and $m_{A[T']}(\lambda) \geq 2$. Applying induction to T' , we have a vertex u in T' such that $m_{A[T'-u]}(\lambda) = m_{A[T']}(\lambda) + 1$ and λ is an eigenvalue of at least three components of $T' - u$. Observe that $m_{A(\{v,u\})}(\lambda) = m + 2$; thus, by interlacing, $m_{A(u)}(\lambda) = m + 1$. Consider the (unique) shortest path between v and u in T , P_{vu} , and let (v, u) denote the component of $T - \{v, u\}$ containing vertices of P_{vu} . Note that (v, u) is one of the components of $T' - u$ (if not empty). If there are three components of $T' - u$ having λ as an eigenvalue and none of these is (v, u) , then these three components are also components of $T - u$, and we are done. If there are only three components of $T' - u$

having λ as an eigenvalue and one of them is (v, u) then, by interlacing applied to the component of $T - u$ containing v , since $T - v$ has another component with λ as an eigenvalue, $T - u$ still has three components with λ as an eigenvalue.

In case 2, there is a component T' of $T - v$ with λ as an eigenvalue of $A[T']$ and $m_{A[T']}(\lambda) = m_A(\lambda) + 1$. Applying induction to T' , we have a vertex u in T' such that $m_{A[T'-u]}(\lambda) = m_{A[T']}(\lambda) + 1$ and λ is an eigenvalue of at least three components of $T' - u$. By interlacing, $m_{A(u)}(\lambda) = m + 1$. Thus, if there are three components of $T' - u$ having λ as an eigenvalue and none of these is (v, u) , then these three components are also components of $T - u$ and we are done. If there are only three components of $T' - u$ having λ as an eigenvalue and one of these components is (v, u) , we may apply case 1 to complete the consideration of case 2.

For part (c), the only contrary possibility is that λ is an eigenvalue of multiplicity 2 of one of the direct summands of $A(v')$. But, in this event, we may replace v' with the vertex adjacent to it in the corresponding branch of $T - v'$ and continue such replacement until a v' of the desired sort is found. \square

COROLLARY 6. *Let T be a tree and A be a matrix of $\mathcal{S}(T)$. If for some vertex v of T , λ is an eigenvalue of $A(v)$, then there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$.*

Proof. Suppose that λ is an eigenvalue of $A(v)$ for some vertex v of T . If λ is not an eigenvalue of A , then setting $v' = v$, $m_{A(v')}(\lambda) = m_A(\lambda) + 1$. If λ is also an eigenvalue of A , by Theorem 2, there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$. \square

It has been mentioned in several prior works (e.g., [1], [4], [5]) that for a tree, the multiplicities of the largest and smallest eigenvalues are 1. It is an interesting question to characterize those trees for which there is a matrix with just two eigenvalues of multiplicity 1, and to determine for each tree the minimum number of eigenvalues of multiplicity 1 that can occur (it may be much more than two). Here, we give another (simple) proof about the multiplicities of the largest and smallest eigenvalues.

COROLLARY 7. *If T is a tree, the largest and smallest eigenvalues of each $A \in \mathcal{S}(T)$ have multiplicity 1. Moreover, the largest or smallest eigenvalue of a matrix $A \in \mathcal{S}(T)$ cannot occur as an eigenvalue of a submatrix $A(v)$, for any vertex v of T .*

Proof. Let T be a tree and λ be the smallest (largest) eigenvalue of a matrix $A \in \mathcal{S}(T)$. Suppose that there is a vertex v of T such that λ is an eigenvalue of $A(v)$. By Theorem 2, there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$. But, from the interlacing inequalities, since λ is the smallest (largest) eigenvalue of A , for any vertex i of T , $m_{A(i)}(\lambda) \leq m_A(\lambda)$, which gives a contradiction. Thus, λ cannot occur as an eigenvalue of any submatrix $A(i)$ of A . Therefore, $m_A(\lambda) = 1$. \square

Lemma 5 indicates that a neighboring vertex, in whose branch the multiplicity goes down, is important for the existence of a Parter vertex. We call a branch at v in the direction of u_0 , satisfying the requirement $m_{A[T_0]}(\lambda) = m_{A[T_0-u_0]}(\lambda) + 1$, of Lemma 5 a *downer branch* at v for the eigenvalue λ ; the vertex u_0 is called a *downer vertex*. According to Lemma 5, the existence of a downer branch is sufficient for a vertex to be Parter. Importantly, the existence of a downer branch is also necessary for a vertex to be Parter, which provides a precise structural mechanism for recognition of Parter vertices. Notice that, even when $m_A(\lambda) = 0$, if there is a downer branch at v , then $m_{A(v)}(\lambda) = 1$. It cannot be more by interlacing, nor less because $A[T_0]$ is a direct summand of $A(v)$.

THEOREM 8. *For $A \in \mathcal{S}(T)$, T a tree, and v a vertex of T , $m_{A(v)}(\lambda) = m_A(\lambda) + 1$*

if and only if there is a downer branch at v for λ .

Proof. The sufficiency follows from Lemma 5 and the comment preceding the statement of this theorem.

For necessity, return to (1). Suppose that none of u_0, u_1, \dots, u_k is a downer vertex. Then, the number of times λ is a root of the second term on the right is at least the number of times that λ is a root of $p_{A(v)}(t)$ (i.e., $\prod_{i=0}^k p_{A[T_i]}(t)$). Thus, $m_A(\lambda)$ is, at least, $m_{A(v)}(\lambda)$, and v could not be Parter. \square

By Corollary 7, a branch of T at v having λ as the smallest (largest) eigenvalue is automatically a downer branch, so that we may make the following observation using Theorem 8.

COROLLARY 9. *Let A be a Hermitian matrix whose graph is a tree T . If λ is the smallest (largest) eigenvalue of at least one of the direct summands of $A(v)$, then $m_{A(v)}(\lambda) = m_A(\lambda) + 1$.*

We note that, extending the divisibility argument of the proof of Theorem 8, if each neighbor of v is Parter in its branch, then v cannot be Parter. We note also that if u_i is Parter in its branch, then it is Parter in T , as its downer branch within its branch will be a downer branch in T .

Let T be a path on n vertices and $A \in \mathcal{S}(T)$. Theorem 2 allows us to give information about the relationship between the eigenvalues of A and those of a principal submatrix of size one smaller. A path on n vertices admits a labeling $1, 2, \dots, n$ such that, for $i = 1, \dots, n-1$, $\{i, i+1\}$ is an edge. Without loss of generality, if T is a path, we shall assume this labeling of the vertices, giving an irreducible tridiagonal matrix, in terms of which, for convenience, we now make several observations. The first is a classical fact that now follows here in quite a different way.

COROLLARY 10. *If A is an $n \times n$ irreducible tridiagonal Hermitian matrix, then the eigenvalues of $A(1)$ and $A(n)$ strictly interlace those of A .*

Proof. We induct on n . For $n \leq 3$, the validity of the claim was mentioned in the proof of Lemma 4. Assume now the claim for tridiagonal matrices of size less than n . By symmetry, we need only verify the claim for $A(n)$. Suppose to the contrary that $\lambda \in \sigma(A) \cap \sigma(A(n))$. By the induction hypothesis, $\lambda \notin \sigma(A(\{n-1, n\}))$, so that $n-1$ is a downer vertex for λ at n . By Theorem 8, then, $m_{A(n)}(\lambda) = 1 + 1$, a contradiction to Lemma 3, as the graph of $A(n)$ is again a path. \square

By Corollary 10, a pendant path with λ as an eigenvalue is also a downer branch, so that we may make the following observation using Theorem 8.

COROLLARY 11. *Let A be a Hermitian matrix whose graph is a tree T . If at least one of the direct summands of $A(v)$ has λ as an eigenvalue, and its graph is a path and a neighbor of v is a pendant vertex of this path, then $m_{A(v)}(\lambda) = m_A(\lambda) + 1$.*

A new observation is now immediate.

COROLLARY 12. *If A is an $n \times n$ irreducible, tridiagonal, Hermitian matrix, then $\lambda \in \sigma(A) \cap \sigma(A(i))$ if and only if $1 < i < n$ and $m_{A(i)}(\lambda) = 2$, with $\lambda \in \sigma(A[\{1, \dots, i-1\}])$ and $\lambda \in \sigma(A[\{i+1, \dots, n\}])$.*

From Corollary 12 and Lemma 3, we immediately have the following.

COROLLARY 13. *Let A be an $n \times n$ irreducible, tridiagonal, Hermitian matrix. Then there are at most $\min\{i-1, n-i\}$ different eigenvalues that are common to both A and $A(i)$, i.e., at most $\min\{i-1, n-i\}$ equalities in the interlacing inequalities.*

We note that Corollary 13 is sharp. If $A[\{1, \dots, i-1\}]$ and $A[\{i+1, \dots, n\}]$ have $\min\{i-1, n-i\}$ eigenvalues in common (which may always be arranged), then the upper bound on the number of the interlacing inequalities will be attained. Smaller numbers also may be designed.

Remark. We note that if A is an irreducible, tridiagonal, Hermitian matrix, then an interpretation of Corollary 12 is the following. If any common eigenvalues of A and $A(i)$ are deleted from $\sigma(A)$ and $\sigma(A(i))$ (only once each in the latter case), then the latter strictly interlaces the former.

We now turn to a more detailed discussion of the structure and size of Parter sets.

THEOREM 14. *Let A be a Hermitian matrix whose graph is a tree T and let λ be an eigenvalue of A . Then, there is a vertex v of T such that $\lambda \in \sigma(A) \cap \sigma(A(v))$ if and only if there is a Parter set S of cardinality $k \geq 1$ such that λ is an eigenvalue of $m_A(\lambda) + k$ direct summands of $A(S)$.*

Proof. Let λ be an eigenvalue of A . Suppose that $S = \{v_1, \dots, v_k\}$, $k \geq 1$, is a Parter set of λ such that λ is an eigenvalue of $m_A(\lambda) + k$ direct summands of $A(S)$. By the interlacing inequalities, for the multiplicity to increase by k , it would have to increase by 1 with the removal of each vertex, starting with any one; i.e., each vertex of S is a Parter vertex. Thus if $v \in S$, then $m_{A(v)}(\lambda) = m_A(\lambda) + 1$. Therefore, there is a vertex v of T such that $\lambda \in \sigma(A) \cap \sigma(A(v))$.

For the converse, suppose that v is a vertex of T such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. By Theorem 2, there is a vertex v_1 of T such that $m_{A(v_1)}(\lambda) = m_A(\lambda) + 1$ and, if $m_A(\lambda) = 1$, λ is an eigenvalue of two direct summands of $A(v_1)$ or, if $m_A(\lambda) \geq 2$, then λ is an eigenvalue of at least three direct summands of $A(v_1)$. So, if $m_A(\lambda) = 1$ or $m_A(\lambda) = 2$, the claimed result follows directly from Theorem 2. Now, suppose that $m_A(\lambda) \geq 3$. If λ is an eigenvalue of $m_A(\lambda) + 1$ direct summands of $A(v_1)$, we are done. If not, λ is an eigenvalue of less than $m_A(\lambda) + 1$ direct summands of $A(v_1)$. This means that λ is still a multiple eigenvalue of some direct summands of $A(v_1)$. Since each direct summand of $A(v_1)$ is a Hermitian matrix whose graph is a subtree of T , applying recursively Theorem 2 we find vertices v_2, \dots, v_k of T such that $m_{A(\{v_1, \dots, v_k\})}(\lambda) = m_{A(\{v_1, \dots, v_{k-1}\})}(\lambda) + 1$ and λ is not a multiple eigenvalue of any direct summands of $A(\{v_1, \dots, v_k\})$; i.e., setting $S = \{v_1, \dots, v_k\}$, $m_{A(S)}(\lambda) = m_A(\lambda) + k$ and λ is an eigenvalue of $m_A(\lambda) + k$ direct summands of $A(S)$. \square

In Corollary 11, we noted that if $\lambda \in \sigma(A)$, $A \in \mathcal{S}(T)$, and there is a pendant path in T with λ as an eigenvalue, then that pendant path is a downer branch for λ in T . Of course, by Lemma 3, λ has multiplicity 1 in this downer branch. It is possible to show by example that there may be no multiplicity 1 downer branch in T that is a path, but it is not difficult to show, using Theorem 14 and induction, that there is always a multiplicity 1 downer branch for λ in T , $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A) \cap \sigma(A(v))$. We have the following.

COROLLARY 15. *Let $A \in \mathcal{S}(T)$ and suppose that there is a vertex v of T such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. Then, there is a Parter vertex v' of T such that for at least one of its downer branches T_0 for λ at v' , $m_{A[T_0]}(\lambda) = 1$.*

If λ is a multiple eigenvalue of A , there is a Parter vertex v for λ such that $m_{A(v)}(\lambda) = m_A(\lambda) + 1$. It can occur that λ is an eigenvalue of $m_A(\lambda) + 1$ direct summands of $A(v)$ but, for example, if $\deg v < m_A(\lambda) + 1$, necessarily λ is an eigenvalue of less than $m_A(\lambda) + 1$ direct summands of $A(v)$.

COROLLARY 16. *Let A be a Hermitian matrix whose graph is a tree T and let λ be an eigenvalue of A . If S is a Parter set for λ of cardinality k such that λ is an eigenvalue of $m_A(\lambda) + k$ direct summands of $A(S)$, and $v \in S$ is a Parter vertex for λ of degree less than $m_A(\lambda) + 1$, then $k > 1$.*

THEOREM 17. *Let A be a Hermitian matrix whose graph is a tree T and let λ be an eigenvalue of A . Also, let $d_1 \geq \dots \geq d_n$ be the vertex degree sequence of T and S be a Parter set for λ of cardinality k such that λ is an eigenvalue of $m_A(\lambda) + k$ direct*

summands of $A(S)$ (each exactly once). Then, for $1 \leq p \leq r$, in which $d_r > 1$ and $d_{r+1} = 1$,

$$\sum_{i=1}^p d_i \leq m_A(\lambda) + 2(p-1)$$

implies $k > p$.

Proof. By hypothesis, λ is an eigenvalue of $m_A(\lambda) + k$ direct summands of $A(S)$. This means that the number of components of $T - S$ is, at least, $m_A(\lambda) + k$. Let v_1, \dots, v_k be the vertices of S . The number of components of $T - S$, c_S , is $1 + \sum_{i=1}^k (\deg v_i - 1) - e$, in which e is the number of edges in the subgraph of T induced by S . It is clear that $0 \leq e \leq k - 1$. Therefore, $c_S \leq 1 + \sum_{i=1}^k (\deg v_i - 1)$. Recall that c_S must be, at least, $m_A(\lambda) + k$ and, observe that, since $d_1 \geq \dots \geq d_n$, $1 + \sum_{i=1}^k (\deg v_i - 1) \leq 1 + \sum_{i=1}^k (d_i - 1)$. Thus, if for $p \geq 1$ ($d_p > 1$),

$$1 + \sum_{i=1}^p (d_i - 1) \leq m_A(\lambda) + p - 1,$$

i.e.,

$$\sum_{i=1}^p d_i \leq m_A(\lambda) + 2(p-1),$$

then $k > p$. \square

We conclude with a general lower bound for the cardinality of a Parter set of the special type guaranteed in Theorem 14.

COROLLARY 18. *Let A be a Hermitian matrix whose graph is a tree T and let λ be an eigenvalue of A . Let $d_1 \geq \dots \geq d_n$ be the degree sequence of the vertices of T and S be a Parter set for λ of cardinality k such that λ is an eigenvalue of $m_A(\lambda) + k$ direct summands of $A(S)$. Then, $k \geq q$, in which q is the first integer such that $\sum_{i=1}^q d_i > m_A(\lambda) + 2(q-1)$.*

If we let K_q be a maximum number of components remaining after removal of q vertices, then in the language of [5], a lower bound on the cardinality of such a Parter set is the first value of q such that $K_q \geq m_A(\lambda) + q$.

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