CONTROL AND ESTIMATION ALGORITHMS FOR MULTIPLE-AGENT SYSTEMS

BY

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DISSERTATION

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Abstract

In this thesis we study crucial problems within complex, large scale, networked control systems and mobile sensor networks. The first one is the problem of decomposition of a large-scale system into several interconnected subsystems, based on the imposed information structure constraints. After associating an intelligent agent with each subsystem, we face with a problem of formulating their local estimation and control laws and designing inter-agent communication strategies which ensure stability, desired performance, scalability and robustness of the overall system. Another problem addressed in this thesis, which is critical in mobile sensor networks paradigm, is the problem of searching positions for mobile nodes in order to achieve optimal overall sensing capabilities.

Novel, overlapping decentralized state and parameter estimation schemes based on the consensus strategy have been proposed, in both continuous-time and discrete-time. The algorithms are proposed in the form of a multi-agent network based on a combination of local estimators and a dynamic consensus strategy, assuming possible intermittent observations and communication faults. Under general conditions concerning the agent resources and the network topology, conditions are derived for the stability and convergence of the algorithms. For the state estimation schemes, a strategy based on minimization of the steady-state mean-square estimation error is proposed for selection of the consensus gains; these gains can also be adjusted by local adaptation schemes. It is also demonstrated that there exists a connection between the network complexity and efficiency of denoising, *i.e.*, of suppression of the measurement noise influence. Several numerical examples serve to illustrate characteristic properties of the proposed algorithm and to demonstrate its applicability to real problems.

Furthermore, several structures and algorithms for multi-agent control based on a dynamic consensus strategy have been proposed. Two novel classes of structured, overlapping decentralized control algorithms are presented. For the first class, an agreement between the agents is implemented at the level of control inputs, while the second class is based on the agreement at the state estimation level. The proposed control algorithms have been illustrated by several examples. Also, the second class of the proposed consensus based control scheme has been applied to decentralized overlapping tracking control of planar formations of UAVs. A comparison is given with the proposed novel design methodology based on the expansion/contraction paradigm and the inclusion principle.

Motivated by the applications to the optimal mobile sensor positioning within mobile sensor networks, the perturbation-based extremum seeking algorithm has been modified and extended. It has been assumed that the integrator gain and the perturbation amplitude are time varying (decreasing in time with a proper rate) and that the output is corrupted with measurement noise. The proposed basic, one dimensional, algorithm has been extended to two dimensional, hybrid schemes and directly applied to the planar optimal mobile sensor positioning, where the vehicles can be modeled as velocity actuated point masses, force actuated point masses, or nonholonomic unicycles. The convergence of all the proposed algorithms, with probability one and in the mean square sense, has been proved. Also, the problem of target assignment in multi-agent systems using multi-variable extremum seeking algorithm has been addressed. An algorithm which effectively solves the problem has been proposed, based on the local extremum seeking of the specially designed global utility functions which capture the dependance among different, possibly conflicting objectives of the agents. It has been demonstrated how the utility function parameters and agents' initial conditions impact the trajectories and destinations of the agents. All the proposed extremum seeking based algorithms have been illustrated with several simulations.

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Contents

Chapter 1

Introduction

Recent technological advances and integrated communications have critically influenced standard control systems to evolve to, so called, *networked control systems.* These systems are, in general, distributed, large scale, complex systems which comprise of sensors, actuators, controllers and processes which may all operate in an asynchronous manner and are all connected through some form of communication network. Applications are numerous, such as space and terrestrial exploration, formations of robots, aircraft or automobiles, teleoperation, remote diagnostics and troubleshooting, remote surgery, collaborations over the Internet etc. (relevant survey is provided *e.g.*, in [5]).

It is desirable to approach networked control systems related problems in a decentralized way and treat them by decomposing a large scale complex system into many (possibly overlapping) interconnected subsystems, where each subsystem has a decision maker (intelligent agent) associated with it. The decentralized approach is imposed naturally in the networked control systems, having in mind that local agents, nowadays, can have great processing power and can locally implement estimation, control and other calculations. The agents usually coordinate and communicate only with a small subset of other agents. This way, there is no need for sending large amount of data through the network, which is usually prone to delays, losses, quantization effects, noise, etc. Other desirable properties of decentralized systems are their *modularity, scalability, adaptability, flexibility and robustness.*

The case when the agents are mobile and their interconnections are time-varying can be considered in the context of mobile sensor networks. These networks typically consist of a large number of mobile nodes deployed in the environment being sensed and controlled. Recent technological advances will allow fabrication and commercialization of inexpensive very small scale autonomous, potentially mobile electromechanical devices containing a wide range of sensors. When grouped together, these sensors can offer access to a great quantity of information about our environment, which can bring a revolution in the amount of control an individual has over his environment, with numerous applications (*e.g.* [26],[5],[85]).

1.1 Literature Review

Decentralized or distributed state or parameter estimation is of fundamental importance for large scale, complex, networked systems, representing one of the key factors for their proper functioning in numerous contexts. Depending on the available resources, agents have access to different measurements, different *a priori* information, such as system models and sensor characteristics, and different inter-agent communication channels. A class of decentralized estimators has been directly obtained starting from parallelization of the globally optimal Kalman filter; typically, such estimators possess a fusion center which generates the global estimates (*e.g.*, see [8, 29, 116]). An insight into the basic principles and structures of decentralized estimation can be found in *e.g.* [74, 75, 80, 103, 84, 112]. Also, different aspects of decentralized, *multi-agent control* systems are covered by a vast literature within the frameworks of computer science, artificial intelligence, network and system theory; for some aspects of multi-agent control systems see *e.g.* [26, 17, 112, 71].

One of the general design methodologies for overlapping decentralized estimation and control has been derived from the *inclusion principle*, using the *expansion/contraction* paradigm, where a complex system is expanded, decomposed into subsystems, and contracted back into the original system space after designing local estimators or controllers for the extracted subsystems, *e.g.*, [33, 35, 36, 80, 90].

Many deterministic and stochastic iterative algorithms naturally admit a distributed parallel implementation, where a number of agents perform computations and exchange of messages with a certain common goal. As early as in the 1980s, important results were obtained in the area of distributed asynchronous iterations in parallel computation and distributed optimization (*e.g.* [105, 13, 107, 9, 15, 46]). The majority of the cited references share a common general methodology: they all use some kind of *agreement* or dynamic *consensus* strategy. The decentralized state estimation problem itself is deeply embedded in this line of thought either implicitly, through the very definition of the consensus algorithms (*e.g.*, see [72]), or explicitly, where a dynamic consensus averaging strategy between multiple agents is used to obtain the required estimates (*e.g.*, see [56, 110, 111]).

One application of the mentioned methodologies that has received increasing interest for conducting research is the analysis and control of formations of Unmanned Autonomous Vehicles (UAVs). Recently, a number of important results in this area has been reported in various publications (*e.g.*, see [11, 23, 25, 39, 47, 101, 104, 68, 70, 82, 3, 112] and references reported therein).

Within mobile sensor networks paradigm, the critical problem is the problem of searching optimal sensing positions for mobile nodes, where the *extremum seeking* (ES) methodology can be directly applied. Extremum seeking represents a nonmodel based method for adaptive control which deals with systems where the reference-to-output map is uncertain but is known to have an extremum. In 1950s and 1960s this approach was popular as "extremum control" or "self-organizing control" (see *e.g.* [41, 51, 52]). A significant contribution to this field has been made in the last years by Krstić and his co-workers, who succeeded both to clarify the main conceptual aspects of this methodology and to present interesting and useful applications (see [7, 20, 42, 109, 40, 115, 114]). They presented stability analysis for the extremum seeking systems with sinusoidal perturbations in both continuous and discrete-time case using averaging and singular perturbations providing sufficient conditions for the plant output to converge to a neighborhood of the extremum value. In [50] some stability results have been presented for the case when the sinusoidal perturbation is replaced with a stationary stochastic process. The problem of multi-target assignment, addressed in Section 4.7, based on designing global utility functions ([2, 1, 102]) involves the multi-variable ES algorithm proposed and analyzed in [6].

There is a vast literature related to the problems of performance and stability limitations of control/estimation over unreliable communication links/networks. It has been treated using several tools and models involving coding/decoding over band-limited channels, quantization effects, delays, packet dropouts, etc. (for a relevant survey see *e.g.* [5])

1.2 Dissertation Outline and Contributions

The focus of this thesis is on two aspects of the mentioned problems: a) decomposing a complex/large-scale system into (possibly overlapping) subsystems and formulating local estimation and control laws, which, along with suitably defined inter-agent communication schemes (possibly over wireless, sensor networks), ensure stability, acceptable performance and robustness of the overall system; b) developing algorithms, suitable for mobile sensor networks, for placement of mobile nodes to the positions which enable optimal sensing/communication capabilities.

In Chapter 2 novel decentralized overlapping state and parameter estimation algorithms are presented. In Section 2.1 a state estimation algorithm for complex systems, in both continuous and discrete-time ([98], [97], [96]), is proposed on the basis of: 1) structured, overlapping system decomposition; 2) implementation of local state estimators by intelligent agents, according to their own sensing and computing resources; 3) application of a consensus strategy providing the global state estimates to all the agents in the network. In discrete-time case, lossy inter-agent communication network is assumed, *i.e.*, intermittent observations and communication faults are allowed. Stability of the proposed algorithms is analyzed. A strategy aimed at obtaining the consensus gains on the basis of minimization of the overall mean-square error is proposed. It is also shown, by using characteristic network topologies, that asymptotic denoising, *i.e.*, measurement noise elimination when the number of nodes is large, can be achieved in the case of the network connectivity increasing at a sufficient rate with the number of nodes. A number of characteristic examples are given within all the sections in order to illustrate the theoretically derived conclusions.

Section 2.2 is devoted to decentralized parameter estimation by consensus based stochastic approximation ([94], [95]). The proposed algorithm is based on: (a) local recursive estimation schemes of stochastic approximation type which utilize local measurements; (b) a consensus strategy aimed at improving reliability and noise immunity of the estimates.

The asymptotic behavior of the algorithm is analyzed, including different choices of the algorithm gains, different probabilities of getting local measurements and sending inter-agent messages, network connectedness ensuring convergence, as well as important aspects of consensus-based denoising.

Chapter 3 is devoted to the problem of *overlapping decentralized control* of complex systems by using a multi-agent strategy, where the agents (subsystems) communicate in order to achieve agreement upon a control action by using a dynamic consensus methodology [86]. Several new control structures are proposed based on the agreement between the agents upon the control variables. In the most general setting, it is assumed that each agent is able to formulate its local feedback control law starting from the local information structure constraints in the form of a general four-term dynamic output controller. The subsystem inputs generated by the agents by means of the local controllers enter the consensus process which generates the control signals to be applied to the system by some *a priori* specified agents. In the general case, the consensus scheme, determining, in fact, the control law for the whole system, is constructed on the basis of an *aggregation* of the local dynamic controllers. It is shown how the proposed scheme can be adapted to either static local output feedback controllers, or static local state feedback controllers. Also, an alternative to this approach is proposed, based on the introduction of a dynamic consensus at the level of state estimation introduced in Section 2.1. The control signal is obtained by applying the known global LQ optimal state feedback gain to the locally available estimates. A number of selected examples illustrate the applicability of all the proposed consensus based control schemes. In Section 3.4 a novel design methodology for decentralized overlapping tracking control of planar formations of UAVs based on the expansion/contraction paradigm [100] is presented and compared with the proposed consensus based control scheme applied to the formations control problem. The benefits of the consensus based scheme are verified having in mind much better responses and tracking performance.

Motivated by the critical problem within mobile sensor networks paradigm of searching optimal sensing positions, the extremum seeking algorithm with sinusoidal perturbation is analyzed in Chapter 4. The standard discrete-time ES algorithm has been extended and modified in the following way ([87], [89], [88]): a) the amplitudes of the sinusoidal perturbation signals, as well as the gains of the integrator blocks, are time varying and tend to zero at a pre-specified rate; b) the output of the system is corrupted with measurement noise. In general, the first assumption opens up a possibility to obtain convergence of the whole scheme to a unique extremum point and not to its neighborhood which depends on the perturbation amplitude even in the deterministic context. The second assumption, *i.e.*, the inclusion of the additive stochastic component in the extremum seeking loop, allows important generalizations and applications of the extremum seeking methodology to a large number of real adaptation problems in control and signal processing. Conditions for the local convergence to the extremum point in the mean-square sense and with probability one are derived. It is also shown how the extremum seeking scheme can be applied to *noise source localization* problems and an *adaptive state estimation* problem where the observation noise influence is minimized and, thus, can be used for the *optimal positioning of mobile sensors*. Using a generalization of the methodology developed for the 1D case, the convergence to the extremal points has been proved for the planar, hybrid ES algorithms, adopted for the control of: a) velocity actuated vehicles; b) force actuated vehicles; c) nonholonomic vehicles (unicycles). Section 4.7 is devoted to the problem of multi-target assignment in multi-agent systems where the agents need to cover the minima of all the measured functions. An algorithm based on designing a global utility function, which would capture the dependence among different agents' objectives, and finding it's local extremum is proposed. It is shown that the scheme can be considered as a multi-variable ES algorithm where the agents seek the local extremum of the proposed global utility function (the closest one to the agents' initial positions, taking into account parameters of the applied utility function). All the proposed ES based schemes have been illustrated through several examples.

Finally, in Chapter 5 we review the results presented in this thesis and give some directions for the future research.

Chapter 2

Consensus Based State and Parameter Estimation

In this chapter consensus based state and parameter estimation algorithms are presented. Section 2.1 is devoted to decentralized overlapping state estimation schemes while in Section 2.2 decentralized overlapping parameter estimation scheme based on stochastic approximation is presented.

2.1 Consensus Based Decentralized Overlapping State Estimator in Lossy Network

In this section both continuous-time and discrete-time consensus based decentralized overlapping state estimation schemes are proposed. First, the main definitions of the problems, together with the description of the proposed estimation algorithm are given. Formally speaking, the algorithm is composed of a set of overlapping decentralized Kalman filters put together within a multi-agent network by using a first-order dynamic consensus strategy. Stability of the proposed schemes is discussed. It is proved that it is possible to find, under general conditions concerning the local estimators and the network topology, such a consensus scheme which ensures asymptotic stability of the whole estimator. A strategy aimed at obtaining the gains of the consensus scheme by minimizing the total mean-square estimation error with respect to the unknown consensus gains is also described. The problem of denoising of the obtained estimates with respect to the measurement noise is presented, with an emphasis on the connection between the suppression of the measurement noise influence and complexity of the multi-agent network.

2.1.1 Continuous-Time Case

Let us first consider the continuous-time case, where we assume that the inter-agent network is perfect (without any losses) and that the local measurements are not interrupted.

2.1.1.1 Problem and Algorithm Definition

We represent a continuous-time large scale linear stochastic system in standard form

$$\mathbf{S}: \qquad \dot{x} = Ax + \Gamma e, \\ y = Cx + v, \tag{2.1}$$

where $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_p)^T$, $e = (e_1, \ldots, e_m)^T$ and $v = (v_1, \ldots, v_p)^T$ are its state, output, input and measurement noise vectors, respectively, while A, Γ and C are constant $n \times n$, $n \times m$ and $p \times n$ matrices, respectively. It is assumed that e and v are mutually independent white zero-mean stochastic processes with covariances $E\{e(t)e(\tau)^T\} = Q\delta(t - \tau)$ and $E\{v(t)v(\tau)^T\} = R\delta(t-\tau)$, respectively.

We will consider the problem of decentralized estimation in which N autonomous agents have the goal to generate their estimates ξ^i of the state x of $\mathbf{S}$, $i = 1, \ldots, N$, on the basis of: (1) locally available measurements; (2) specific *a priori* knowledge they possess about the system; and (3) real-time communication between the agents.

Formally, we assume that the i-th agent has a possibility to observe the p_i-dimensional vector $y^{(i)} = (y_{l_1^i}, \ldots, y_{l_{p_i}^i})^T$, composed of the components of y with indices specified by the agent's *output index set* $I_i^y = \{l_1^i, \ldots, l_{p_i}^i\}$, $l_1^i, \ldots, l_{p_i}^i \in \{1, \ldots p\}$, $l_1^i < \ldots < l_{p_i}^i$, $p_i \leq p$. According to (2.1), $y^{(i)} = C^{(i)}x^{(i)} + v^{(i)}$, where $x^{(i)}$ is an n_i-dimensional vector composed of the components of x selected by the agent's *state index set* $I_i^x = \{j_1^i, \ldots, j_{n_i}^i\}$, $j_1^i, \ldots, j_{n_i}^i \in \{1, \ldots n\}$, $j_1^i < \ldots < j_{n_i}^i$, $n_i \leq n$, $C^{(i)}$ is a constant $p_i \times n_i$ matrix and $v^{(i)}$ the measurement noise vector with covariance $R^{(i)}\delta(t-\tau)$, representing a part of v. Accordingly, we define the $n_i \times n_i$ matrix $A^{(i)}$ which contains the elements of A selected by the pairs of indices specified by $I_i^x \times I_i^x$, and the matrix $\Gamma^{(i)}$, composed of n_i rows of Γ selected by I_i^x. Consequently, the *local system models* available to the agents are defined by

$$\begin{aligned}\mathbf{S_i}: \qquad \dot{x}^{(i)} &= A^{(i)}x^{(i)} + \Gamma^{(i)}e,\\ y^{(i)} &= C^{(i)}x^{(i)} + v^{(i)},\end{aligned} \tag{2.2}$$

$i = 1, \ldots, N$; systems $\mathbf{S_i}$ represent *overlapping subsystems* of $\mathbf{S}$. Notice that decomposition of $\mathbf{S}$ into overlapping subsystems does not have to rely necessarily on a decomposition of the matrices from (2.1): the parameter matrices in (2.2) can also be obtained as a result of approximate modelling and local identification, approximate aggregation, etc. (see *e.g.* [80, 99, 19]).

We will assume that the agents are able to generate the overlapping local state estimates $\hat{x}^{(i)}$ of the vectors $x^{(i)}$ using steady state Kalman filters [4]. Since the final goal of all the agents is to get the estimates of the entire state vector x of $\mathbf{S}$, additional strategies can be added to the local estimators (*e.g.*, see [75, 103, 80, 29, 8]). However, all such approaches require a kind of centralized strategy or special, model dependent communications between the agents.

We propose an estimation algorithm based on the introduction of a consensus scheme specifying communications between the agents (see *e.g.* [107, 105, 23, 39, 49, 51, 58, 73, 72]). Namely, the estimate ξ^i of x generated by the i-th agent is given by

$$\mathbf{E_i}: \qquad \dot{\xi}^i = A_i\xi^i + \Sigma_{\substack{j=1\\ j\neq i}}^N K_{ij}(\tilde{\xi}^{i,j} - \xi^i) + L_i(y^{(i)} - C_i\xi^i), \tag{2.3}$$

$i = 1, \ldots, N$, where: A_i is an $n \times n$ matrix whose $n_i \times n_i$ elements are equal to those of $A^{(i)}$, but are placed at the indices specified by $I_i^x \times I_i^x$, while the remaining elements are zeros; C_i is, similarly, a $p_i \times n$ matrix with $p_i \times n_i$ elements equal to those of $C^{(i)}$, placed at row-indices specified by I_i^y (notice that $C^{(i)}x^{(i)} = C_ix$); L_i is an $n \times p_i$ matrix whose $n_i \times n_i$ elements are equal those of the steady state gains $L^{(i)}$ in the local Kalman filters for $\mathbf{S_i}$, placed at row-indices specified by I_i^x; K_{ij} are constant $n \times n$ gain matrices; $\tilde{\xi}^{i,j}$ represents the noisy estimate ξ^j communicated by the j-th node, *i.e.* $\tilde{\xi}^{i,j} = \xi^j + w_{ij}$, where w_{ij} is the n-dimensional zero-mean white *communication noise* between the nodes j and i, with covariance $E\{w_{ij}(t)w_{ij}(\tau)^T\} = W_{ij}\delta(t-\tau)$, $i, j = 1, \ldots, N$.

It is possible to observe that the proposed algorithm represents a combination of decentralized overlapping Kalman filters and a first order consensus scheme which tends to make the local estimates ξ^i as close as possible (*e.g.* [73, 72, 58]). Notice that the estimator $\mathbf{E_i}$ reminds structurally of the distributed optimization algorithm proposed in [107, 105, 13], and the parallel estimator proposed in [82].

Furthermore, we will assume that $K_{ij} = \text{diag}\,\{k_1^{ij}, \ldots, k_n^{ij}\}$, where $k_\nu^{ij} \geq 0$, $\nu = 1, \ldots, n$, $i, j = 1, \ldots, N$, and that $k_\nu^{ij} = h_\nu^{ij} g_\nu^{ij}$, $h_\nu^{ij}, g_\nu^{ij} \geq 0$, where g_ν^{ij} directly reflects structural properties of $\mathbf{S}$ and $\mathbf{S_j}$ and the uncertainty in the local estimates $\hat{x}^{(j)}$, while h_ν^{ij} reflects properties of communication links.

Therefore, the whole multi-agent network can be represented as a collection of n directed graphs (digraphs) with N nodes corresponding to the agents and edges with gains k_ν^{ij}, specifying transmission of particular components of the vectors ξ^i between the nodes. Let $\mathcal{G}_\nu$ represent the digraph connected to the ν-th component x_ν of x, $\nu = 1, \ldots, n$; its Laplacian L^{G_ν} is defined as $L^{G_\nu} = [L_{ij}^{G_\nu}]$, $L_{ij}^{G_\nu} = k_\nu^{ij}$, $i \neq j$, $L_{ii}^{G_\nu} = -\sum_{j, j\neq i} k_\nu^{ij}$, $i, j = 1, \ldots, N$ [27].

2.1.1.2 Stability

Let $\Xi = ((\xi^1)^T, \ldots, (\xi^N)^T)^T$; then, from (2.3) we have

$$\mathbf{E}: \qquad \dot{\Xi} = \Phi\Xi + \Lambda Y + K_\Xi \Sigma, \tag{2.4}$$

where $\Phi = [\Phi_{ij}], \Phi_{ij} = K_{ij}$, $i \neq j$, $\Phi_{ii} = A_i - L_i C_i - \sum_{j, j\neq i} K_{ij}$, $\Lambda = \text{diag}\{L_1, \ldots, L_N\}$, $K_\Xi = \text{diag}\{\tilde{K}^1, \ldots, \tilde{K}^N\}$, $\tilde{K}^i = \left[\, K_{i1} \,\vdots\, K_{i2} \,\vdots \cdots \vdots\, K_{iN} \,\right]$, $K_{ii} = 0$, $Y = ((y^1)^T, \ldots, (y^N)^T)^T$, $\Sigma = (w_{11}^T, \ldots, w_{1N}^T,\ w_{21}^T, \ldots, w_{2N}^T, \ldots, w_{N1}^T, \ldots, w_{NN}^T)^T$, $w_{ii} = 0$, $i, j = 1, \ldots, N$,. We will investigate stability of $\mathbf{E}$ in the sense of stability of Φ. The basic starting assumptions are:

(A.2.1.1) the local estimators $\bar{\mathbf{E}}_\mathbf{i}$ are asymptotically stable, *i.e.*, the matrices $A^{(i)} - L^{(i)}C^{(i)}$ are Hurwitz, $i = 1, \ldots, N$.

(A.2.1.2) For each $\mathcal{G}_\nu$, $\nu = 1, \ldots, n$, there is at least one *center node* μ (from which every node is reachable, *e.g.* [48]), satisfying $\nu \in I_\mu^x$.

In order to demonstrate stabilizability of $\mathbf{E}$ by a proper choice of the consensus gains, we will introduce the following notation: $h_\nu^{ij} = h'_{ij} \geq 0$ for $\nu \in I_i^x$, and $h_\nu^{ij} = h''_{ij} \geq 0$ for $\nu \in \bar{I}_i^x = \{1, \ldots, n\} \setminus I_i^x$, $\nu = 1, \ldots, n$. We will also introduce $G_1^{ij} = \mathrm{diag}\{g_{\nu_1^i}^{ij}, \ldots, g_{\nu_{n_i}^i}^{ij}\}$ and $G_2^{ij} = \mathrm{diag}\{g_{\bar{\nu}_1^i}^{ij}, \ldots, g_{\bar{\nu}_{n-n_i}^i}^{ij}\}$, where $\nu_1^i, \ldots, \nu_{n_i}^i \in I_i^x$ and $\bar{\nu}_1^i, \ldots, \bar{\nu}_{n-n_i}^i \in \bar{I}_i^x$, as well as $K_{ij}^{1,0} = h'_{ij} G_1^{ij}$ and $K_{ij}^{2,0} = h''_{ij} G_2^{ij}$.

We will also adopt the following additional assumptions:

(A.2.1.3) $\bigcup_{i=1}^N I_i^x = \{1, 2, \ldots, n\}$;

(A.2.1.4) $\bigcup_{i,j=1,\ldots,N; i \neq j} (I_i^x \bigcap I_j^x) \neq \emptyset$.

Assumptions (A.2.1.3) and (A.2.1.4) imply that all the components of the state vector x of $\mathbf{S}$ are estimated, and that there is at least one component estimated by more than one local estimator.

Theorem 2.1.1 Let the assumptions (A.2.1.1), (A.2.1.2), (A.2.1.3) and (A.2.1.4) hold. Then, for any given $h''_{ij} \geq 0$ and $g_\nu^{ij} \geq 0$, it is possible to find such $h'_{ij} \geq 0$ that the estimator $\mathbf{E}$ is asymptotically stable, $i, j = 1, \ldots, N$, $\nu = 1, \ldots, n$.

Proof: Matrix Φ in (2.4) is cogredient to $\Phi' = \begin{bmatrix} \Phi^{11} & \Phi^{12} \\ \Phi^{21} & \Phi^{22} \end{bmatrix}$, in which the blocks containing $A^{(i)} - L^{(i)} C^{(i)}$, $i = 1, \ldots, N$, are grouped together at the main block-diagonal in such a way that $\Phi^{11} = [\Phi_{ij}^{11}]$, $\Phi_{ij}^{11} = K_{ij}^{1,1}$, $i \neq j$, $\Phi_{ii}^{11} = A^{(i)} - L^{(i)} C^{(i)} - \sum_{j, j \neq i} K_{ij}^{1,0}$, so that $\Phi^{12} = [\Phi_{ij}^{12}]$, $\Phi_{ij}^{12} = K_{ij}^{1,2}$, $i \neq j$, $\Phi_{ii}^{12} = 0$, $\Phi^{21} = [\Phi_{ij}^{21}]$, $\Phi_{ij}^{21} = K_{ij}^{2,1}$, $i \neq j$, $\Phi_{ii}^{21} = 0$ and $\Phi^{22} = [\Phi_{ij}^{22}]$, $\Phi_{ij}^{22} = K_{ij}^{2,2}$, $i \neq j$, $\Phi_{ii}^{22} = -\sum_{j, j \neq i} K_{ij}^{2,0}$; $i, j = 1, \ldots, N$; by $K_{ij}^{1,1}$, $K_{ij}^{1,2}$, $K_{ij}^{2,1}$ $K_{ij}^{2,2}$ we denote the submatrices of K_{ij} obtained by deleting its elements with indices from $\bar{I}_i^x \times \bar{I}_j^x$, $\bar{I}_i^x \times I_j^x$, $I_i^x \times \bar{I}_j^x$ and $I_i^x \times I_j^x$, respectively.

Take such $h''_{ij} \geq 0$, $i, j = 1, \ldots, N$, that (A.2.1.2) is satisfied, and analyze the submatrix Φ^{22} (which depends on h''_{ij}, and not on h'_{ij}). Assumption (A.2.1.2) implies that each digraph $\bar{\mathcal{G}}_\nu$ opposite to $\mathcal{G}_\nu$, $\nu = 1, \ldots, n$ (obtained by reversing the direction of the arcs of $\mathcal{G}_\nu$), has only one closed strong component (a maximal induced strongly connected subdigraph with no arcs leaving its node set [27, 48]). Consequently, those submatrices of Φ' which represent Laplacians of $\mathcal{G}_\nu$, $\nu = 1, \ldots, n$, are cogredient to lower-block-triangular matrices with two diagonal blocks, where the first is an irreducible Metzler matrix which has one eigenvalue at the origin and the remaining ones in the left-half plane, and the second is a diagonally dominant Metzler matrix, which is, therefore stable [48, 49, 79]. The center nodes of $\mathcal{G}_\nu$ (or the globally reachable nodes of $\bar{\mathcal{G}}_\nu$) have to belong to the set of nodes of the unique closed strong component of $\bar{\mathcal{G}}_\nu$. Therefore, one concludes that Φ^{22} is composed of the submatrices of Φ' that are obtained from the irreducible Metzler matrices by deleting their rows and columns with indices corresponding to the nodes of the strong components of $\bar{\mathcal{G}}_\nu$. These irreducible submatrices are, in general, cogredient to

$$L_\nu^D = \begin{bmatrix} -\sum\limits_{j,j\neq 1}\alpha_{1j} & \alpha_{12} & \dots & \alpha_{1\tilde{N}} \\ \alpha_{21} & -\sum\limits_{j,j\neq 2}\alpha_{2j} & \dots & \alpha_{2\tilde{N}} \\ & \cdots & & \\ \alpha_{\tilde{N}1} & \alpha_{\tilde{N}2} & \dots & -\sum\limits_{j,j\neq \tilde{N}}\alpha_{\tilde{N}j} \end{bmatrix},$$

where $\tilde{N} \leq N$, $\alpha_{ij} \geq 0$, $\alpha_{21} > 0$ [24, 48, 27]. Deleting the first row and first column of L_ν^D we obtain a matrix in which the first row is strictly diagonally dominant, having in mind that $\alpha_{21} > 0$ as a consequence of irreducibility of L_ν^D. Consequently, this matrix is Metzler and quasidominant diagonal, which implies that it is Hurwitz (see *e.g.* [79]). Therefore, the whole matrix Φ^{22} is Hurwitz, having in mind assumptions (A.2.1.3) and (A.2.1.4).

Assuming now that $h'_{ij} = 0$, $i, j = 1, \dots, N$, we obtain that $\Phi^{12} = 0$ and that Φ^{11} is asymptotically stable, having in mind that the matrices $A^{(i)} - L^{(i)}C^{(i)}$, $i = 1, \dots, N$, are Hurwitz by assumption; this implies that the whole matrix Φ is Hurwitz. Retaining the same h''_{ij} as above and choosing such $h'_{ij} \geq 0$ that (A.2.1.2) is satisfied, we directly conclude that there exists such $\varepsilon > 0$ that the system $\mathbf{E}$ is asymptotically stable as long as $h'_{ij} \leq \varepsilon$, having in mind the continuous dependence between of eigenvalues of Φ on the values of h'_{ij} [30, 24]. Therefore, a stabilizing consensus scheme exists, and the theorem is proved. ■

Remark 2.1.1 It is straightforward to prove that in the case of nonoverlapping subsystems the proposed estimator is stable under the conditions (A.2.1.1), (A.2.1.2) and (A.2.1.4) for all nonnegative h_ν^{ij}.

Example 2.1.1 Consider as an illustration the estimator $\mathbf{E}$ with the state matrix $\Phi =$

$\begin{bmatrix} A_1 - h_{12}I & h_{12}I \\ h_{21}I & A_2 - h_{21}I \end{bmatrix}$, where A_1 and A_2 are $n \times n$ Hurwitz matrices and $h_{12}, h_{21} \geq 0$ (in this case $g_\nu^{ij} = 1$, $i, j = 1, 2, \nu = 1, \ldots, n$). According to Theorem 2.1.1, for $h_{12} = 0$, Φ is Hurwitz for any $h_{21} > 0$; it remains Hurwitz for h_{12} positive and small enough.

If $A_1 = A_2$, it follows directly that $\det\{\Phi + j\omega I\} \neq 0$ for all real ω and all $h_{12}, h_{21} > 0$. This implies stability, having in mind that Φ is stable for $h_{12} = h_{21} = 0$ and that the mapping of the parameters into the eigenvalues is continuous.

If $A_2 = 0$, the same determinant condition requires $\det\{(-h_{21} + j\omega)A_1 - \omega^2 I - j\omega(h_{12} + h_{21})I\} \neq 0$. If $-\sigma + j\Omega$ is any eigenvalue of A_1, this condition gives $h_{21}\sigma - jh_{21}\Omega - j\omega\sigma - \omega\Omega - \omega^2 - j\omega(h_{12} + h_{21}) \neq 0$, which is true for any h_{12}, $h_{21} > 0$ and all real ω. Therefore, Φ is again stable.

In general, when $A_1 \neq A_2$ and $A_1, A_2 \neq 0$, such a direct analysis becomes more complicated, but it is possible to conclude that only special structures of A_1 and A_2 can impose important restrictions on the stabilizing values of h_{12} and h_{21}.

2.1.1.3 Optimization

In this section we will demonstrate that the consensus parameters can be determined by minimizing the steady-state mean-square estimation error.

Inserting $Y = \Psi X + V$ in (2.4), where $X = (x^T, \ldots, x^T)^T$, $\Psi = \text{diag}\{C_1, \ldots, C_N\}$ and $V = ((v^{(1)})^T, \ldots, (v^{(N)})^T)^T$ is a white noise term with zero mean and covariance matrix R_V (which can be derived from R), one obtains from (2.1) and (2.4)

$$\textbf{SE}: \quad \dot{Z} = \begin{bmatrix} A & 0 \\ \tilde{\Phi} & \Phi \end{bmatrix} Z + B_Z\Theta = \Phi_Z Z + B_Z\Theta \tag{2.5}$$

where $Z = (x^T, (\Xi - X)^T)^T$, $\tilde{\Phi} = \text{col}\{(A_1 - A), \ldots, (A_N - A)\}$ (col$\{.\}$ denotes the block-column matrix composed of the listed elements), $B_Z = \text{diag}\{-\tilde{\Gamma}, \Lambda, K_\Xi\}$, $\tilde{\Gamma} = \text{col}\{\Gamma, \ldots, \Gamma\}$ and $\Theta = (e^T, V^T, \Sigma^T)^T$. Obviously, **SE** represents a stochastic system with the white noise Θ as a stochastic input. We will distinguish two cases. If Φ_Z is Hurwitz, the steady-state covariance P_Z of Z is defined by the positive semi-definite solution of the Lyapunov equation

$$\Phi_Z P_Z + P_Z \Phi_Z^T + B_Z R_Z B_Z^T = 0, \tag{2.6}$$

where R_Z is the covariance matrix of Θ, which can easily be derived. If $\tilde{\Phi} = 0$, the system itself can be unstable, but the steady-state covariance P of $\Xi - X$ can be directly obtained in the case of stable Φ by an appropriate splitting of (2.6).

If we define the vector H containing all the unknown parameters of the consensus scheme in $\mathbf{E}$, we can formulate the following optimization problem:

$$\min_H J = \min_H \mathrm{Tr}\, P; \tag{2.7}$$

solutions to this problem (which is, in general, not convex) can provide convenient consensus parameters for the proposed estimation scheme. The problem can be simplified by composing H only from the weights h_ν^{ij}, assuming that the parameters g_ν^{ij} for $\nu \in I_j^x, j = 1, \ldots, N$, are proportional to some measure of the accuracy of the ν-th component of the j-th agent's state vector estimate. It has been found to be convenient to adopt that g_ν^{ij} is proportional to the ν-th diagonal element of the inverse of the estimation error covariance matrix of the corresponding local Kalman filter.

Example 2.1.2 Let $\mathbf{S}$ be represented by a fourth order model with $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where $A_{11} = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix}$, with $\Gamma = I$ and $Q = I$. Assume that Agent 1 gets the measurements using $C = C_1 = [1\ 0\ 0\ 0]$ with $R = R_1$, and Agent 2 using $C = C_2 = [0\ 0\ 0\ 1]$ with $R = R_2$, and that both agents possess the knowledge of the entire state model; the communication noise is characterized by $W_{12} = 0.01$ and $W_{21} = 0.01$. Assuming that the consensus gains are $K_{12} = h_{12}G_{12}$ and $K_{21} = h_{21}G_{21}$, where G_{12} and G_{21} are diagonal matrices composed of the diagonal elements of the steady state estimation error covariances $P^{(2)}$ and $P^{(1)}$ of the local Kalman filters, parameters $h_{12} \geq 0$ and $h_{21} \geq 0$ are to be determined by optimization. Table 2.1 shows the results obtained for $R_2 = 1$ and different values of R_1. The criterion values J show high robustness of the proposed estimator. Both gains are higher for lower measurement noise levels; however, h_{21} decreases much more rapidly, and for high values of R_1 becomes close to zero, having in mind that the mean-square error of the local estimator $\bar{\mathbf{E}}_1$ becomes high.

Consider now three agents, the first two being the same as above (with $R_1 = R_2 = 1$),

	h_{12}	h_{21}	J
R_1=1	1521.9	855.5	1.9819
R_1=10	898.4	49.56	2.0102
R_1=100	170.2	1.927	2.0109
R_1=1000	110.2	0.026	2.0110

Table 2.1: Optimization results for different measurement noise levels

while the third observes the system using $C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $R_3 = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$. Optimization provides now six parameters, two per agent; the obtained results are: $k_{12} = 0.155$, $k_{13} = 0.355$, $k_{21} = 0.460$, $k_{23} = 0.300$, $k_{31} \approx 0$ and $k_{32} \approx 0$, taking, as above, diagonal matrices G_{ij} equal to the main diagonals of the corresponding local estimation error covariance inverses. Obviously, the scheme behaves as predicted: Agent 3, with the globally optimal Kalman estimator, does not need any help, so that the weights of the edges leading to it are approximately zero. On the other hand, Agents 1 and 2 take the more accurate estimates obtained from Agent 3 with higher gains.

When the local estimators are built using the local second-order state models defined only by the submatrices A_{11} and A_{22}, respectively, we obtain $h_{12} = 0.6311$ and $h_{21} = 0.8088$, with $J = 2.0271$, assuming $R_1 = R_2 = 1$, leading to the conclusion that the estimator is robust also with respect to modelling errors (see Table 2.1). Figure 2.1 depicts the form of the corresponding criterion function, which is in this case obviously convex.

Example 2.1.3 In this example we consider two agents in two situations: in the first, the subsystem models are disjoint, while in the second the subsystem models are of third order, and are, obviously, overlapping. We assume now that $\mathbf{S}$ is composed of $A_{11} = \begin{bmatrix} 1 & 1 \\ -1 & 0.2 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_{22} = \begin{bmatrix} -0.1 & 1 \\ -0.3 & -5 \end{bmatrix}$, and that in situation I Agent 1 utilizes A_{11}, and Agent 2 utilizes A_{22}. In situation II, we assume overlapping subsystems, with $A_1 = \left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ -1 & 0.2 & -1 & 0 \\ 0.1 & 0 & -0.1 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right]$ and $A_2 = \left[\begin{array}{c:ccc} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0.2 & -1 & 0 \\ 0 & 0 & -0.1 & 1 \\ 0 & 1 & -0.3 & -5 \end{array}\right]$. With the same noise levels as above, we obtained for situation I $k_1 = 0.001$ and $k_2 = 0.1791$, with $J = 35.43$, and for situation II $k_1 = 2.5421$ and $k_2 = 8.1781$, with $J = 7.8621$. This example shows possible advantages

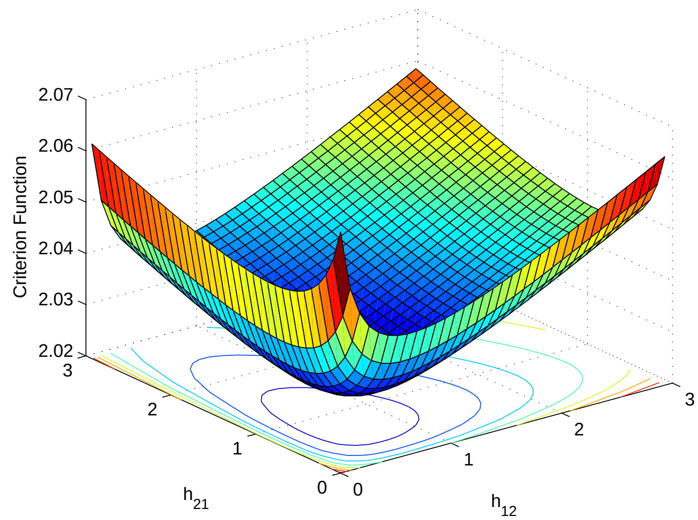

Figure 2.1: Criterion function

of overlapping decompositions with respect to the disjoint ones.

Example 2.1.4 In this case we consider the problem not explicitly addressed in this thesis: we will assume that the system has two deterministic inputs u_1 and u_2, so that we have in $\mathbf{S}$ (model (4.12)) the additional term $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. We take disjoint case as in the Example 2.1.2, and assume that Agent 1 knows only u_1 (square wave), and that Agent 2 knows only u_2 (sine wave). The estimator $\mathbf{E}$ is applied, with the usual modification taking care of the locally known deterministic inputs within the local Kalman filters (the consensus scheme remains unaltered) [4].

The given Figures 2.2 and 2.3 represent the estimation errors of ξ_1 and ξ_2 as functions of time in the case when $k_1 = k_2 = 0$ (Fig. 2.2), and in the case when the consensus scheme exists with $k_1 = k_2 = 10$ (Fig. 2.3). It is obvious that the consensus scheme efficiently reduces the estimation error in spite of the lack of the *a priori* knowledge about the inputs.

2.1.1.4 Denoising by Consensus

The aim of this subsection is to give an insight into an interesting aspect of the proposed algorithm: its capability to reduce the measurement noise influence as a function of both the number of nodes and the network connectivity. Convergence rate of the schemes for

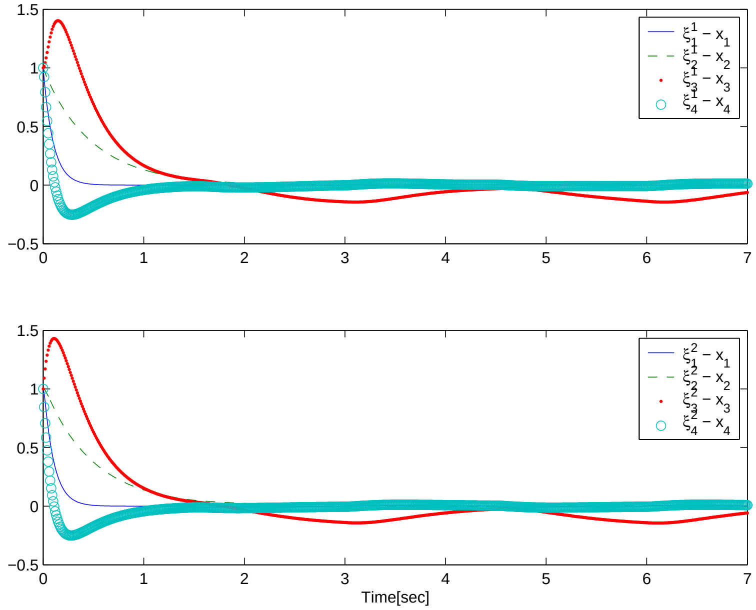

Figure 2.2: Estimation error for $k_1 = k_2 = 10$

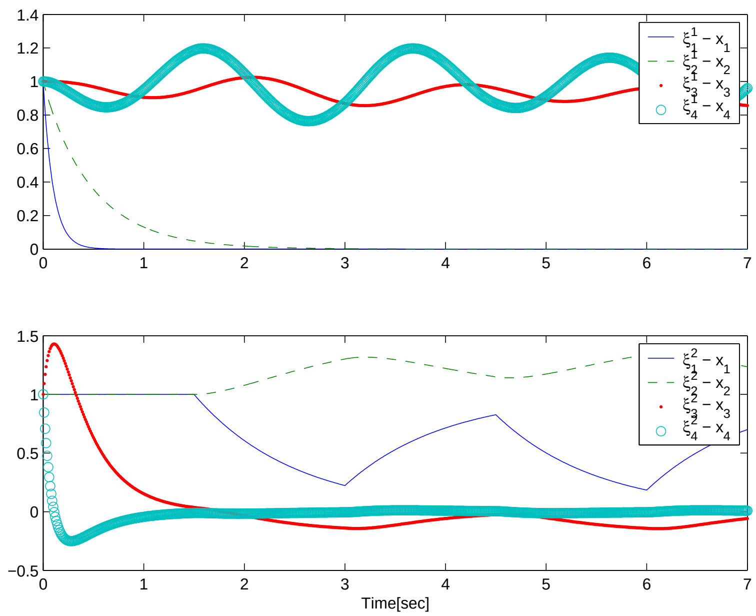

Figure 2.3: Estimation error for $k_1 = k_2 = 0$

consensus averaging has been studied in [110, 111]. Pursuing another line of thought, we will analyze asymptotic denoising capabilities of the proposed estimator using simple, yet representative examples of networks with different connectivities.

Case A) Consider first the case when the algorithm $\mathbf{E}$ in (2.4) consists of N identical local estimators of the state x of $\mathbf{S}$ and a consensus scheme with $\tilde{K} = [K_{ij}]$, $K_{ij} = kI$, $i \neq j$, $K_{ii} = -(N-1)kI$, $i, j = 1, \ldots, N$, $k > 0$ (fully connected graphs). Assuming that $w_{ij} = 0$, the steady-state estimation error covariance matrix P_N satisfies

$$\Phi^1 P_N + P_N \Phi^{1T} + Q_N = 0, \tag{2.8}$$

where $\Phi^1 = \tilde{A} + \tilde{K}$, $\tilde{A} = \mathrm{diag}\{\bar{A}, \ldots, \bar{A}\}$, $\bar{A} = A - LC$ (L is the local steady-state optimal Kalman gain), and $Q_N = [Q_{N,ij}]$, $Q_{N,ij} = \Gamma Q \Gamma^T$, $i \neq j$, $Q_{N,ii} = \Gamma Q \Gamma^T + LRL^T$, $i, j = 1, \ldots, N$. If $T_1 = \begin{bmatrix} I & I & \cdots & I \\ I & -(N-1)I & \cdots & I \\ & & \cdots & \\ I & & \cdots & -(N-1)I \end{bmatrix}$, we have $T_1^{-1} \tilde{K} T_1 = \mathrm{diag}\{0, -NkI, \ldots, -NkI\}$. Applying T_1^{-1} and T_1 to (2.8), we obtain for N large enough that the diagonal $n \times n$ blocks $P_N^{D,i}$ of $P_N^D = T_1^{-1} P_N T_1$, $i = 1, \ldots, N$ become: $P_N^{D,1} \approx N\hat{P}$, where $\hat{P}$ is the solution of the Lyapunov equation

$$\bar{A}\hat{P} + \hat{P}\bar{A}^T + \Gamma Q \Gamma^T = 0, \tag{2.9}$$

and $P_N^{D,i} \approx \frac{1}{2kN} LRL^T$, $i = 2, \ldots, N$.

If the *average performance index* of an estimator is defined as $\bar{J}_N = \frac{1}{N}\mathrm{Tr}P_N$, we obtain

$$\bar{J}_N = \frac{1}{N}\mathrm{Tr}P_N^D \approx \mathrm{Tr}\hat{P} + \frac{1}{2kN}\mathrm{Tr}(LRL^T),$$

so that $\bar{J} = \lim_{N\to\infty} \bar{J}_N = \mathrm{Tr}\hat{P}$. Obviously, the estimation scheme, averaging different realizations of the measurement noise, is able to achieve complete asymptotic denoising, since, according to (2.9), the term LRL^T is eliminated from the standard local Lyapunov equation

$$\bar{A}P^* + P^*\bar{A}^T + \Gamma Q \Gamma^T + LRL^T = 0 \tag{2.10}$$

for the covariance matrix P^* of one independent local estimator.

Case B) In the case of the consensus matrix with minimal connectivity which still satisfies (A.2.1.2), we have $\tilde{K} = \left.\begin{bmatrix} -kI & kI & & \cdots & 0 \\ 0 & -kI & kI & \cdots & 0 \\ & & \cdots & & \\ kI & 0 & \cdots & 0 & -kI \end{bmatrix}\right\} N$ (directed ring). Matrix T_2 transforming $\tilde{K}$ to its Jordan form retains from T_1 only the first block-column block, so that $P_N^{D,1}$ is the same as in Case A). However, we have

$$(\bar{A} + \lambda_i I)P_N^{D,i} + P_N^{D,i}(\bar{A} + \lambda_i I)^* + LRL^T = 0, \tag{2.11}$$

for $i = 2, \ldots, N$, where F^* denotes the conjugate transpose of F and λ_i are the nonzero distinct eigenvalues of the consensus matrix (which all lie on a circle with center at $(-k, 0)$ and radius k). According to [78], we have

$$\bar{J}_N = \frac{1}{N}\mathrm{Tr}P_N \geq \mathrm{Tr}\hat{P} + n\frac{\lambda_{min}(LRL^T)}{2\sigma_{max}(\bar{A} + \lambda_i I)}, \tag{2.12}$$

where $\lambda_{min}(.)$ denotes the minimal eigenvalue and $\sigma_{max}(.)$ the maximal singular value of an indicated matrix. Consequently, the estimator does not ensure complete asymptotic denoising, in spite of the fact that the underlying graph is strongly connected. This conclusion can be readily extended to double directed rings, as well as to all graphs with Laplacians in the form of *circulant matrices* with a predefined fixed number M of edges entering each node. Namely, in this case we have that $\max_i |\lambda_i| \leq 2kM$, so that the conclusions related to (2.11) can still be applied (see [28] for properties of circulant matrices). However, the criterion $\bar{J}_N$ still decreases with k. It is interesting that in the simple case of consensus averaging treated in [110, 111], asymptotic denoising is achieved whenever the underlying undirected graph is connected.

Case C) In general, it appears that the problem of defining the relationship between the network connectivity and the asymptotic denoising achievable by the proposed estimator is difficult. Consider here, however, a special case in which

$$\mathrm{Tr}\, P_N^{D,i} \leq \frac{\kappa_i}{|\lambda_i|}, \tag{2.13}$$

where $0 \leq \kappa_i \leq \kappa < \infty$, $i = 2, \ldots, N$. Then,

$$\bar{J} \leq \text{Tr}\hat{P} + \lim_{N \to \infty} \frac{\kappa}{(N-1)} \sum_{i=2}^{N} \frac{1}{|\lambda_i|}. \tag{2.14}$$

Let $1/|\lambda_{m_1}| \geq \ldots \geq 1/|\lambda_{m_i}| \geq \ldots \geq 1/|\lambda_{m_N}|$, $i = 1, \ldots, N$. Assuming that for any $\varepsilon > 0$ there exists a positive integer N_0 such that $1/|\lambda_{m_i}| < \varepsilon$ for all $i > N_0$, we obtain $\bar{J} = \text{Tr}\hat{P}$ as in Case A), since the second term in (2.14) tends to zero when $N \to \infty$.

In particular, in the case of Laplacians in the form of circulant matrices treated already in Case B), we have that

$$\lambda_i = k(-M(N) + \sum_{l=1}^{M(N)} e^{-j\frac{2\pi}{N}(i-1)l}), \tag{2.15}$$

$i = 2, \ldots, N$, and that (2.11) holds, where $M(N)$ represents the number of edges entering each node, which is here supposed to depend on N [28]. It is possible to see that in the case when $\lim_{N\to\infty} M(N) = \infty$ we have that $\max_i |\lambda_i| = \infty$, and that the above assumption about the nature of the sequence $\{1/|\lambda_{m_i}|\}$ holds, so that, accordingly, (2.14) implies complete asymptotic denoising.

The given examples show that complete asymptotic denoising results from sufficient graph connectedness.

Communication noise. When the communication noise exists, the Lyapunov equation for the estimation error covariance contains an additional term depending on the matrix $W_{ij} = W$. Then, for example, one can show that in the case of the fully connected graph (Case A))

$$\bar{J} = \lim_{N\to\infty} \frac{1}{N} \text{Tr} P_N = \text{Tr}\hat{P}_1 + \frac{k}{2}\text{Tr}W.$$

where

$$\bar{A}\hat{P}_1 + \hat{P}_1\bar{A}^T + \Gamma Q \Gamma^T + k^2 W = 0.$$

The whole scheme works better than the set of N independent local Kalman filters in spite of the communication noise if $\text{Tr}\hat{P}_1 + \frac{k}{2}\text{Tr}W < \text{Tr}P^*$, where P^* is the solution of the Lyapunov equation (2.10).

Example 2.1.5 The estimator in this example consists of a set of identical local Kalman filters estimating the whole state of the fourth order system described in Example 2.1.2 ($C = I$), connected by a consensus scheme. The average criterion $\bar{J}_N = \frac{1}{N}\mathrm{Tr}P_N$ has been calculated as a function of the number of nodes N for the network topologies from Case A) (solid lines) and Case B) (dotted lines); the consensus gain k has been taken as a parameter. The horizontal line corresponds to $\mathrm{Tr}\hat{P}$, the lower bound obtained by using (2.9). The presented results (Fig. 2.4) fully confirm the given theoretical analysis.

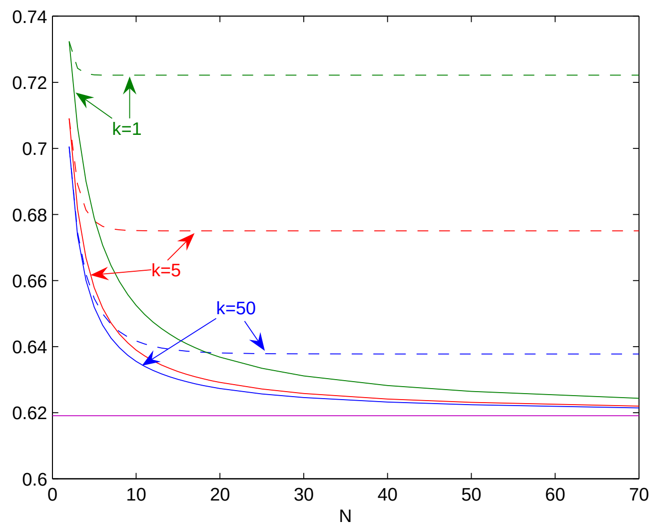

Figure 2.4: Average criterion as a function of N

2.1.2 Discrete-Time Case

Now, we will consider a discrete-time version of the proposed consensus based estimation scheme, where we will assume that the inter-agent network is lossy, *i.e.* that communication faults can happen, with some predefined probabilities, and that the local measurements are intermittent.

2.1.2.1 Problem Definition

Let a finite-dimensional discrete-time stochastic system be represented by

$$\begin{aligned} \mathbf{S}: \qquad & x(t+1) = Fx(t) + Ge(t), \\ & y(t) = Hx(t) + v(t) \end{aligned} \tag{2.16}$$

where t is the discrete-time instant, $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_p)^T$, $e = (e_1, \ldots, e_m)^T$ and $v = (v_1, \ldots, v_p)^T$ are its state, output, input and measurement noise vectors, respectively, while F, G and H are constant $n \times n$, $n \times m$ and $p \times n$ matrices, respectively. It is assumed that $\{e(t)\}$ and $\{v(t)\}$ are white zero-mean sequences of independent vector random variables with covariance matrices Q and R, respectively.

Similarly as in the continuous-time case, we will assume that the i-th agent has a possibility to observe the p_i-dimensional vector $y^{(i)} = (y_{l_1^i}, \ldots, y_{l_{p_i}^i})^T$, composed of the set of components of y with indices contained in the agent's output index set $I_i^y = \{l_1^i, \ldots, l_{p_i}^i\}$, $l_1^i, \ldots, l_{p_i}^i \in \{1, \ldots p\}$, $l_1^i < \ldots < l_{p_i}^i$, $p_i \leq p$. We will assume further that the i-th agent possesses the local system model $\mathbf{S_i}$ defined as

$$\begin{aligned} \mathbf{S_i}: \qquad & x^{(i)}(t+1) = F^{(i)}x^{(i)}(t) + G^{(i)}e(t), \\ & y^{(i)}(t) = H^{(i)}x^{(i)}(t) + v^{(i)}(t), \end{aligned} \tag{2.17}$$

$i = 1, \ldots, N$, where $x^{(i)}$ is an n_i-dimensional vector composed of the components of x selected by the agent's state index set $I_i^x = \{j_1^i, \ldots, j_{n_i}^i\}$, $j_1^i, \ldots, j_{n_i}^i \in \{1, \ldots n\}$, $j_1^i < \ldots < j_{n_i}^i$, $n_i \leq n$, and $v^{(i)}$ is a measurement noise vector containing the components of v selected by I_i^y, having the covariance matrix $R^{(i)}$ (which can be readily obtained from R); $F^{(i)}$, $G^{(i)}$ and $H^{(i)}$ are $n_i \times n_i$, $n_i \times m$ and $p_i \times n_i$ matrices, respectively.

Starting from the local model $\mathbf{S_i}$ and the accessible measurements $y^{(i)}$, the i-th agent is supposed to be able to generate autonomously the local estimate $\hat{x}^{(i)}$ of the vector $x^{(i)}$. The following local estimator will be assumed to be implementable by the i-th agent:

$$\bar{\mathbf{E}}_\mathbf{i}: \ \hat{x}^{(i)}(t+1|t) = F^{(i)}\hat{x}^{(i)}(t|t-1) + \gamma_i(t)F^{(i)}L^{(i)}[y^{(i)}(t) - H^{(i)}\hat{x}^{(i)}(t|t-1)], \tag{2.18}$$

where $L^{(i)}$ is the steady state Kalman gain given by $L^{(i)} = P^{(i)}H^{(i)T}[H^{(i)}P^{(i)}H^{(i)T}+R^{(i)}]^{-1}$, $P^{(i)}$ is a solution of the algebraic Riccati equation

$$P^{(i)} = F^{(i)}[P^{(i)} - L^{(i)}H^{(i)}P^{(i)}]F^{(i)T} + G^{(i)}QG^{(i)T}, \tag{2.19}$$

while $\gamma_i(t)$ is a scalar equal to 1 when the i-th agent receives measurements $y^{(i)}$, and 0 otherwise ([81]). It is natural to assume that subsystems $\mathbf{S_i}$ are defined in such a way that the pairs $(F^{(i)}, G^{(i)}Q^{\frac{1}{2}})$ are stabilizable and the pairs $(F^{(i)}, H^{(i)})$ detectable, so that the matrices $F^{(i)} - L^{(i)}H^{(i)}$, the state matrices of the estimators (2.18), are asymptotically stable and $P^{(i)} > 0$, $i = 1, \ldots, N$, ([4, 81]). The estimator based on the steady-state gain $L^{(i)}$ has been selected for the sake of clarity of presentation aimed dominantly at structural aspects of the proposed estimator; even better performance can be expected in practice from estimators with time varying gains (see *e.g.* [81]). In general, the local estimators can be designed using any methodology, in such a way that the general requirements formulated below are satisfied (robust estimators, fault detection filters, etc).

In a similar way as in the continuous-time case, we propose the following algorithm, based on the introduction of a discrete-time consensus scheme:

$$\begin{aligned} \mathbf{E_i}: \qquad \xi_i(t|t) &= \xi_i(t|t-1) + \gamma_i(t)L_i[y^{(i)}(t) - H_i\xi_i(t|t-1)], \\ \xi_i(t+1|t) &= \textstyle\sum_{j=1}^{N} C_{ij}(t)F_j\xi_j(t|t) \end{aligned} \tag{2.20}$$

$i = 1, \ldots, N$, where ξ_i is an estimate of x generated by the i-th agent, F_i is an $n \times n$ matrix with $n_i \times n_i$ nonzero elements that are equal to those of $F^{(i)}$, but are placed at the indices defined by $I_i^x \times I_i^x$, while H_i and L_i are $p_i \times n$ and $n \times p_i$ matrices, respectively, obtained from $H^{(i)}$ and $L^{(i)}$ in the same way as F_i is obtained from $F^{(i)}$. We will assume that $C_{ij}(t)$, $i, j = 1, \ldots, N$, are $n \times n$ time-varying gain matrices defining communications between the nodes, given in the form $C_{ij}(t) = k_{ij}(t)K_{ij}(t)$, where $k_{ij}(t) = 1$ when the directed communication link from the node j to the node i exists, and $k_{ij}(t) = 0$ otherwise; $K_{ij}(t)$ are diagonal matrices with nonnegative elements, giving appropriate weights to the estimates communicated between the agents. Furthermore, we will assume that $\{k_{ij}(t)\}$,

$i, j = 1, \ldots, N, i \neq j$, are mutually independent scalar sequences of independent binary random variables, satisfying $P\{k_{ij}(t) = 1\} = p_{ij}$ and $P\{k_{ij}(t) = 0\} = 1 - p_{ij}$ for $i \neq j$, as well as that $k_{ii}(t) = 1$, $i = 1, \ldots, N$. Also, we will assume that $\{\gamma_i(t)\}$ is a sequence of independent binary random variables independent of $\{k_{ij}(t)\}$, $i, j = 1, \ldots, N, i \neq j$, such that $P\{\gamma_i(t) = 1\} = p_{ii}$ and $P\{\gamma_i(t) = 0\} = 1 - p_{ii}$. We will also introduce the random vector Ξ_t composed of N^2 binary components: $N(N-1)$ elements $k_{ij}(t)$ $(i, j = 1, \ldots, N, i \neq j)$ and N elements $\gamma_i(t)$. This vector is, by assumption, generated on the basis of Bernoulli trials, *i.e.*, $\{\Xi_t\}$ represents a sequence of independent random vectors; let π_r be the time invariant probabilities of all possible realizations $\Xi^{[r]}$ of Ξ_t, $r = 1, \ldots, \nu$, $\nu = 2^{N^2}$.

Define the $nN \times nN$ consensus matrix $\tilde{C}(t) = [C_{ij}(t)]$, $i, j = 1, \ldots, N$, and assume that it is *row-stochastic* for all t, *i.e.* $\tilde{C}(t)$ is a non-negative matrix in which the sum of the elements in each row is equal to one ([30]). This assumption is in accordance with the definition of discrete-time consensus schemes presented in *e.g.* [39, 72, 107, 57]. Having in mind uncertainty of the communication links, this assumption practically implies recalculation or re-scaling of the sub-matrices $K_{ij}(t)$ composing the consensus matrix $\tilde{C}(t)$ for each new realization of $k_{ij}(t)$, $i, j = 1, \ldots, N$. This re-scaling does not impose any difficulty and can be easily done locally by each agent in many different ways. One of the straightforward possibilities is to adopt initial diagonal positive semidefinite matrices $K_{ij}(0) = K_{ij}^0$, $i, j = 1, \ldots, N$, according to some predefined criterion (*e.g.* accuracy of the local estimation), and to obtain $\tilde{C}(t)$ for each t by dividing all the elements of each row of the matrix $\tilde{C}^0(t) = [k_{ij}(t)K_{ij}^0]$, $i, j = 1, \ldots, N$, by the sum of all the elements of the same row, *i.e.* $\tilde{C}(t) = \bar{c}(t)\tilde{C}^0(t)$, where $\bar{c}(t) = \mathrm{diag}\{\sum_j k_{1j}(t)(K_{1j}^0)_1, \ldots, \sum_j k_{1j}(t)(K_{1j}^0)_N, \ldots, \sum_j k_{Nj}(t)(K_{Nj}^0)_1, \ldots, \sum_j k_{Nj}(t)(K_{Nj}^0)_N\}^{-1}$ and $(K_{ij}^0)_l$ represents the l-th element at the diagonal of the block $K_{ij}^0 = \mathrm{diag}\{(K_{ij}^0)_1, \ldots, (K_{ij}^0)_N\}$.

The proposed estimator is strictly scalable as far as the calculation of $\xi_i(t|t)$ in (2.20) is concerned, since it does not depend on the number of agents; on the other hand, calculation of $\xi_i(t+1|t)$ remains scalable as long as each agent communicates with a bounded number of neighbors. Consequently, scalability of the algorithm can be violated only when the structure of the consensus matrix $\tilde{C}(t)$ is such that the number of connections per node tends to infinity when N tends to infinity.

Let us introduce the following notation: $\tilde{F}_E = \mathrm{diag}\{F_1, \ldots, F_N\}$, $\tilde{\Phi} = \mathrm{diag}\{\Phi_1, \ldots, \Phi_N\}$, $\Phi_i = F_i - L_i H_i$, and $\tilde{A}(t) = \tilde{C}(t)\tilde{\Phi}$. Introducing $\hat{X}(t|t) = \mathrm{vec}\{\xi_1(t|t), \ldots, \xi_N(t|t)\}$ and $\hat{X}(t+1|t) = \mathrm{vec}\{\xi_1(t+1|t), \ldots, \xi_N(t+1|t))\}$, we can obtain a compact formulation of the proposed algorithm

$$\begin{aligned}\hat{X}(t|t) &= \hat{X}(t|t-1) + \tilde{L}[Y(t) - \tilde{H}\hat{X}(t|t-1)]\\ \hat{X}(t+1|t) &= \tilde{C}(t)\tilde{F}_E\hat{X}(t|t),\end{aligned} \tag{2.21}$$

where $Y(t) = \mathrm{vec}\{y^{(1)}(t), \ldots, y^{(N)}(t)\}$, $\tilde{L} = \mathrm{diag}\{L_1, \ldots, L_N\}$ and $\tilde{H} = \mathrm{diag}\{H_1, \ldots, H_N\}$ ($\mathrm{vec}\{.,.\}$ represents a column vector obtained by concatenation of the column vectors in the braces). Further, for the prediction error $\varepsilon(t+1|t) = \hat{X}(t+1|t) - X(t+1)$, where $X(t) = \mathrm{vec}\{x(t), \ldots, x(t)\}$, we obtain $\varepsilon(t+1|t) = \tilde{A}(t)\varepsilon(t|t-1) + \tilde{C}(t)(\tilde{F}_E - \tilde{F})X(t) + \tilde{C}(t)\tilde{\Gamma}(t)\tilde{L}\tilde{H}V(t) - E(t)$, where $\tilde{F} = \mathrm{diag}\{F, \ldots, F\}$, $V(t) = \mathrm{vec}\{v^{(1)}(t), \ldots, v^{(N)}(t)\}$ and $E(t) = \mathrm{vec}\{e(t), \ldots, e(t)\}$. Consequently, we obtain the following state space system-estimator model:

$$Z(t+1) = \left[\begin{array}{c:c} \tilde{F} & 0 \\ \hdashline \tilde{C}(t)(\tilde{F}_E - \tilde{F}) & \tilde{A}(t) \end{array}\right] Z(t) + \left[\begin{array}{c:c} \tilde{G} & 0 \\ \hdashline -\tilde{G} & \tilde{C}(t)\tilde{L}\tilde{H} \end{array}\right] N(t), \tag{2.22}$$

where $Z(t) = \mathrm{vec}\{X(t), \varepsilon(t|t-1)\}$ and $N(t) = \mathrm{vec}\{E(t), V(t)\}$.

Furthermore, we obtain

$$\bar{Z}(t+1) = \tilde{B}(t)\bar{Z}(t), \tag{2.23}$$

where $\bar{Z}(t) = E\{Z(t)\}$ and $\tilde{B}(t) = \left[\begin{array}{c:c} \tilde{F} & 0 \\ \hdashline \tilde{C}(t)(\tilde{F}_E - \tilde{F}) & \tilde{A}(t) \end{array}\right]$ and

$$\mathrm{col}\{P(t+1)\} = (\tilde{B}(t) \otimes \tilde{B}(t))\mathrm{col}\{P(t)\} + (\tilde{D}_{[r]} \otimes \tilde{D}_{[r]})\mathrm{col}\{W\}] \tag{2.24}$$

where $P(t) = E\{Z(t)Z(t)^T\}$ and $\tilde{D}(t) = \left[\begin{array}{c:c} \tilde{G} & 0 \\ \hdashline -\tilde{G} & \tilde{C}(t)]\tilde{L}\tilde{H} \end{array}\right]$ and $W = E\{N(t)\ N(t)^T\} = \mathrm{diag}\{Q^*, \tilde{R}\}$, where $Q^* = \begin{bmatrix} Q & \cdots & Q \\ \vdots & & \\ Q & \cdots & Q \end{bmatrix}$ and $\tilde{R} = \mathrm{diag}\{R^{(1)}, \ldots, R^{(N)}\}$. ($\mathrm{col}\{.\}$ denotes a vector obtained by concatenating columns of an indicated matrix and the sign $\otimes$ denotes the Kronecker's product).

2.1.2.2 Stability

In the stability analysis of the proposed estimator, we will use the following results from the matrix analysis.

Lemma 2.1.1 [60] Let $f(.)$ be a matrix norm having the property $f(A) \leq f(B)$ for two $n \times n$ matrices A and B satisfying $A \leq B$ ($A \geq 0$ means that all the elements of A are nonnegative). Let $g(.)$ be any matrix norm and let A be partitioned into square blocks A_{ii}. Then, $h(A)$ is a matrix norm, where

$$h(A) = f\left(\begin{bmatrix} g(A_{11}) & \cdots & g(A_{1k}) \\ \vdots & & \vdots \\ g(A_{k1}) & \cdots & g(A_{kk}) \end{bmatrix}\right). \tag{2.25}$$

■

Lemma 2.1.2 ([30], Lemma 5.6.10) Let A be an $n \times n$ matrix and $\varepsilon > 0$. Then, there exists a matrix norm $\|A\|$ such that

$$\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon, \tag{2.26}$$

where $\rho(A)$ is the spectral radius of a matrix A ($\rho(A) = \max_i |\lambda_i(A)|$, where $\lambda_i(A)$ are the eigenvalues of A). ■

A norm satisfying the requirement (2.26) is the norm $\|A\|_\tau = \|D_\tau U^T A U D_\tau^{-1}\|_\infty$, where U is an orthogonal matrix in the representation $A = U \Delta U^T$, where Δ is an upper triangular matrix (according to the Schur's theorem), $D_\tau = \mathrm{diag}\{\tau, \tau^2, \tau^3, \ldots, \tau^n\}$ and $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$ (for $A = [a_{ij}]$, $i, j = 1, \ldots, n$). Inequality (2.26) is satisfied for any given $\varepsilon > 0$ by choosing $\tau \geq 0$ large enough.

The following two theorems give sufficient conditions for stability of the proposed algorithm in the sense of convergence to zero of the estimation error mean and boundedness of the mean-square error. The analysis is based on the definition of a new, specially constructed norm according to Lemma 2.1.1, adapted to the partition of the consensus matrix $\tilde{C}(t)$ into the blocks $C_{ij}(t)$, and the methodology from [81, 54, 55].

Theorem 2.1.2 Let $\tilde{A}_{[r]}$ be partitioned into blocks $\tilde{A}_{jk}^{[r]} = C_{jk}^{[r]} \Phi_j^{[r]}$, where $C_{jk}^{[r]}$ and $\Phi_j^{[r]}$

are realizations of $C_{jk}(t)$ and $\Phi_j(t)$ obtained by choosing $\Xi_t = \Xi^{[r]}$, and let $\rho(\Phi_k^{[r]}) < b_k^{[r]} < \infty$, $k = 1, \ldots, N$, together with

$$\sum_{r=1}^{\nu} \pi_r \max_j \sum_{k=1}^{N} \rho(C_{jk}^{[r]}) b_k^{[r]} < 1. \tag{2.27}$$

Then, $\lim_{t\to\infty} E\{\varepsilon(t|t-1)\} = 0$ if the system (2.16) is asymptotically stable. If the system (2.16) is not asymptotically stable, $\lim_{t\to\infty} E\{\varepsilon(t|t-1)\} = 0$ if, additionally, $\tilde{F}_E = \tilde{F}$.

Proof: Consider the matrix $\tilde{A}_{[r]}$ and define its norm $\|\tilde{A}_{[r]}\|_\star$ in the following way:

$$\|\tilde{A}_{[r]}\|_\star = \left\| \begin{bmatrix} \|C_{11}^{[r]}\Phi_1^{[r]}\|_\tau & \cdots & \|C_{1N}^{[r]}\Phi_N^{[r]}\|_\tau \\ \vdots & & \vdots \\ \|C_{N1}^{[r]}\Phi_1^{[r]}\|_\tau & \cdots & \|C_{NN}^{[r]}\Phi_N^{[r]}\|_\tau \end{bmatrix} \right\|_\infty , \tag{2.28}$$

having in mind properties of the norm $\|.\|_\infty$, and Lemma 2.1.1. For particular terms in (2.28) we have that $\|C_{jk}^{[r]}\Phi_k^{[r]}\|_\tau \leq \rho(C_{jk}^{[r]})\|\Phi_k^{[r]}\|_\tau$, having in mind that $\|C_{jk}^{[r]}\|_\tau = \rho(C_{jk}^{[r]})$ for diagonal matrices $C_{jk}^{[r]}$. Moreover, it is always possible to find such a $\bar{\tau} > 0$ that for any $\tau > \bar{\tau}$ we have $\|\Phi_k^{[r]}\|_\tau \leq \rho(\Phi_k^{[r]}) + \varepsilon$, for any given $\varepsilon > 0$. Making ε small enough we always have that $\rho(\Phi_k^{[r]}) + \varepsilon \leq b_k^{[r]}$ (having in mind that the assumption $\rho(\Phi_k^{[r]}) < b_k^{[r]}$ is in the form of a strict inequality). Therefore, $\|\Phi_k^{[r]}\|_\tau \leq b_k^{[r]}$, and consequently,

$$\|\tilde{A}_{[r]}\|_\star \leq \max_j \sum_{k=1}^{N} \rho(C_{jk}^{[r]}) b_k^{[r]},$$

so that the matrix $\sum_{r=1}^{\nu} \pi_r \tilde{A}_{[r]}$ is Hurwitz if (2.27) holds, implying that the model for the mean (2.23) is asymptotically stable if $\tilde{F}$ is Hurwitz. The second statement of the Theorem follows trivially from the definition of the matrix $\tilde{B}_{[r]}$, since $E\{X(t)\}$ and $E\{\varepsilon(t|t-1)\}$ become decoupled. Thus the result. ■

Theorem 2.1.3 The proposed estimator provides $\|S(t)\| < \infty$, where $S(t) = E\{\varepsilon(t|t-1)\varepsilon(t|t-1)^T\}$, $\forall t \in \mathcal{I}$, ($\mathcal{I}$ is the set of all integers), if $\rho(\Phi_k^{[r]}) < b_k^{[r]} < \infty$, $k = 1, \ldots, N$,

$$\sum_{r=1}^{\nu} \pi_r [\max_j \sum_{k=1}^{N} \rho(C_{jk}^{[r]}) b_k^{[r]}]^2 < 1 \tag{2.29}$$

and the system (2.16) is asymptotically stable. If the system (2.16) is not asymptotically stable, $\|S(t)\| < \infty$ if, additionally, $\tilde{F}_E = \tilde{F}$.

Proof: If $\tilde{A}_{[r]}$ is partitioned into $n \times n$ blocks $A_{ij}^{[r]}$, $i, j = 1, \ldots, N$, then it is possible to show that the matrix $\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}$ is cogredient to

$$\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P = \begin{bmatrix} A_{11}^{[r]} \otimes A_{11}^{[r]} & \ldots & A_{11}^{[r]} \otimes A_{1N}^{[r]} & \ldots & A_{1N}^{[r]} \otimes A_{1N}^{[r]} \\ A_{11}^{[r]} \otimes A_{21}^{[r]} & \ldots & A_{11}^{[r]} \otimes A_{2N}^{[r]} & \ldots & A_{1N}^{[r]} \otimes A_{2N}^{[r]} \\ \vdots & & & & \\ A_{21}^{[r]} \otimes A_{11}^{[r]} & \ldots & A_{21}^{[r]} \otimes A_{1N}^{[r]} & \ldots & A_{2N}^{[r]} \otimes A_{1N}^{[r]} \\ \vdots & & & & \\ A_{N1}^{[r]} \otimes A_{N1}^{[r]} & \ldots & A_{N1}^{[r]} \otimes A_{NN}^{[r]} & \ldots & A_{NN}^{[r]} \otimes A_{NN}^{[r]} \end{bmatrix}$$

i.e. $\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P = T_p(\tilde{A}_{[r]} \otimes \tilde{A}_{[r]})T_p^T$, where T_p is a permutation transformation. Therefore, the norm $\|\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P\|_\star$ is a norm $\|\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}\|_\circ$ of $\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}$, *i.e.*

$$\|\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}\|_\circ = \|\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P\|_\star = \left\| \begin{bmatrix} \|A_{11}^{[r]} \otimes A_{11}^{[r]}\|_\tau & \ldots & \|A_{1N}^{[r]} \otimes A_{1N}^{[r]}\|_\tau \\ \vdots & & \vdots \\ \|A_{N1}^{[r]} \otimes A_{N1}^{[r]}\|_\tau & \ldots & \|A_{NN}^{[r]} \otimes A_{NN}^{[r]}\|_\tau \end{bmatrix} \right\|_\infty .$$

Majorizing the last expression similarly as in Theorem 2.1.2, one obtains that

$$\|\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}\|_\circ \leq \max_{j,l} \sum_{k=1}^{N} \rho(C_{jk}^{[r]}) b_k^{[r]} \sum_{m=1}^{N} \rho(C_{lm}^{[r]}) b_m^{[r]},$$

so that the matrix $\sum_{r=1}^{\nu} \pi_r(\tilde{A}_{[r]} \otimes \tilde{A}_{[r]})$ is Hurwitz if (2.29) holds. As (2.29) implies (2.27), we also have that both matrices $\sum_{r=1}^{\nu} \pi_r(\tilde{F}_E \otimes \tilde{A}_{[r]})$ and $\sum_{r=1}^{\nu} \pi_r(\tilde{A}_{[r]} \otimes \tilde{F}_E)$ are Hurwitz if $\tilde{F}_E$ is Hurwitz, implying asymptotic stability of the model (2.24), and, therefore, boundedness of $P(t)$ (and, consequently, of $S(t)$). The second statement follows directly, since $\tilde{F}_E = \tilde{F}$ decouples the models of the system and the estimation error. Thus the result. ∎

Remark 2.1.2 A comparison of the above results with the results related to the continuous-time estimator shows basic similarity of the main ideas and some technical differences. The main point of the stability analysis presented therein has been to show the existence of stabilizing consensus gains assuming that the local estimators are asymptotically stable. The above results provide a more specific insight into the the influence of the particular components of the system, supposing intermittent observations and communication faults. It is important to notice that Theorems 2.1.2 and 2.1.3 do not assume explicitly asymptotic stability of the local estimators: parameters of the consensus matrix can be selected in such a way that the conditions (2.27) and (2.29) hold in spite of the fact that $b_k^{[r]} > 1$ for some k and r, *i.e.* when some local estimators are unstable. This is a clear consequence of the eventual instability of some local estimators for $\gamma_i(t) = 0$, having in mind that Theorems 2.1.2 and 2.1.3 deal with the average behavior of the whole estimator. However, instability of some local estimators can be tolerated even in the case of no measurement and communication errors (with probability 1). We can directly observe that in this case the condition (2.27) can be satisfied for some consensus parameters provided for each j there exists a term $\rho(C_{jk}^{[1]})b_k^{[1]} \neq 0$ in which $b_k^{[1]} < 1$ (there is only one realization $\Xi^{[1]}$ in the case of no errors). This condition, requiring, in fact, that each unstable node receives information directly from at least one stable node, is too conservative. Note here only that it is possible to show that there exist stabilizing consensus parameters in more general cases when all the nodes with unstable local estimators are reachable from at least one node with a stable local estimator. ■

Example 2.1.6 Intercommunications between the agents introduced by the consensus matrix increase, in principle, robustness to measurement faults. A clear insight can be obtained by analyzing a simple example with two estimators. Assume that the system is of first order and unstable, with $F = 1.1$; assume also that $F^{(1)} = 1.2$, $L^{(1)} = 0.7$ and $H^{(1)} = 1$ for the first agent, and $F^{(2)} = 1.2$, $L^{(2)} = 0.9$ and $H^{(2)} = 1$ for the second, according to (2.17) and (2.18). Both estimators are stable when the measurements are available (when $\gamma_i = 1$). Assume also that a multi-agent network is implemented with the fixed consensus matrix $\tilde{C} = 0.5 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, according to the proposed algorithm (2.21). Figure 2.5 contains stability boundaries in the $1-p_{11}, 1-p_{22}$-plane in the sense of Theorem 2.1.2 (label (1)) and Theorem 2.1.3 (label (2)) for different values of the communication probability $p = p_{12} = p_{21}$; solid lines are obtained by using the derived conditions (2.27) or (2.29), while the dotted lines correspond to the experimentally obtained real stability boundaries (for $p = 0$ explicit results can be obtained by using [81, 54]). The derived boundaries are based on sufficient conditions and are conservative, as expected; however, the beneficial effects of the consensus scheme are obvious. ■

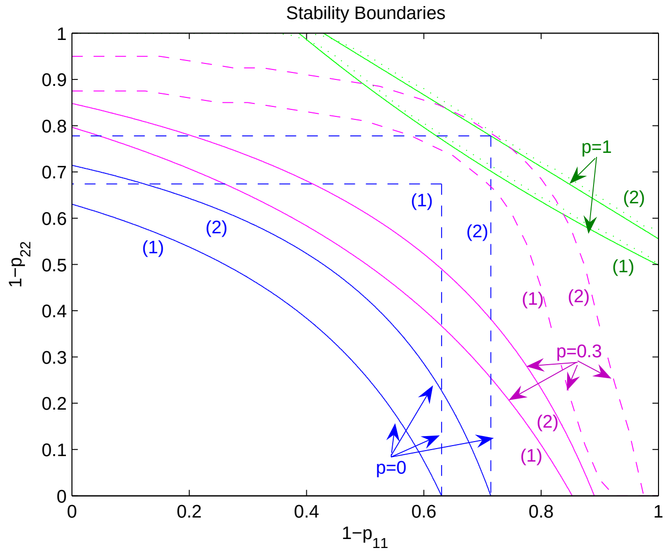

Figure 2.5: Stability boundaries

Example 2.1.7 Basic effects of introducing the consensus scheme in the proposed estimator are further illustrated by the following example. The system **S** is assumed to be represented by (2.16) with $F = \begin{bmatrix} 1/2 & 1 & 0 \\ -1 & -1/5 & -1 \\ 0 & -2/3 & 1/2 \end{bmatrix}$, $G = I$ and $H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = I_3$ and $R = \text{diag}\{R^{(1)}, R^{(2)}\}$. It can be easily seen that this system can be decomposed into two overlapping subsystems $\mathbf{S}_1$ and $\mathbf{S}_2$ according to (2.17), described by $F^{(1)} = \begin{bmatrix} 1/2 & 1 \\ -1 & -1/5 \end{bmatrix}$, $G^{(1)} = I_2$, $H^{(1)} = [\,0\ 1\,]$, $F^{(2)} = \begin{bmatrix} -1/5 & -1 \\ -2/3 & 1/2 \end{bmatrix}$, $G^{(2)} = I_2$, $H^{(2)} = [\,0\ 1\,]$, with the same noise covariances as in the case of **S** (notice that the second subsystem is unstable). According to the exposed methodology, we will design a consensus based estimator for **S** starting from the local Kalman filters $\bar{\mathbf{E}}_\mathbf{1}$ and $\bar{\mathbf{E}}_\mathbf{2}$ for $\mathbf{S}_1$ and $\mathbf{S}_2$, by introducing the consensus matrix $\tilde{C}(t)$ with $C_{11}(t) = \alpha_1(t) I_2$, $C_{12}(t) = (1 - \alpha_1(t)) I_2$, $C_{21}(t) = (1 - \alpha_2(t)) I_2$ and $C_{22}(t) = \alpha_2(t) I_2$, where $0 \le \alpha_1(t), \alpha_2(t) \le 1$ (for $R^{(1)} = R^{(2)} = 0.1$ we have $L^{(1)} = [\,0.8270\ 0.0681\,]^T$ and $L^{(2)} = [\,-0.9517\ 0.4248\,]^T$ in (2.18)). This means that we have two agents, the first having the local model $\mathbf{S}_1$ and having access to the output $y^{(1)}$ (the noisy state component x_2), and the second having the local model $\mathbf{S}_2$ and having access to the output $y^{(2)}$ (the noisy state component x_3). Fig. 2.6 shows the performance of the proposed estimator (curve (2)), of the local Kalman filters $\bar{\mathbf{E}}_1$ and $\bar{\mathbf{E}}_2$ (curves (3) and (4)) and of the globally optimal Kalman filter (curve(1)); the curves represent the experimentally obtained mean-square error for the estimate of x_2 obtained by the first agent on the basis of 200 realizations, assuming $\alpha_1(t) = \alpha_2(t) = 0.5$. As it can be easily seen, performance of the proposed estimator is close to the optimal, while the local estimators alone are obviously inferior. The estimates obtained by the second agent are very close to those obtained by the first, as a consequence of the main tendency of the consensus scheme.

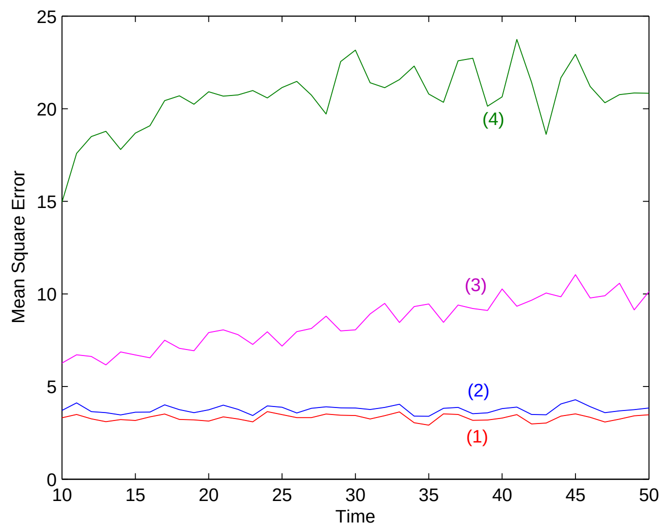

Figure 2.6: Mean square error for different estimators

Example 2.1.8 In the case when two agents get their measurements with different accuracies ($R^{(1)} \neq R^{(2)}$), we have the design problem of determining the coefficients $\alpha_1(t)$ and $\alpha_2(t)$ in the consensus matrix, having in mind that, logically, a larger weight should be given to the agent with higher local estimation accuracy. A heuristic *local adaptive strategy* implementable on line can easily be added to the basic estimation algorithm. Define

$$\zeta_i(t+1) = \delta_i \zeta_i(t) + (1 - \delta_i)(y^{(i)}(t) - H_i \xi_i(t|t))^2,$$

where $0 < \delta_i < 1$, $i = 1, 2$, representing filtered squared residuals obtained by the agents. Then, according to the general ideas exposed above, the consensus coefficients can be defined as

$$\alpha_1(t) = \zeta_1(t)^{-1}/(\zeta_1(t)^{-1} + \zeta_2(t)^{-1}) = \zeta_2(t)/(\zeta_1(t) + \zeta_2(t)),$$

$$\alpha_2(t) = \zeta_2(t)^{-1}/(\zeta_1(t)^{-1} + \zeta_2(t)^{-1}) = \zeta_1(t)/(\zeta_1(t) + \zeta_2(t)),$$

enforcing that the weights in the consensus matrix are inversely proportional to the local estimation accuracy. Table 2.2, containing the average weight $\alpha_1(t)$ obtained after $t = 50$ iterations, gives an illustration of the efficiency of the described adaptation procedure.

	$R^{(1)} = 1$	$R^{(1)} = 10$	$R^{(1)} = 100$
$R^{(2)}$=1	0.4731	0.3577	0.1914
$R^{(2)}$=10	0.5537	0.4927	0.2602
$R^{(2)}$=100	0.7945	0.5789	0.5498

Table 2.2: Adaptive consensus coefficient α_1 for different values of the measurement noise variances

2.1.2.3 Optimization

Optimization of the consensus gains can be done following the approach given in the continuous-time case. Namely, if the optimization criterion is taken to be the steady-state mean-square prediction error of the whole estimator defined as $J = \operatorname{Tr} S = \operatorname{Tr} \lim_{t\to\infty} S(t)$ (where $S(t)$ is defined in Theorem 2.1.3), then, if we collect all the unknown parameters in a vector θ, the following problem can be posed: minimize J with respect to θ, where J is calculated from the solution of the following Lyapunov-like algebraic equation derived from (2.24)

$$P = \sum_{r=1}^{\nu} \pi_r [\tilde{B}_{[r]} P \tilde{B}_{[r]}^T + \tilde{D}_{[r]} W \tilde{D}_{[r]}^T], \tag{2.30}$$

having in mind that S is a block of P. This equation has a solution under the conditions formulated within Theorem 2.1.3. It is to be noticed that intermittent measurements and communication losses make this optimization problem much more difficult and numerically more complex than the optimization problem formulated in continuous-time estimator.

Example 2.1.9 The following example illustrates the above optimization procedure in the case of an unstable system. The system is supposed to be given by (2.16) with

$F = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, Q=1; the eigenvalues of F are at $1.5 \pm j0.866$. There are two agents with two Kalman filters, the first using $H^{(1)} = [\,1\ 0\,]$ with $R^{(1)} = 0.1$, and the second $H^{(2)} = [\,0\ 1\,]$ with $R^{(2)} = 1$, so that $L^{(1)} = \begin{bmatrix} 2.0750 \\ -0.7807 \end{bmatrix}$ and $L^{(2)} = \begin{bmatrix} -1.5448 \\ -2.1632 \end{bmatrix}$, supposing that both estimators possess the information about the system model. Optimization is done with respect to the scalar parameters α_1 and α_2 in $C_{11}(t) = \alpha_1 I$ and $C_{22}(t) = \alpha_2 I$. The results have been found to be sensitive to the initial conditions, having in mind system instability. Fig. 2.7 depicts the dependence of the obtained parameters on the communication probability $p = p_{12} = p_{21}$. As it can be seen, the quality of the first estimator dominates in the case of high communication reliability, since $R^{(2)} > R^{(1)}$; when the communication reliability deteriorates, the relative importance of the second local estimator increases, as expected.

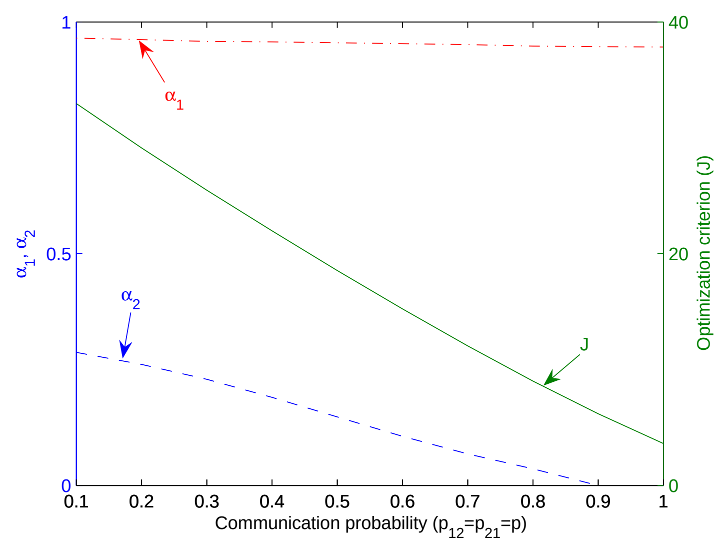

Figure 2.7: Optimal consensus parameters

2.1.2.4 Denoising

We will do the similar analyzes of the denoising effects of the introduced consensus scheme as in the case of the proposed continuous-time algorithm. Hence, we will use characteristic network topologies and assume that all the estimators have the information about the overall system model, and that they observe identical components of the state vector, but with different realizations of the measurement noise with covariance R (generalizations to more complex structures are feasible, although technically more difficult). We will also assume that the measurements are never interrupted, and that there are no communication faults.

Case A) The consensus matrix $\tilde{C}(t)$ is constant and is given in the form $\tilde{C}(t) = \tilde{C}_1^{(N)} = \frac{1}{N}\begin{bmatrix} I & I & \cdots & I \\ I & I & \cdots & I \\ & & \cdots & \\ I & I & \cdots & I \end{bmatrix}$, where I stands for I_n.

The steady-state estimation mean-square error S in the case when the agents possess the exact system models satisfies the following Lyapunov-like algebraic equation:

$$S = \sum_{r=1}^{\nu} \pi_r [\tilde{A}_{[r]} S \tilde{A}_{[r]}^T + \tilde{E}_{[r]} W \tilde{E}_{[r]}^T], \tag{2.31}$$

where $\tilde{E}_{[r]} = \left[-\tilde{G} \,\vdots\, \tilde{C}_{[r]} \tilde{\Gamma}_{[r]}] \tilde{L} \tilde{H} \right]$. The adopted assumptions lead to the following simplified relation

$$S^{(N)} = \tilde{C}_1^{(N)} [\tilde{\Phi} S^{(N)} \tilde{\Phi}^T + \tilde{L}\tilde{H}\tilde{R}\tilde{H}^T \tilde{L}^T] \tilde{C}_1^{(N)T} + \tilde{G} Q^* \tilde{G}^T \tag{2.32}$$

where the superscript $^{(N)}$ is added to emphasize that there are N agents; the block-diagonal matrices $\tilde{\Phi}$, $\tilde{L}$, $\tilde{H}$ and $\tilde{G}$ are composed of identical block-diagonal elements.

We observe now that $\tilde{C}_1^{(N)}$ has n eigenvalues at 1, and $(N-1)n$ eigenvalues at 0. Its diagonalization can be done by

$$T_N = \begin{bmatrix} I & I & \cdots & I \\ I & -(N-1)I & \cdots & I \\ & & \cdots & \\ I & & \cdots & -(N-1)I \end{bmatrix},$$

with $T_N^{-1} = \frac{1}{N}\begin{bmatrix} I & I & \cdots & I \\ I & -I & 0 & \cdots \\ & & \cdots & \\ I & 0 & \cdots & -I \end{bmatrix}$, so that

$$T_N^{-1} \tilde{C}_1^{(N)} T_N = \bar{C}_1^{(N)} = \left[\begin{array}{c:ccc} I & 0 & \cdots & \\ \hdashline 0 & 0 & 0 & \cdots \\ & & \cdots & \\ 0 & 0 & \cdots & 0 \end{array}\right]. \tag{2.33}$$

Applying T_N^{-1} and T_N to equation (2.32), we obtain

$$\bar{S}^{(N)} = \bar{C}_1^{(N)}[\tilde{\Phi}\bar{S}^{(N)}\tilde{\Phi}^T + \tilde{L}\tilde{H}\tilde{R}\tilde{H}^T\tilde{L}^T]\bar{C}_1^{(N)T} + \bar{Q}^{(N)} \tag{2.34}$$

where $\bar{S}^{(N)} = T_N^{-1}S^{(N)}T_N$ and $\bar{Q}^{(N)} = T_N^{-1}\tilde{G}Q^*\tilde{G}^T T_N = \begin{bmatrix} NGQG^T & \cdots & 0 \\ \vdots & & \\ 0 & & \cdots\ 0 \end{bmatrix}$. A solution to this equation is $\bar{S}^{(N)} = \left[\begin{array}{c:ccc} \hat{S}^{(N)} & 0 & \cdots & \\ \hdashline 0 & 0 & 0 & \cdots \\ & & \cdots & \\ 0 & 0 & \cdots & 0 \end{array}\right]$, where $\hat{S}^{(N)}$ is obtained from the Lyapunov equation

$$\hat{S}^{(N)} = \Phi\hat{S}^{(N)}\Phi^T + LHRH^TL^T + NGQG^T. \tag{2.35}$$

Obviously, the mean-square error for the whole estimator is $J = \operatorname{Tr}\bar{S}^{(N)} = \operatorname{Tr}\hat{S}^{(N)}$. Having in mind that N independent estimators have the mean-square error equal to $N\hat{J}$, where $\hat{J} = \operatorname{Tr}\hat{S}$ and $\hat{S}$ is a solution to the standard local Lyapunov equation

$$\hat{S} = \Phi\hat{S}\Phi^T + LHRH^TL^T + GQG^T, \tag{2.36}$$

we take $\bar{J} = \frac{1}{N}J$ as the *average criterion* "per agent", and obtain that for N large enough $\bar{J} \approx \operatorname{Tr} S^*$, where S^* is a solution of the Lyapunov equation

$$S^* = \Phi S^*\Phi^T + GQG^T. \tag{2.37}$$

Comparing (2.36) and (2.37), one concludes that for large N the consensus scheme asymptotically achieves complete denoising in the sense that it reduces the mean-square error from the level defined by (2.36) to the level defined by (2.37) where the term depending on R is eliminated.

Case B) $\tilde{C}(t) = \tilde{C}_2^{(N)} = \frac{1}{2}\begin{bmatrix} I & I & 0 & \cdots & 0 \\ 0 & I & I & 0 & \cdots \\ & & \cdots & & \\ 0 & 0 & \cdots & I & I \\ I & 0 & \cdots & 0 & I \end{bmatrix}$, *i.e.*, the network graph forms a directed ring.

Reasoning as in Case A), we obtain for the diagonal blocks $(\bar{S}^{(N)})_i$ of $\bar{S}^{(N)}$, $i = 2, \ldots, N$, the following relations:

$$(\bar{S}^{(N)})_i = |\lambda_i^{(N)}|^2[\Phi(\bar{S}^{(N)})_i\Phi^T + LHRL^TH^T]; \tag{2.38}$$

where $\lambda_i^{(N)}$, $i = 1, \ldots, N$, are N distinct eigenvalues of the consensus matrix $\tilde{C}_2^{(N)}$, uniformly distributed on a circle in the complex plane, with radius $\frac{1}{2}$ and the center at $(\frac{1}{2}, 0)$; the first block $(\bar{S}^{(N)})_1$ is the same as in Case A). It is obvious that now $\lim_{t\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathrm{Tr}(\bar{S}^{(N)})_i \neq 0$, so that denoising in the above sense is not achievable in spite of the fact that all the nodes are reachable from any other node; a similar phenomenon has been observed in the case of the continuous-time algorithm proposed in the previous subsection.

However, the relation (2.38) indicates how complete asymptotic denoising can be achieved in the case of graphs with complexity lying between the above two extremes. Assuming that (2.38) holds, we can easily conclude that

$$\|\Sigma_{i=2}^N(\bar{S}^{(N)})_i\| \leq \beta \sum_{i=1}^{N} |\lambda_i^{(N)}|^2$$

for some finite $\beta > 0$. Therefore, it comes out that the condition

$$\sum_{i=1}^{N} |\lambda_i^{(N)}|^2 = o(N) \tag{2.39}$$

is sufficient for successful denoising in this case. In general, any rigorous analysis is here faced with considerable technical difficulties; however, in some special cases the condition (2.39) can be more directly related to structural properties of the corresponding graphs, like in the following examples.

Example 2.1.10 We assume that the graph that describes the network is undirected (i.e. all the links between the agents are bidirectional). Under this assumption, it is easy to show that

$$\sum_{i=1}^{N} (\lambda_i^{(A)})^2 = 2M, \tag{2.40}$$

where $\lambda_i^{(A)}$, $i = 1, ..., N$, are the eigenvalues of the graph's adjacency matrix A, defined as $A = [a_{ij}]$, where $a_{ij} = k_{ij}$, $i \neq j$ and $a_{ii} = 0$, $i, j = 1, \ldots, N$; matrix A is constant in the case of no communication faults ([22]). Next, we assume that the number of links per node in the graph is the same, and that all the weights K_{ij} in the consensus matrix are the same and equal to $\frac{1}{\mu+1}I$, where $\mu = \mu(N)$ is the number of links per node. Under these assumptions, it is obvious that the consensus matrix $\tilde{C}$ is equal to $\frac{1}{\mu+1}(A \otimes I_n + I)$. Since we assumed an undirected graph structure, matrices $\tilde{C}$ and A are symmetric and all their eigenvalues are real. Thus, it is easy to show (using (2.40) and the fact that $\sum_{i=1}^{N} \lambda_i^{(A)} = 0$ and $M = N\mu(N)/2$), that the condition (2.39) reduces to

$$\frac{N}{\mu(N)} = o(N). \tag{2.41}$$

Therefore, for the assumed network structure, the sufficient condition for complete denoising is that the number of links per node $\mu(N)$ tends to infinity with the number of nodes N (compare with analogous results for the continuous time case).

Example 2.1.11 This example illustrates the denoising capabilities of the proposed estimator for different network topologies. We assume that all the agents have identical models of a fourth order system, with $F = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0.8 & 0.7 & 0 & 0 \\ 0.5 & 0.3 & 0.8 & 0.3 \\ 0 & 0.5 & 0.1 & 0.7 \end{bmatrix}$, $H = I_4$, $Q = 0.5I_4$, $R = 0.5I_4$, but with different realizations of the measurement noise. Average values of the criterion ($\bar{J}$) have been calculated for five network topologies: a) fully connected network; b) directed ring; c) undirected ring; d) random graph with fixed probability 0.2 of connection between every two nodes; e) random graph for which probability of connection decreases as $1/d_{ij}^2$, where d_{ij} is the distance between the nodes i and j. The results are shown in Fig. 2.8. The horizontal dashed line correspond to the criterion lower bound $\mathrm{Tr}(S^*)$, where S^* is obtained by using (2.37). The presented results fully confirm the above analysis. In the case of the fully connected graph, the curve converges exactly to the lower bound of the criterion when N tends to infinity. As expected, in the case of the directed and undirected rings the limit values of the criterion are higher than $\mathrm{Tr}(S^*)$, since the complete denoising is not achievable. We have the same situation in the case e) of random graphs for which the probability of connection decreases as $1/d_{ij}^2$, since the average number of links per node converges to a constant when N tends to infinity. However, when the probability of connection is constant for all the pairs of nodes (case d)), we can see that the complete asymptotic denoising is achieved. The explanation lies in the fact that for case d) (fixed probability) the average number of links per node grows linearly with N (it is equal to pN where p is probability of connection between two nodes).

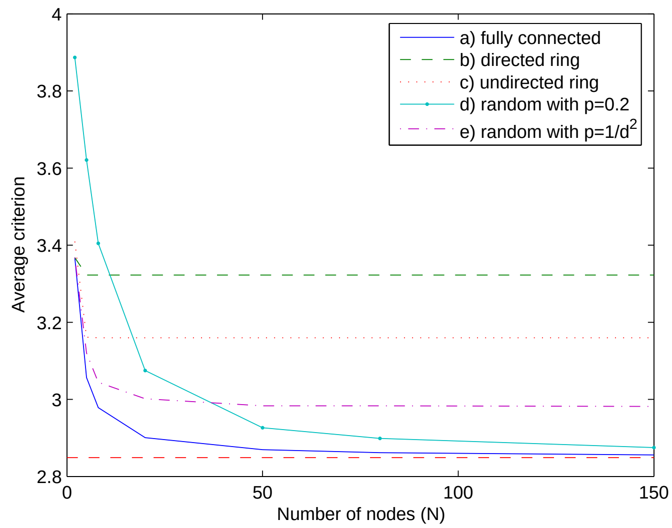

Figure 2.8: Average criterion as a function of N

Remark 2.1.3 According to the above results, complete asymptotic denoising is achievable in the case of networks with the number of edges $E = O(N^2)$, but scalability of the algorithm becomes violated. However, the algorithm is still capable of achieving complete asymptotic denoising when the number of edges satisfies $E = O(N\mu(N))$, where $\mu(N)$ tends to infinity at a much slower rate than the linear function (*e.g.* $\mu(N) = O(\log N)$), ensuring a better scalability. In the case of a bounded number of branches entering each node, strict scalability holds, but complete asymptotic denoising is not achievable; in practice, however, the algorithm can still efficiently suppress the noise influence and provide reliable overall results.

2.2 Decentralized Parameter Estimation by Consensus Based Stochastic Approximation

This section deals with the problem of distributed parameter estimation. In the first part of this section the formulation of the algorithm and the main definitions are presented. The second part is devoted to the convergence analysis of the proposed algorithm. It starts with five main lemmas, providing tools for further derivations. Lemma 2.2.1 determines the main structure of the network resulting from general requirements for measurement availability and inter-agent communications, while Lemma 2.2.2 deals more specifically with characteristic properties of the matrix generating the network graph. Lemmas 2.2.3 and 2.2.4 treat the basic convergence problem of recursions involving matrix gain structures typical for the proposed algorithm, while Lemma 2.2.5 proves the existence of solution of a Lyapunov-type linear matrix equation which appears in the subsequent derivations. The first convergence theorem deals with the case of asymptotically nonvanishing gains and provides the resulting estimation error covariance matrix. It represents a generalization of the classical results of Polyak (*e.g.* [61, 62]) to the multi-agent environment based on consensus, including intermittent observations and inter-agent communications treated on the basis of the corresponding probabilities like in [77, 81, 54, 55]. The following theorem deals with asymptotically vanishing gains, giving conditions for the mean-square convergence of the parameter estimates. Theorem 2.2.3 provides an estimate of the rate of convergence of the algorithm under specific conditions. Finally, a discussion of the required network topology, a treatment of the problem of additive communication noise, as well as a brief presentation of the important problem of denoising, analogously to the result presented in the context of state estimation.

2.2.1 Problem Formulation and Algorithm Definition

Consider the situation in which N autonomous agents perform *real-time estimation of parameters* in the following local *regression models*:

$$y_i(t) = \theta^T \varphi_i(t) + \xi_i(t), \tag{2.42}$$

$i = 1, \ldots, N$, where t is the discrete-time instant, θ represents the unknown parameter vector ($\dim \theta = m$), $\varphi_i(t)$ are the vectors of regressors locally accessible to the agents, $y_i(t)$ are the local scalar output measurements and $\xi_i(t)$ the local measurement noises.

We will assume that in the case of no connection between the agents local estimation of the parameter vector θ is done by the *gradient-type stochastic approximation algorithm*

$$\hat{\theta}_i^L(t+1) = \hat{\theta}_i^L(t) + \gamma_i^L(t)[y_i(t) - \hat{\theta}_i^L(t)^T \varphi_i(t)]\varphi_i(t), \tag{2.43}$$

$i = 1, \ldots, N$, where $\hat{\theta}_i^L(t)$ is the local estimate of θ and $\{\gamma_i^L(t)\}$ a positive number sequence [53, 61, 62, 64, 65, 108].

We will assume that the agents are, in general, connected by directed communication links aimed at transmitting the current parameter estimates. We will denote by $\tilde{C}_{ij}(t)$ $m \times m$ time-varying diagonal matrix gains with nonnegative entries, defining transmission gains of the parameter estimates from the j-th to the i-th node (agent), $i, j = 1, \ldots, N$. Based on the local estimation algorithms (2.43) and the introduced communication links between the agents, we propose the following consensus based parameter estimation algorithm (compare with the similar discrete-time consensus based state estimation algorithm presented in the previous section):

$$\begin{aligned} \hat{\theta}_i(t) &= \tilde{\theta}_i(t) + \tilde{\gamma}_i(t)[y_i(t) - \tilde{\theta}_i(t)^T \varphi_i(t)]\varphi_i(t), \\ \tilde{\theta}_i(t+1) &= \textstyle\sum_{j=1}^N \tilde{C}_{ij}(t)\hat{\theta}_j(t), \end{aligned} \tag{2.44}$$

where $\tilde{\theta}_i(t)$ is the estimate of θ generated by the i-th agent, $i = 1, \ldots, N$. Obviously, for $\tilde{C}_{ii}(t) = I$ and $\tilde{C}_{ij}(t) = 0$, $i \neq j$, the algorithm (2.44) reduces to (2.43). Defining $\tilde{\theta}(t) = \left[\tilde{\theta}_1(t)^T \cdots \tilde{\theta}_N(t)^T\right]^T$, $Y(t) = \left[y_1(t) \cdots y_N(t)\right]^T$, $\Phi(t) = \mathrm{diag}\{\varphi_1(t), \ldots, \varphi_N(t)\}$, $\tilde{C}(t) = [\tilde{C}_{ij}(t)]$, $i, j = 1, \ldots, N$, and $\tilde{\Gamma}(t) = \mathrm{diag}\{\tilde{\gamma}_1(t), \ldots, \tilde{\gamma}_N(t)\} \otimes I_m$, where $\otimes$ denotes the Kronecker's product, we obtain the following compact representation of the whole estimation algorithm:

$$\tilde{\theta}(t+1) = \tilde{C}(t)\tilde{\theta}(t) + \tilde{C}(t)\tilde{\Gamma}(t)\Phi(t)[Y(t) - \Phi(t)^T \tilde{\theta}(t)]. \tag{2.45}$$

The algorithm represents, in fact, a combination of the local estimation algorithms of stochastic gradient type (2.43) and a first order discrete-time consensus scheme, already introduced in the previous section. It can be considered as a practical way of achieving asymptotic agreement upon the parameter estimates, which overcomes the needs for both prior and posterior distributions inherent to general treatments of the distributed decision making problem (*e.g.* [16, 106]). More specifically, the consensus scheme is aimed at: (a) reducing the number of nodes performing measurements and local estimation by distributing the estimates throughout the network; (b) increasing reliability of the estimates in the case of missing observations; (c) contributing to the estimation accuracy and reduction of measurement noise influence.

Remark 2.2.1 Notice that the algorithm (2.44) is formulated in accordance with the usual split of state estimation algorithms into their "filtering" and "prediction" parts; in our case the "filtering" part corresponds to the local stochastic gradient algorithms, and the "prediction" part to convex combinations of the available local estimates. Alternative structures are possible, in accordance with [107, 105, 13, 44, 45, 113]. Starting, for example, from [107, 105], one can obtain

$$\tilde{\theta}_i(t+1) = \sum_{j=1}^{N} \tilde{C}_{ij}(t)\tilde{\theta}_j(t) + \tilde{\gamma}_i(t)[y_i(t) - \tilde{\theta}_i(t)^T\varphi_i(t)]\varphi_i(t). \tag{2.46}$$

Averaging can be applied only to the increment of the estimates $\tilde{\gamma}_i(t)[y_i(t) - \tilde{\theta}_i(t)^T\varphi_i(t)]\varphi_i(t)$, as in [44]. It could be expected that these algorithms have similar properties as the proposed one. However, it will be seen later that (2.45) have some advantages, including a more transparent formulation of the convergence conditions.

Remark 2.2.2 The consensus scheme introduces implicitly averaging of the estimates generated by the agents, since the available measurements contain local outputs containing different measurement noise realizations. Such an *ensemble averaging* is essentially different from the *time averaging* done on one measurement realization, introduced in the Polyak's stochastic approximation with averaging [63]. The Polyak's scheme can, obviously, be introduced locally within the proposed algorithm, with the aim to improve the overall convergence rate of the algorithm. ∎

In order to encompass the important case of intermittent measurements and unreliable communication links, we will adopt that the introduced matrices $\tilde{\Gamma}(t)$ and $\tilde{C}(t)$ are random, satisfying the following general assumptions:

- the constituent blocks of $\tilde{C}(t)$ are in the form $\tilde{C}_{ij}(t) = k_{ij}(t)C_{ij}(t)$, where $\{k_{ij}(t)\}$, $i, j = 1, \ldots, N$, are mutually independent scalar sequences of independent binary random variables, such that $P\{k_{ij}(t) = 1\} = p_{ij}$ and $P\{k_{ij}(t) = 0\} = 1-p_{ij}$ for $i \neq j$, while matrices $C_{ij}(t)$ are $m \times m$ diagonal weighting matrices with positive entries which can reflect local estimation uncertainty (for example, the elements of $C_{ij}(t)$ can be chosen to be higher in the case of higher accuracy obtainable by the local estimator at the j-th node);

- the diagonal elements of $\tilde{\Gamma}(t)$ are in the form $\tilde{\gamma}_i(t) = \kappa_i(t)\gamma_i(t)$, where $\{\gamma_i(t)\}$ is a predefined deterministic sequence and $\{\kappa_i(t)\}$ a sequence of independent binary random variables, such that $\kappa_i(t) = 1$ in the case when the local measurement is available to the i-th agent at time t, and $\kappa_i(t) = 0$ in the opposite case; we will adopt that $P\{\kappa_i(t) = 1\} = p_{ii}$ and $P\{\kappa_i(t) = 0\} = 1 - p_{ii}$;

- X_t is the random vector composed of N^2 binary components: $N(N-1)$ elements $k_{ij}(t)$, $i, j = 1, \ldots, N, i \neq j$, and N elements $\kappa_i(t)$, $i = 1, \ldots, N$, so that $\{X_t\}$ represents a sequence of independent random vectors; let π_i be the probabilities of all possible realizations $X^{(i)}$ of X_t, $i = 1, \ldots, 2^{N^2}$ (superscript $^{(i)}$ will denote in the sequel the i-th realization of an indicated variable);

- $K(t) = [k_{ij}(t)]$ is a matrix with binary elements, where the off-diagonal elements $k_{ij}(t)$, $j \neq i$, are determined by the current realization of X_t, while the diagonal elements are fixed in such a way that $k_{ii}(t) = 1$ for all indices for which $p_{ii} > 0$, and for the remaining indices $k_{ii}(t)$ are fixed to either 1 or 0;

- $K^{(i)}$, $\tilde{C}^{(i)}$ and $\tilde{\Gamma}^{(i)}(t)$, $i = 1, \ldots, \tilde{N} = 2^{N^2}$, will denote all possible realizations of $K(t)$, $\tilde{C}(t)$ and $\tilde{\Gamma}(t)$ resulting from different realizations $X^{(i)}$ of X_t; matrices $\tilde{C}^{(i)}$ will be assumed to be time invariant;

- K^* and $\tilde{C}^*$ represent the "full" realizations of the random matrices $K(t)$ and $\tilde{C}(t)$, obtained by introducing $k_{ij}(t) = 1$ if $p_{ij} > 0$ and $k_{ij}(t) = 0$ if $p_{ij} = 0$, $i, j = 1, \ldots, N$, $i \neq j$;

- similarly, $\tilde{\Gamma}^*(t)$ is the "full" realization of $\tilde{\Gamma}(t)$, obtained by introducing $\kappa_i(t) = 1$ if $p_{ii} > 0$ and $\kappa_i(t) = 0$ if $p_{ii} = 0$;

- the consensus matrix $\tilde{C}(t)$ is row-stochastic for all t, *i.e.* the sum of the elements of each of its rows is equal to 1.

Remark 2.2.3 The assumption about the diagonal elements of $K(t)$ implies that the estimate $\tilde{\theta}_i(t+1)$ explicitly depends on $\hat{\theta}_i(t)$ when the i-th agent has the access to measurements with positive probability. When this probability is equal to 0, the choice $k_{ii}(t) = 1$ enables incorporation of a local *a priori* estimate of the parameter vector, and prevents from forgetting previously received estimates.

Remark 2.2.4 As in the discrete-time state estimation scheme, the assumption that $\tilde{C}(t)$ is row stochastic requires its recalculation for each new realization of X_t. This can be done by re-normalization of its rows as in the Subsection 2.1.2. ■

As above, we will represent the instantaneous inter-agent connections in the network by a directed graph $\mathcal{G}(K(t)) = \{\mathcal{N}(K(t)\}, \mathcal{E}(K(t)\}$ associated with the matrix $K(t)$, where $\mathcal{N}(K(t))$ is the node set ($|\mathcal{N}(K(t))| = N$) and $\mathcal{E}(K(t))$ the arc set, where the arc from node j to node i exists if $k_{ij}(t) > 0$ ($|.|$ denotes the cardinality of an indicated set). Obviously, $K(t)$ represents at the same time the adjacency matrix of the graph $\mathcal{G}(K(t))$. Graphs $\mathcal{G}(P)$ and $\mathcal{G}(K^*)$ associated with matrices P and K^* will have an important role in the sequel. These matrices have the same off-diagonal positions of positive entries. At the diagonal, matrix K^* has positive entries for all indices for which $p_{ii} > 0$; in addition, it can have positive entries for some indices for which $p_{ii} = 0$. The inverse graph of $\mathcal{G}(.)$ will be denoted by $\bar{\mathcal{G}}(.)$: it is obtained by reversing the direction of the arcs in $\mathcal{G}(.)$

2.2.2 Convergence Analysis

We will study convergence properties of the proposed algorithm starting from the following basic assumptions:

(A.2.2.1) $\{\varphi_i(t)\}$, $i = 1, \ldots, N$, are sequences of independent equally distributed random vectors with the following properties:

(a) $E\{\varphi_i(t)\varphi_i(t)^T\} = B = [b_{kj}]$ ($\|B\| < \infty$); B satisfies the *strict diagonal dominance* condition

$$|b_{kk}| > \sum_{j=1, j\neq k}^{m} |b_{kj}|, \tag{2.47}$$

$k, j = 1, \ldots, m$ (see [79]);

(b) fourth order moments of $\{\varphi_i(t)\}$, are finite, so that it is possible to find such an $mN \times mN$ matrix $\bar{B}$, $\|\bar{B}\| < \infty$, that $E\{\Phi(t)\Phi(t)^T U \Phi(t)\Phi(t)^T\} \leq \bar{B} U \bar{B}^T$ for any symmetric $mN \times mN$ matrix $U \geq 0$ (for symmetric positive semidefinite matrices A and B, $A \geq B$ means that $A - B$ is positive semidefinite);

(A.2.2.2) $\{\xi_i(t)\}$ are sequences of independent zero-mean random variables with var $\xi_i(t)$ $= q_i$, $i = 1, \ldots, N$;

(A.2.2.3) sequences $\{X_t\}$, $\{\varphi_i(t)\}$ and $\{\xi_i(t)\}$, $i = 1, \ldots, N$, are mutually independent;

(A.2.2.4) $\Gamma(t) = \mathrm{diag}\{\gamma_1(t), \ldots, \gamma_N(t)\} > 0$, $\forall t \geq 0$; $\lim_{t \to \infty} \Gamma(t) = \Gamma_\infty \geq 0$;

(A.2.2.5) The set $\mathcal{N}^* \subset \mathcal{N}(P)$ containing all the nodes of the graph $\mathcal{G}(P)$ which have the indices i corresponding to $p_{ii} > 0$ is nonempty, and each node in $\mathcal{N}(P)$ is reachable from at least one node from $\mathcal{N}^*$.

Remark 2.2.5 Assumption (A.2.2.1) is stronger than the analogous assumptions for the stochastic approximation procedures in linear system identification, *e.g.* [62, 76]; this is a direct consequence of the introduction of the consensus scheme. The assumption that the covariance B does not depend on i logically follows from the structure of (2.42). Assumptions (A.2.2.2) - (A.2.2.4) are classical for the stochastic approximation algorithms. Assumption (A.2.2.5) deals explicitly with the network structure: graphs $\mathcal{G}(P)$ and $\mathcal{G}(K^*)$ contain loops at the nodes characterized by positive probabilities of getting measurements, while the outgoing branches from these nodes ensure adequate distribution of the parameter estimates throughout the network [27, 48]. Obviously, the network evolves stochastically, and concrete links depend on realizations of $\{X_t\}$. ■

Convergence analysis of the proposed algorithm requires some preliminary results, presented in the form of five lemmas.

Lemmas 2.2.1 and 2.2.2 deal with the main structural properties of the network graph.

Lemma 2.2.1 Let $P = [p_{ij}]$, $i, j = 1, \ldots, N$, according to the above definitions, and let assumption (A.2.2.5) be satisfied. Then the matrix P is cogredient to

$$\bar{P} = \begin{bmatrix} P_1 & \cdots & & 0 \\ 0 & P_2 & \cdots & \\ 0 & \cdots & P_k & 0 \\ Q_1 & \cdots & Q_k & P_0 \end{bmatrix} \tag{2.48}$$

where P_0 is an $r_0 \times r_0$ matrix, $0 \leq r_0 < N$, P_i are *irreducible* $r_i \times r_i$ matrices satisfying $0 < r_i < N$, Q_i are $r_0 \times r_i$ matrices, $i = 1, \ldots, k$, and $1 \leq k \leq |\mathcal{N}^*|$.

Proof: We will prove the lemma by construction. Take any node i_1 from the node set $\mathcal{N}^*$ defined in (A.2.2.5), together with all the nodes from $\mathcal{G}(P)$ reachable from this node, and construct the corresponding subdigraph in which i_1 is a center node (a node from which every node in the subdigraph is reachable [48]). Consequently, the inverse digraph of this subdigraph contains one and only one closed strong component (a maximal induced subdigraph which is closed and strongly connected), and, therefore, this subdigraph can be associated to a nonnegative matrix $R^*_{i_1}$ cogredient to $\bar{R}_{i_1} = \begin{bmatrix} R_{i_1} & 0 \\ S_{i_1} & R^0_{i_1} \end{bmatrix}$, where R_{i_1} is irreducible. If $\mathcal{N}^* - \mathcal{N}(R^*_{i_1}) = \emptyset$, we have the result, since $\bar{R}_{i_1}$ has the structure of $\bar{P}$ in (2.48). If $\mathcal{N}^* - \mathcal{G}(R^*_{i_1}) \neq \emptyset$, we take a node i_2 from $\mathcal{N}^* - \mathcal{G}(R^*_{i_1})$, and construct, analogously as above, the subdigraph $\mathcal{G}(R^*_{i_2})$, where $R^*_{i_2}$ is cogredient to $\bar{R}_{i_2}$, which has the same lower block triangular structure as $\bar{R}_{i_1}$. Continuing until exhaustion of all the nodes from $\mathcal{N}^*$, one obtains $J \leq |\mathcal{N}^*|$ subdigraphs $\mathcal{G}(R^*_{i_1}), \ldots, \mathcal{G}(R^*_{i_J})$ and matrices $R^*_{i_1}$, ..., $R^*_{i_J}$ to which they are asssociated, together with the corresponding cogredient matrices $\bar{R}_{i_1}, \ldots, \bar{R}_{i_J}$, respectively. By assumption (A.2.2.5), the *whole node set* $\mathcal{N}(P)$ is decomposed by the above procedure into J overlapping subsets. By construction, the node set that represents the union of the *non-overlapping* parts of the node sets $\mathcal{N}(R^*_{i_j})$, $j = 1, \ldots, J$, contains k, $1 \leq k \leq J$, closed strong components of the inverse subdigraph $\bar{\mathcal{G}}(P)$, associated to k of J irreducible submatrices R_{i_j} of $\bar{R}_{i_j}$, $j = 1, \ldots, J$; denote these submatrices by P_l, $l = 1, \ldots, k$. Therefore, according to [48] (Theorem 2.7), P is cogredient to $\bar{P}$ in (2.48). Thus the result. ■

Lemma 2.2.2 Let the assumption (A.2.2.5) be satisfied and let $Q^{[l]} = \left[\, Q^{[l]}_1 \,\vdots\, \cdots \,\vdots\, Q^{[l]}_k \,\right]$ be defined by

$$\bar{P}^l = \left[\begin{array}{c:c} \mathrm{diag}\{P^l_1, \ldots, P^l_k\} & 0 \\ \hdashline Q^{[l]} & P^l_0 \end{array}\right], \tag{2.49}$$

where $\bar{P}$ is given by (2.48) (A^l with no brackets around the superscript will denote in the sequel the l-th power of a matrix A). Then:

(a) there exists an integer l_j such that for $l \geq l_j$ each block matrix $Q^{[l]}_j$, $j = 1, \ldots, k$, contains at least one row whose all elements are positive;

(b) there exists an integer l' such that for $l \geq l'$ each row of $Q^{[l]}$ contains at least one entire row belonging to $Q_j^{[l]}$, $j = 1, \ldots, k$, whose all elements are positive.

Proof: The proof is based on the arguments exposed in [48]. Consider the inverse digraph $\bar{\mathcal{G}}(\bar{P})$, and identify the overlapping subsets $\mathcal{N}(P_0)_1, \ldots, \mathcal{N}(P_0)_k$ of $\mathcal{N}(P_0)$, containing those nodes that are connected to the nodes from $\mathcal{N}(P_1), \ldots, \mathcal{N}(P_k)$, respectively. According to [48] (Theorem 2.7), for each j, $j = 1, \ldots, k$, there is a walk from $\mathcal{N}(P_0)_j$ to *some* node from $N(P_j)$ of length $m_j \geq N - k - |\bigcup_{i=1, i\neq j}^{k} \mathcal{N}(P_0)_i|$. Moreover, having in mind strong connectedness of $\bar{G}(P_j)$, we conclude that there is a walk of length $l_j \geq m_j + r_j$ from any node in $\mathcal{N}(P_0)_j$ to *any* node from $\mathcal{N}(P_j)$. Therefore, matrix $Q_j^{[l]}$, generated according to (2.49) by $Q_j^{[r+1]} = Q_j P_j^r + P_0 \, Q_j^{[r]}$, $r = 1, \ldots, k-1$, contains the rows composed of positive elements at the row indices corresponding to the elements of $\mathcal{N}(P_0)_j$, and the remaining elements are equal to zero. This fact proves assertion (a). Assertion (b) results from the same way of reasoning applied to the whole matrix $Q^{[l]}$, *i.e.* to all the nodes from $\mathcal{N}(\bar{P})$, having in mind that each element of $\mathcal{N}(P_0)$ is connected to at least one subset $\mathcal{N}(P_j)$, according to assumption (A.2.2.5). Then, we can simply take $l^* = \max_j l_j$. ∎

Lemmas 2.2.3 and 2.2.4 treat the basic matrix recursions appearing within the subsequent theorems. Lemma 2.2.3 is introductory, and assumes scalar parameters. Lemma 2.2.4 provides an important generalization to the case of vector parameters, involving positive definite matrix gains, in accordance with the definition of the algorithm.

Lemma 2.2.3 Let $R = [r_{ij}]$, $i, j = 1, \ldots, N$, and its cogredient matrix $\bar{R}$ be nonnegative row-stochastic matrices having the same structure as P and $\bar{P}$, respectively, *i.e.* $R \sim P$ and $\bar{R} \sim \bar{P}$ (for nonnegative matrices A and B, $A \sim B$ if both matrices have the same associated digraphs). Let $\bar{R}_D = \bar{R}D$, where $D = \mathrm{diag}\{D_1, \ldots, D_k, D_0\}$, $D_i = \mathrm{diag}\{d_{i,1}, \ldots, d_{i,\rho_i}\}$, where either $d_{i,j} = 1$ or $d_{i,j} = d < 1$, $j = 1, \ldots, \rho_i$, $D_0 = I_{\rho_0}$ ($\rho_0 \times \rho_0$ identity matrix), and let $\mathrm{Tr}\, D_i < \rho_i$, $i = 1, \ldots, k$ (the last assumption means that at least one element $d_{i,j}$ is strictly less than 1 for each i). Let $Z(t)$ be an $N \times N$ symmetric positive semidefinite matrix generated by the recursion

$$Z(t+1) = R_D \, Z(t) \, R_D^T, \tag{2.50}$$

starting from some $Z(0) \geq 0$, where R_D is cogredient to $\bar{R}_D$. Then, $Z(t) \to 0$ when $t \to \infty$.

Proof: Matrices P_i, $i = 1, \ldots, k$, in (2.48) are *primitive* matrices since each subdigraph $\bar{\mathcal{G}}_{P_i}$ is strongly connected and aperiodic by construction, according to Lemma 2.2.1 (see [27, 48, 30]). Define $\bar{R}$ using $\bar{P}$ by replacing P_i by R_i, $i = 0, \ldots, k$, and Q_i by S_i, $i = 1, \ldots, k$ (obviously, $P_i \sim R_i$ and $Q_i \sim S_i$). Therefore, matrices R_i, $i = 1, \ldots, k$, are primitive and, in addition, row-stochastic by assumption. Therefore, there exists such an integer l_i that $R_i^{l_i} \succ 0$ [24, 12, 30] ($A \succ 0$ denotes that all the elements of a matrix A are positive). Moreover, we have that $(R_i D_i)^{l_i - 1} R_i \succ 0$ having in mind that $R_i \sim R_i D_i$, according to the properties of D_i. Also, $\|(R_i D_i)^{l_i - 1} R_i\|_\infty \leq 1$, having in mind that R_i is row-stochastic and D_i is diagonal with positive entries not greater than 1 ($\|A\|_\infty = \max_i \sum_j |a_{ij}|$ for a matrix $A = [a_{ij}]$). In the same way, $(R_i D_i)^{l_i} \succ 0$, but, in addition, we have now that $\|(R_i D_i)^{l_i}\|_\infty < 1$, having in mind that and that at least one column of the matrix $(R_i D_i^{l_i - 1}) R_i$ consisting entirely of positive elements becomes multiplied by a positive number less than 1 (at least one element $d_{i,j}$, $j = 1, \ldots, \rho_i$, is strictly less than 1 by assumption).

Take $l \geq l' = \max_i (N - k + l_i)$, $i = 1, \ldots, k$, and write $(\bar{R}D)^l$ as:

$$(\bar{R}D)^l = \begin{bmatrix} \mathrm{diag}\{(R_1 D_1)^l, \ldots, (R_k D_k)^l\} & 0 \\ S_D^{[l]} & R_0^l \end{bmatrix}, \tag{2.51}$$

where $S_D^{[l]}$ is a matrix readily obtained from $S^{[l]} \sim Q^{[l]}$ (in accordance with the definition of $\bar{R} \sim \bar{P}$ - see (2.49) and the proof of Lemma 2.2.2), by replacing R_j with $R_j D_j$ and S_j with $S_j D_j$, $j = 1, \ldots, k - 1$, *i.e.* after multiplying at least one nonzero element of each of its rows by a positive number less than one (see assertion (b) from Lemma 2.2.2). Having in mind that $\|(R_i D_i)^{l_i}\|_\infty < 1$, one concludes that $\|(\bar{R}D)^l\|_\infty < 1$, implying that $\|R_D^l\|_\infty < 1$.

Iterating (2.50) l steps backwards, one obtains

$$Z(t+1) = R_D^l Z(t - l + 1)(R_D^T)^l.$$

Moreover, if $\mathrm{vec}(Z(t))$ denotes the vector obtained from $Z(t)$ by concatenating its column vectors, we have the following equivalent representation

$$\mathrm{vec}(Z(t+1)) = [R_D^l \otimes R_D^l]\,\mathrm{vec}(Z(t-l+1)). \tag{2.52}$$

According to the above analysis, $\|[R_D^l \otimes R_D^l]\|_\infty < 1$, implying that $\|\mathrm{vec}(Z(t))\| \to 0$, or $Z(t) \to 0$, when $t \to \infty$. Thus, the result. ■

Lemma 2.2.4 Let $R^{(B)}$ and $\bar{R}^{(B)}$ be row-stochastic $Nm \times Nm$ matrices obtained from R and $\bar{R}$, respectively, by replacing their scalar elements r_{ij} with matrix blocks $r_{ij}M_{ij}$, where M_{ij} are $m \times m$ diagonal matrices with positive entries, $i,j = 1,\dots,N$. Let $D^{(B)} = \{D_1^{(B)},\dots,D_k^{(B)},D_0^{(B)}\}$ be an $Nm\times Nm$ matrix obtained from D (Lemma 2.2.2) by replacing in D_i scalars $d_{i,j}$ with $m \times m$ symmetric positive definite matrices $\Delta_{i,j}$ in such a way that $\Delta_{i,j} = I$ when $d_{ij} = 1$ and $\Delta_{i,j} = \Delta > 0$, $\|\Delta\|_\infty \leq 1$, when $d_{ij} = d < 1$, and let $\sum_{j=1}^{\rho_i} \|\Delta_{i,j}\|_\infty < \rho_i$, $i = 1,\dots,k$ ($\|A\|_\infty = \max_i \sum_j |a_{ij}|$ for a matrix $A = [a_{ij}]$). Then, the matrix $Z^{(B)}(t+1)$ generated by the recursion $Z^{(B)}(t+1) = R_D^{(B)} Z^{(B)}(t) R_D^{(B)T}$ (starting from some $Z^{(B)}(0) \geq 0$) converges to zero, where $R_D^{(B)}$ is a matrix cogredient to $\bar{R}_D^{(B)} = \bar{R}^{(B)}D^{(B)}$.

Proof: If $R_i^{(B)}$, $i = 1,\dots,k$, is constructed using R_i (see the proof of Lemma 2.2.3) in the same way as $R^{(B)}$ is constructed using R, we first conclude that there exists such an integer l_i that $(R_i^{(B)})^{l_i}$ is composed of $N \times N$ diagonal $m \times m$ blocks with positive entries, as a consequence of primitiveness of R_i ($R_i^{l_i} \succ 0$) and the properties of the constituent blocks of $R_i^{(B)}$. Reasoning as in the proof of Lemma 2.2.3, we analyze matrix $(R_i^{(B)}D_i^{(B)})^{l_i-1}R_i^{(B)}$. It is straightforward to conclude that all of its $m\times m$ blocks are in the form $\sum_q F_1^{(q)}\bar{\Delta}_1 \cdots F_{l_i-1}^{(q)}\bar{\Delta}_{l_i-1}F_{l_i}^{(q)}$, where $F_j^{(i)}$ are diagonal $m \times m$ matrices with positive entries and $\bar{\Delta}_j$ are symmetric and positive definite $m \times m$ matrices, by assumption (equal either to I or to Δ), $j = 1,\dots,l_i$. These blocks are never zero matrices, having in mind Lemma 2.2.3. Moreover, none of their rows can be equal to the zero row vector, having in mind that $F_j^{(i)}\Delta_j$ has the same positions of positive entries, negative entries and zero entries as Δ_j, and that, therefore, their diagonal elements always remain positive. Therefore, matrix $(R_i^B D_i^{(B)})^{l_i}$ contains at least one block-column from $(R_i^{(B)}D_i^{(B)})^{l_i-1}R_i^{(B)}$ (which entirely consists of nonzero blocks having no zero rows), in which each element is multiplied by Δ, which satisfies $\|\Delta\|_\infty < 1$, according to the assumption. Having in mind that $R_i^{(B)}$

is row stochastic by assumption, this fact immediately implies that $\|(R_i^{(B)}D_i^{(B)})^{l_i}\|_\infty < 1$. Now, it is possible to extend directly the reasoning of the proof of Lemma 2.2.3, and to conclude further that $\|(\bar{R}_D^{(B)})^l\|_\infty < 1$; the rest of the proof represents a direct extension of the proof of Lemma 2.2.3. ■

Lemma 2.2.5 establishes the existence of a solution of the Lyapunov-like matrix equation used for describing asymptotic covariance of the estimates in Theorem 2.2.1.

Lemma 2.2.5 Let for some positive integers μ and ν

$$W = \sum_{i=1}^{\nu} \alpha_i A_i W A_i^T + \gamma \sum_{j=1}^{\mu} \beta_j B_j W B_j^T + Q, \tag{2.53}$$

where W is a square matrix, $\alpha_i \geq 0$ and $\sum_{i=1}^{\nu} \alpha_i = 1$. Assume that $\|A_i\|_\infty \leq 1$, $i = 1, \ldots, \nu$, and that there exists such a positive integer l that $\min_i \|A_i^l\|_\infty < 1$. Then for any constants β_j, matrices B_j, $j = 1, \ldots, \mu$ and a square matrix Q there exists such a $\bar{\gamma} > 0$ that for $0 \leq \gamma < \bar{\gamma}$ the matrix equation (2.53) has a unique solution.

Proof: We will apply the methodology of successive approximations, see *e.g.* [61]. Let $W_0 = 0$; then, we define W_{n+1} by

$$W_{n+1} - \sum_{i=1}^{\nu} \alpha_i A_i W_{n+1} A_i^T = \gamma \sum_{j=1}^{\mu} \beta_j B_j W_n B_j^T + Q. \tag{2.54}$$

We also have

$$\Delta W_{n+1} - \sum_{i=1}^{\nu} \alpha_i A_i \Delta W_{n+1} A_i^T = \gamma \sum_{j=1}^{\mu} \beta_j B_j \Delta W_n B_j^T, \tag{2.55}$$

where $\Delta W_n = W_n - W_{n-1}$. The assumptions of the Lemma imply that $\|[\sum_{i=1}^{\nu} \alpha_i(A_i \otimes A_i)]^l\|_\infty < 1$ for some positive integer l, and that, therefore, $\rho(\sum_{i=1}^{\nu} \alpha_i(A_i \otimes A_i)) < 1$ ($\rho(.)$ denotes the spectral radius of an indicated matrix). Having in mind that $\sum_{i=1}^{\nu} \alpha_i A_i \Delta W_{n+1} A_i^T = \sum_{i=1}^{\nu} \alpha_i \; (A_i \otimes A_i) \; \mathrm{vec}(\Delta W_{n+1})$, we conclude that, consequently, the equation (2.55) has a unique solution for ΔW_{n+1} satisfying

$$\|\Delta W_{n+1}\|_\infty \leq \zeta\gamma\Sigma_{j=1}^{\mu}|\beta_j|\|B_j\|_\infty^2\|\Delta W_n\|_\infty \tag{2.56}$$

for some finite $\zeta > 0$ ([10, 61]). Choosing $\gamma < \bar{\gamma} = \zeta^{-1}(\sum_{j=1}^{\mu}|\beta_j|\|B_j\|_\infty^2)^{-1}$, we have

$\|\Delta W_{n+1}\|_\infty \leq a\|\Delta W_n\|_\infty$ with $a < 1$, so that W_n converges to some limit W_∞, which represents the solution to (2.53). Hence the result. ■

The following theorem deals with the asymptotic behavior of (2.45) in the case of non-vanishing gains $\Gamma(t)$.

Theorem 2.2.1 Let assumptions (A.2.2.1) - (A.2.2.5) be satisfied and let $\Gamma_\infty > 0$. Then it is possible to find a scalar $\bar{\gamma} > 0$ such that for $\Gamma_\infty \leq \bar{\gamma} I_{mN}$

$$U(t) \leq U_\infty + V(t) \tag{2.57}$$

where $U(t) = E\{(\tilde{\theta}(t) - \bar{\theta})(\tilde{\theta}(t) - \bar{\theta})^T\}$ (by $E\{.\}$ we denote the mathematical expectation) $\bar{\theta} = \left[\,\theta^T \ldots \theta^T\,\right]^T$, $\lim_{t\to\infty} V(t) = 0$ and U_∞ is the solution of the following matrix equation

$$U_\infty = \textstyle\sum_{i=1}^{\tilde{N}} [\tilde{C}^{(i)}(I - \tilde{\Gamma}_\infty^{(i)}\tilde{B})U_\infty(I - \tilde{\Gamma}_\infty^{(i)}\tilde{B})\tilde{C}^{(i)T} +$$
$$\tilde{C}^{(i)}\tilde{\Gamma}_\infty^{(i)}\bar{B}U_\infty\bar{B}^T\,\tilde{\Gamma}_\infty^{(i)}\tilde{C}^{(i)T} - \tilde{C}^{(i)}\tilde{\Gamma}_\infty^{(i)}\tilde{B}U_\infty\tilde{B}\,\tilde{\Gamma}_\infty^{(i)}\tilde{C}^{(i)T} + \tilde{C}\tilde{\Gamma}_\infty^{(i)}\bar{Q}\tilde{\Gamma}_\infty^{(i)}\tilde{C}^{(i)T}]\pi_i, \tag{2.58}$$

where $\tilde{\Gamma}_\infty^{(i)}$ are constant matrices obtained from the generally time varying matrices $\tilde{\Gamma}^{(i)}(t)$ by inserting $\Gamma(t) = \Gamma_\infty = \mathrm{diag}\{\gamma_{\infty,1}, \ldots, \gamma_{\infty,N}\}$, *i.e.* $\tilde{\Gamma}_\infty^{(i)} = \tilde{\Gamma}^{(i)}(t)|_{\Gamma(t)=\Gamma_\infty}$,$Q = \mathrm{diag}\{q_1, \ldots, q_N\}$, $\tilde{B} = \mathrm{diag}\{B, \ldots, B\}$ and $\bar{Q} = E\{\Phi(t)Q\Phi(t)^T\} = Q \otimes B$.

Proof: From (2.45) we have immediately

$$\Delta\tilde{\theta}(t+1) = \tilde{C}(t)[I_{mN} - \tilde{\Gamma}(t)\Phi(t)\Phi(t)^T]\Delta\tilde{\theta}(t) + \tilde{C}(t)\tilde{\Gamma}(t)\Phi(t)\Xi(t) \tag{2.59}$$

where $\Delta\tilde{\theta}(t) = \tilde{\theta}(t) - \bar{\theta}$ and $\Xi(t) = \left[\,\xi_1(t)\cdots\xi_N(t)\,\right]^T$, having in mind that $I_m - \tilde{C}_{ii}(t) = \sum_{j=1,j\neq i}^N \tilde{C}_{ij}(t)$ (since $\tilde{C}(t)$ is row-stochastic, by definition), and that, therefore, $\tilde{C}(t)\tilde{\theta}(t) - \bar{\theta} = \tilde{C}(t)\Delta\tilde{\theta}(t)$. After multiplying (2.59) with $\Delta\tilde{\theta}(t+1)^T$ from the right and taking the conditional expectation given the communication and measurement links at time t determined by the realization X_t, we obtain

$$U(t+1|X_t) \leq \tilde{C}(t)U(t)\tilde{C}(t)^T - \tilde{C}(t)\tilde{\Gamma}(t)\tilde{B}U(t)\tilde{C}(t)^T -$$
$$-\tilde{C}(t)U(t)\tilde{B}\tilde{\Gamma}(t)\tilde{C}(t)^T + \tilde{C}(t)\tilde{\Gamma}(t)\bar{B}U(t)\bar{B}^T\tilde{\Gamma}(t)\tilde{C}(t)^T + \tilde{C}(t)\tilde{\Gamma}(t)\bar{Q}\tilde{\Gamma}(t)\tilde{C}(t)^T, \tag{2.60}$$

where $U(t+1|X_t) = E\{\Delta\tilde{\theta}(t+1)\Delta\tilde{\theta}(t+1)^T|X_t\}$, having in mind that $\{X_t\}$ is a sequence of independent random vectors. After averaging (2.60) with respect to X_t and applying assumption (A.2.2.1 b)), we obtain

$$U(t+1) \leq \sum_{i=1}^{\tilde{N}}[\tilde{C}^{(i)}U(t)\tilde{C}^{(i)T} - \tilde{C}^{(i)}\tilde{\Gamma}^{(i)}(t)\tilde{B}U(t)\tilde{C}^{(i)T} - \tilde{C}^{(i)}U(t)\tilde{B}\tilde{\Gamma}^{(i)}(t)\,\tilde{C}^{(i)T}+$$
$$+\tilde{C}^{(i)}\,\tilde{\Gamma}^{(i)}(t)\bar{B}U(t)\bar{B}^T\,\tilde{\Gamma}^{(i)}(t)\tilde{C}^{(i)T} + \tilde{C}^{(i)}\tilde{\Gamma}^{(i)}(t)\bar{Q}\tilde{\Gamma}^{(i)}(t)\tilde{C}^{(i)T}]\pi_i. \tag{2.61}$$

Define the bounding sequence $\bar{U}(t)$ satisfying $U(t) \leq \bar{U}(t)$ for all $\bar{U}(0) = U(0)$ by

$$\bar{U}(t+1) = \sum_{i=1}^{\tilde{N}}[\tilde{C}^{(i)}\bar{U}(t)\tilde{C}^{(i)T} - \tilde{C}^{(i)}\tilde{\Gamma}^{(i)}(t)\tilde{B}\bar{U}(t)\tilde{C}^{(i)T} - \tilde{C}^{(i)}\bar{U}(t)\tilde{B}\tilde{\Gamma}^{(i)}(t)\,\tilde{C}^{(i)T}+$$
$$+\tilde{C}^{(i)}\,\tilde{\Gamma}^{(i)}(t)\bar{B}\bar{U}(t)\bar{B}^T\,\tilde{\Gamma}^{(i)}(t)\tilde{C}^{(i)T} + \tilde{C}^{(i)}\tilde{\Gamma}^{(i)}(t)\bar{Q}\tilde{\Gamma}^{(i)}(t)\tilde{C}^{(i)T}]\pi_i. \tag{2.62}$$

Following methodologically [62], we will now replace $\Gamma(t)$ with $\Gamma_\infty + \Delta\Gamma(t)$, where $\|\Delta\Gamma(t)\| = o(1)$ ($o(1)$ denotes a sequence tending to zero when $t \to \infty$), and $\bar{U}(t)$ with $U_\infty + V(t)$, where U_∞ is a constant matrix.

We will focus the analysis first on the terms depending on U_∞ and Γ_∞. The following set of conclusions is important for further derivations:

(1.a) according to assumption (A.2.2.1 a)), for sufficiently small values of a positive scalar γ, matrix $I - \gamma B$ is positive definite, strictly diagonally dominant and satisfies the condition $\|I - \gamma B\|_\infty < 1$, having in mind that $\min_i[1 + \gamma(|b_{i1}| + \cdots - b_{ii} + \cdots + |b_{in}|)] < 1$, according to (2.47).

(1.b) matrix $\tilde{C}^*$, the "full" realization of the consensus matrix, is structurally equivalent to the matrix $R^{(B)}$ in Lemma 2.2.4, in the same way as K^* is structurally equivalent to P (except for some loops corresponding to the nodes which do not have access to measurements), in such a way that the blocks C^*_{ij} in $\tilde{C}^*$ correspond to the blocks $r_{ij}M_{ij}$ in R^B;

(1.c) both matrices $\tilde{C}^*$ and $R^{(B)}$ are row-stochastic by assumption;

(1.d) matrix $\tilde{C}^*(I - \tilde{\Gamma}^*_\infty\tilde{B})$, where $\tilde{\Gamma}^*_\infty$ is defined as $\tilde{\Gamma}^*(t)|_{\Gamma(t)=\Gamma_\infty}$, represents a realization $\tilde{C}^{(i_1)}(I - \tilde{\Gamma}^{(i_1)}_\infty\tilde{B})$ for some $i_1 \in \{1,\ldots,\tilde{N}\}$, having a positive probability π_{i_1};

(1.e) $\tilde{C}^*(I - \tilde{\Gamma}^*_\infty\tilde{B})$ is structurally equivalent to the matrix $R^{(B)}_D$ in Lemma 2.2.4 for $\gamma_{i\infty}$

small enough, $i = 1, \ldots, N$ (matrix $I - \gamma_{i\infty} B$ corresponds to Δ in Lemma 2.2.4), having in mind the assumed diagonal dominance of B, so that matrix $I - \tilde{\Gamma}^*_\infty \tilde{B}$ has the properties of $D^{(B)}$ in Lemma 2.2.4 (notice that the algorithm gains $\tilde{\gamma}_i(t)$ have nonzero values for the indices i which correspond to the indices of all the nonzero probabilities p_{ii});

(1.f) according to Lemma 2.2.4, $\|(\tilde{C}^*(I - \tilde{\Gamma}^*_\infty \tilde{B}))^l\| < 1$ for some integer $l \geq 1$, having in mind (A.2.2.1 a)) and (A.2.2.5);

(1.g) according to Lemma 2.2.5 and under the assumptions of the theorem, the matrix equation (2.58) has a unique solution for sufficiently small values of $\gamma_{i\infty}$, $i = 1, \ldots, N$.

Conclusion (1.g) allows using (2.58) to eliminate all the terms containing U_∞ and Γ_∞ from (2.61), so that we obtain

$$V(t+1) = \sum_{i=1}^{\tilde{N}} [\tilde{C}^{(i)} V(t) \tilde{C}^{(i)T} - \tilde{C}^{(i)} \tilde{B} \tilde{\Gamma}^{(i)}_\infty V(t) \tilde{C}^{(i)T} -$$

$$- \tilde{C}^{(i)} V(t) \tilde{\Gamma}^{(i)}_\infty \tilde{B} \tilde{C}^{(i)T} + \tilde{C}^{(i)} \tilde{\Gamma}^{(i)}_\infty \bar{B} V(t) \bar{B}^T \tilde{\Gamma}^{(i)}_\infty \tilde{C}^{(i)T}] \pi_i + F(t), \tag{2.63}$$

where $F(t)$ contains the terms depending on $\Delta\Gamma(t)$, such that their norms are in the form of $o(1)\|U_\infty\|$, $o(1)\|V(t)\|$, etc.

Having in mind continuous dependence of the eigenvalues of a matrix upon its elements, for any given $\tilde{\Gamma}^{(i)}_\infty$ satisfying (A.2.2.4) and (2.63) it is possible to find such a $\bar{\Gamma}^{(i)}_\infty = \tilde{\Gamma}^{(i)}(t)|_{\Gamma(t) = \bar{\gamma}^{(i)}_\infty I}$, where $\bar{\gamma}^{(i)}_\infty > 0$ is small enough, that

$$\tilde{C}^{(i)} V(t) \tilde{C}^{(i)T} - \tilde{C}^{(i)} \tilde{\Gamma}^{(i)}_\infty \tilde{B} V(t) \tilde{C}^{(i)T} - \tilde{C}^{(i)} V(t) \tilde{B} \tilde{\Gamma}^{(i)}_\infty \tilde{C}^{(i)T} +$$

$$+ \tilde{C}^{(i)} \tilde{\Gamma}^{(i)}_\infty \bar{B} V(t) \bar{B}^T \tilde{\Gamma}^{(i)}_\infty \tilde{C}^{(i)T} \leq \tilde{C}^{(i)} (I - \bar{\Gamma}^{(i)}_\infty \tilde{B}) V(t) (I - \bar{\Gamma}^{(i)}_\infty \tilde{B}) \tilde{C}^{(i)T}. \tag{2.64}$$

Therefore, we define, similarly as above, a bounding matrix sequence $\bar{V}(t)$ satisfying $V(t) \leq \bar{V}(t)$, generated by

$$\bar{V}(t+1) = \sum_{i=1}^{\tilde{N}} [\tilde{C}^{(i)} (I - \bar{\Gamma}^{(i)}_\infty \tilde{B}) \bar{V}(t) (I - \bar{\Gamma}^{(i)}_\infty \tilde{B}) \tilde{C}^{(i)T}] \pi_i + F(t), \tag{2.65}$$

for some $\bar{V}(0) = V(0)$. Iterating now (2.65) l steps backwards, we obtain

$$\bar{V}(t+1) = \sum_{i_1=1}^{\tilde{N}} \cdots \sum_{i_l=1}^{\tilde{N}} \Pi^{[l]}_{i_1,\cdots,i_l} \bar{V}(t-l+1)(\Pi^{[l]}_{i_1,\cdots,i_l})^T \pi_{i_1} \cdots \pi_{i_l} + F_l(t) \tag{2.66}$$

where $\Pi^{[l]}_{i_1,\cdots,i_l} = \tilde{C}^{(i_1)}(I - \bar{\Gamma}^{(i_1)}_\infty \tilde{B}) \cdots \tilde{C}^{(i_l)}(I - \bar{\Gamma}^{(i_l)}_\infty \tilde{B})$, $i_1, \ldots, i_l = 1, \ldots, \tilde{N}$, while the term $F_l(t)$ depends on $F(t)$ and $\Pi^{[m]}_{i_1,\cdots,i_m}$ for $m < l$. Define $\Pi^{[l]*} = (\tilde{C}^*(I - \bar{\Gamma}^*_\infty \tilde{B}))^l$, according to the above definitions and (1.d). We infer now, using Lemmas 2.2.3 and 2.2.4, together with the conclusion (1.g), that $\|\Pi^{[l_0]*}\|_\infty < 1$ for some $l_0 \geq 1$. Therefore, starting from (1.d) we have immediately from (2.66) that for $l > l_0$

$$\|\bar{V}(t+1)\|_\infty \leq (1 - \lambda_1)\|\bar{V}(t-l+1)\|_\infty + o(1), \tag{2.67}$$

where $0 < \lambda_1 < 1$ (notice that the important condition $\lambda_1 > 0$ is ensured by the properties of the "full" realization defined by the structure of measuring nodes and communications links having positive probabilities). According to *e.g.* [61] (Lemmas 1 and 4), we derive directly that $\lim_{t\to\infty} \bar{V}(t) = 0$, implying that $\lim_{t\to\infty} V(t) = 0$. Thus the result. ■

Now we will analyze convergence of the parameter estimates in the mean-square sense in the case when $\Gamma(t)$ asymptotically tends to zero.

(A.2.2.6) $\lim_{t\to\infty} \gamma_i(t) = 0$, $\sum_{t=1}^{\infty} \gamma_i(t) = \infty$, $i = 1, \ldots, N$, $\max_i \gamma_i^2(t) = o(\min_i \gamma_i(t))$.

Theorem 2.2.2 Let assumptions (A.2.2.1)-(A.2.2.6) be satisfied. Then $\lim_{t\to\infty} U(t) = 0$, *i.e.*, $\tilde{\theta}(t)$ converges to $\bar{\theta}$ in the mean square sense.

Proof: We immediately get (2.62) from (2.59) in the same way as in the case of Theorem 2.2.1. We rewrite (2.62) using (2.64) and directly construct a bounding matrix sequence $\bar{U}(t)$ satisfying $U(t) \leq \bar{U}(t)$ defined by

$$\bar{U}(t+1) = \sum_{i=1}^{\tilde{N}} [\tilde{C}^{(i)}(I - \tilde{\Gamma}^{(i)}(t)\tilde{B})\bar{U}(t)(I - \tilde{\Gamma}^{(i)}(t)\tilde{B})\tilde{C}^{(i)T} + G(t)]\pi_i, \tag{2.68}$$

where $G(t) = \sum_{i=1}^{\tilde{N}} \tilde{C}^{(i)}\tilde{\Gamma}^{(i)}(t)\bar{Q}\tilde{\Gamma}^{(i)}(t)\tilde{C}^{(i)T}$.

Convergence properties of (2.68) depend primarily, as in the case of (2.65), on the properties of the matrix $\tilde{C}^*(I - \tilde{\Gamma}^*(t)\tilde{B})$. As matrix K^* remains the same as in Theorem

2.2.1, for sufficiently high values of t we can extend the basic conclusions (1.a) - (1.h) to the time varying case. Technically, we iterate (2.68) l times backwards and analyze the resulting time varying terms analogous to those in (2.66). Reasoning as in Theorem 2.2.1, we can derive, using (A.2.2.1a), (A.2.2.5) and (A.2.2.6), the following basic conclusions:

(2.a) having in mind the general property that $A^l \succ 0 \iff \prod_{i=1}^{l} A_i \succ 0$ if the places of positive elements are the same for nonnegative primitive matrices $A, A_1, \ldots, A_l$ (see also the proofs of Lemmas 2.2.3 and 2.2.4), we conclude, similarly as in Lemma 2.2.4 and Theorem 2.2.1, that for each $t \geq t_0$ there exists such an integer $l \geq 1$ that the matrix $\Pi^*(t, t-l+1) = \tilde{C}^*[I - \tilde{\Gamma}^*(t)\tilde{B}]\tilde{C}^*[I - \tilde{\Gamma}^*(t-1)\tilde{B}] \cdots \tilde{C}^*[I - \tilde{\Gamma}^*(t-l+1)\tilde{B})]$ is composed of $N \times N$ nonzero $m \times m$ blocks (notice that $\Pi^*(t, t-l+1)$ is analogous to $\Pi^{[l]*}$ in Theorem 2.2.1);

(2.b) matrix $\Pi^*(t, t-l+1)$ can be expressed as $\Pi^*(t, t-l+1) = \tilde{C}^{*l} - \tilde{\Gamma}^*(t)\tilde{B}\tilde{C}^{*l} - \tilde{C}^*\tilde{\Gamma}^*(t-1)\tilde{B}\tilde{C}^{*(l-1)} - \ldots - \tilde{C}^{*l}\tilde{\Gamma}^*(t-l+1)\tilde{B} + higher\ \ order\ \ terms\ \ in\ \ \tilde{\Gamma}$; therefore, we have for $t > t_1$ and some $l > l_1 \geq 1$

$$\|\Pi^*(t, t-l+1)\|_\infty \leq 1 - \lambda_2 \min_i \gamma_i(t-l+1), \tag{2.69}$$

where $\lambda_2 > 0$, as a consequence of the diagonal dominance of B, and of the fact $\tilde{C}^*$ is row stochastic, according to the assumptions (A.2.2.1 a)) and (A.2.2.6) and the results of Lemmas 2.2.2 and 2.2.4 (notice that it is possible to find such a sufficiently small $\lambda_2' > 0$ that for t large enough all the higher order terms in $\gamma_i(t)$ can be maximized by $\lambda_2'\gamma_i(t)$ in (2.69)).

Consequently, we readily obtain from (2.68) that asymptotically

$$\|\bar{U}(t+1)\|_\infty \leq [1 - \lambda_2 \min_i \gamma_i(t-l+1)]\|\bar{U}(t-l+1)\|_\infty + \mu_2 \max_i \gamma_i(t-l+1)^2 \tag{2.70}$$

for some $l \geq 1$ and $0 < \mu_2 < \infty$, according to (A.2.2.2), (A.2.2.3) and (A.2.2.6). We can now use the classical results from the field of stochastic approximation (see *e.g.* [61], Lemma 1 and Corollary 2, or [83], Final Value Theorem) and conclude directly that $\lim_{t\to\infty} \|\bar{U}(t)\|_\infty = 0$, implying $\lim_{t\to\infty} U(t) = 0$. Hence the result. ■

Theorem 2.2.2 gives rise directly to an estimate of the convergence rate of the algorithm for a specific form of the weighting sequence $\tilde{\Gamma}(t)$.

Corollary 2.2.1 Let the assumptions of Theorem 2.2.2 be satisfied, and let $\gamma_i(t) = \gamma_i/t$, $\gamma_i > 1/\lambda_2, \lambda_2 > 0$, $i = 1, \ldots, N$. Then, we have from (2.70) that

$$\|U(t)\|_\infty \leq \frac{l}{t} \frac{\mu_2 \max_i \gamma_i}{\lambda_2 \min_i \gamma_i - 1} + o(\frac{1}{t}) \tag{2.71}$$

Proof: The proof follows directly from (2.70) after applying Chung's lemma (see *e.g.* [66], Theorem 2, [21], Lemma 1). ∎

An estimate of the convergence rate in the matrix form, analogous to the one from Theorem 2.2.1, can be derived under a set of additional assumptions.

Theorem 2.2.3 Let assumptions (A.2.2.1) - (A.2.2.3) and (A.2.2.5) be satisfied. Let $\tilde{C}(t) = \tilde{C}$ be deterministic and time invariant, $\tilde{\Gamma}(t) = \Gamma(t) = \frac{1}{t}\Gamma$, where $\Gamma > 0$ is a constant diagonal matrix, and matrix $\Gamma\tilde{B} - \frac{1}{2}I$ be positive definite and diagonally dominant. Then, asymptotically

$$U(t) \leq \frac{1}{t}[V_\infty + V(t)], \tag{2.72}$$

provided a positive semidefinite solution for V_∞ exists simultaneously for $(I - \tilde{C})V_\infty^{\frac{1}{2}} = 0$ and

$$\tilde{C}V_\infty\tilde{C}^T - \tilde{C}\Gamma\tilde{B}V_\infty\tilde{C}^T - \tilde{C}V_\infty\tilde{B}\Gamma\tilde{C}^T + \tilde{C}\Gamma\bar{Q}\Gamma\tilde{C}^T = 0, \tag{2.73}$$

and $\|V(t)\| = o(1)$.

Proof: The proof follows methodologically [62], taking into account specific properties of the proposed algorithm. We start again from (2.61) and obtain

$$\begin{aligned} U(t+1) \leq \tilde{C}U(t)\tilde{C}^T - \tfrac{1}{t}\tilde{C}\Gamma\tilde{B}U(t)\tilde{C}^T - \tfrac{1}{t}\tilde{C}U(t)\tilde{B}\Gamma\,\tilde{C}^T + \\ + \tfrac{1}{t^2}\tilde{C}\,\Gamma\bar{B}U(t)\bar{B}^T\,\Gamma\tilde{C}^T + \tfrac{1}{t^2}\tilde{C}\Gamma\bar{Q}\Gamma\tilde{C}^T, \end{aligned} \tag{2.74}$$

taking into account the assumptions of the theorem. After replacing $U(t)$ by $\frac{1}{t}[V_\infty + V(t)]$, similarly as in Theorem 2.2.1, we first analyze the terms depending on V_∞ in the resulting inequality. The first important conclusion is that $V_\infty = \tilde{C}V_\infty\tilde{C}^T$ under the assumptions of the theorem, so that the corresponding terms can be eliminated from both sides of (2.74). In the next step, we apply (2.73) to the terms containing $\frac{1}{t}$, provided a solution to both (2.73) and $(I-\tilde{C})V_\infty^{\frac{1}{2}}=0$ exists. The problem is not trivial, in general. However, if we assume that $V_\infty = XWX^T$, matrix X can be directly obtained from $(I-\tilde{C})X=0$; this matrix is, obviously, singular. Replacing $V_\infty = XWX^T$ in (2.73), we obtain a singular equation for W, which has either an infinite number of solutions or no solutions. In general, supposing that a solution for W exists, we obtain further, after eliminating the terms depending on V_∞ like in Theorem 2.2.1, that $V(t) \leq \bar{V}(t)$, where $\bar{V}(t)$ satisfies

$$\bar{V}(t+1) = \tilde{C}[I - \frac{1}{t}(\Gamma\tilde{B} - \frac{1}{2}I)]\bar{V}(t)[I - \frac{1}{t}(\Gamma\tilde{B} - \frac{1}{2}I)]^T\tilde{C}^T + G(t), \tag{2.75}$$

with $\|G(t)\|_\infty = o(\frac{1}{t}) + o(\frac{1}{t})\|\bar{V}(t)\|_\infty$. Following the methodology of Theorem 2.2.2, we obtain for t large enough that

$$\|\bar{V}(t+1)\|_\infty \leq (1 - \frac{\lambda_3}{t-l+1})\|\bar{V}(t-l+1)\|_\infty + o(\frac{1}{t-l+1}), \tag{2.76}$$

where $\lambda_3 > 0$, having in mind the assumed properties of the matrix $\Gamma\tilde{B} - \frac{1}{2}I$ (these properties are required from the matrix $\tilde{B}$ itself in (A.2.2.1 a)). Consequently, $\lim_{t\to\infty}\|\bar{V}(t)\|_\infty = 0$, according to e.g. [61, 62]. Thus the result. ■

Remark 2.2.6 Obviously, the above methodology of convergence analysis can be applied to the similar algorithms mentioned in Remark 2.2.1 [107, 105, 44, 113]. We will only remark here that matrix $\tilde{C}^*(I - \tilde{\Gamma}^*(t)\tilde{B})$, playing the main role in the above analysis, becomes $\tilde{C}^* - \tilde{\Gamma}^*(t)\tilde{B}$ in the case of the algorithm based on "convexification" of the previous estimates (see [105, 107, 44, 113]), and becomes $I - \tilde{C}^*\tilde{\Gamma}^*(t)\tilde{B}$ in the case of the algorithm based on "convexification" of the increments, presented in [44], paragraph 7.6. It is evident that in both cases a delineation between the influence of the terms resulting from the local stochastic approximation schemes and from the network properties alone cannot be so clearly achieved as in the case of the proposed algorithm. Moreover, it seems, according to simulations, that the proposed algorithm possesses superior asymptotic properties. Without drawing any resolute conclusion here, we will demonstrate a typical performance of all three algorithms through a simple example. Let $\tilde{C} = \begin{bmatrix} 0.5 & 0.5 \\ c & 1-c \end{bmatrix}$, $\tilde{\Gamma} = \begin{bmatrix} 0.1 & 0 \\ 0 & g \end{bmatrix}$, $B = 1$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, where c and g are scalars. In this simple case, it is possible to calculate exactly the asymptotic covariance of the estimates according to $P = APA^T + Q'$, where $A = \tilde{C}(I - \tilde{\Gamma})$ and $Q' = \tilde{C}Q\tilde{C}^T$ in the case of the proposed algorithm (AL1), $A = \tilde{C} - \tilde{\Gamma}$ and $Q' = Q$ in the case of the algorithm in (2.46) (AL2), and $A = I - \tilde{C}\tilde{\Gamma}$ and $Q' = \tilde{C}Q\tilde{C}^T$ in the case of the third mentioned algorithm ([44]) (AL3). Table 2.3 gives the eigenvalues λ_1 and λ_2 of A and $J = \mathrm{Tr}P$ for all three algorithms and several values of the parameters c and g. It is evident that the proposed algorithm AL1 gives the best performance; AL3 is

	c=0.25,g=0.1		c=0.25,g=0.3		c=0.75,g=0.1		c=0.75, g=0.3	
	λ_1, λ_2	J	λ_1, λ_2	J	λ_1, λ_2	J	λ_1, λ_2	J
AL1	0.9000 0.2250	5.8514	0.2044 0.7706	2.7343	0.9000 -0.2250	5.5588	0.8176 -0.1926	3.2371
AL2	0.9000 0.1500	6.7277	0.0706 0.7794	3.7316	0.9000 -0.3500	6.5980	0.8274 -0.4774	4.5570
AL3	0.9750 0.9000	6.8009	0.9693 0.7557	3.8721	0.9000 1.0250	-	1.0443 0.8397	-

Table 2.3: Performance of different algorithms

in two cases even unstable. ■

2.2.3 Discussion

2.2.3.1 Network Topology

Formulation of the problem introduces random communication links and random access to measurements; also, assumption (A.2.2.5) does not require from all the agents to receive measurements with positive probabilities. The results of Lemmas 2.2.1 and 2.2.2 formalize the whole setting in terms of the properties of the network graph related to the "full" realization. Lemma 2.2.1 shows that the graph $\bar{\mathcal{G}}_P$ is composed of a certain number of closed strong components. Consequently, $\rho(P_i) = 1$ is a simple eigenvalue of P_i, $i = 1, \ldots, k$ in (2.48), and we immediately realize connections with the first order discrete time consensus scheme discussed in *e.g.* [23, 39, 49, 51, 57, 73]. However, this result is insufficient for direct conclusions about the convergence of the proposed algorithm, having in mind the dynamics of the local parameter estimation algorithms themselves. Lemmas 2.2.3 and 2.2.4 provide necessary prerequisites for the convergence analysis. Lemma 2.2.4 contains an important generalization encompassing "consensus matrices" with positive definite blocks. The proofs of the Lemmas 2.2.1-4 are derived mainly using the results presented in [48].

A comparison of the above results with those related to the overlapping decentralized state estimation algorithm presented in Section 2.1 shows that the stability results derived there in the sense of keeping the mean-square estimation error bounded do not explicitly rely on so strict assumptions concerning the network structure. This is a consequence of the fact that all the local estimators in the state estimation problem are assumed to receive measurements with positive probabilities, relaxing the corresponding requirements.

2.2.3.2 Additive Communication Noise

Assume that the uncertainty of inter-agent communications in the network is modelled as additive zero-mean white communication noise, in such a way that the i-th agent receives $\tilde{\theta}_j(t) + \eta_{ij}(t)$ instead of $\tilde{\theta}_j(t)$ from the j-th agent, $i, j = 1, \ldots, N, i \neq j$, where $\eta_{ij}(t)$ is the noise term. Assuming that the rest of the whole setting is the same as above, we obtain from (2.44) an additional additive term at the right hand side of (2.45) in the form $\tilde{C}_\eta(t)\eta(t)$, where $\tilde{C}_\eta(t) = \mathrm{diag}\{ \left[\tilde{C}_{12}(t) \vdots \ldots \vdots \tilde{C}_{1N}(t) \right], \cdots, \left[\tilde{C}_{N1}(t) \vdots \ldots \vdots \tilde{C}_{N,N-1}(t) \right] \}$ and $\eta(t) = \left[\eta_{12}(t)^T \cdots \eta_{1N}(t)^T \ldots \eta_{N1}(t)^T \ldots \eta_{N,N-1}(t)^T \right]^T$. In general, it is not possible in this case to achieve convergence of the parameter estimates generated by the proposed algorithm only by adopting a gain sequence $\Gamma(t)$ tending to zero, like in Theorem 2.2.2. An idea of how to overcome this problem is based on adopting vanishing communication gains, according to the main ideas of stochastic approximation procedures. Notice first that in the case when all the agents have permanent access to the measurements, the mean-square convergence can be achieved by the local algorithms themselves after disconnecting the network. In general, when the consensus scheme is introduced, convergence of the parameter estimates can be achieved in the general case by adopting $\tilde{C}(t) = I + \tilde{L}(t)$, where

$$\tilde{L}(t) = \begin{bmatrix} -\sum_{j,j\neq 1} \tilde{C}_{1j}(t) & \tilde{C}_{12}(t) & \cdots & \tilde{C}_{1N}(t) \\ \tilde{C}_{21}(t) & -\sum_{j,j\neq 2} \tilde{C}_{2j}(t) & \cdots & \\ & \cdots & & \\ \tilde{C}_{N1}(t) & \tilde{C}_{N2}(t) & \cdots & -\sum_{j,j\neq N} \tilde{C}_{Nj}(t) \end{bmatrix},$$

with $\|\tilde{L}(t)\|_\infty \leq \delta(t)$, where $\{\delta(t)\}$ is a positive number sequence satisfying $\lim_{t\to\infty} \delta(t) = 0$ and $\delta(t)^2 = o(\min_i \gamma_i(t))$. After some technicalities similar to those presented in Theorem 2.2.2, we obtain the following basic inequality analogous to (2.70):

$$\|\bar{U}(t+1)\|_\infty \leq [1-\lambda_4 \min_i \gamma_i(t-l+1)]\|\bar{U}(t-l+1)\|_\infty + \mu_2 \max_i \gamma_i(t-l+1)^2 + \mu_3\delta(t-l+1)^2, \tag{2.77}$$

where $0 < \lambda_4, \mu_3 < \infty$, wherefrom the conclusion $\lim_{t\to\infty} \|U(t)\|_\infty = 0$ follows in the same way as in Theorem 2.2.2.

Additive communication noise is certainly not the most adequate model for uncertainty in modern communications, so that this case has dominantly a theoretical significance. However, the idea to apply stochastic approximation type algorithms for consensus seeking in a noisy environment has appeared recently in [31, 32]. Our case is, however, different, having in mind that the consensus scheme obeying the above time varying law represents only a part of the proposed decentralized estimation scheme, and that we are not looking for the conditions ensuring asymptotic consensus, but for the conditions ensuring the mean-square convergence of the parameter estimates.

2.2.3.3 Denoising

According to Remark 2.2.2, the proposed scheme, containing a specific ensemble averaging, can directly contribute to the overall suppression of measurement noise influence, having in mind that the local outputs are corrupted by different noise realizations. However, the efficiency of noise suppression depends on the network complexity. Following similar line of thought as in the state estimation algorithms, it can be shown that the condition

$$\sum_{i=1}^{N} |\lambda_i^{(N)}|^2 = o(N), \tag{2.78}$$

where $\lambda_i^{(N)}$, $i = 1, \dots, N$, are N distinct eigenvalues of the consensus matrix $\tilde{C}^{(N)}$ (N denotes the number of nodes) is sufficient for achieving asymptotic denoising in the sense of reducing the asymptotic mean square error bound to zero (for the case of non-vanishing gains). As in the state estimation case, it is possible to show that the condition (2.78) holds in the case of undirected graphs when the number of connections per node tends to infinity when N tends to infinity, but at a rate which can be much slower than the linear function.

In the case of the gains tending to zero, when t tends to infinity, the denoising effect contributes to the rate of convergence of the algorithm.

Chapter 3

Multi-Agent Consensus Based Control Structures

In this chapter we will address the problem of structured, multi-agent control of complex networked systems [86]. Two consensus based algorithms will be proposed; one is based on the consensus at the control input level, and the second algorithm is based on the consensus at the state estimation level (described in Chapter 2).

3.1 Problem Formulation

Let a complex system be represented by a linear model

$$\begin{aligned} \mathbf{S}: \qquad \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{3.1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and $y \in \mathcal{R}^\nu$ are the state, input and output vectors, respectively, while A, B and C are constant matrices of appropriate dimensions.

Assume that N agents have to control the system $\mathbf{S}$ according to their own resources.

The agents have their local models of parts of $\mathbf{S}$

$$\mathbf{S}_i: \qquad \dot{\zeta}^{(i)} = A^{(i)}\zeta^{(i)} + B^{(i)}v^{(i)}$$
$$y^{(i)} = C^{(i)}\zeta^{(i)} \tag{3.2}$$

where $\zeta^{(i)} \in \mathcal{R}^{n_i}$, $v^{(i)} \in \mathcal{R}^{m_i}$ and $y^{(i)} \in \mathcal{R}^{\nu_i}$ are the corresponding state, input and output vectors, and $A^{(i)}$, $B^{(i)}$ and $C^{(i)}$ constant matrices, $i = 1, \ldots, N$. Components of the input vectors $v^{(i)} = (v_1^{(i)}, \ldots, v_{m_i}^{(i)})^T$ represent subsets of the global input vector u of $\mathbf{S}$, so that $v_j^{(i)} = u_{p_j^i}$, $j = 1, \ldots, m_i$, and $p_j^i \in \mathcal{V}^i$, where $\mathcal{V}^i = \{p_1^i, \ldots, p_{m_i}^i\}$ is the input index set defining $v^{(i)}$. Similarly, for the outputs $y^{(i)}$ we have $y_j^{(i)} = y_{q_j^i}$, $j = 1, \ldots, \nu_i$, and $q_j^i \in \mathcal{Y}^i$, where $\mathcal{Y}^i = \{q_1^i, \ldots, q_{p_i}^i\}$ is the output index set; according to these sets, it is possible to find such constant $p_i \times n$ matrices C_i that $y^{(i)} = C_i x$, $i = 1, \ldots, N$. The state vectors $\zeta^{(i)}$ do not necessarily represent parts of the global state vector x. They can be chosen, together with the matrices $A^{(i)}$, $B^{(i)}$ and $C^{(i)}$, according to the local criteria for modelling the input-output relation $v^{(i)} \to y^{(i)}$. In the particular case when $\zeta^{(i)} = x^{(i)}$, $x_j^{(i)} = x_{r_j^i}$, $j = 1, \ldots, n_i$, $n_i \leq n$ and $r_j^i \in \mathcal{X}^i$, where $\mathcal{X}^i = \{r_1^i, \ldots, r_{n_i}^i\}$ is the state index set defining $x^{(i)}$. In the last case, models $\mathbf{S}_i$, in general, represent overlapping subsystems of $\mathbf{S}$ in a more strict sense; matrices $A^{(i)}$, $B^{(i)}$ and $C^{(i)}$ can represent in this case submatrices of A, B and C.

The task of the i-th agent is to generate the control vector $v^{(i)}$ and to implement the control action $u^{(i)} \in \mathcal{R}^{\mu_i}$, satisfying $u_j^{(i)} = u_{s_j^i}$, $j = 1, \ldots, \mu_i$, and $s_j^i \in \mathcal{U}^i$, where $\mathcal{U}^i = \{s_1^i, \ldots, s_{\mu_i}^i\}$ is the *control index set* defining $u^{(i)}$. It is assumed that $\mathcal{U}^i \subseteq \mathcal{V}^i$ and $\mathcal{U}^i \cap \mathcal{U}^j = \emptyset$, so that $\sum_{i=1}^N \mu_i = m$, that is, the control vector $u^{(i)}$ of the i-th agent is a part of its input vector $v^{(i)}$, while one and only one agent is responsible for generation of each component of u within the considered control task. Consequently, all agents include the entire vectors $v^{(i)}$ of $\mathbf{S}_i$ in their control design considerations, but they have to implement only those components of $v^{(i)}$ for which they are responsible.

In the case when the inputs $v^{(i)}$ do not overlap, the agents perform their task autonomously, without interactions with each other; that is we have the case of decentralized control of $\mathbf{S}$, when the control design is based entirely on the local models $\mathbf{S}_i$. However, in the case when the model inputs $v^{(i)}$ overlap, more than one model $\mathbf{S}_i$ can be used for calculation of a particular component of the input vector u. Obviously, it would be beneficial for the agent responsible for implementation of that particular input component to use different suggestions about the control action and to calculate the numerical values of the control signal to be implemented on the basis of an agreement between the agents. The agents that do not implement any control action ($\mathcal{U}^i = \emptyset$) could, in this context, represent "advisors" to the agents responsible for control implementation. Our aim is to propose several overlapping decentralized feedback control structures for $\mathbf{S}$ based on a consensus between multiple agents.

We will classify different control structures which can be used for solving the above problem in two main groups: (1) the structures based on the consensus at the control input level; (2) the structures based on the consensus at the state estimation level.

3.2 Structures Based on Consensus at the Control Input Level

3.2.1 Algorithms Derived from the Local Dynamic Output Feedback Control Laws

We assume that all the agents are able to design their own local dynamic controllers which generate the input vectors $v^{(i)}$ in $\mathbf{S}_i$ according to

$$\mathbf{C}_i : \quad \begin{aligned} \dot{w}^{(i)} &= F^{(i)} w^{(i)} + G^{(i)} y^{(i)} \\ v^{(i)} &= K^{(i)} w^{(i)} + H^{(i)} y^{(i)} \end{aligned} \tag{3.3}$$

where $w^{(i)} \in \mathcal{R}^{\rho_i}$ represents the controller state, and matrices $F^{(i)}$, $G^{(i)}$, $K^{(i)}$ and $H^{(i)}$ are constant, with appropriate dimensions. Local controllers are designed according to the local models and local design criteria, $i = 1, \ldots, N$. Assuming that the agents can communicate between each other, the goal is to generate the control signal u for $\mathbf{S}$ based on mutual agreement, starting from the inputs $v^{(i)}$ generated by $\mathbf{C}_i$. The idea about reaching an agreement upon the components of u stems from the fact that the index sets $\mathcal{V}^{(i)}$ are, in general, overlapping, so that the agents responsible for control implementation according to the index sets $\mathcal{U}^{(i)}$ can improve their local control laws by getting "suggestions" from the other agents.

Algorithm 1 The second relation in (3.3) gives rise to $\dot{v}^{(i)} = K^{(i)}\dot{w}^{(i)} + H^{(i)}\dot{y}^{(i)}$, wherefrom we get

$$\begin{aligned} \dot{v}^{(i)} &= K^{(i)}[F^{(i)}w^{(i)} + G^{(i)}y^{(i)}] + H^{(i)}C^{(i)}[A^{(i)}\zeta^{(i)} + B^{(i)}v^{(i)}] \\ &= K^{(i)}F^{(i)}w^{(i)} + K^{(i)}G^{(i)}y^{(i)} + H^{(i)}C^{(i)}A^{(i)}\zeta^{(i)} + H^{(i)}C^{(i)}B^{(i)}v^{(i)}. \end{aligned} \tag{3.4}$$

Since $y^{(i)}$ are the available signals, and $v^{(i)}$ vectors to be locally generated for participation in the agreement process, we will use the following approximation

$$\begin{aligned} \dot{v}^{(i)} &\approx [K^{(i)}F^{(i)}K^{(i)+} + H^{(i)}C^{(i)}B^{(i)}]v^{(i)} \\ &+ [K^{(i)}G^{(i)} + H^{(i)}C^{(i)}A^{(i)}C^{(i)+} - K^{(i)}F^{(i)}K^{(i)+}H^{(i)}]y^{(i)}, \end{aligned} \tag{3.5}$$

where $F_*^{(i)} = K^{(i)}F^{(i)}K^{(i)+}$ and $A_*^{(i)} = C^{(i)}A^{(i)}C^{(i)+}$ are approximate solutions of the *aggregation* relations $K^{(i)}F^{(i)} = F_*^{(i)}K^{(i)}$ and $C^{(i)}A^{(i)} = A_*^{(i)}C^{(i)}$, respectively, where A^+ denotes the pseudoinverse of a given matrix A [80, 38].

We will assume, for the sake of presentation clarity, that all the agents can have their "suggestions" for all the components of u; that is, we assume that the vector $U_i \in \mathcal{R}^m$ is a "local version" of u proposed by the i-th agent to the other agents. Furthermore, we introduce $m \times \rho_i$ and $m \times \nu_i$ constant matrices K_i and H_i, obtained by taking the rows of $K^{(i)}$ and $H^{(i)}$ at the row indices defined by the index set $\mathcal{V}^{(i)}$ and leaving zeros elsewhere, and $n_i \times m$ matrix B_i obtained from $B^{(i)}$ by taking its columns at the indices defined by $\mathcal{V}^i$. Let $U = \mathrm{col}\{U_1, \ldots, U_N\}$, $Y = \mathrm{col}\{y^{(1)}, \ldots, y^{(N)}\}$, $\tilde{K} = \mathrm{diag}\{K_1, \ldots, K_N\}$ and $\tilde{H} = \mathrm{diag}\{H_1, \ldots, H_N\}$. Similarly, let $\tilde{A} = \mathrm{diag}\{A^{(1)}, \ldots, A^{(N)}\}$, $\tilde{B} = \mathrm{diag}\{B_1, \ldots, B_N\}$, $\tilde{C} = \mathrm{diag}\{C^{(1)}, \ldots, C^{(N)}\}$, $\tilde{F} = \mathrm{diag}\{F^{(1)}, \ldots, F^{(N)}\}$, and $\tilde{G} = \mathrm{diag}\{\tilde{G}^{(1)}, \ldots, \tilde{G}^{(N)}\}$. Assume that the agents communicate between each other in such a way that they send current values of U_i to each other. Accordingly, we define the consensus matrix as $\tilde{\Gamma} = [\Gamma_{ij}]$, where Γ_{ij}, $i, j = 1, \ldots, N$, $i \neq j$, are $m \times m$ diagonal matrices with positive entries and

$\Gamma_{ii} = -\sum_{i=1,i\neq j}^{N} \Gamma_{ij}$, $i = 1, \ldots, N$. Then, the algorithm for generating U, *i.e.* the vector containing all the agent input vectors U_i, $i = 1, \ldots, N$, representing the result of the overall consensus process, is given by

$$\dot{U}_i = \sum_{j=1,j\neq i}^{N} \Gamma_{ij}(U_j - U_i) + [K_i F^{(i)} K_i^+ + H_i C^{(i)} B^{(i)}]U_i + \\ +[K_i G^{(i)} + H_i C^{(i)} A^{(i)} C^{(i)+} - K_i F^{(i)} K_i^+]y^{(i)}, \tag{3.6}$$

$i = 1, \ldots, N$, or

$$\dot{U} = [\tilde{\Gamma} + \tilde{K}\tilde{F}\tilde{K}^+ + \tilde{H}\tilde{C}\tilde{B}]U + [\tilde{K}\tilde{G} + \tilde{H}\tilde{C}\tilde{A}\tilde{C}^+ - \tilde{K}\tilde{F}\tilde{K}^+\tilde{H}]Y. \tag{3.7}$$

The vector U generated by (3.7) is used for control implementation in such a way that the i-th agent picks up the components of U_i selected by the index set $\mathcal{U}^{(i)}$ and applies them to the system $\mathbf{S}$. If Q is an $m \times mN$ matrix with zeros everywhere except one place in each row, where it contains 1; for the j-th row with $j \in \mathcal{U}^{(i)}$, 1 is placed at the column index $(i-1)m + j$. Then, we have $u = QU$, and system (3.1) can be written as

$$\dot{x} = Ax + BQU. \tag{3.8}$$

Also, according to the adopted notation, $y^{(i)} = C_i x$, so that $Y = \bar{C}x$, where $\bar{C}^T = \left[C_1^T \vdots \cdots \vdots C_N^T \right]$. Therefore, the whole closed-loop system is represented by

$$\begin{bmatrix} \dot{U} \\ \dot{x} \end{bmatrix} = \left[\begin{array}{c:c} \tilde{\Gamma} + \tilde{K}\tilde{F}\tilde{K}^+ + \tilde{H}\tilde{C}\tilde{B} & (\tilde{K}\tilde{G} + \tilde{H}\tilde{C}\tilde{A}\tilde{C}^+ - \\ & -\tilde{K}\tilde{F}\tilde{K}^+\tilde{H})\bar{C} \\ \hdashline BQ & A \end{array}\right] \begin{bmatrix} U \\ x \end{bmatrix}. \tag{3.9}$$

Obviously, the system is stabilized by the controller (3.7) if the state matrix in (3.9) is asymptotically stable.

Algorithm 2 One alternative for the above algorithm is the algorithm depending explicitly on the regulator state $w^{(i)}$. It has the disadvantage of being of higher order than Algorithm 1; however, it does not utilize any approximation of $w^{(i)}$ with $v^{(i)}$. Recalling

(3.4), we obtain equation

$$\dot{v}^{(i)} \approx K^{(i)}F^{(i)}w^{(i)} + H^{(i)}C^{(i)}B^{(i)}v^{(i)} + [K^{(i)}G^{(i)} + H^{(i)}C^{(i)}A^{(i)}C^{(i)+}]y^{(i)},$$

since $w^{(i)}$ is generated by the first relation in (3.3). If $W = \mathrm{col}\{w^{(1)}, \dots, w^{(N)}\}$, then we have, similarly as in the case of (3.7), that

$$\dot{U} = [\tilde{\Gamma} + \tilde{H}\tilde{C}\tilde{B}]U + \tilde{K}\tilde{F}W + [\tilde{K}\tilde{G} + \tilde{H}\tilde{C}\tilde{A}\tilde{C}^{+}]Y. \tag{3.10}$$

The whole closed-loop system can be represented as

$$\begin{bmatrix} \dot{U} \\ \dot{W} \\ \dot{x} \end{bmatrix} = \left[\begin{array}{c:c:c} \tilde{\Gamma} + \tilde{H}\tilde{C}\tilde{B} & \tilde{K}\tilde{F} & (\tilde{K}\tilde{G} + \tilde{H}\tilde{C}\tilde{A}\tilde{C}^{+})\bar{C} \\ \hdashline 0 & \tilde{F} & \tilde{G}\bar{C} \\ \hdashline BQ & 0 & A \end{array}\right] \begin{bmatrix} U \\ W \\ x \end{bmatrix}. \tag{3.11}$$

Both control algorithms 1 and 2 have the structure which reduces to the local controllers when $\tilde{\Gamma} = 0$. In the case of Algorithm 1, the local controllers are derived from $\mathbf{C}_i$ after aggregating (3.3) to one vector-matrix differential equation for $v^{(i)}$, while in the case of Algorithm 2 the differential equation for $v^{(i)}$ contains explicitly the term $w^{(i)}$, generated by the local observer in $\mathbf{C}_i$. The form of these controllers is motivated by the idea to introduce a first order dynamic consensus scheme. Namely, without the local controllers, relation $\dot{U} = \tilde{\Gamma}U$ provides asymptotically a weighted sum of the initial conditions $U_i(t_0)$, if the graphs corresponding to the particular components of U_i have a center node (see *e.g.* [71, 98]). Combination of the two terms provides a possibility to improve the overall performance by exploiting potential advantages of each local controller. However, the introduction of additional dynamics required by the consensus scheme may deteriorate the performance, and makes the choice of the local controller parameters dependable upon the overall control scheme.

Example 3.2.1 An insight into the possibilities of the proposed algorithms can be obtained from a simple example in which the system $\mathbf{S}$ is represented by (3.1), with $A = \begin{bmatrix} 0.8 & 2 & 0 \\ -2.5 & -5 & -0.3 \\ 0 & 10 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Assume that we have two agents characterized by $\mathbf{S}_1$ with $A^{(1)} = \begin{bmatrix} 0.8 & 2 \\ -2.5 & -5 \end{bmatrix}$, $B^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C^{(1)} = [\,1\ 0\,]$, and $\mathbf{S}_2$ with $A^{(2)} = \begin{bmatrix} -5 & -0.3 \\ 10 & -2 \end{bmatrix}$, $B^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $C^{(2)} = [\,0\ 1\,]$. Obviously, there is only one control signal u. Assume that the second agent is responsible for control implementation, so that $u = u^{(2)} = v^{(2)}$, according to the adopted notation. Assume that both agents have their own controllers $\mathbf{C}_1$ and $\mathbf{C}_2$, obtained by the LQG methodology, assuming a low measurement noise level, so that $F^{(1)} = \begin{bmatrix} 1.6502 & 2.0000 \\ -2.4717 & -2.8223 \end{bmatrix}$, $G^{(1)} = \begin{bmatrix} -0.8502 \\ -0.26970 \end{bmatrix}$, $K^{(1)} = [\,0.7414\ 0.82231\,]$ and $H^{(1)} = 0$, and $F^{(2)} = \begin{bmatrix} -2.2361 & -24.3071 \\ 0.1000 & 1.1200 \end{bmatrix}$, $G^{(2)} = \begin{bmatrix} 24.2068 \\ -3.1200 \end{bmatrix}$, $K^{(2)} = [\,0.2361\ 0.0003\,]$ and $H^{(2)} = 0$. The system $\mathbf{S}$ with the local controller $\mathbf{C}_2$ is unstable. Algorithm 1 has been applied according to (3.7), after introducing $Q = [\,0\ 1\,]$ and $\Gamma_{12} = \Gamma_{21} = 100I_2$. Fig. 3.1 presents the impulse response for all three components of the state vector x for $\mathbf{S}$. Algorithm 2 has then been applied according to (3.11); the corresponding responses are presented in Fig. 3.2.

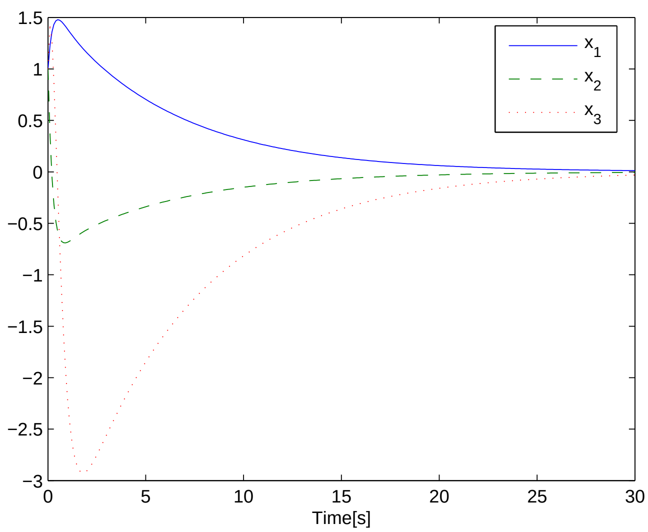

Figure 3.1: Impulse response for Algorithm 1

It is to be emphasized that the consensus scheme puts together two local controllers, influencing in such a way both performance and robustness. Here, the role of the first controller is only to help the second controller in defining the control signal. The importance of the consensus effects can be seen from Fig. 3.3 in which the responses in the case when $\tilde{\Gamma} = 0$ is presented for the Algorithm 1. It is obvious that the response is worse than in Fig.

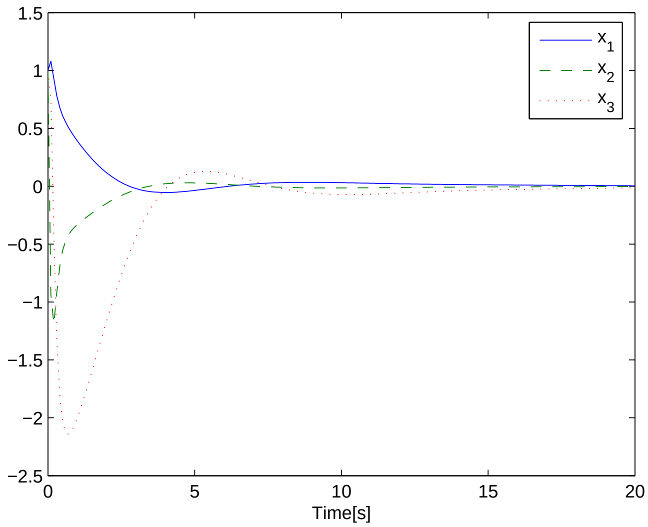

Figure 3.2: Impulse response for Algorithm 2

3.1. In the case of Algorithm 2 the system without consensus is even unstable (Fig. 3.4). ■

The control algorithms can be made more flexible by introducing some adjustable parameters, so that, for example, the terms $\tilde{K}\tilde{F}\tilde{K}^+$ in (3.7) and $\tilde{K}\tilde{F}$ in (3.10) are multiplied by a parameter α, and the term $\tilde{K}\tilde{G}$ in both algorithms by β; it has been found to be beneficial to have $\alpha > 1$ and $\beta < 1$. ■

The problem of stabilizability of $\mathbf{S}$ by the proposed algorithms is, in general, very difficult having in mind the supposed diversity of local models and dynamic controllers. Any analytic insight from this point of view into the system matrices in (3.9) and (3.11) seems to be very complicated. It is, however, logical to expect that the introduction of the consensus scheme can, in general, contribute to the stabilization of $\mathbf{S}$. Selection of the elements of $\tilde{\Gamma}$ can, obviously, be done in accordance with the expected performance of the local controllers and the confidence in their suggestions (see, for example, an analogous reasoning related to the estimation problem addressed in Chapter 2). In this sense, connectedness of the agents network contributes, in general, to the overall control performance.

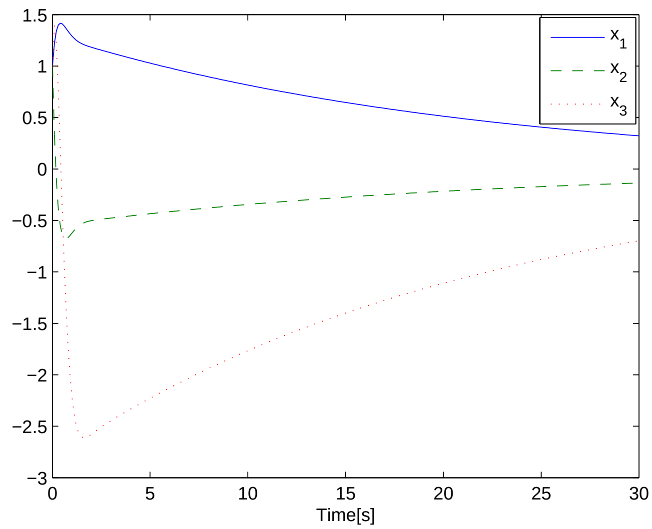

Figure 3.3: Algorithm 1: local controllers without consensus

3.2.2 Algorithms Derived from Local Static Feedback Control Laws

Algorithm 3 Assume now that we have static local output controllers, obtained from $\mathbf{C}_i$ in (3.3) by introducing $F^{(i)} = 0$, $G^{(i)} = 0$ and $K^{(i)} = 0$, so that we have $v^{(i)} = H^{(i)}y^{(i)}$. Both algorithms 1 and 2 give in this case

$$\dot{U} = \tilde{\Gamma}U + \tilde{H}\tilde{C}[\tilde{B}U + \tilde{A}\tilde{C}^{+}Y]. \tag{3.12}$$

The closed-loop system is now given by

$$\begin{bmatrix} \dot{U} \\ \dot{x} \end{bmatrix} = \left[\begin{array}{c:c} \tilde{\Gamma} + \tilde{H}\tilde{C}\tilde{B} & \tilde{H}\tilde{C}\tilde{A}\tilde{C}^{+}\bar{C} \\ \hdashline BQ & A \end{array}\right] \begin{bmatrix} U \\ x \end{bmatrix}. \tag{3.13}$$

A special case of the above controller deserves particular attention. Assume in the Algorithm 3 that $C^{(i)} = I_{n_i}$ and that that $\zeta^{(i)} = x^{(i)}$, $y^{(i)} = x^{(i)}$ represents a part of the vector x. In the special case when all the agents possess the knowledge about the entire model of $\mathbf{S}$, $y^{(i)} = x$, and the agents can differ by their control laws and responsibilities for control actions. Under these assumptions, algorithm 3 becomes

$$\dot{U} = \tilde{\Gamma}U + \tilde{H}[\tilde{B}U + \tilde{A}\tilde{x}], \tag{3.14}$$

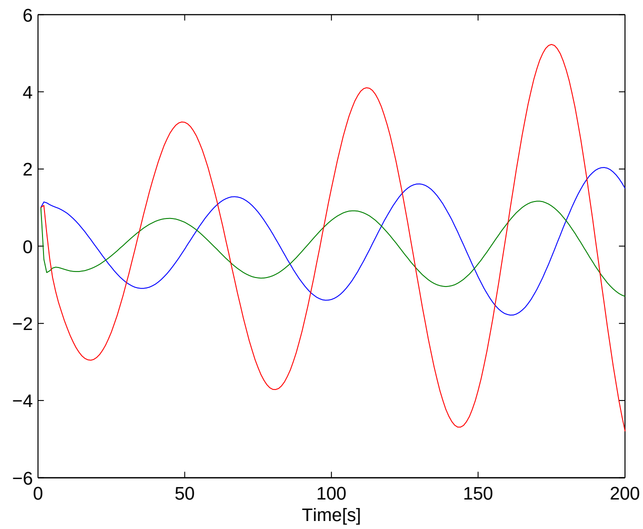

Figure 3.4: Algorithm 2: local controllers without consensus

where $\tilde{x} = \text{col}\{x^{(1)}, \ldots, x^{(N)}\}$, $\dim\{\tilde{x}\} = n_i$ represents the expanded vector x, available through measurements. Notice that it is always possible to find a full rank $\sum_{i=1}^{N} n_i \times n$ matrix V that $\tilde{x} = Vx$ (for a general discussion about state expansion, see [80]). The closed-loop system is now

$$\begin{bmatrix} \dot{U} \\ \dot{x} \end{bmatrix} = \left[\begin{array}{c:c} \tilde{\Gamma} + \tilde{H}\tilde{B} & \tilde{H}\tilde{A}V \\ \hdashline BQ & A \end{array}\right] \begin{bmatrix} U \\ x \end{bmatrix}. \tag{3.15}$$

Remark 3.2.1 The proposed multi-agent control schemes can be compared to those overlapping decentralized control schemes for complex systems that are derived by using the expansion/contraction paradigm and the inclusion principle (especially in the case of Algorithm 3) *e.g.* [80, 38, 35, 36, 99, 101], having in mind that both approaches follow analogous lines of thought, starting from similar information structure constraints (the above presented approach is, however, much more general). From this point of view, formulation of the local controllers connected to the agents corresponds to the controller design in the expanded space in the case of inclusion based systems, and the application of a dynamic consensus strategy to the contraction to the original space for control action implementation, see *e.g.* [80, 35, 36]. The proposed methodology offers, evidently, much more flexibility (local model structure, agreement strategy), at the expense of additional closed loop dynamics introduced by the consensus scheme itself. Moreover, it is interesting to notice that numerous numerical simulations show a pronounced advantage of the proposed scheme (smoother and even faster responses). The reason could be found in the advantage of the consensus strategy over the contraction transformation, which seems to be overly simplified and unsatisfactory for putting together locally designed overlapping decentralized controllers. In Section 3.4 an application of the mentioned expansion/contraction methodology to the control of formations of UAVs will be presented and compared with the proposed consensus based methodology.

3.3 Structures Based on Consensus at the State Estimation Level

The previous section was devoted to general structures with consensus at the input level in systems where multiple agents with overlapping resources and different competences participate in defining the global control law. The algorithms start from the local models and the local controllers, and the consensus scheme tends to make equal the overlapping components of the local input vectors. It is possible to approach the problem in a different way, where the consensus strategy is introduced at the level of state estimation. This estimation scheme itself has been proposed in Section 2.1.

Algorithm 4 Assume that the local models are such that $\zeta^{(i)} = x^{(i)}$, so that the dynamic systems $\mathbf{S_i}$ are overlapping subsystems of $\mathbf{S}$. Therefore, we have the same system decomposition as in Section 2.1, continuous-time case. We will assume that all the agents have the a priori knowledge about the optimal state feedback for $\mathbf{S}$, expressed as $u = K^o x$. Using this knowledge and the estimation scheme (2.3), the agents can calculate the corresponding inputs $U_i = K^o \xi_i$; implementation of the control signals is done according to the index sets $\mathcal{U}^i$.

The decentralized overlapping estimation scheme with consensus, providing state estimates of the whole state vector x to all the agents, together with the globally optimal control law, represents a control algorithm, denoted as Algorithm 4, which provides a solution to the posed multi-agent control problem of $\mathbf{S}$.

Defining $\tilde{K}^o = \text{diag}\{K^o, \ldots, K^o\}$, we have, according to the above given notation, that $u = Q\tilde{K}^o\Xi$, so that the whole closed-loop system becomes

$$\begin{bmatrix} \dot{\Xi} \\ \dot{x} \end{bmatrix} = \left[\begin{array}{c:c} \tilde{\Gamma} + \tilde{A}^* - \tilde{L}^*\tilde{C}^* + \tilde{B}^*\bar{K} & \bar{L} \\ \hdashline BQ\tilde{K}^o & A \end{array}\right] \begin{bmatrix} \Xi \\ x \end{bmatrix}, \tag{3.16}$$

where $\bar{K} = \text{col}\{QK^o, \ldots, QK^o\}$ and $\bar{L} = \text{col}\{L_1C_1, \ldots, L_NC_N\}$. A simplified version of the above algorithm, from the point of view of communications, is obtained by replacing the actual input u by the local estimates of the input vector $U_i = K^o\xi_i$, having in mind the local availability of ξ_i.

Example 3.3.1 In this example the performance of the above algorithm is demonstrated on the same system as in the Example 3.2.1. The local estimators are performing the local state estimation using the gains $L_1 = [-4 \;\; 9]^T$ and $L_2 = [2 \;\; -7]^T$. The consensus gains in the matrix $\tilde{\Gamma}$ are selected to be $\Gamma_{12} = \Gamma_{21} = 1000I_2$. The global LQ optimal control matrix K^o is implemented by both agents. Since only the second agent implements the input u, we assume that the first one uses the estimate $U_i = K^o\xi_i$ in the local state estimation algorithm. The impulse response of the proposed control algorithm, which is shown in Figure 3.5, is comparable to the the impulse response of the globally LQ optimal controller shown in the same figure. ■

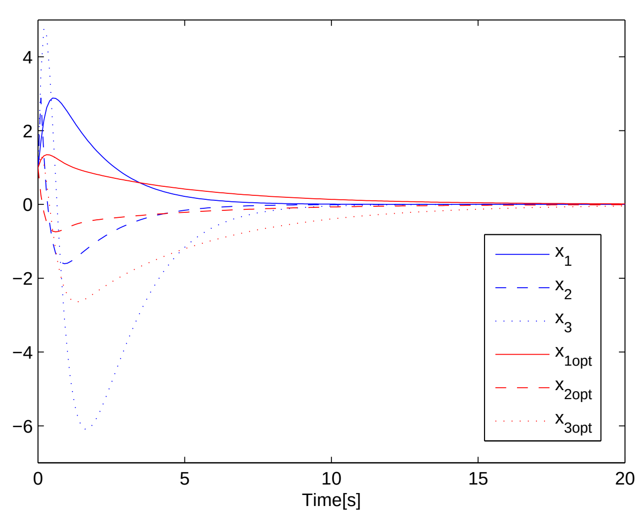

Figure 3.5: Algorithm 4 and globally LQ optimal controller

Stability analysis of Algorithm 4 represents in general a very complex task. It is possible to apply the methodology of [82] under very simplifying assumptions, and to show that the eigenvalues of (3.16) are composed of the eigenvalues of $\tilde{A}^* - \tilde{L}^*\tilde{C}^*$, $\tilde{A}^* + \tilde{B}^*\bar{K}$ and $\tilde{A}^* - \tilde{L}^*\tilde{C}^* + \tilde{B}^*\bar{K}$ modified by a term depending on the eigenvalues of the Laplacian of the network and the consensus gain matrices. However, the underlying assumptions include the one that all the agents have the exact system model, as well as that the control inputs are transmitted throughout the network; in the overlapping decentralized case, which is in the focus of this work, these assumptions are violated, making the stability problem much more complex, dependent on the accuracy of the local models and the related estimators.

3.4 Decentralized Overlapping Tracking Control of a Formation of UAVs

3.4.1 Introduction

In this section we present a novel design methodology for decentralized overlapping tracking control law of planar formations based on the expansion/contraction paradigm and the inclusion principle [100]. In Subsection 3.4.2, a specific formation state-space model is formulated on the basis of the assumed information structure, using the initial results presented in [99, 101]; this approach enables treating formation as an interconnection of subsystems formally attached to each vehicle. Section 3.4.3 deals with a general control design methodology for a formation to track given references of velocity and relative distances of the vehicles with respect to their neighbors, which allows local application of diverse controller design methodologies, like LQ or LMI design. As the resulting overall feedback and feedforward matrix gains do not allow proper contraction to the original system space for implementation, a special attention is paid to the contractibility issue. It is shown that suitably modified feedback and feedforward gains can be constructed. Section 3.4.4 is devoted to the stability issue. It is proved, starting from a digraph representation of the information flow, that asymptotic convergence of all the states to the desired constant references can be achieved provided the underlying digraph has a spanning tree. This result, derived directly on the basis of the proposed formation state model and the expansion/contraction methodology, is in accordance with the recent results related to the second order consensus scheme [68, 70, 69]. In Section 3.4.5 a dynamic output scheme with local observers is presented in the case when the velocities of the neighboring vehicles are not known [80, 99, 35]. Section 3.4.6 contains simulation results and comparison of the proposed control scheme based on the inclusion principle with the consensus based control scheme proposed in Section 3.3.

3.4.2 Formation Model

Consider a set of N vehicles moving in a plane, in which the i-th vehicle is represented by the linear double integrator model

$$\dot{z}_i = A_v z_i + B_v u_i = \begin{bmatrix} 0_{2\times 2} & I_2 \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} z_i + \begin{bmatrix} 0_{2\times 2} \\ I_2 \end{bmatrix} u_i, \tag{3.17}$$

($i = 1, \dots, N$), where $z_i \in \mathcal{R}^4$ and $u_i \in \mathcal{R}^2$ are the state and the control input vectors, respectively ($0_{m\times n}$ denotes the $m \times n$ zero matrix, and I_n the $n \times n$ identity matrix). The state z_i and the input u_i are related to the physical state and input through standard transformations, *e.g.* [101]. We will assume that the i-th vehicle is provided with the information about the set of neighboring vehicles, indices of which define the set of sensing indices $S_i = \{s_1^i, \dots, s_{m_i}^i\}$; this information includes velocities and relative distances of the neighboring vehicles with respect to the i-th vehicle, the velocity of the vehicle itself, as well as the relative distance references and the velocity reference (which is supposed to be the same for all the vehicles). Decomposing z_i as $z_i = \begin{bmatrix} z_i'^T & z_i''^T \end{bmatrix}^T$, where $z_i' = \begin{bmatrix} z_{i,1}' & z_{i,2}' \end{bmatrix}^T = [\, z_{i,1} \;\; z_{i,2} \,]^T$ and $z_i'' = \begin{bmatrix} z_{i,1}'' & z_{i,2}'' \end{bmatrix}^T = [\, z_{i,3} \;\; z_{i,4} \,]^T$, we introduce the following change of variables

$$x_i' = \sum_{j\in S_i} \alpha_j^i z_j' - z_i', \qquad x_i'' = z_i'', \tag{3.18}$$

where $\alpha_j^i \geq 0$ and $\sum_{j\in S_i} \alpha_j^i = 1$; $x_i' = \begin{bmatrix} x_{i,1}' & x_{i,2}' \end{bmatrix}^T$ represents the distance between the i-th vehicle and a "centroid" of the set of vehicles selected by S_i, with *a priori* selected weights

α_j^i (in the case of formation leaders when $S_i = \emptyset$, $x_i' = -z_i'$). Therefore, one obtains

$$\dot{x}_i' = \sum_{j \in S_i} \alpha_j^i z_j'' - z_i'' = \sum_{j \in S_i} \alpha_j^i x_j'' - x_i'', \qquad \dot{x}_i'' = u_i, \tag{3.19}$$

$i = 1, \ldots, N$, using the fact that $z_i'' = \dot{z}_i'$, so that $x_i'' = \begin{bmatrix} x_{i,1}'' & x_{i,2}'' \end{bmatrix}^T = z_i'' = \begin{bmatrix} z_{i,1}'' & z_{i,2}'' \end{bmatrix}^T = \dot{z}_i' = \begin{bmatrix} \dot{z}_{i,1}' & \dot{z}_{i,2}' \end{bmatrix}^T$ [101].

Defining the formation state and control input vectors x and u as concatenations of all the vehicle state and control input vectors $x_i = \begin{bmatrix} x_i'^T & x_i''^T \end{bmatrix}^T$ and u_i, $i = 1, \ldots, N$, we obtain the following formation state model

$$\mathbf{S}: \quad \dot{x} = Ax + Bu = [(G - I) \otimes A_v]x + [I \otimes B_v]u, \tag{3.20}$$

where $\otimes$ denotes the Kronecker's product. We will assume that each vehicle has the information about the reference state trajectories $r_i = \begin{bmatrix} r_i'^T & r_i''^T \end{bmatrix}^T$, so that the control task to be considered is the task of tracking the desired references.

The above described set of N vehicles with their sensing indices and the corresponding weights can be considered as a directed weighted graph $\mathcal{G}$ in which each vertex represents a vehicle, and an arc with the weight α_j^i leads from vertex j to vertex i if $j \in S_i$. Consequently, the weighted adjacency matrix $G = [G_{ij}]$ is an $N \times N$ square matrix defined by $G_{ij} = \alpha_j^i$ for $j \in S_i$, and $G_{ij} = 0$ otherwise. We will define the weighted Laplacian of the graph as $L = [L_{ij}]$, $L_{ij} = G_{ij}$, $i \neq j$, $L_{ii} = -\sum_j \alpha_j^i$ (*e.g.*, see [23]).

3.4.3 Decentralized Tracking Design by Expansion/Contraction

The structure of the formulated model (3.20) indicates that it is possible to consider the formation as an interconnection of N overlapping subsystems. Extending the reasoning successfully applied within the platooning problem (*e.g.*, [91, 99, 101]), we will assign to the i-th vehicle in a formation a formally defined subsystem $\tilde{\mathbf{S}}_i$ with the state vector containing the vehicle state coordinates x_i' and x_i'', together with the second components x_j'' (velocity components) of the state vectors of all the vehicles sensed by the i-th vehicle, and the input vector $\tilde{u}_i$ containing the vehicle control vector u_i, together with the control vectors u_j associated with all the vehicles sensed by the i-the vehicle, *i.e.* $\tilde{x}_i = \left[\, x''^T_{s^i_1} \;\cdots\; x''^T_{s^i_{m_i}} \;\, x'^T_i \;\, x''^T_i \,\right]^T$ and $\tilde{u}_i = \left[\, u^T_{s^i_1} \;\cdots\; u^T_{s^i_{m_i}} \;\, u^T_i \,\right]^T$. Consequently, the subsystem models are:

$$\tilde{\mathbf{S}}_i : \quad \dot{\tilde{x}}_i = \tilde{A}_i \tilde{x}_i + \tilde{B}_i \tilde{u}_i, \tag{3.21}$$

where $\tilde{A}_i = \left[\begin{array}{c:c} 0_{2m_i\times 2m_i} & 0_{2m_i\times 4} \\ \hdashline A^i_\alpha & -A_v \end{array}\right]$, $A^i_\alpha = \left[\begin{array}{c:c:c} \alpha_{s^i_1} I_2 & \cdots & \alpha_{s^i_{m_i}} I_2 \\ \multicolumn{3}{c}{0_{2\times 2m_i}} \end{array}\right]$ and $\tilde{B}_i = \left[\begin{array}{c:c} I_{2m_i\times 2m_i} & 0_{2m_i\times 2} \\ \hdashline 0_{4\times 2m_i} & B_v \end{array}\right]$.

We define the expansion $\tilde{\mathbf{S}}$ of $\mathbf{S}$ as a system whose state and input vectors are defined as concatenations of the subsystem state and input vectors, that is, $\tilde{x} = \left[\, \tilde{x}^T_1 \,\ldots\, \tilde{x}^T_N \,\right]^T$ and $\tilde{u} = \left[\, \tilde{u}^T_1 \,\ldots\, \tilde{u}^T_N \,\right]^T$. Consequently,

$$\tilde{\mathbf{S}} : \quad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \tag{3.22}$$

where $\tilde{A} = \mathrm{diag}\{\tilde{A}_1, \ldots, \tilde{A}_N\}$ and $\tilde{B} = \mathrm{diag}\{\tilde{B}_1, \ldots, \tilde{B}_N\}$.

The expanded state and control vectors $\tilde{x}$ and $\tilde{u}$ can be represented as full rank linear transformations of the original state and control vectors x and u, *i.e.* $\tilde{x} = Vx$ and $\tilde{u} = Ru$, where $V^T = \left[\begin{array}{c:c:c} V^T_1 & \cdots & V^T_N \end{array}\right]^T$ and $R^T = \left[\begin{array}{c:c:c} R^T_1 & \cdots & R^T_N \end{array}\right]^T$, with $V_i = \left[\begin{array}{c} V'_i \\ \hdashline V''_i \end{array}\right]$ and $R_i = \left[\begin{array}{c} R'_i \\ \hdashline R''_i \end{array}\right]$, where V'_i is an $m_i \times 2N$ (2×2)-block matrix containing I_2 in j-th row at the column index $2s^i_j$, $j = 1, \ldots, m_i$ and zeros elsewhere, V''_i a $2 \times 2N$ (2×2)-block matrix containing I_2 at the $(2i-1)$-st place in the first tow and at the $2i$-th place in the second row, R'_i is an $m_i \times N$ block matrix containing I_2 in j-th row at the column index s^i_j, $j = 1, \ldots, m_i$ and zeros elsewhere and R''_i a $1 \times N$ block matrix containing I_2 at the i-th place.

It is not difficult to verify on the basis of the structure of $\mathbf{S}$, $\tilde{\mathbf{S}}$, V and R, that $\mathbf{S}$ and $\tilde{\mathbf{S}}$ satisfy, in general, the following conditions:

$$\tilde{A}V = VA, \quad \tilde{B}R = VB. \tag{3.23}$$

According to the inclusion principle, the original model $\mathbf{S}$ is a restriction of $\tilde{\mathbf{S}}$ (see *e.g.* [37, 38, 35, 34, 80] for basic results related to the inclusion principle). Consequently, stability of $\tilde{\mathbf{S}}$ implies stability of $\mathbf{S}$.

Once $\tilde{\mathbf{S}}$ is defined and the subsystems $\tilde{\mathbf{S}}_i$ extracted, the next task is to design the local control laws for the subsystems. If $\tilde{r}_i(t)$ represents a given reference signal for the i-th subsystem (the desired state trajectory of $\tilde{\mathbf{S}}_i$), then we have to determine pairs of constant feedback and feedforward matrices $(\tilde{K}^i, \tilde{M}^i)$ in the local tracking control laws for (3.22)

$$\tilde{\mathbf{F}}_i : \qquad \tilde{u}_i = \tilde{K}^i \tilde{x}_i + \tilde{M}^i \tilde{r}_i, \tag{3.24}$$

$i = 1, \ldots, N$. Notice that the references for x_i', denoted as r_i^d, $i = 1, \ldots, N$, are related to the set of references for individual inter-vehicle distances with respect to the sensed vehicles, denoted as $r^d_{i-s^i_j}$, simply by $r_i^d = \sum_{j=1}^{m_i} \alpha^i_{s^i_j} r^d_{i-s^i_j}$.

In the case when $S_i = \emptyset$ (formation leaders), $\tilde{u}_i = u_i$, and we have only the velocity feedback, so that $\tilde{K}^i = \begin{bmatrix} 0 & K^{Li} \end{bmatrix}$ and $\tilde{M}^i = \begin{bmatrix} 0 & M^{Li} \end{bmatrix}$, where K^{Li} and M^{Li} are 2×2 matrices.

When $S_i = \{s^i_1, \ldots, s^i_{m_i}\} \neq \emptyset$, we assume that the control signals are $u_j = \hat{K}^i_j x''_j + \hat{M}_j r^v$ for all $j \in S_i$, where r^v is the velocity reference; the design of $\hat{K}^i_j$ and $\hat{M}^i_j$ can, in principle, be done as in the case of the vehicles with $S_i = \emptyset$. However, the control vector u_i is obtained using all the measurements available in $\tilde{\mathbf{S}}_i$, *i.e.*, $u_i = \bar{K}^i \tilde{x}_i + \bar{M}^i \tilde{r}_i$, where both $\bar{K}^i$ and $\bar{M}^i$ can be decomposed as $\bar{K}^i = \left[\bar{K}^i_1 \,\vdots\, \ldots \,\vdots\, \bar{K}^i_{m_i} \,\vdots\, \bar{K}^i_{m_i+1} \,\vdots\, \bar{K}^i_{m_i+2} \right]$ and $\bar{M}^i = \left[\bar{M}^i_1 \,\vdots\, \ldots \,\vdots\, \bar{M}^i_{m_i} \,\vdots\, \bar{M}^i_{m_i+1} \,\vdots\, \bar{M}^i_{m_i+2} \right]$, having in mind the structure of $\tilde{x}_i$ (and $\tilde{r}_i$).

Therefore, the tracking control law for $\tilde{\mathbf{S}}_i$ given by (3.24) is characterized by matrices $\tilde{K}^i = \begin{bmatrix} \hat{K}^i & 0_{2m_i \times 4} \\ \hline \multicolumn{2}{c}{\bar{K}^i} \end{bmatrix}$ and $\tilde{M}^i = \begin{bmatrix} \hat{M}^i & 0_{2m_i \times 4} \\ \hline \multicolumn{2}{c}{\bar{M}^i} \end{bmatrix}$, where $\hat{K}^i = \mathrm{diag}\{\hat{K}^i_{s^i_1}, \ldots, \hat{K}^i_{s^i_{m_i}}\}$, and $\hat{M}^i = \mathrm{diag}\{\hat{M}^i_{s^i_1}, \ldots, \hat{M}^i_{s^i_{m_i}}\}$; the structure of $\tilde{K}^i$ and $\tilde{M}^i$ reflects the fact that the i-th vehicle senses the vehicles selected by S_i.

The overall control law $\tilde{\mathbf{F}}$ for the whole expanded system $\tilde{\mathbf{S}}$ is characterized by the pair $(\tilde{K}, \tilde{M})$, where $\tilde{K} = \mathrm{diag}\{\tilde{K}^1, \ldots, \tilde{K}^N\}$ and $\tilde{M} = \mathrm{diag}\{\tilde{M}^1, \ldots, \tilde{M}^N\}$, so that

$$\tilde{\mathbf{F}} : \qquad \tilde{u} = \tilde{K}\tilde{x} + \tilde{M}\tilde{r}, \tag{3.25}$$

where $\tilde{r}$ is the desired trajectory of $\tilde{x}$.

The final step in the formation control design is the contraction of the obtained tracking controller for the expansion $\tilde{\mathbf{S}}$ to the controller for the original system $\mathbf{S}$, given by

$$\mathbf{F}: \quad u = Kx + Mr, \tag{3.26}$$

where r is the desired trajectory of x ($\tilde{r} = Vr$). The contractibility conditions given by

$$RK = \tilde{K}V, \quad RM = \tilde{M}V, \tag{3.27}$$

ensure that the closed-loop system $(\mathbf{S}, \mathbf{F})$ represents a restriction of the closed-loop system $(\tilde{\mathbf{S}}, \tilde{\mathbf{F}})$ (for more details on contractibility, see [93]). However, relations (3.27) do not have any solutions for K and M in the case when $\tilde{K}$ and $\tilde{M}$ are in the form of block diagonal matrices [35, 37, 36, 80].

One way to overcome this problem is to suitably modify both $\tilde{K}$ and $\tilde{M}$ in such a way as to achieve contractibility [35, 101]. We define $\tilde{K}_m$ (or $\tilde{M}_m$) by $\tilde{K}_m = \tilde{K}_{m1} + \tilde{K}_{m2}$, where $\tilde{K}_{m1} = RR^T\tilde{K}$, while $\tilde{K}_{m2}$ is constructed in such a way as to reduce the number of off-block-diagonal terms in $\tilde{K}_{m1}$, and to satisfy, at the same time, the restriction condition $\tilde{K}_{m1}V = 0$. More specifically, in order to construct the l-th block-row of $\tilde{K}_{m2}$ ($l = 1, \ldots, N + \sum_{i=1}^{N} m_i$), we first locate the part of the l-th block-row in $\tilde{K}_{m1}$ which belongs to some $\tilde{K}^i$, $i = 1, \ldots, N$ (diagonal blocks), and then identify the block-column index ν_l in the following way: a) when $S_i = \emptyset$, ν_l is the column index of K^{Li}; b) when $S_i \neq \emptyset$, this is the block-column index of either $\hat{K}^i_j$ ($j = 1, \ldots, m_i$) within the first m_i block-rows in $\tilde{K}^i$, or of $\bar{K}^i_{m_i+2}$ in the last block-row of $\tilde{K}_i$. Then, we identify the block-column of V having "I" at its ν_l-th block-row; the block-row indices of the remaining "I"'s in the same block-column compose a set V_l^{NZ}. Then, the nonzero terms in l-th block-row of $\tilde{K}_{m2}$ are taken to be the blocks from the l-th row of $\tilde{K}_{m1}$ at the block-column indices defined by V_l^{NZ} with the reversed sign, while the sum of these blocks is put at the column index ν_l. Therefore, the resulting contracted gains are

$$K = R^+\tilde{K}_mV, \quad M = R^+\tilde{M}_mV. \tag{3.28}$$

The vehicle control u_i in the case when $S_i \neq \emptyset$ is generated by

$$u_i = \left[\begin{array}{c|c|c|c} \bar{K}_1^i \cdots & \bar{K}_{m_i}^i & \bar{K}_{m_i+1}^i & \bar{K}_{m_i+2}^i + \sum\limits_{k \in \bar{S}_i} \hat{K}_i^k \end{array} \right] \tilde{x}_i +$$
$$\left[\begin{array}{c|c|c|c} \bar{M}_1^i \cdots & \bar{M}_{m_i}^i & \bar{M}_{m_i+1}^i & \bar{M}_{m_i+2}^i + \sum\limits_{k \in \bar{S}_i} \hat{M}_i^k \end{array} \right] \tilde{r}_i. \tag{3.29}$$

3.4.4 Stability

The resulting closed-loop system is represented by

$$\mathbf{S_{cl}} : \quad \dot{x} = A_{cl} x + B_{cl} r \tag{3.30}$$

where $A_{cl} = [(G - I) \otimes A_v + [(I \otimes B_v) R^+ \tilde{K}_m V]$ and $B_{cl} = [(I \otimes B_v) R^+ \tilde{M}_m V]$. Both matrices $K = R^+ \tilde{K}_m V$ and $M = R^+ \tilde{M}_m V$ are composed of $N \times N$ (4×4)-blocks, such that for $S_i \neq 0$ we have the block $\left[\begin{array}{c|c} 0 & 0 \\ \hline \bar{K}_{m_i+1}^i & \bar{K}_{m_i+2}^i + \sum\limits_{k \in \bar{S}_i} \hat{K}_i^k \end{array} \right]$ at the corresponding block diagonal and the blocks $\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{K}_j^i \end{array} \right]$, $j = 1, \ldots, m_i$, at the block indices (i, s_j^i) determined by S_i; for $S_i = 0$ we have in the i-th block row only the diagonal block $\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K^{Li} \end{array} \right]$, $i = 1, \ldots, N$. Therefore, the state matrix A_{cl} contains in the i-th block row for $S_i \neq 0$ the diagonal block $\left[\begin{array}{c|c} 0 & -I \\ \hline \bar{K}_{m_i+1}^i & \bar{K}_{m_i+2}^i + \sum\limits_{k \in \bar{S}_i} \hat{K}_i^k \end{array} \right]$ and the blocks $\left[\begin{array}{c|c} 0 & \alpha_{s_j^i}^i I \\ \hline 0 & \bar{K}_j^i \end{array} \right]$, $j = 1, \ldots, m_i$, at the block indices (i, s_j^i), and $\left[\begin{array}{c|c} 0 & -I \\ \hline 0 & K^{Li} \end{array} \right]$ at the diagonal for $S_i = 0$, $i = 1, \ldots, N$. The indices of the nonzero (4×4)-blocks in A_{cl} are the same as the indices of the nonzero elements in the adjacency matrix G of the formation graph. Therefore, the matrix A_{cl} is cogredient (amenable by permutation transformations) to the following matrix

$$A_{cl}^P = \left[\begin{array}{c|c} 0 & (G - I) \otimes I_2 \\ \hline \mathrm{diag}\{\bar{K}_{m_1+1}^1, \ldots, \bar{K}_{m_N+1}^N\} & K_{cl} \end{array} \right], \tag{3.31}$$

where K_{cl} contains (2×2)-blocks $\bar{K}_{m_i+2}^i + \sum_{k \in \bar{S}_i} \hat{K}_i^k$ at the block diagonal and $\bar{K}_j^i$, $j = 1, \ldots, m_i$, at the block indices (i, s_j^i), $i = 1, \ldots, N$. The eigenvalues of A_{cl} are the solutions of the equation $\det(\lambda I_{4N} - A_{cl}) = 0$, or, equivalently, of $\det(\lambda I_{4N} - A_{cl}^P) = 0$, which gives rise to

$$\det(\lambda^2 I_{2N} - \lambda K_{cl} - \mathrm{diag}\{\bar{K}^1_{m_1+1}, \ldots, \bar{K}^N_{m_N+1}\}((G-I)\otimes I_2)) = 0. \tag{3.32}$$

We will assume that all the constituent (2×2)-blocks of K are diagonal with nonnegative entries, so that K (and, consequently, A_{cl}) can be decomposed into two components K^I and K^{II} (A^I_{cl} and A^{II}_{cl}) which correspond to the components $x'_{i,I}$, and $x'_{i,II}$ (or $x''_{i,I}$ and $x''_{i,II}$) of the two-dimensional distance and velocity vectors in **S** (in the sequel, it is understood that the assumptions and conclusions about K^I and A^I_{cl} hold analogously for K^{II} and A^{II}_{cl}). We will analyze solutions of (3.32) under simplifying assumptions emphasizing structural properties of the formation:

(A.3.4.1) Matrices K^I_{cl} and $(\mathrm{diag}\{\bar{K}^1_{m_1+1}, \ldots, \bar{K}^N_{m_N+1}\})^I(G-I)$ can be transformed into the triangular form by the same unitary matrix W (Schur transformation [30]).

(A.3.4.2) If $\mu_1, \ldots, \mu_N$ and $\nu_1, \ldots, \nu_N$ are the eigenvalues of K^I_{cl} and $(\mathrm{diag}\{\bar{K}^1_{m_1+1}, \ldots, \bar{K}^N_{m_N+1}\})^I$ $(G-I)$, respectively, then there are such real numbers $\gamma_i > 0$ and $\varepsilon_i > 0$ that $\mu_i = \gamma_i\nu_i - \varepsilon_i$, $i = 1, \ldots, N$.

Theorem 3.4.1 Let assumptions (A.3.4.1-2) be satisfied, and let the formation digraph $\mathcal{G}$ have a directed spanning tree. Then, for γ_i large enough matrix A^I_{cl} has one simple eigenvalue at 0, and all the remaining eigenvalues have negative real parts.

Proof: Applying W^T and W to (3.32), one obtains

$$\det(\lambda I_{2N} - A^P_{cl}) = \prod_{i=1}^{N}(\lambda^2 - (\gamma_i\nu_i - \varepsilon_i)\lambda - \nu_i) = 0, \tag{3.33}$$

wherefrom the eigenvalues of A^I_{cl} are

$$\lambda_{i\pm} = \frac{\gamma_i\nu_i - \varepsilon_i \pm \sqrt{(\gamma_i\nu_i - \varepsilon_i)^2 + 4\nu_i}}{2}, \tag{3.34}$$

$i = 1, \ldots, N$.

When $S_i \neq \emptyset$, $i = 1, \ldots, N$, we have $G - I = L$, where L is the weighted Laplacian of the formation digraph $\mathcal{G}$. If this digraph has a directed spanning tree, L has one simple zero eigenvalue and the other eigenvalues have negative real parts, so that for $\nu_1 = 0$, one obtains $\lambda'_{1+} = 0$ and $\lambda_{1-} = -\gamma_i$. For the remaining ν_i, $i = 2, \ldots, N$, a simple geometric reasoning based on [70, 68] shows that the corresponding $\lambda_{i\pm}$ have negative real parts for γ_i large enough. Remark only that the condition $\gamma_i > \sqrt{\frac{2}{\mathrm{Re}\{\nu_i\}}}$ which can be derived from the results in [70, 68] is overly conservative: it is possible to check the case of real ν_i, when, in fact, $\mathrm{Re}\{\lambda_{i\pm}\} < 0$ for all positive γ_i.

If there is one vehicle satisfying $S_i = \emptyset$, $G-I$ is nonsingular if the digraph has a spanning tree. However, in this case $\bar{K}^i_{m_1+1} = 0$, and, therefore, matrix $(\mathrm{diag}\{\bar{K}^1_{m_1+1}, \ldots, \bar{K}^N_{m_N+1}\})^I$ $((G-I)))$ has one simple eigenvalue at the origin, *i.e.* for $\nu_1 = 0$, one obtains again $\lambda_{1+} = 0$ and $\lambda_{1-} = -\gamma_i$, etc. Thus the result. ■

We will adopt further simplifying assumptions implying assumptions (A.3.4.1-2) in order to make clear the main structural properties of the analyzed formation control law.

(A.3.4.3) (a) $(\bar{K}^i_{m_i+1})^I = \kappa > 0$, (b) $(\bar{K}^i_j)^I = \rho > 0$, (c)$(\bar{K}^i_{m_i+2} + \sum_{k\in\bar{S}_i} \hat{K}^k_i)^I = -m_i\rho - \varepsilon$, $\varepsilon > 0$, $i = 1, \ldots, N$.

Theorem 3.4.2 Let assumption (A.3.4.3) be satisfied and let the underlying graph $\mathcal{G}$ have a directed spanning tree. Then A^I_{cl} has a single eigenvalue at zero and all the remaining eigenvalues have negative real parts for $\rho\kappa^{-1}$ large enough.

Proof: The proof is entirely based on Theorem 3.4.1, with $\rho\kappa^{-1}$ playing the role of γ_i.

■

The main result of this section, connecting the results of Theorems 3.4.1 and 3.4.2 with the specific structure of the proposed formation model, is given in the following theorem.

Theorem 3.4.3 Let $\tilde{M} = -\tilde{K}$. Then, under the assumptions of Theorem 3.4.2, for $\rho\kappa^{-1}$ large enough:

(a) when $S_i \neq \emptyset$, $i = 1, \ldots, N$, $\lim_{t\to\infty}[x'_i(t) - \bar{r}^d_i] = 0$ and $\lim_{t\to\infty}[x''_i(t) - \bar{r}^v] = 0$, $i = 1, \ldots, N$, where $\bar{r}^d = \begin{bmatrix} \bar{r}^d_1 \cdots \bar{r}^d_N \end{bmatrix}$ satisfies $\bar{r}^d = L\bar{r}^z$ and $\bar{r}^z$ and $\bar{r}^v$ are arbitrary predefined constant $2N$-dimensional and 2-dimensional vectors, respectively;

(b) when $S_j = \emptyset$ for some $j \in \{1, \ldots, N\}$, $x_j(t) \to_{t\to\infty} \bar{r}^v t$, $\lim_{t\to\infty}[x'_i(t) - \bar{r}^d_i] = 0$, $i = 1, \ldots, N$, $i \neq j$, and $\lim_{t\to\infty}[x''_i(t) - \bar{r}^v] = 0$, $i = 1, \ldots, N$, where $\bar{r}^d_i$, $i = 1, \ldots, N$, $i \neq j$, and $\bar{r}^v$ are arbitrary predefined 2-dimensional vectors.

Proof: Assume first that $S_i \neq \emptyset$, $i = 1, \ldots, N$. Then, according to Theorem 3.4.2,

$$e^{(A^P_{cl})^I t} = P \left[\begin{array}{c:c} 1 & 0 \\ \hdashline 0 & e^{Jt} \end{array}\right] P^{-1}, \tag{3.35}$$

where $P = \begin{bmatrix} r_1 \vdots \cdots \vdots r_{2N} \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} s_1^T \\ \cdots \\ s_{2N}^T \end{bmatrix}$, r_i representing the right eigenvectors (or generalized eigenvectors) and s_i the left eigenvectors (or generalized eigenvectors) of $(A_{cl}^P)^I$ and where the $(2N-1)\times(2N-1)$ matrix J is Hurwitz. Without loss of generality, we choose $r_1^T = \left[\, \mathbf{1}^T \;\; \kappa\varepsilon^{-1}\mathbf{1}^T \,\right]$ and $s_1^T = \begin{bmatrix} p_1^T & 0 \end{bmatrix}$, where $\mathbf{1}^T = [\,1 \cdots 1\,]$ and p_1 is a nonnegative vector such that $p_1^T L = 0$ and $p_1^T \mathbf{1} = 0$ as a consequence of the fact that L has a simple zero eigenvalue; also, $s_1^T r_1 = 1$. Consequently, we obtain, having in mind that $\tilde{M} = -\tilde{K}$, that when $t \to \infty$

$$\begin{bmatrix} X_1^I(t) \\ \hdashline X_2^I(t) \end{bmatrix} \to \begin{bmatrix} \mathbf{1} \\ \hdashline \kappa\varepsilon^{-1}\mathbf{1} \end{bmatrix} \begin{bmatrix} p_1^T \vdots 0 \end{bmatrix} \begin{bmatrix} X_1^I(0) \\ \hdashline X_2^I(0) \end{bmatrix}, \tag{3.36}$$

where $X_1(t)^{IT} = [(x'_{1,I} - \bar{r}^d_{1,I})^T \cdots (x'_{N,I} - \bar{r}^d_{N,I})^T]$ and $X_2(t)^{IT} = [(x''_{1,I} - \bar{r}^v_I)^T \cdots (x''_{N,I} - \bar{r}^v_I)^T]$ ($x'_{j,I}$ denotes the first component of x'_j, $x''_{j,I}$ the first component of x''_j, etc., $j = 1, \ldots N$). Obviously, $X_1^I(t) \to \mathbf{1}p_1^T X_1^I(0)$ and $X_2^I(t) \to \kappa\varepsilon^{-1}\mathbf{1}p_1^T X_1^I(0)$. However, according to the model definition in Section 2, we have the transformation $\begin{bmatrix} x'_{1,I}(t) \\ \vdots \\ \hdashline x''_{1,I}(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z'_{1,I}(t) \\ \vdots \\ \hdashline z''_{1,I}(t) \\ \vdots \end{bmatrix}$, so that, according to the assumption of the theorem that $\bar{r}^d = L\bar{r}^z$ for some $\bar{r}^z$, we obtain

$$\begin{bmatrix} X_1^I(t) \\ \hdashline X_2^I(t) \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_1^I(t) \\ \hdashline Z_2^I(t) \end{bmatrix}, \tag{3.37}$$

where $Z_1(t)^{IT} = [(z'_{1,I} - \bar{r}^z_{1,I})^T \cdots (z'_{N,I} - \bar{r}^z_{N,I})^T]$ and $Z_2(t)^{IT} = [(z''_{1,I} - \bar{r}^v_I)^T \cdots (z''_{N,I} - \bar{r}^v_I)^T]$. Introducing $\begin{bmatrix} X_1^I(0) \\ \hdashline X_2^I(0) \end{bmatrix}$ back into (3.36), one obtains that $\lim_{t\to\infty} X_1^I(t) = \lim_{t\to\infty} X_2^I(t) = 0$ for any $\bar{r}^z$ and $\bar{r}^v$, having in mind that $p_1^T L = 0$.

Suppose now, without loss of generality, that $S_1 = \emptyset$. According to Theorem 3.4.2, $(A_{cl}^P)^I$ has a simple zero eigenvalue and $G - I$ is nonsingular. It is straightforward to deduce that now $r_1^T = [\,1\ 0\ \cdots\ 0\,]$ and $s_1^T = \begin{bmatrix} 1\ 0\ \cdots\ 0 \vdots (\varepsilon-\rho)^{-1}\ 0\ \cdots\ 0 \end{bmatrix}$, so that $x_1(t) \to x'_{1,I}(0) + (\varepsilon - \rho)x''_{1,I}(0) + \bar{r}^v_I t$ when $t \to \infty$, and that the remaining components of the vector $\begin{bmatrix} X_1^I(t) \\ \hdashline X_2^I(t) \end{bmatrix}$ tend to zero for any $\bar{r}^d_i$, $i = 2, \ldots, N$, and $\bar{r}^v$. Hence the result. ■

3.4.5 Output Feedback with Decentralized Observers

Assume that the measurements available to the vehicles do not contain the velocities of the sensed vehicles, so that y_i, the measurement vector of the i-th vehicle, is composed of the distances with respect to the sensed vehicles and its own velocity, *i.e.* $y_i = \begin{bmatrix} (z'_{s^i_{m_1}} - z'_i)^T & \cdots & (z'_{s^i_{m_i}} - z'_i)^T & (z''_i)^T \end{bmatrix}^T$. If our task is to construct local state estimators, we will attach to the vehicles specific subsystem models $\mathbf{\Xi_i}$ having the form

$$\mathbf{\Xi_i}: \quad \dot{\xi}_i = A_i^* \xi_i + B_i^* \tilde{u}_i \tag{3.38}$$

with the state vectors $\xi_i = \begin{bmatrix} (z''_{s^i_1})^T & \cdots & (z''_{s^i_{m_i}})^T & (z'_{s^i_1} - z'_i)^T & \cdots & (z'_{s^i_{m_i}} - z'_i)^T & (z''_i)^T \end{bmatrix}^T$, where $A_i^* = \begin{bmatrix} 0_{2m_i \times 2N} \\ \bar{A}_i^* \\ 0_{2 \times 2N} \end{bmatrix}$, in which $\bar{A}_i^*$ is a $m_i \times N$ (2×2)-block matrix in which all block rows contain $-I_2$ at the last column index and I_2 at the column index $s^i_{m_j}$, $j = 1, \ldots, m_i$, and $B_i^* = \begin{bmatrix} I_{2m_i} & 0_{2m_i \times 2} \\ 0_{2m_i \times 2m_i} & 0_{2m_i \times 2} \\ 0_{2 \times 2m_i} & I_2 \end{bmatrix}$ ($\tilde{u}_i$ is defined as $\tilde{u}_i = \begin{bmatrix} u^T_{s^i_1} & \cdots & u^T_{s^i_{m_i}} & u^T_i \end{bmatrix}^T$). Subsystem models $\tilde{\mathbf{S}}_\mathbf{i}$ used for control design in the previous sections can be easily obtained from $\mathbf{\Xi_i}$ as aggregations, *i.e.* $\tilde{x}_i = U\xi_i$ where U is a full rank $(2m_i + 4) \times (4m_i + 2)$ matrix of the form $U = \begin{bmatrix} I_{2m_i} & & & & \\ & \alpha^i_{s^i_1} I_2 & \cdots & \alpha^i_{s^i_{m_i}} I_2 & \\ & & & & I_2 \end{bmatrix}$, so that we have the aggregation conditions $UA_i^* = \tilde{A}_i U$. Notice that $\mathbf{\Xi_i}$ cannot be used for control design purposes, having in mind that it is uncontrollable from $\tilde{u}_i$. However, it can used as a basis for defining the following local observers of Luenberger type

$$\mathbf{E_i^*}: \quad \dot{\hat{\xi}}_i = A_i^* \hat{\xi}_i + B_i^* \tilde{u}_i + L^*[y_i - C^* \hat{\xi}_i], \tag{3.39}$$

where L^* is the estimator gain (*e.g.* Kalman gain) and $C^* = \begin{bmatrix} 0_{2(m_i+1) \times 2m_i} & I_{2(m_i+1)} \end{bmatrix}$. Essentially, the main problem related to $\mathbf{E_i^*}$ is how to define the control vector $\tilde{u}_i$, since the real control inputs of the neighboring vehicles are generally unknown at the i-th vehicle. We will adopt here approximations, motivated by the idea to generate $\tilde{u}_i$ by using the subsystem control law $\tilde{\mathbf{F}}_i$ in (3.24) in which $\tilde{x}_i$ is replaced by its estimate obtained by using $\mathbf{E}_{\mathbf{i}}^{*}$ in such a way that $\hat{\tilde{x}}_i = U\hat{\xi}_i$, where $\hat{\xi}_i$ is generated by (3.39), so that $\tilde{u}_i = \tilde{u}_i^* = \begin{bmatrix} u_{s_1^i}^{*T} & \cdots & u_{s_{m_i}^i}^{*T} & u_i^{*T} \end{bmatrix}^T = \tilde{K}^i \hat{\tilde{x}}_i + \tilde{M}^i \tilde{r}_i$. According to the description of the structure of $\tilde{\mathbf{F}}_i$ given in Subsection 3.4.4, the control vector components $u_{s_1^i}^*, \ldots, u_{s_{m_i}^i}^*$ are generated by the local feedback designed for the leading vehicles as $u_j^* = \hat{K}_j^i \hat{z}_j'' + \hat{M}_j r^v$, $j = s_1^i, \ldots, s_{m_i}^i$, where $\hat{z}_j''$ is a part of the state estimation vector $\hat{\xi}_i$. According to (3.29), the last component u_i^* in $\tilde{u}_i^*$ is defined by

$$u_i^* = \left[\bar{K}_1^i \,\vdots \cdots \vdots\, \bar{K}_{m_i}^i \,\vdots\, \bar{K}_{m_i+1}^i \,\vdots\, \bar{K}_{m_i+2}^i + \sum_{k \in \bar{S}_i} \hat{K}_i^k \right] U\hat{\xi}_i + \left[\bar{M}_1^i \,\vdots \cdots \vdots\, \bar{M}_{m_i}^i \,\vdots\, \bar{M}_{m_i+1}^i \,\vdots\, \bar{M}_{m_i+2}^i + \sum_{k \in \bar{S}_i} \hat{M}_i^k \right] \tilde{r}_i, \tag{3.40}$$

where $\hat{x}_i'$ is easily obtained from $\hat{x}_i^*$ according to the definition of the vector x_i as a function of the distances with respect to the sensed vehicles (this mapping is incorporated in the transformation U).

3.4.6 Global LQ Optimal State Feedback with the Consensus Based Estimator

We can attach the following global quadratic criterion to (3.17)

$$J = \int_0^\infty (x^T Q x + u^T R u) dt, \tag{3.41}$$

where $Q \geq 0$ and $R > 0$ are appropriately defined matrices. However, direct construction of the LQ optimal state regulator is not directly possible, since (3.17) is in general not completely controllable.

The main observation in this respect is that the part of the state vector of (3.17) which corresponds, for example, to the relative positions with respect to the first axis $x_1' = \begin{bmatrix} x_{i,1}' & \cdots & x_{N,1}'' \end{bmatrix}^T$ satisfies the relation $x_1' = (I - G)p_1$, where p_1 is the vector of absolute vehicle positions with respect to a reference frame. Now, assuming that the graph $\mathcal{G}$ has a spanning tree, we recollect that the Laplacian L has one eigenvalue at the origin, and the rest in the open left-half plane. This means that when $I - G = L$ we have that $r^T x = 0$, where r^T is the left eigenvector of L corresponding to the zero eigenvalue. On the other hand, it is straightforward to realize that $\mathbf{S}$ is not controllable, since rank $[\,B \vdots AB\,] = 2(N-1)$ (having in mind that $A^2 = 0$). However, it is possible to see that the system is in this case controllable for the admissible initial conditions satisfying $r^T x_0 = 0$. The given observations do not hold in the case when $I - G \neq L$, which corresponds to a formation having a leader; namely, the matrix $I - G$ is then nonsingular provided $\mathcal{G}$ has a spanning tree. Having in mind that, physically, a real formation always satisfies the imposed initial conditions, a way of solving the above problem of the controllability of $\mathbf{S}$ can be seen after applying to x a nonsingular transformation $T = \begin{bmatrix} r^T \\ \hdashline U \end{bmatrix}$ (U is a full rank matrix, and r^T is linearly independent of the rows of U). Namely, it is possible to realize that $\mathbf{S}$ is controllable for all the admissible initial conditions provided the system is controllable with respect to $v = Ux$: a model for v represents an aggregation of $\mathbf{S}$. Therefore, we will construct an aggregated formation model

$$\mathbf{S_a}: \quad \dot{v} = \bar{A}v + \bar{B}u, \tag{3.42}$$

where $v = Ux$, and the system matrices satisfy the aggregation conditions $UA = \bar{A}U$ and $\bar{B} = UB$. In order to take care of optimality, we will attach to (3.42) the following criterion

$$\bar{J} = \int_0^\infty (v^T \bar{Q} v + u^T R u) dt. \tag{3.43}$$

Obviously, the criterion J includes the criterion $\bar{J}$, *i.e.* $J = \bar{J}$, if $U^T \bar{Q} U = Q$. If one starts from J, an approximate solution to the posed optimization problem can be found by formulating $\bar{J}$ using the approximate relation $\bar{Q} = U^{+T} Q U^{+}$ (where U^{+} denotes the pseudoinverse of U) and solving the optimization problem for $\mathbf{S_a}$. If $\bar{K}$ is the corresponding optimal feedback gain matrix obtained by the standard design procedure, the feedback gain matrix K for $\mathbf{S}$ can be found simply by applying the relation $K = \bar{K}U$, since in this case the closed-loop system $(\mathbf{S_a}, \bar{K})$ is an aggregation of the closed-loop system $(\mathbf{S}, K)$ and $J = \bar{J}$. Notice that U can be chosen in many different ways: a simple choice is, for example, $U = \begin{bmatrix} 0.5 & 0 & 0.5 & & \cdots \\ 0 & 1 & 0 & & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ & & & \cdots & \\ & & \cdots & & 1 \end{bmatrix}$, which does not substantially change the structure of $\bar{Q}$ with respect to Q. Also, for a given U, different choices of $\bar{A}$ are possible; having in mind the sparsity of A, $\bar{A}$ can be found for adequate choices of U by simple linear combinations of the rows of A and deletions of its columns.

By using a consensus based estimation algorithm exposed in Section 3.3 each agent is supplied with the global state estimates and, hence, each agent can implement globally optimal feedback control law calculated using the above exposed methodology.

3.4.7 Controller Realizations and Experiments

In this subsection we illustrate the proposed methods for control of formations of UAVs. First, we give example of the formation control of five vehicles with a leader, using the proposed design method based on the inclusion principle. Then we give an example of the formation control without the leader, using the proposed consensus based estimator and the globally LQ optimal feedback. A comparison with the method based on the inclusion principle demonstrates the advantage and much better performance of the proposed consensus based scheme.

3.4.7.1 Example 1

The above exposed methodology for formation tracking control design based on the inclusion principle has been implemented by using the suboptimal hierarchical LQ strategy for local controller design and Kalman filters as local observers, based on the results presented in [59, 92, 91, 99]. A formation of five vehicles has been simulated, assuming that one vehicle plays the role of the formation leader. It has been assumed that the second vehicle observes the first, the third vehicle observes the first, the fourth observes the second and the third and the fifth vehicle observes the third. The proposed design methodology has been applied for both, the case of perfect state measurements and the case of dynamic output feedback controller design, assuming that the measurements of the local velocity and the distances to the neighboring vehicles are available in the vehicles. The references of the distances (with respect to the centroid of the neighboring vehicles) and velocities have been composed in such a way as to obtain reconfiguration of the formation starting from the "V" form and ending with a line (platoon). Figures 3.6 and 3.7 represent the x-components of the distances and velocities of four vehicles in the formation, excluding

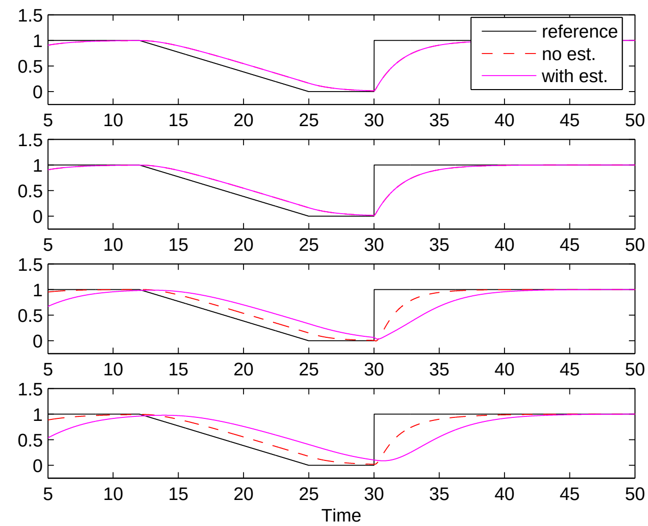

Figure 3.6: Distance plots

the leader. Obviously, tracking is very successful, even in the regime of fast changes of the references. It is important to emphasize that the presented curves correspond to a specific choice of the weighting matrices in the quadratic criterion; different choices of these matrices provide different tracking properties. This has been illustrated in Fig. 3.8 and Fig 3.9, where the formation response is shown for the case when the suboptimal hierarchical LQ controllers are obtained using larger weights for the velocity tracking. Hence, in this case, the tracking of the velocity is better compared with the Fig 3.7, at the expense of worst distance tracking, compared to Fig 3.6.

3.4.7.2 Example 2

In this example, a formation of four vehicles without a formation leader has been simulated. It has been assumed that the second vehicle observes the first, the third vehicle observes the first, the fourth observes the second and the third and the first vehicle observes the fourth. We applied the methodology exposed in the Section 3.5 for finding the globally LQ optimal feedback gains. The consensus based estimator, proposed in Chapter 2 and Section 3.3 has been implemented by each agent, assuming the same information flaw between the agents defined by the formation structure. The consensus gains are all set to be the same,

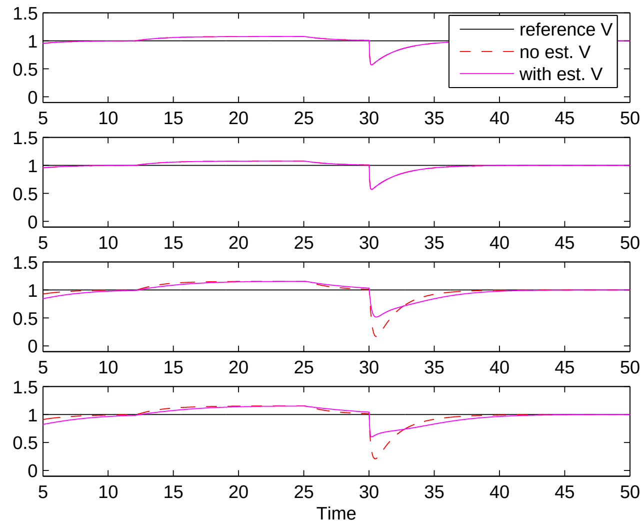

Figure 3.7: Velocity plots

equal to 100. In Fig. 3.10 and Fig. 3.11 x-components of the distances and velocities of all four vehicles in the formation are depicted, assuming step distance reference change. On the other hand, Figures 3.12 and 3.13 represent the responses of the same formation with the controllers designed using the inclusion principle with local estimators, exposed in Subsections 3.4.2-5. It is obvious that better performance is obtained using the consensus based control structure, at the expense of additional communications between the vehicles in the formation.

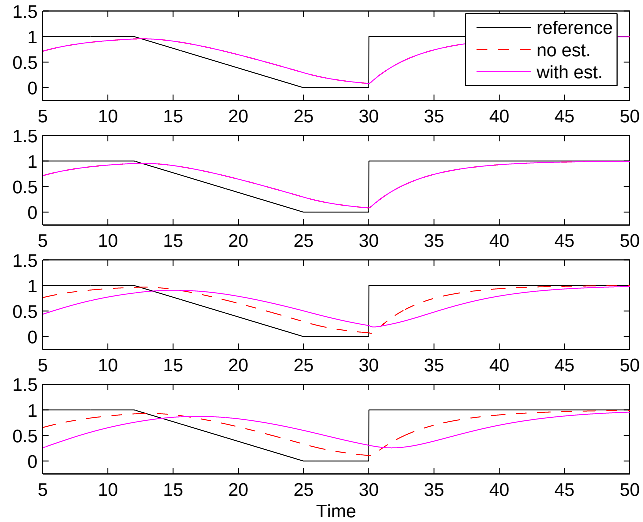

Figure 3.8: Distance plots

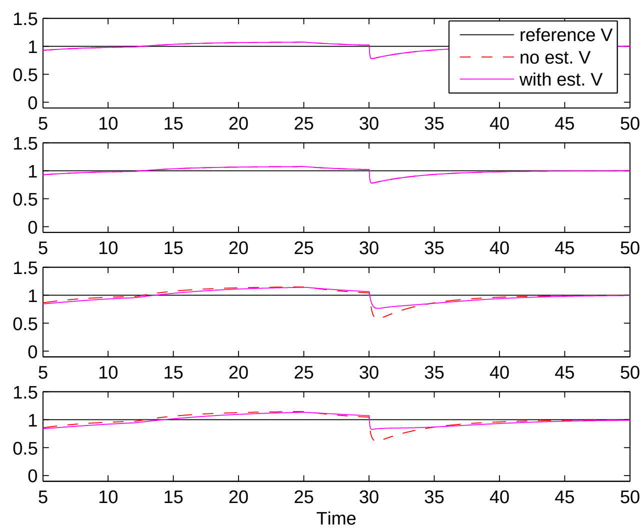

Figure 3.9: Velocity plots

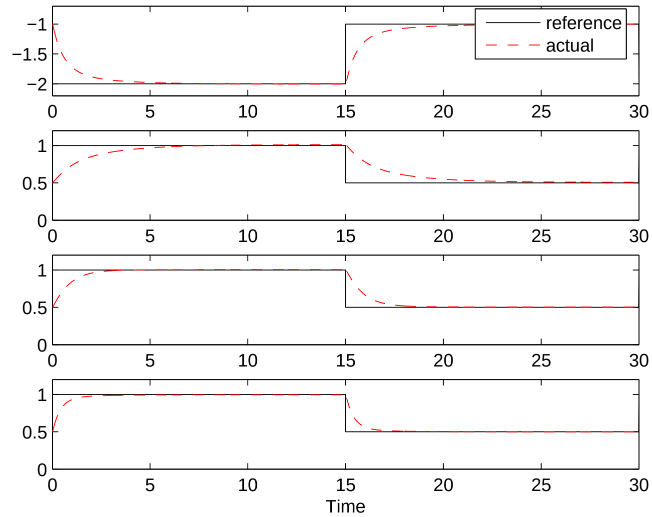

Figure 3.10: Distance plots: consensus based controllers

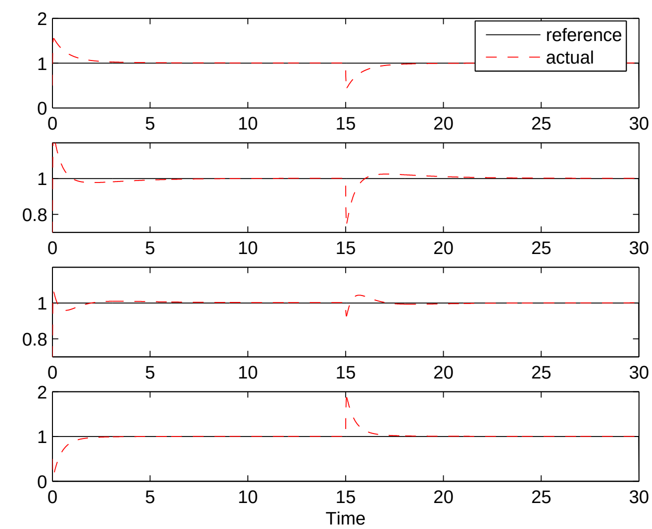

Figure 3.11: Velocity plots: consensus based controllers

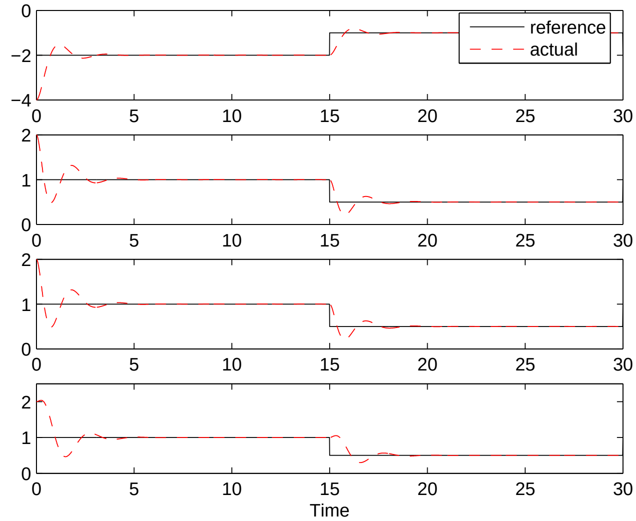

Figure 3.12: Distance plots: expansion/contraction based controllers

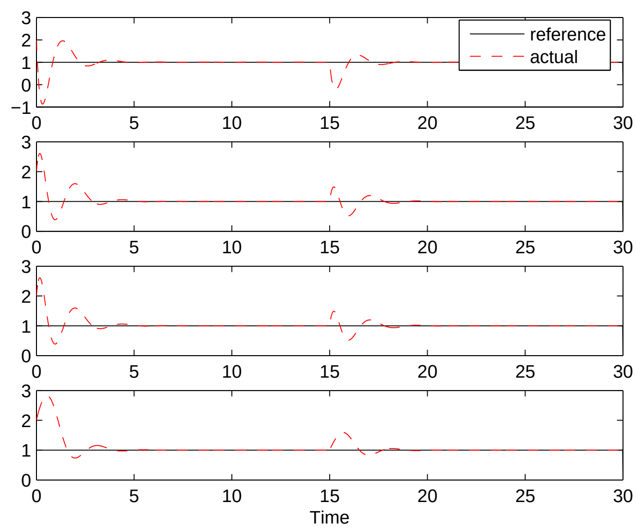

Figure 3.13: Velocity plots: expansion/contraction based controllers

Chapter 4

Stochastic Extremum Seeking with Applications to Mobile Sensor Networks

As already mentioned, the proposed consensus based (state or parameters) estimation algorithms are highly robust to local model uncertainties, noise influence, measurement and communication faults, and, at the same time, provide highly accurate estimates. Hence, they can be naturally applied by mobile (wireless) sensor networks in numerous scenarios. In this chapter a stochastic extremum seeking algorithm will be proposed and rigorously analyzed, motivated by its effective applications within mobile sensor networks, for searching the points in the plane where the optimal sensing capabilities can be achieved.

Section 4.1 contains the problem definition. Section 4.2 is devoted to the convergence analysis of the main, one dimensional stochastic ES algorithm. It is proved that the system converges under the specified conditions to the extremum point in the mean square sense and with probability one. In Section 4.3 applications of the proposed scheme to noise source localization and adaptive state estimation, where the measurement noise influence is minimized are presented. In Section 4.4 the proposed basic 1D scheme is extended to the two dimensional case and a scheme for the planar autonomous vehicle target localization is proposed, where the vehicle is modeled as a single integrator. In Section 4.5 the scheme is further generalized to the case when the vehicle is modeled as a double integrator. Finally, in Section 4.6 a scheme involving unicycle vehicle model is proposed and the convergence analysis is given. All the proposed schemes are illustrated with several simulations.

4.1 Discrete-Time Extremum Seeking Algorithm with Time-Varying Gains

We will consider a discrete-time extremum seeking algorithm with sinusoidal perturbation, as shown in Figure 4.1. The basic idea is as follows. Since we cannot measure the gradient of an *unknown* function f, whose unique extremum we are seeking, a slow sinusoidal perturbation (compared to the dynamics of the stable systems $F_i(z)$ and $F_o(z)$), with frequency $\omega = a\pi$, $0 < a < 1$, a is a rational number, is added to the system input in order to observe its effects to the output $y(k)$. In the further analysis we will assume that $f(\theta)$ has a minimum at $\theta = \theta^*$ and that locally it can be approximated with the quadratic form:

$$f(\theta) = f^* + (\theta - \theta^*)^2 \tag{4.1}$$

where f^* is a constant. Possible cubic and higher order terms can be neglected in the local convergence analysis; hence we are omitting them here. The sinusoidal perturbation, going through the mapping f, will be modulated by its local slope. Therefore, we use a high pass filter $\frac{z-1}{z+h}$, $0 < h < 1$, which filters out a DC component of the measurements $y(k)$ corrupted by noise $\zeta(k)$. Then, the resulting noisy sinusoidal signal is being demodulated (by the multiplication with the same frequency sinusoid). Hence, the input to the integrator $-\frac{1}{z-1}$ is proportional to the slope of the function $f(\theta)$ and it will drive θ to the extremal value (for which the slope of the function $f(\theta)$ is zero).

In the next section we will prove convergence of $\theta(k)$ to the extremal point θ^* (with probability one and in the mean square sense) in the presence of the measurement noise $\zeta(k)$. What makes this possible is, similarly as in the stochastic approximation algorithms (*e.g.* [61, 21, 62]), the introduction of the time varying, vanishing gains $\epsilon(k)$ and $\alpha(k)$ which make the system capable of eliminating noise. Note that in the case of similar algorithm

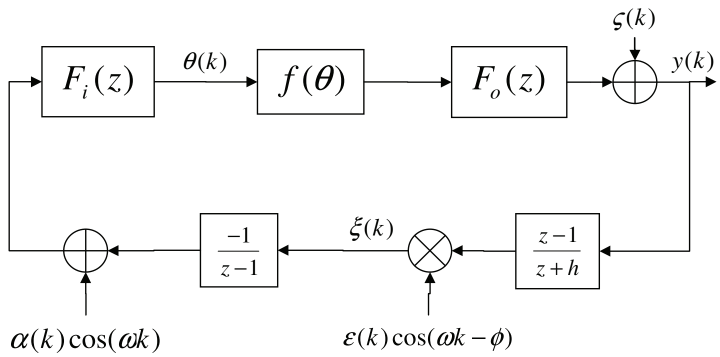

Figure 4.1: Discrete-time extremum seeking scheme

whose local stability has been analyzed in [20, 42] noisy measurements and time-varying gains have not been assumed; hence $\theta(k)$ in their case was proved to converge only to some $O(\alpha)$ neighborhood of the extremal point. Also, because of the time varying gains, the averaging theory can not be applied directly, as in [20, 42], what makes the analysis much more complicated. For the clarity of presentation, we will assume that dynamics of the systems $F_i(z)$ and $F_o(z)$ are fast enough so that they can be neglected in the convergence analysis.

In the derivation of the tracking error equation we will use the following lemmas, which can be found in [20].

Lemma 4.1 ([20], Lemma 2) If the transfer functions $G(z)$ and $H(z)$ have all of their poles inside the unit circle, the following statement is true for any real ϕ and any uniformly bounded $v(k)$:

$$G(z)[(H(z)[\cos(\omega k-\phi)])v(k)] = \mathrm{Re}\{e^{j(\omega k-\phi)}H(e^{j\omega})G(e^{j\omega}z)[v(k)]\} + \varepsilon^{-k}. \tag{4.2}$$

Lemma 4.2 ([20], Lemma 3) For any two rational functions $A(\cdot)$ and $B(\cdot,\cdot)$, the following is true:

$$\mathrm{Re}\{e^{j(\omega k-\psi)}A(e^{j\omega})\}\mathrm{Re}\{e^{j(\omega k-\phi)}B(z,e^{j\omega})[v(k)]\} =$$
$$= \tfrac{1}{2}\mathrm{Re}\{e^{j(\psi-\phi)}A(e^{-j\omega})B(z,e^{j\omega})[v(k)]\} + \tfrac{1}{2}\mathrm{Re}\{e^{j(2\omega k-\psi-\phi)}A(e^{j\omega})B(z,e^{j\omega})[v(k)]\}. \tag{4.3}$$

Lemma 4.3 ([20], Lemma 4) For any rational function $B(\cdot,\cdot)$ the following is true:

$$\mathrm{Re}\{e^{j(\omega k-\phi)}B(z,e^{j\omega})[v(k)]\} =$$
$$= \cos(\omega k-\phi)\mathrm{Re}\{B(z,e^{j\omega})[v(k)]\} - \sin(\omega k-\phi)\mathrm{Im}\{B(z,e^{j\omega})[v(k)]\}. \tag{4.4}$$

The following equations model the behavior of the above described system:

$$y(k) = f^* + (\theta(k)-\theta^*)^2 + \zeta(k) \tag{4.5}$$

$$\theta(k) = \alpha(k)\cos(\omega k) - \frac{1}{z-1}[\xi(k)] \tag{4.6}$$

$$\xi(k) = \epsilon(k)\cos(\omega k-\phi)\frac{z-1}{z+h}[y(k)] \tag{4.7}$$

where $\zeta(k)$ is the measurement noise, and, throughout the chapter, the expression $H(z)[x(k)]$ denotes a time domain signal obtained as the output of the transfer function $H(z)$ when the input is $x(k)$.

We define the tracking error as:

$$\tilde{\theta}(k) = \theta^* - \theta(k) + \alpha(k)\cos(\omega k). \tag{4.8}$$

By substituting (4.6) into (4.8) we obtain

$$\tilde{\theta}(k) = \theta^* + \frac{1}{z-1}[\xi(k)] \tag{4.9}$$

which can be written as a difference equation:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + \xi(k) \tag{4.10}$$

Consequently, we substitute (4.5) in (4.7) and then in (4.10) and obtain

$$\begin{aligned}
\tilde{\theta}(k+1) - \tilde{\theta}(k) &= \epsilon(k)c(\omega k)\frac{z-1}{z+h}[f^* + (\theta(k) - \theta^*)^2 + \zeta(k)] \\
&= \epsilon(k)c(\omega k)\{\frac{z-1}{z+h}[\tilde{\theta}(k)^2] + \frac{z-1}{z+h}[-2\alpha(k)\cos(\omega k)\tilde{\theta}(k)] \\
&+ \frac{z-1}{z+h}[f^* + \alpha(k)^2\cos^2(\omega k)] + \frac{z-1}{z+h}[\zeta(k)]\}. \qquad (4.11)
\end{aligned}$$

where $c(\omega k) = \cos(\omega k - \phi)$. After applying Lemmas 4.1-3 to the linear term in (4.11), containing $2\alpha(k)\cos(\omega k)\tilde{\theta}(k)$, we finally obtain the following equation which describes the evolution of the tracking error:

$$\tilde{\theta}(k+1) - \tilde{\theta}(k) = \epsilon(k)\{L(z)[\alpha(k)\tilde{\theta}(k)] + \Phi_1(k) + \Phi_2(k) + \Phi_3(k) + u(k)\} \qquad (4.12)$$

where

$$L(z) = -\tfrac{1}{2}[e^{j\phi}M(z, e^{j\omega}) + e^{-j\phi}M(z, e^{-j\omega})], \qquad (4.13)$$

$$\Phi_1(k) = s(2\omega k)\mathrm{Im}\{M(z, e^{j\omega})[\alpha(k)\tilde{\theta}(k)]\}, \qquad (4.14)$$

$$\Phi_2(k) = -c(2\omega k)\mathrm{Re}\{M(z, e^{j\omega})[\alpha(k)\tilde{\theta}(k)]\}, \qquad (4.15)$$

$$\Phi_3(k) = c(\omega k)\tfrac{z-1}{z+h}[\tilde{\theta}(k)^2], \qquad (4.16)$$

$$u(k) = d(k) + c(\omega k)\tfrac{z-1}{z+h}[\zeta(k)] \qquad (4.17)$$

$$d(k) = c(\omega k)\tfrac{z-1}{z+h}[f^* + \alpha(k)^2\cos^2(\omega k)] + \varepsilon^{-k}, \qquad (4.18)$$

$s(2\omega k) = \sin(2\omega k - \phi)$, $c(2\omega k) = \cos(2\omega k - \phi)$, ε^{-k} denotes exponentially decaying terms and $M(z, e^{j\omega}) = (e^{j\omega}z - 1)/(e^{j\omega}z + h)$. Hence, all the terms in equation (4.12) are time-varying; the first four terms depend on $\tilde{\theta}$ ($\Phi_3(k)$ is nonlinear), while the input term $u(k)$ is composed of the deterministic part $d(k)$ and the stochastic part $n(k) = c(\omega k)\frac{z-1}{z+h}[\zeta(k)]$.

4.2 Convergence Analysis

In the convergence analysis we will assume that the following basic assumptions are satisfied:

(A.4.1) The sequence $\{\zeta(k)\}$ is a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ with a specified sequence of σ-algebras $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$, such that the variables $\zeta(k)$ are measurable with respect to $\mathcal{F}_k$ and they satisfy

$$E\{\zeta(k)^2\} = \sigma(k)^2 < M < \infty, \; k = 1, 2, ... \tag{4.19}$$

(A.4.2) The sequence $\epsilon(k)$ is decreasing, $\epsilon(k) > 0$, $k = 1, 2, ...$ and $\lim_{k\to\infty} \epsilon(k) = 0$

(A.4.3) The sequence $\alpha(k)$ is decreasing, $\alpha(k) > 0$, $k = 1, 2, ...$ and $\lim_{k\to\infty} \alpha(k) = 0$

(A.4.4) $\sum_{k=1}^{\infty} \epsilon(k)\alpha(k) = \infty$

(A.4.5) $\sum_{k=1}^{\infty} \epsilon(k)^2 < \infty$

(A.4.6) $\sum_{k=1}^{\infty} \epsilon(k)\alpha(k)^2 < \infty$

(A.4.7) $-\frac{\pi}{2} < \phi + \mathrm{Arg}\{\frac{e^{j\omega}-1}{e^{j\omega}+h}\} < \frac{\pi}{2}$

The following theorem deals with the asymptotic behavior of the algorithm.

Theorem 4.1 Let the assumptions (A.4.1-7) be satisfied. Then $\theta(k)$ converges to θ^* almost surely (a.s.) and in the mean square sense under the condition that $\sup_k(|\tilde{\theta}(k)|) < K$ (a.s.), $0 < K < \infty$.

Proof. We will analyze the right hand side of equation (4.12) term by term.

Thus, we start with the first term, by writing

$$\epsilon(k)L(z)[\alpha(k)\tilde{\theta}(k)] = \rho(k)L(z)[\tilde{\theta}(k)] + \epsilon(k)\delta l(k), \tag{4.20}$$

where $\delta l(k) = L(z)[\alpha(k)\tilde{\theta}(k)] - \alpha(k)L(z)[\tilde{\theta}(k)]$ and $\rho(k) = \epsilon(k)\alpha(k)$. If $l(k), k = 0, 1, ...$ is the impulse response of the system $\mathbf{S}$ with transfer function $L(z)$, we have

$$\begin{aligned} \delta l(k) &= l(0)[\alpha(k) - \alpha(k)]\tilde{\theta}(k) + l(1)[\alpha(k-1) - \alpha(k)]\tilde{\theta}(k-1) + \cdots \\ &+ l(k-1)[\alpha(1) - \alpha(k)]\tilde{\theta}(1) \end{aligned} \tag{4.21}$$

so that

$$\delta l(k) = [\alpha(k-1) - \alpha(k)]y_1(k) \tag{4.22}$$

where $y_1(k)$ can be considered as the output of a time varying system $\mathbf{S}_1$ with the impulse response $h_1(k,j) = l(j)\frac{\alpha(k-j)-\alpha(k)}{\alpha(k-1)-\alpha(k)}$ and input $\tilde{\theta}(k)$, *i.e.*, $y_1(k) = \sum_{j=0}^{k-1} h_1(k,j)\tilde{\theta}(k-j)$.

System $\mathbf{S}_1$ is bounded-input, bounded-output (b.i.b.o.) stable, having in mind that $h_1(k,j)$ is absolutely summable under the formulated assumptions ($\mathbf{S}$ is exponentially stable and $\alpha(k)$ satisfies (A.4.3-4)).

In the further analysis we define $\kappa = \frac{1}{2}\mathrm{Re}\{e^{j\phi}\frac{e^{j\omega}-1}{e^{j\omega}+h}\}$. Notice that we also have $\kappa = -\sum_{j=0}^{\infty} l(j)$, according to the above notation. Also notice that $\kappa > 0$ having in mind assumption (A.4.7). It will turn out that the linear term $-\kappa\tilde{\theta}(k)$ will be dominant in the right hand side of the tracking error difference equation and, thus, crucial for proving the almost sure convergence of the algorithm.

Hence, we write $L(z)[\tilde{\theta}(k)] = -\kappa\tilde{\theta}(k) + \delta l_\kappa(k)$ and obtain

$$\delta l_\kappa(k) = L(z)[\tilde{\theta}(k)] + \kappa\tilde{\theta}(k) = \sum_{j=0}^{k-1} l(j)[\tilde{\theta}(k-j) - \tilde{\theta}(k)] + [\sum_{i=0}^{k-1} l(i) + \kappa]\tilde{\theta}(k) \quad (4.23)$$

where the last term is equal to $\lambda(k)\tilde{\theta}(k)$, with $\lambda(k) = -\sum_{i=k}^{\infty} l(i)$. After iterating (4.12) back to the initial condition and plugging into the first term in (4.23), we obtain

$$\begin{aligned} \delta l_\kappa(k) = \; & - \; l(1)\{\epsilon(k-1)[L(z)[\alpha(k-1)\tilde{\theta}(k-1)] + \Phi(k-1) + u(k-1)]\} \\ & - \; l(2)\{\epsilon(k-2)[L(z)[\alpha(k-2)\tilde{\theta}(k-2)] + \Phi(k-2) + u(k-2)] \qquad (4.24) \\ & + \; \epsilon(k-1)[L(z)[\alpha(k-1)\tilde{\theta}(k-1)] + \Phi(k-1) + u(k-1)]\} + ... + \lambda(k)\tilde{\theta}(k) \end{aligned}$$

where $\Phi(k) = \Phi_1(k) + \Phi_2(k) + \Phi_3(k)$. After regrouping the terms in (4.24), we obtain

$$\begin{aligned} \delta l_\kappa(k) \; = \; & \sum_{j=1}^{k-1}[-\sum_{i=j}^{k-1} l(i)]\epsilon(k-j)\{L(z)[\alpha(k-j)\tilde{\theta}(k-j)] \\ + \; & \Phi(k-j) + u(k-j)\} + \lambda(k)\tilde{\theta}(k) \qquad (4.25) \end{aligned}$$

Defining a time-varying system $\mathbf{S}_2$ with the impulse response $h_2(k,j) = \bar{l}(k,j)\frac{\epsilon(k-j)}{\epsilon(k-1)}$, where $\bar{l}(k,j) = -\sum_{i=j}^{k-1} l(i)$, we can write

$$\delta l_\kappa(k) = \epsilon(k-1)y_2(k) + \lambda(k)\tilde{\theta}(k) \quad (4.26)$$

where $y_2(k) = \sum_{j=0}^{k-1} h_2(k,j)\{L(z)[\alpha(k-j)\tilde{\theta}(k-j)] + \Phi(k-j) + u(k-j)\}$ is the output of

$\mathbf{S}_2$. One can easily verify that $\mathbf{S}_2$ is b.i.b.o. stable under the adopted assumptions, while $\lambda(k)$ is exponentially decaying.

Now, we focus on $\Phi_i(k)$, $i = 1, 2, 3$, terms in equation (4.12).

Considering first $\Phi_1(k)$ defined by (4.14), we form, similarly as before, the difference

$$\delta l_1(k) = \alpha(k)s(2\omega k)\mathrm{Im}\{M(z, e^{j\omega})[\tilde{\theta}(k)]\} - s(2\omega k)\mathrm{Im}\{M(z, e^{j\omega})[\alpha(k)\tilde{\theta}(k)]\} \tag{4.27}$$

and obtain that

$$\delta l_1(k) = \alpha(k)s(2\omega k)[\alpha(k-1) - \alpha(k)]y_3(k) \tag{4.28}$$

where $y_3(k)$ is the output of a b.i.b.o. stable system $\mathbf{S}_3$ with the input $\tilde{\theta}(k)$ and with the impulse response sequence $h_3(k, j) = m_1(j)\frac{\alpha(k-j)-\alpha(k)}{\alpha(k-1)-\alpha(k)}$, where $\{m_1(j)\}$ is the impulse response of $\mathrm{Im}\{M(z, e^{j\omega})\}$ which is exponentially stable.

Further, we write $\mathrm{Im}\{M(z, e^{j\omega})[\tilde{\theta}(k)]\} = \kappa_1\tilde{\theta}(k) + \delta_\kappa^1(k)$, where $\kappa_1 = \mathrm{Im}\{\frac{e^{j\omega}-1}{e^{j\omega}+h}\}$, and, following the methodology of deriving (4.24) and (4.25), we obtain

$$\begin{aligned} \delta_\kappa^1(k) &= \sum_{j=1}^{k-1}[-\sum_{i=j}^{k-1} m_1(i)]\epsilon(k-j)\{L(z)[\alpha(k-j)\tilde{\theta}(k-j)] \\ &+ \Phi(k-j)u(k-j)\} + \mu_1(k)\tilde{\theta}(k) \end{aligned} \tag{4.29}$$

where $\mu_1(k) = -\sum_{i=k}^{\infty} m_1(i)$ is decaying exponentially. Following further an analogous reasoning as above, we obtain

$$\delta_\kappa^1(k) = \epsilon(k-1)y_4(k) + \mu_1(k)\tilde{\theta}(k) \tag{4.30}$$

where $y_4(k)$ is the output of a b.i.b.o. stable system $\mathbf{S}_4$ with impulse response $h_4(k, j) = \bar{m}_1(k, j)\frac{\epsilon(k-j)}{\epsilon(k-1)}$, where $\bar{m}_1(k, j) = -\sum_{i=j}^{k-1} m_1(i)$, and with the input $L(z)[\alpha(k)\tilde{\theta}(k)] + \Phi(k) + u(k)$. Consequently, we have

$$\Phi_1(k) = \alpha(k)s(2\omega k)[\kappa_1\tilde{\theta}(k) + \delta_\kappa^1(k)] + \delta l_1(k) \tag{4.31}$$

Again, using analogous arguments we obtain that

$$\Phi_2(k) = -\{\alpha(k)c(2\omega k)[\kappa_2\tilde{\theta}(k) + \delta_\kappa^2(k)] + \delta l_2(k)\} \tag{4.32}$$

where $\kappa_2 = \mathrm{Re}\{\frac{e^{j\omega}-1}{e^{j\omega}+h}\}$, while

$$\delta l_2(k) = \alpha(k)c(2\omega k)[\alpha(k-1) - \alpha(k)]y_5(k) \tag{4.33}$$

and

$$\delta_\kappa^2(k) = \epsilon(k-1)y_6(k) + \mu_2(k)\tilde{\theta}(k) \tag{4.34}$$

where $y_5(k)$ is the output of a b.i.b.o. stable system $\mathbf{S}_5$ with the input $\tilde{\theta}(k)$ and with the impulse response sequence $h_5(k,j) = m_2(j)\frac{\alpha(k-j)-\alpha(k)}{\alpha(k-1)-\alpha(k)}$. $\{m_2(j)\}$ is the impulse response of $\mathrm{Re}\{M(z, e^{j\omega})\}$ which is exponentially stable, $\mu_2(k) = -\sum_{i=k}^{\infty} m_2(i)$ is exponentially decaying. Furthermore, $y_6(k)$ is the output of a b.i.b.o. stable system $\mathbf{S}_6$ with the impulse response $h_6(k,j) = \bar{m}_2(k,j)\frac{\epsilon(k-j)}{\epsilon(k-1)}$, where $\bar{m}_2(k,j) = -\sum_{i=j}^{k-1} m_2(i)$, and with the input $L(z)[\alpha(k)\tilde{\theta}(k)] + \Phi(k) + u(k)$.

Therefore, after replacing the obtained expressions for $L(z)[\alpha(k)\tilde{\theta}(k)] + \Phi_1(k) + \Phi_2(k) + \Phi_3(k)$ in (4.12), we obtain

$$\tilde{\theta}(k+1) = [1 - \kappa\rho(k) + \eta(k)]\tilde{\theta}(k) + \pi(k) + \epsilon(k)u(k) \tag{4.35}$$

where

$$\eta(k) = [\kappa_1 s(2\omega k) - \kappa_2 c(2\omega k)]\rho(k) \tag{4.36}$$

and

$$\begin{aligned}\pi(k) = \quad & \epsilon(k) \quad \delta l(k) + \rho(k)\delta l_\kappa(k) + \epsilon(k)c(2\omega k)\delta_\kappa^1(k) \\ & + \quad \epsilon(k)\delta l_1(k) + \epsilon(k)s(2\omega k)\delta_\kappa^2(k) \\ & + \quad \epsilon(k)\delta l_2(k) + \epsilon(k)\Phi_3(k)\end{aligned} \tag{4.37}$$

Considering first the term $\eta(k)$ in (4.36), we can easily derive that $\eta(k) = \rho(k)\sin(2\omega k+$

ψ), where ψ depends on ϕ and $\phi_M = \text{Arg}\{\frac{e^{j\omega}-1}{e^{j\omega}+h}\}$. If N is the integer period of $\sin(2\omega k)$, we have further that

$$|\sum_{k=1}^{\infty} \eta(k)| \leq \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} b_j \sum_{k=0}^{\infty} [\rho(j+kN) - \rho(j + \lfloor \frac{N}{2} \rfloor + kN)] < \infty \tag{4.38}$$

where $b_j \geq 0$, $j = 1, \ldots, \lfloor \frac{N}{2} \rfloor$, having in mind that $\omega = a\pi$, where a is a rational number. Therefore, having in mind that $\sum_{k=1}^{\infty} \rho(k) = \infty$ (A.4.4), from (4.35) we obtain for k large enough that

$$\tilde{\theta}(k+1) = \prod_{j=1}^{k} (1 - \kappa' \rho(j)) \tilde{\theta}(1) + \sum_{j=1}^{k} [\pi(j) + \epsilon(j) u(j)] \prod_{i=j+1}^{k} (1 - \kappa' \rho(i)) \tag{4.39}$$

where $0 < \kappa' < \kappa$. Now, using the inequality $1 - x \leq e^{-x}$ it is easy to see that $\prod_{j=1}^{k}(1 - \kappa'\rho(j)) \to 0$ when $k \to \infty$, having in mind the condition (A.4.4). Furthermore, after applying the Kronecker's lemma to the second term at the right hand side of (4.39), we conclude that $\tilde{\theta}(k)$ converges to zero almost surely if $\sum_{j=1}^{\infty}[\pi(j) + \epsilon(j)u(j)]$ converges (a.s.).

In order to show that the last condition holds, we will decompose $\pi(j)$ as $\pi(j) = \sum_{i=1}^{3} \pi_i(j)$, where $\pi_2(j)$ and $\pi_3(j)$ contain only those components of $y_2(j)$, $y_4(j)$ and $y_6(j)$ (outputs of b.i.b.o. stable linear systems $\mathbf{S}_2$, $\mathbf{S}_4$ and $\mathbf{S}_6$) that are responses to the inputs $\epsilon(j)d(j)$ and $\epsilon(j)n(j)$, respectively; $\pi_1(j)$ contains all the remaining terms of $\pi(j)$.

According to the Assumptions (A.4.2-5), boundedness of $\tilde{\theta}(k)$ guarantees the property that $\sum_{k=1}^{\infty} \pi_1(k)$ converges. This is evident for all the terms in $\pi_1(k)$ except the last one, where we need to verify that $\sum_{k=1}^{\infty} \epsilon(k)\Phi_3(k)$ converges. To this end, we follow a similar approach as in deriving (4.38). By defining $s(k) = \epsilon(k)c(\omega k)\frac{z-1}{z+h}[\tilde{\theta}(k)^2] = \epsilon(k)c(\omega k)\sum_{i=0}^{k-1} l^*(i)\tilde{\theta}(k-i)^2$ and $r(k) = \sum_{i=0}^{k-1} l^*(i)\tilde{\theta}(k-i)$, where $l^*(i)$ is the impulse response of $\frac{z-1}{z+h}$, we have that

$$\begin{aligned}
\sum_{k=1}^{\infty} s(k) \quad &\leq \quad \sum_{j=1}^{N} b_j \sum_{k=0}^{\infty} [\epsilon(j+2kN) r(j+2kN) - \epsilon(j+N+2kN) r(j+N+2kN)] \\
&= \quad \sum_{j=1}^{N} b_j \sum_{k=0}^{\infty} \{[\epsilon(j+2kN) - \epsilon(j+N+2kN)] r(j+2kN) \\
&- \quad \epsilon(j+N+2kN)[r(j+N+2kN) - r(j+2kN)]\}
\end{aligned} \tag{4.40}$$

for some $b_j \geq 0$, $j = 1, ..., N$, where $2N$ is the integer period of $\cos(\omega k)$. The first term in equation (4.40) converges, having in mind boundedness of $r(k)$. For the second one, we can write:

$$
\begin{aligned}
& r(j+N+2kN) - r(j+2kN) = \\
= & \sum_{i=0}^{j+N+2kN-1} l^*(i)\tilde{\theta}(j+N+2kN-i)^2 - \sum_{i=0}^{j+2kN-1} l^*(i)\tilde{\theta}(j+2kN-i)^2 \\
= & \sum_{i=0}^{j+2kN-1} l^*(i)[\tilde{\theta}(j+N+2kN-i)^2 - \tilde{\theta}(j+2kN-i)^2] \\
+ & \sum_{i=j}^{j+N+2kN-1} l^*(i)\tilde{\theta}(j+2kN-i)^2
\end{aligned}
\tag{4.41}
$$

The second term in the above equation exponentially goes to zero when $k \to \infty$. The first term can be written as

$$
\begin{aligned}
& \sum_{i=0}^{j+2kN-1} l^*(i)[\tilde{\theta}(j+N+2kN-i)^2 - \tilde{\theta}(j+2kN-i)^2] \\
= & \sum_{i=0}^{j+2kN-1} l^*(i)[\tilde{\theta}(j+N+2kN-i) - \tilde{\theta}(j+2kN-i)] \\
& \cdot[\tilde{\theta}(j+N+2kN-i) + \tilde{\theta}(j+2kN-i)]
\end{aligned}
\tag{4.42}
$$

By treating the difference $\tilde{\theta}(j+N+2kN-i) - \tilde{\theta}(j+2kN-i)$ the same way as in deriving (4.23), (4.24) and (4.25), that is, by substituting and iterating equation (4.12), having in mind the condition that $\tilde{\theta}(j+N+2kN-i) + \tilde{\theta}(j+2kN-i)$ is bounded, one can conclude that the absolute value of the whole sum in (4.42) can be bounded by $k_1\epsilon(j+2kN-i)$, for some $k_1 > 0$. Therefore, using condition (A.4.5) we can conclude that the sum (4.40) converges.

The analysis proceeds with the terms in (4.35) depending on $d(k)$. Using the identity $\cos^2(\omega k) = \frac{1}{2}(1+\cos(2\omega k))$, we obtain that $d(k) = d_1(k) + d_2(k) + \varepsilon^{-k}$, where

$$
d_1(k) = c(\omega k)\frac{z-1}{z+h}[f^* + \frac{1}{2}\alpha(k)^2] \tag{4.43}
$$

$$d_2(k) = \frac{1}{2}c(\omega k)\frac{z-1}{z+h}[\alpha(k)^2\cos(2\omega k)] \tag{4.44}$$

Considering the term $d_1(k)$ we first conclude that $\frac{z-1}{z+h}[f^*] = 0$ (high pass filter). Furthermore,

$$\frac{z-1}{z+h}[\alpha(k)^2] = \alpha(k)^2\sum_{j=0}^{k-1} l^*(j)\frac{\alpha(k-j)^2}{\alpha(k)^2} \tag{4.45}$$

where sequence $\{l^*(j)\}$ is the impulse response of the system $\frac{z-1}{z+h}$. The summation in (4.45) can be considered as the output of a b.i.b.o. stable time varying system with the impulse response $h_7(k,j) = \frac{l^*(j)}{\alpha(k)^2}$ and with the input $\alpha(k)^2$. Therefore, we conclude that $|d_1(k)| \leq k_2\alpha(k)^2$, where $k_2 > 0$ is a constant. Similarly, for $d_2(k)$ we have

$$d_2(k) = \frac{1}{2}c(\omega k)\alpha(k)^2\sum_{j=0}^{k-1} l^*(j)\cos(2\omega(k-j))\frac{\alpha(k-j)^2}{\alpha(k)^2} \tag{4.46}$$

which leads, as above, to the conclusion that $|d_2(k)| \leq k_3\alpha(k)^2$, where $k_3 > 0$ is a constant. Therefore, we have

$$|d(k)| \leq k_4\alpha(k)^2 \tag{4.47}$$

for some constant $k_4 > 0$.

Consequently, it follows clearly that $\sum_{j=1}^{\infty}[\epsilon(j)d(j)+\pi_2(j)]$ converges, under the adopted assumption (A.4.6).

The last part of (4.35) to be analyzed is the stochastic component, obtained as a consequence of $\epsilon(k)n(k)$. We will first demonstrate that

$$\sum_{k=1}^{\infty}\epsilon(k)n(k) \text{ converges a.s.} \tag{4.48}$$

To do so, we will use the results from [67] (Theorem 1) which state that the sufficient conditions for (4.48) to be satisfied, both with probability 1 and in the mean square sense, are

$$\text{(a)} \qquad r(k) = \sum_{j=k+1}^{\infty} \epsilon(j)\Psi_{j,k} \to 0,\ k \to \infty \tag{4.49}$$

$$\text{(b)} \qquad \sum_{k=1}^{\infty} \epsilon(k)^2 \sigma(k)^2 < \infty \tag{4.50}$$

$$\text{(c)} \qquad \sum_{k=1}^{\infty} \epsilon(k)\sigma(k) r(k) < \infty \tag{4.51}$$

where $\Psi_{k,m} = \|E\{n(k)|\mathcal{F}_m\}\|_2$ with $k > m$, $\|\cdot\|_2 = (E\{|\cdot|^2\})^{\frac{1}{2}}$ and $\mathcal{F}_k$ is a sequence of σ-algebras such that the variables $n(k)$ are measurable with respect to $\mathcal{F}_k$. These conditions specify a class of noise with a sufficiently slowly increasing second moment and a sufficiently fast decreasing correlation.

For condition (a) we have

$$\begin{aligned}
E\{n(j)|\mathcal{F}_k\} &= E\{\sum_{i=1}^{j} l^*(j-i)\zeta(i)|\mathcal{F}_k\}c(\omega j) \\
&= \sum_{i=1}^{j} l^*(j-i)E\{\zeta(i)|\mathcal{F}_k\}c(\omega j) \\
&= c(\omega j)[\sum_{i=1}^{k} l^*(j-i)\zeta(i) + \sum_{s=k+1}^{j} l^*(j-i)E\{\zeta(s)|\mathcal{F}_k\}] \\
&= c(\omega j)\sum_{i=1}^{k} l^*(j-i)\zeta(i) \qquad (4.52)
\end{aligned}$$

where we used the fact that $E\{\zeta(s)|\mathcal{F}_k\} = 0$ for $s > k$ and $E\{\zeta(s)|\mathcal{F}_k\} = \zeta(s)$ for $s \le k$ (since $\zeta(i)$ is a martingale difference sequence), $\{l^*(i)\}$ is the impulse response sequence of $\frac{z-1}{z+h}$. Furthermore, from (4.49) and (4.52), we have

$$\begin{aligned}
r(k) &= \sum_{j=k+1}^{\infty} \epsilon(j)|c(\omega j)|E\{(\sum_{i=1}^{k} l^*(j-i)\zeta(i))^2\}^{\frac{1}{2}} \\
&\le K' \sum_{j=k+1}^{\infty} \epsilon(j) \sum_{i=1}^{k} l^*(j-i)^2 \qquad (4.53)
\end{aligned}$$

for some positive constant K', where we used the fact that $E\{\zeta(i)\zeta(j)\} = 0$ for $i \ne j$

and $E\{\zeta(i)\zeta(j)\} = \sigma(i)^2$ for $i = j$. The last term in (4.53) goes to zero when $k \to \infty$ having in mind that $\epsilon(k) \to 0$ and $l^*(k) \to 0$ exponentially, when $k \to \infty$. Therefore, the condition (4.49) is satisfied. Condition (4.50) follows directly from the Assumptions (A.4.1) and (A.4.5). To prove condition (4.51) we have

$$\begin{aligned}\sum_{k=1}^{\infty}\epsilon(k)\sigma(k)r(k) &\leq K'\sum_{k=1}^{\infty}\epsilon(k)\sigma(k)\sum_{j=k+1}^{\infty}\epsilon(j)\sum_{i=1}^{k}l^*(j-i)^2 \\ &= K'\sum_{k=1}^{\infty}\epsilon(k)^2\sigma(k)\sum_{j=k+1}^{\infty}\frac{\epsilon(j)}{\epsilon(k)}\sum_{i=1}^{k}l^*(j-i)^2 \end{aligned} \quad (4.54)$$

The last term converges having in mind conditions (A.4.2) and (A.4.5). Therefore, the property (4.48) holds.

Using the above arguments, it follows directly that $\sum_{j=1}^{\infty}\pi_3(j)$ converges a.s., and in the mean square sense.

Therefore, $\sum_{j=1}^{\infty}[\pi(j)+\epsilon(j)u(j)]$ converges almost surely, and in the mean square sense, and we have the result. ■

Remark 4.1 The results of Theorem 4.1 hold under the general condition that $|\tilde{\theta}(k)|$ is bounded a.s.. Such an assumption is realistic for practical applications; it represents a frequent assumption for convergence analysis of diverse stochastic approximation based schemes (see, *e.g.*, [43]). It could be eliminated by introducing fixed or expanding truncations as in, *e.g.*, [18]. Also, if we are interested in the probability $P(|\tilde{\theta}(k)| < K$ for all $k \geq k_0)$, where K is a preselected constant, it is possible to follow the line of thought in [61] based on the Kolmogorov's inequality for semi-martingales.

Remark 4.2 The main assumption for proving the almost sure convergence to the extremal point and, hence, for complete measurement noise elimination, was that gains $\alpha(k)$ and $\epsilon(k)$ tend to zero at a pre-specified rate. However, it might be the case that the extremal point has some constant drift and is slowly changing in time. In this situation, in order to achieve tracking of the extremal point, we can define positive lower bounds for the time varying coefficients $\alpha(k)$ and $\epsilon(k)$, at the expense of not being able to completely eliminate the noise influence any more. The values of the lower bounds would reflect the compromise between the tracking capabilities of the algorithm and the noise immunity. It is also possible to apply adaptive procedures similar to those used in the neural network training algorithms, based on the observations of the noisy criterion function, e.g. [14].

4.3 An Application to Mobile Sensors

In this section some direct applications of the stochastic extremum seeking scheme to the optimal positioning of mobile sensors will be presented.

4.3.1 Noise Source Localization

Assume that we have a noise source which generates an independent zero-mean sequence $\{\xi(k)\}$ with variance which depends on a parameter θ, *i. e.*, $E\{\xi(k)^2\} = R(\theta)$, where $R(\theta)$ is assumed to be a convex function of θ. Our goal can be to find the optimal θ^* which minimizes (or maximizes) $R(\theta)$ by measuring $\{\xi(k)\}$ generated for different values of θ, having in mind that θ can define the physical position of the noise source. According to the above results, we can apply the extremum seeking (ES) scheme for this purpose. Assume that the measurements fed to the ES scheme in Fig. 4.1 are defined as $y(k) = \xi(k)^2$, and write $y(k) = R(\theta) + \zeta(k)$, where $\zeta(k) = \xi(k)^2 - R(\theta)$. The sequence $\{\zeta(k)\}$ is white and zero-mean with finite variance, assuming that the fourth-order moment of $\xi(k)$ is finite. Therefore, according to Fig. 4.1, $y(k)$ represents the noisy output, $R(\theta(k))$ the noiseless output and $\zeta(k)$ the measurement noise, according to the above notation. Using Theorem 4.1, it is straightforward to prove the following Corollary.

Corollary 4.1 Assume that $y(k) = \xi(k)^2$, where $\{\xi(k)\}$ is an independent zero-mean sequence with variance $R(\theta)$ which depends on a parameter θ and satisfies (4.1), and $E\{\xi(k)^4\} < \infty$. Then, under the Assumptions (A.4.1-7) the ES scheme depicted in Fig. 4.1 generates $\theta(k)$ which converges to θ^* a.s. and in the mean square sense under the condition that $\sup_k(|\tilde{\theta}(k)|) < K$ (a.s.), $0 < K < \infty$.

The Corollary has a great practical importance, since it represents a basis for either noise source localization or finding a position with the lowest noise influence. In the next section, the scheme in Fig. 4.1 is generalized to the two dimensional case, using orthogonal sinusoidal perturbations. Then, utilizing the above approach, the ES scheme becomes able to find the position in the plane corresponding to the extremum of the noise variance.

4.3.2 Optimal Observer Positioning for State Estimation

Assume now that we are faced with a more complex problem of state estimation in which the Kalman filter is applied, and that it is necessary to find the best place in the plane for an observer, assuming that the measurement noise variance is coordinate dependent. This problem is fundamental in applications related to mobile sensor networks. Recall that in the optimal steady state regime of the estimator the innovation sequence $\{\nu(k)\} = \{z(k) - C\hat{x}(k|k)\}$ is white (under appropriate assumptions), where $z(k)$ is the system output, $\hat{x}(k|k)$ is the state estimate and C is the output matrix of the system (assuming that we have a scalar output). Assume that $\{v(k)\}$ is the filter measurement noise, which is white, with variance depending on the position of the observer in a plane, *i.e.*, $E\{v(k)^2\} = R(\theta_1(k), \theta_2(k))$ ($\theta_1(k)$ and $\theta_2(k)$ are the observer's coordinates). Then, we have

$$E\{\nu(k)^2\} = R_\nu(\theta_1(k), \theta_2(k)) = CP(\theta_1(k), \theta_2(k))C^T + R(\theta_1(k), \theta_2(k)) \tag{4.55}$$

where $P(\theta_1(k), \theta_2(k))$ is the steady state estimation error covariance matrix which satisfies the algebraic Riccati equation

$$P = \Phi P \Phi^T - \Phi P C^T [CPC^T + R]^{-1} C P \Phi^T + Q \tag{4.56}$$

where Φ is the state matrix of the system model and Q is the input driving noise covariance. We can calculate $p = CPC^T$ by assuming that $C\Phi P \Phi^T C^T \approx ap$ and $C\Phi P C^T \approx bp$, for some constants a and b. From (4.56) we obtain that p, which is scalar, is a solution of the quadratic equation

$$b^2 p^2 + (1-a)p(p+R) - q(p+R) = 0 \tag{4.57}$$

where $q = CQC^T$. It is easy to verify that for R small enough $p \approx p^* + a^* R$, where p^* and $a^* > 0$ are constants depending on the parameters a, b and q. Therefore, from (4.55) we derive

$$R_\nu(\theta_1(k), \theta_2(k)) \approx R_\nu^* + a_1^*(\theta_1(k) - \theta_1^*)^2 + a_2^*(\theta_2(k) - \theta_2^*)^2 \tag{4.58}$$

for some constants R_ν^*, a_1^* and a_2^*, assuming that $R(\theta_1(k), \theta_2(k))$ can be approximated by a quadratic function, where (θ_1^*, θ_2^*) is the optimal position. From this result we conclude that the observer position can be asymptotically optimized by applying the ES scheme as in Corollary 1. Namely, we take the realizations $\nu^2(k)$ as measurements (instead of $\xi^2(k)$) and apply the ES scheme from Fig. 1, for one dimensional case; the scheme asymptotically provides the optimal observer position. Two dimensional case will be analyzed in detail in the next three sections.

One practical modification of this scheme is to take $\frac{1}{T}\sum_{i=k-T+1}^{k}\nu(i)^2$ instead of $\nu^2(k)$ in order to reduce the equivalent noise variance (by the factor T).

4.4 Velocity Actuated Vehicles

In this section, the proposed one dimensional ES scheme will be generalized to the two dimensional, hybrid case. We will model an autonomous vehicle, moving in the plane, as a velocity actuated point mass such that

$$\dot{x} = v_x, \qquad \dot{y} = v_y \tag{4.59}$$

where (x, y) is the position of the point mass and v_x and v_y are the velocity inputs. We will consider a stochastic, two dimensional, discrete-time extremum seeking algorithm with sinusoidal perturbation connected to (4.59), as shown in Figure 4.2. The nonlinear map represents the signal being tracked. As in the 1D case, we will assume that the nonlinear map $J = f(x, y)$ has a local minimum and our goal is to position the vehicle at this minimal point. For simplicity, we will assume that this nonlinear map is quadratic and its Hessian is diagonal

$$J = f(x, y) = f^* + q_x(x - x^*)^2 + q_y(y - y^*)^2 \tag{4.60}$$

where (x^*, y^*) is the unknown maximizer, f^* is the unknown minimum and q_x and q_y unknown positive constants. The discrete integrator from the 1D scheme shown in Fig. 4.1 is now contained in the vehicle model (4.59). Notice that this is a hybrid system: the continuous part contains zero order hold circuits (ZOH) and integrators for the two channels, and the discrete part contains the whole ES algorithm.

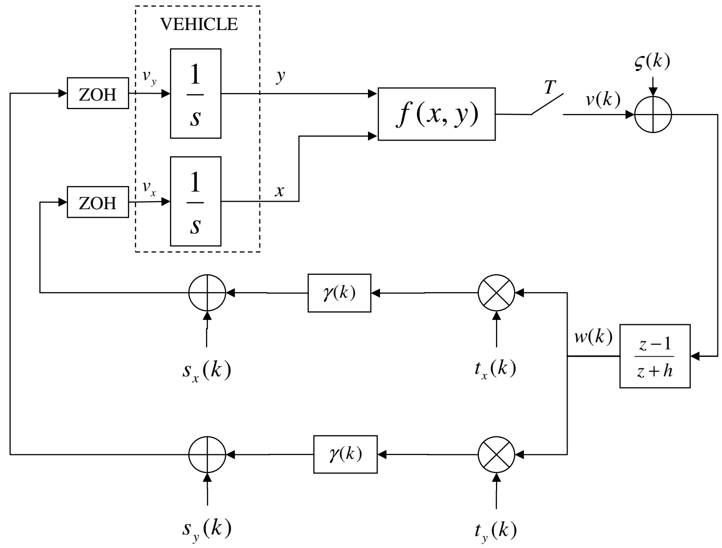

Figure 4.2: Extremum seeking scheme for the velocity driven vehicle

The following equations model the behavior of the described system:

$$v(k) = f^* + q_x(x(k) - x^*)^2 + q_y(y(k) - y^*)^2 \tag{4.61}$$

$$x(k) = (1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}\{\frac{1}{s^2}\}|_{t=kT}\}[\xi(k) + s_x(k)] \tag{4.62}$$

$$y(k) = (1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}\{\frac{1}{s^2}\}|_{t=kT}\}[\eta(k) + s_y(k)] \tag{4.63}$$

$$s_x(k) = \alpha(k+1)\cos(k+1)\omega - \alpha(k)\cos k\omega \tag{4.64}$$

$$s_y(k) = \alpha(k+1)\sin(k+1)\omega - \alpha(k)\sin k\omega \tag{4.65}$$

$$\xi(k) = -\gamma(k)t_x(k)w(k) \tag{4.66}$$

$$\eta(k) = -\gamma(k)t_y(k)w(k) \tag{4.67}$$

$$w(k) = M(z)[v(k) + \zeta(k)] \tag{4.68}$$

$$t_x(k) = \beta(k)\cos(\omega k - \varphi) \tag{4.69}$$

$$t_y(k) = \beta(k)\sin(\omega k - \varphi) \tag{4.70}$$

where $\zeta(k)$ is the measurement noise, $M(z) = \frac{z-1}{z+h}$, $x(k)$ and $y(k)$ are coordinates of the vehicle in discrete time.

The additive sinusoidal signals $s_x(k)$ and $s_y(k)$ can easily be mapped to the vehicle output, when we simply obtain

$$s_x^*(k) = T \sum_{j=1}^{k-1} s_x(j) = T\alpha(k)\cos\omega k, \tag{4.71}$$

$$s_y^*(k) = T \sum_{j=1}^{k-1} s_y(j) = T\alpha(k)\sin\omega k. \tag{4.72}$$

Similarly as in the 1D case we define the tracking error as:

$$\tilde{x}(k) = x^* - x(k) + s_x^*(k) \tag{4.73}$$

$$\tilde{y}(k) = y^* - y(k) + s_y^*(k) \tag{4.74}$$

and obtain the following compact vector-matrix representation:

$$\tilde{Z}(k+1) = \tilde{Z}(k) + \varepsilon(k)C(k)w(k) \tag{4.75}$$

where $\tilde{Z}(k) = [\tilde{x}(k),\ \tilde{y}(k)]^T$, $\varepsilon(k) = T\gamma(k)\beta(k)$ and $C(k) = [\cos(\omega k - \varphi),\ \sin(\omega k - \varphi)]^T$.

In the sequel, we will assume that the assumptions (A.4.1-6) are satisfied. We will analyze the scheme from Fig. 4.2 under the above assumptions term by term, following the decomposition introduced in the proof of the Theorem 4.1. First, we focus on the essential terms allowing an adequate approximation of the gradient of the function $f(x, y)$, and, consequently, convergence to its minimum. We substitute (4.73) and (4.74) in (4.61) and extract the linear part of $v(k)$ given by $-2[q_x s_x^*(k)\tilde{x}(k) + q_y s_y^*(k)\tilde{y}(k)]$, and concentrate on the corresponding part of the right hand side of (4.75), which is given by

$$L(k) = \varepsilon(k)C(k)S(z,k)[\tilde{Z}(k)], \tag{4.76}$$

where $S(z,k)[\tilde{Z}(k)] = q_x M(z)[s_x^*(k)\tilde{x}(k)] + q_y M(z)[s_y^*(k)\tilde{y}(k)]$. The vector $L(k)$ can be analyzed element by element. For the first element, one can obtain, using Lemmas 4.1-3, that

$$\begin{aligned}
2c(\omega k)M(z)[\sin \omega k \alpha(k)\tilde{y}(k)] &= s(2\omega k)\mathrm{Re}\{M(e^{j\omega}z)[\alpha(k)\tilde{y}(k)]\} \\
&+ c(2\omega k)\mathrm{Im}\{M(e^{j\omega}z)[\alpha(k)\tilde{y}(k)]\} \\
&- \mathrm{Im}\{e^{j\varphi}M(e^{j\omega}z)[\alpha(k)\tilde{y}(k)]\}, \qquad (4.77)
\end{aligned}$$

where $s(2\omega k) = \sin(2\omega k - \varphi)$ and $c(2\omega k) = \cos(2\omega k - \varphi)$. Other elements of the matrix $L(k)$ can be treated similarly. Therefore, for the final form for $L(k)$ we get

$$L(k) = \varepsilon(k)[-A(k) + B_1(k) + B_2(k)]1_2, \qquad (4.78)$$

where $1_2 = [1,\ 1]^T$,

$$A(k) = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\{e^{j\varphi}m_\alpha(k)\} & \mathrm{Im}\{e^{j\varphi}n_\alpha(k)\} \\ -\mathrm{Im}\{e^{j\varphi}m_\alpha(k)\} & \mathrm{Re}\{e^{j\varphi}n_\alpha(k)\} \end{bmatrix}$$

$$B_1(k) = \frac{1}{2}c(2\omega k)\begin{bmatrix} \mathrm{Re}\{m_\alpha(k)\} & \mathrm{Im}\{n_\alpha(k)\} \\ \mathrm{Im}\{m_\alpha(k)\} & -\mathrm{Re}\{n_\alpha(k)\} \end{bmatrix}$$

$$B_2(k) = \frac{1}{2}s(2\omega k)\begin{bmatrix} -\mathrm{Im}\{m_\alpha(k)\} & \mathrm{Re}\{n_\alpha(k)\} \\ \mathrm{Re}\{m_\alpha(k)\} & \mathrm{Im}\{n_\alpha(k)\} \end{bmatrix},$$

$m_\alpha(k) = M(e^{j\omega}z)[\alpha(k)\tilde{x}(k)]$ and $n_\alpha(k) = M(e^{j\omega}z)[\alpha(k)\tilde{y}(k)]$.

Following methodologically the proof of the Theorem 4.1, we decompose the terms with $\alpha(k)\tilde{x}(k)$ and $\alpha(k)\tilde{y}(k)$ as inputs in the following way:

$$\varepsilon(k)\mathrm{Re}\{e^{j\varphi}M(e^{j\omega}z)[\alpha(k)\tilde{x}(k)]\} = \rho(k)\mathrm{Re}\{e^{j\varphi}M(e^{j\omega}z)[\tilde{x}(k)]\} + \varepsilon(k)\delta l(k) \qquad (4.79)$$

where $\rho(k) = \varepsilon(k)\alpha(k)$. Using the fact that $M(z)$ is asymptotically stable, we can derive that

$$\delta l(k) = [\alpha(k-1) - \alpha(k)]y_1(k). \qquad (4.80)$$

where $y_1(k)$ is the output of a stable linear time varying system with $\tilde{x}(k)$ as input. Analogous conclusions can be derived for $\varepsilon(k)\mathrm{Im}\{e^{j\varphi}M(e^{j\omega}z)[\alpha(k)\tilde{x}(k)]\}$, and for the case when

$\tilde{x}(k)$ is replaced by $\tilde{y}(k)$.

The crucial point is defining the matrix

$$K = \begin{bmatrix} \cos(\varphi+\psi) & \sin(\varphi+\psi) \\ -\sin(\varphi+\psi) & \cos(\varphi+\psi) \end{bmatrix} |M(e^{j\omega})|. \tag{4.81}$$

where $\psi = \text{Arg}\{M(e^{j\omega})\}$. Notice that we also have that $\kappa_1 = \sum_{j=0}^{\infty} l_R(j)$ and $\kappa_2 = \sum_{j=0}^{\infty} l_I(j)$, where $l_R(j)$ and $l_I(j)$ are the impulse responses of $\text{Re}\{e^{j\varphi}M(e^{j\omega}z)\}$ and $\text{Im}\{e^{j\varphi}$ $M(e^{j\omega}z)\}$, respectively.

It is possible to demonstrate that the first term in (4.79) can be written as

$$\text{Re}\{e^{j\varphi}M(e^{j\omega}z)[\tilde{x}(k)]\} = -\kappa_1\tilde{x}(k) + \delta^x_{\kappa_1}(k) \tag{4.82}$$

and

$$\text{Im}\{e^{j\varphi}M(e^{j\omega}z)[\tilde{x}(k)]\} = -\kappa_2\tilde{x}(k) + \delta^x_{\kappa_2}(k), \tag{4.83}$$

where

$$\delta^x_{\kappa_1}(k) = \rho(k-1)y_2(k) + \varepsilon(k-1)y_3(k) \tag{4.84}$$

and

$$\delta^x_{\kappa_2}(k) = \rho(k-1)y_4(k) + \varepsilon(k-1)y_5(k). \tag{4.85}$$

$y_2(k), y_3(k), y_4(k)$ and $y_5(k)$ represent outputs of asymptotically stable linear time varying systems with $\tilde{x}(k)$ and $\tilde{y}(k)$ as inputs. The same reasoning is applicable to the similar terms in (4.78) depending on $\tilde{y}(k)$.

Matrices $B_1(k)$ and $B_2(k)$ can be treated term by term. For example, we can show that

$$\text{Re}\{M(e^{j\omega}z)[\alpha(k)\tilde{x}(k)]\} = \alpha(k)\kappa_1\tilde{x}(k) + \sigma_R(k) \tag{4.86}$$

or

$$\text{Im}\{M(e^{j\omega}z)[\alpha(k)\tilde{x}(k)]\} = \alpha(k)\kappa_2\tilde{x}(k) + \sigma_I(k) \tag{4.87}$$

where $\sum_{k=1}^{\infty}\sigma_R(k) < \infty$ and $\sum_{k=1}^{\infty}\sigma_I(k) < \infty$. Then, one can demonstrate that $\sigma_R(k) = \rho(k-1)y_6(k)+\varepsilon(k-1)y_7(k)$ and $\sigma_I(k) = \rho(k-1)y_8(k)+\varepsilon(k-1)y_9(k)$ where $y_6(k), y_7(k), y_8(k)$

and $y_9(k)$ are outputs of asymptotically stable LTV systems with $\tilde{Z}(k)$ as input.

Finally, after substituting the obtained expressions back into (4.75), we obtain

$$\tilde{Z}(k+1) = [I - K\rho(k) + \Gamma(k)]\tilde{Z}(k) + \Pi(k) + \Phi(k) + U(k) \tag{4.88}$$

where $\Gamma(k)$ is a matrix sequence having the form $[C_1 s(2\omega k) + C_2 c(2\omega k)]\varepsilon(k)$, where C_1 and C_2 are constant matrices. $\Pi(k)$ is a matrix sequence containing the terms $\delta l(k), \delta^x_{\kappa_1}(k), \delta^x_{\kappa_2}(k)$, etc., described earlier and analyzed in detail in Section 4.2, while

$$\Phi(k) = \varepsilon(k)C(k)M(z)[\tilde{x}(k)^2 + \tilde{y}(k)^2] \tag{4.89}$$

and $U(k)$ is the "external" input term

$$U(k) = \varepsilon(k)C(k)M(z)[f^* + s^*_x(k)^2 + s^*_y(k)^2 + \zeta(k)] \tag{4.90}$$

We first realize the crucial fact that $K > 0$ if and only if $\cos(\varphi + \psi) > 0$, that is

$$-\frac{\pi}{2} < \varphi + \psi < \frac{\pi}{2} \tag{4.91}$$

(notice also that ψ is close to $\frac{\pi}{2}$ for small values of ω). Furthermore, it is possible to show (using the arguments exposed in the proof of the Theorem 4.1) that

$$|\sum_{k=1}^{\infty}\Gamma(k)| = \sum_{j=1}^{\frac{N}{2}} B_j \sum_{k=0}^{\infty} |\varepsilon(j+kN) - \varepsilon(j + \frac{N}{2} + kN)| < \infty \tag{4.92}$$

where B_j, $j = 1, \ldots, \frac{N}{2}$ are constant matrices with positive elements, N is the integer period of $[C_1 s(2\omega k) + C_2 c(2\omega k)]$. Therefore, for the recursion $\bar{Z}(k+1) = [I - K\rho(k) + \Gamma(k)]\bar{Z}(k)$ (which represents a part of (4.88)) we have

$$\|\bar{Z}(k)\| \leq c_0 \exp\{-c_1 \sum_{j=1}^{\infty} k\rho(j)\} \tag{4.93}$$

where $c_0, c_1 > 0$, implying $\lim_{k\to\infty} \|\bar{Z}(k)\| = 0$, having in mind (A.4.4). Also, similarly as in the proof of the Theorem 4.1, we can conclude that $\sum_{j=1}^{\infty} \Phi(j) < \infty$. Since $\sum_{k=1}^{\infty} \rho(k) = \infty$, by applying Kronecker's lemma, we conclude that $\tilde{Z}(k) \to 0$ almost surely (a.s) if $\sum_{k=1}^{\infty} U(k) < \infty$ (a.s.) which has been already proved in Theorem 4.1. Therefore, we proved the following theorem:

Theorem 4.2 Let the assumptions (A.4.1-6) and (4.91) be satisfied. Then, for the scheme from Fig. 4.2, $x(k)$ converges to x^* and $y(k)$ converges to y^* almost surely and in the mean square sense under the condition that $\sup_k(||\tilde{Z}(k)||) < B$ (a.s.), $0 < B < \infty$.

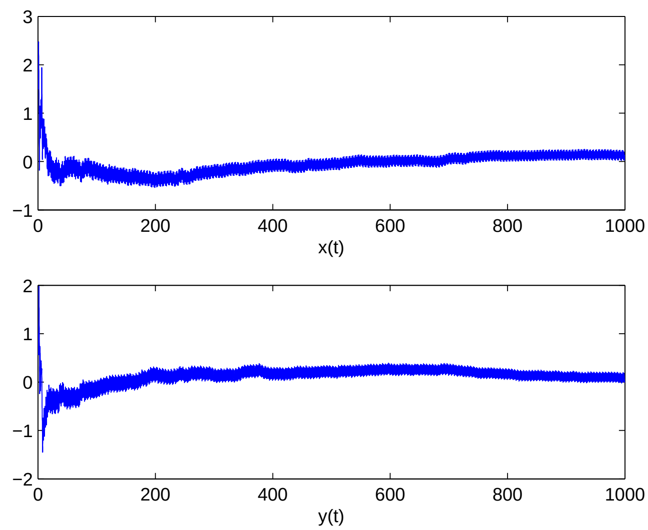

Figure 4.3: Velocity driven vehicle coordinates

Example 4.1 In this example we will apply the described ES algorithm to the adaptive state estimation problem described in the Subsection 4.3.2. We assume the following model for the discrete-time Kalman state estimator: $F = \begin{bmatrix} 0.5 & -0.1 \\ 0.2 & 0.2 \end{bmatrix}$, $G = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$, $H = [0\ 1]$, where F is the system matrix, G is the input matrix, H is the output matrix, input noise covariance matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the measurement noise variance depends on the coordinates of the vehicle x and y as the quadratic function $R(x, y) = 0.5 + 5x^2 + 5y^2$. The goal is to position the vehicle (modeled as the single integrator) at the minimum variance point $(0, 0)$, without the knowledge of the function $R(x, y)$. According to the above discussion, we can apply the scheme in Fig. 4.2, using estimator's squared residuals as the criterion function which is to be minimized. We set the parameters in the scheme to be $h = 0.07$, $\omega = 0.6\pi$,

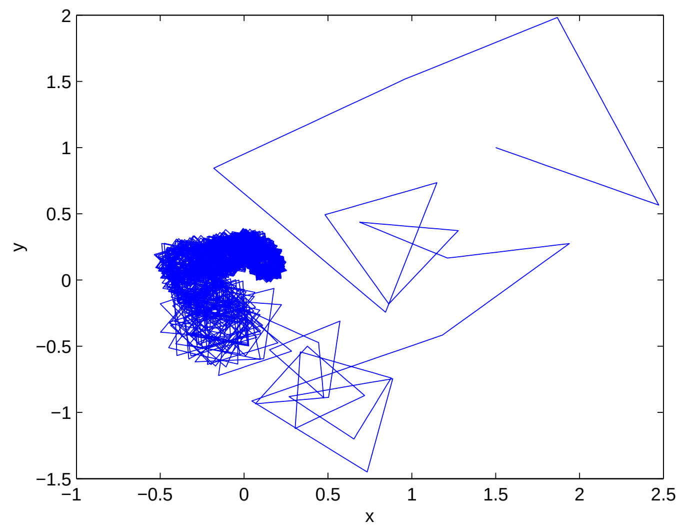

Figure 4.4: Velocity driven vehicle trajectory

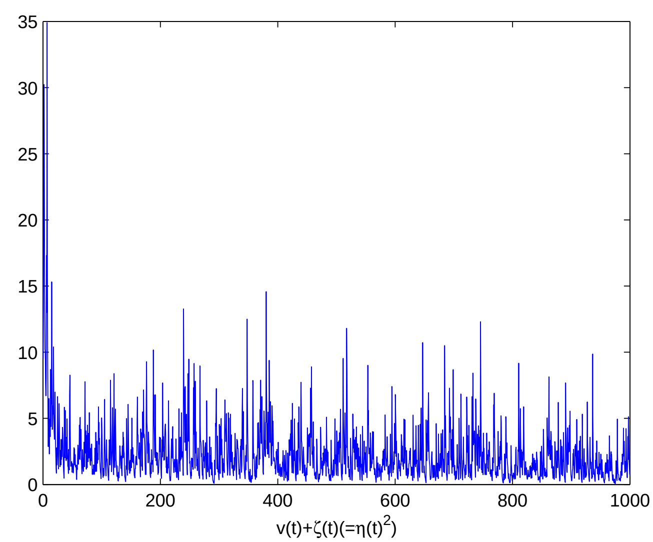

Figure 4.5: Noisy criterion function measurements (Kalman filter squared residuals)

$\alpha(k) = \frac{1}{k^{0.25}}$, $\epsilon(k) = \frac{1}{k^{0.75}}$, $\varphi_x = \varphi_y = 0$, which satisfy the convergence conditions. The coordinates $x(t)$ and $y(t)$ are shown in Fig. 4.3, for the initial conditions $x(0) = 1.5$ and $y(0) = 1$. The trajectory of the vehicle is shown in Fig. 4.4. The exact convergence to the optimal point is achieved in spite of the fact that the variance of the noise is time-varying and very large when the vehicle is far from the optimal point (since it depends quadratically on the vehicle coordinates). This is illustrated in Fig. 4.5, where the noisy measurements (squared residuals), as a function of time, are shown.

4.5 Force Actuated Vehicles

In this section we are going to present a modified ES scheme with discrete-time control for force actuated point mass models based on double integration in the analog part of the system. Both the analog vehicle model and the digital ES scheme are presented in Fig. 4.6. One can observe that the control scheme in Fig. 4.6 differs from the one in Fig. 4.2

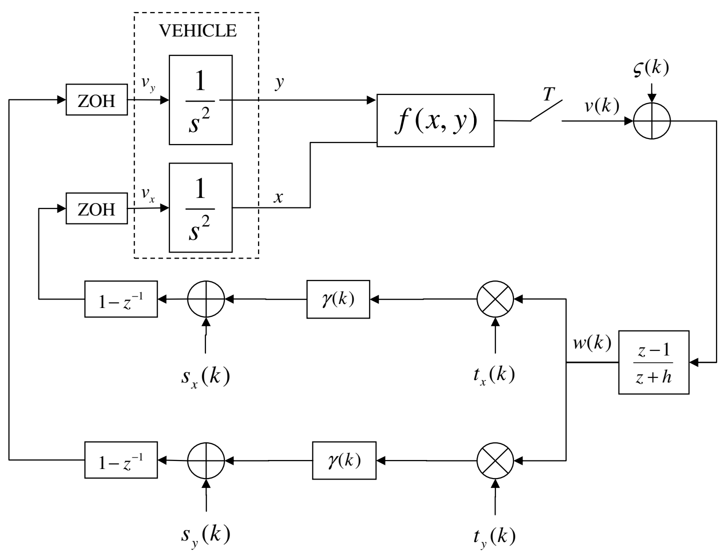

Figure 4.6: Extremum seeking scheme for the force actuated vehicle

by the introduction of the ideal discrete time differentiators $1 - z^{-1}$. Compare this with [115], where a purely analog scheme is considered, and where phase-lead compensators are introduced in order to recover some of the phase in the feedback loop lost due to the addition of the second integrator. The equations modelling the behavior of the scheme are similar to the ones for the scheme in Fig. 4.2. The only difference is that we have now

$$x(k) = (1 - z^{-1})^2 \mathcal{Z}\{\mathcal{L}^{-1}\{\frac{1}{s^3}\}|_{t=kT}\}[\xi(k) + s_x(k)] \tag{4.94}$$

$$y(k) = (1 - z^{-1})^2 \mathcal{Z}\{\mathcal{L}^{-1}\{\frac{1}{s^3}\}|_{t=kT}\}[\eta(k) + s_y(k)]. \tag{4.95}$$

The sinusoidal signals $s_x(k)$, $s_y(k)$, $t_x(k)$ and $t_y(k)$ are taken to have the same form as in the case of single integrators. First notice that we have

$$(1 - z^{-1})^2 \mathcal{Z}\{\mathcal{L}^{-1}\{\frac{1}{s^3}\}|_{t=kT}\} = \frac{1}{2}T^2(1 - z^{-1})\frac{z+1}{(z-1)^2} = \frac{1}{2}T^2(1 + z^{-1})\frac{1}{z-1}. \tag{4.96}$$

Consequently, the sinusoidal signals $s_x(k)$ and $s_y(k)$ become

$$\begin{aligned} s_x^*(k) &= \frac{1}{2}T^2[\alpha(k)\cos\omega k + \alpha(k-1)\cos\omega(k-1)] \\ s_y^*(k) &= \frac{1}{2}T^2[\alpha(k)\sin\omega k + \alpha(k-1)\sin\omega(k-1)] \end{aligned} \tag{4.97}$$

when mapped to the discrete-time outputs of the vehicle, *i.e.*, to the inputs of the nonlinearity. Consequently, we again have the relations (4.73) and (4.74), and the following new system model:

$$\tilde{Z}(k+1) = \tilde{Z}(k) + N(z)[\varepsilon(k)C(k)w(k)], \tag{4.98}$$

where $N(z) = \frac{1}{2}T(1 + z^{-1})$.

Stability of this hybrid scheme can be studied using the same methodology as in Section 4.3 and Section 4.4. The main point is again the influence of the linear term $L(k)$ (see (4.78)), which becomes now

$$L(k) = N(z)[\varepsilon(k)C(k)S(z,k)[\tilde{Z}(k)]], \tag{4.99}$$

where $S(z,k)$ has the same form as in (4.78). Now, we have a more complicated case than in Section 4.4. We have, for example,

$$N(z)[c(\omega k)\beta(k)M(z)[s_x^*(k)\tilde{x}(k)]] = \mathrm{Re}\{e^{j(\omega k - \varphi)}N(e^{j\omega}z)[h(k)]\} \tag{4.100}$$

where $h(k) = \mathrm{Re}\{e^{j(\omega k-\tau)}M(e^{j\omega}z)[\alpha(k)\tilde{x}(k)]\}$, τ is either 0 or ω (according to (4.97)). However, our focus is on the term analogous to $N(z)[\varepsilon(k)A(k)]$, where $A(k)$ is defined in (4.78). Noticing that, simply, $\cos(\omega k-\varphi)[\cos\omega k+\cos\omega(k-1)] = \cos\frac{\omega}{2}[\cos(2\omega k-\varphi-\frac{\omega}{2})+\cos(\varphi-\frac{\omega}{2})]$, we select the terms not depending on ωk and obtain the matrix $A'(k)$ analogous to $A(k)$ (the difference is in the expressions for $s_x^*(k)$ and $s_y^*(k)$) and K' analogous to K:

$$\frac{K'}{|M(e^{j\omega})|} = \cos\frac{\omega}{2}\begin{bmatrix} \cos(\varphi+\psi+\frac{\omega}{2}) & \sin(\varphi+\psi+\frac{\omega}{2}) \\ -\sin(\varphi+\psi+\frac{\omega}{2}) & \cos(\varphi+\psi+\frac{\omega}{2}) \end{bmatrix} \tag{4.101}$$

According to the arguments used in Section 4.4, we conclude that the system is asymptotically stable under the conditions (A.4.1-6) and if

$$-\frac{\pi}{2} < \varphi + \frac{\omega}{2} + \psi < \frac{\pi}{2}. \tag{4.102}$$

The rest of the stability analysis can be conducted following the proof of Theorem 4.2.

Example 4.2 In this example, we present the simulation results for the force actuated vehicle seeking the minimum of the (unknown) function $J = f(x,y) = 1+\frac{1}{2}(x+1)^2+\frac{1}{2}(y+0.5)^2$. The measured output of this nonlinear map is corrupted with the white noise with variance $\sigma^2 = 0.4$. The other parameters of the scheme in Fig. 4.6 are set to be $h = 0.07$, $\omega = 0.6\pi$, $\alpha(k) = \frac{1}{k^{0.25}}$, $\epsilon(k) = \frac{1}{k^{0.75}}$, $\varphi_x = \pi$, $\varphi_y = -\pi$, satisfying the conditions (A.4.2-6) and (4.102). The coordinates of the vehicle $x(t)$ and $y(t)$ are shown in Fig. 4.7 and the trajectory of the vehicle is shown in Fig. 4.8, for the initial position $x(0) = 1$, $y(0) = 0.6$. Both coordinates converge exactly to the minimal point $(-1, -0.5)$, in spite of the presence of the strong noise which can be seen in Fig. 4.9, where the noisy measurements of the criterion function J are depicted.

4.6 Nonholonomic Vehicles

In this section, we consider the unicycle model of a mobile robot with a sensor which is collocated at the center of the vehicle (the case when the sensor is located at some distance r from the center of the vehicle can be treated as in [114]). The equations of motion of the

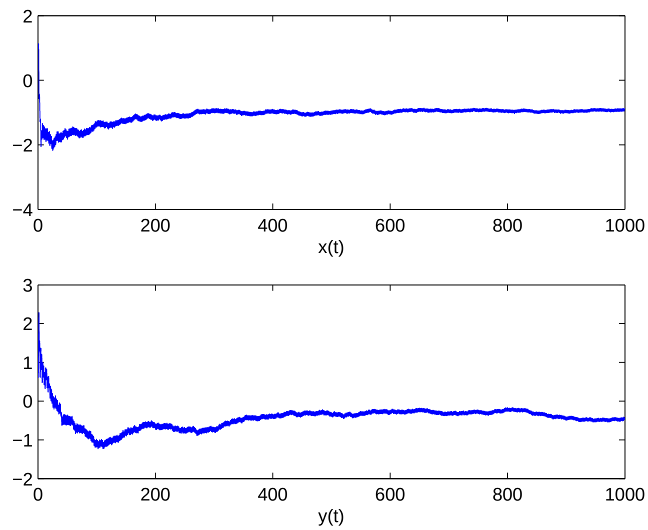

Figure 4.7: Force actuated vehicle coordinates

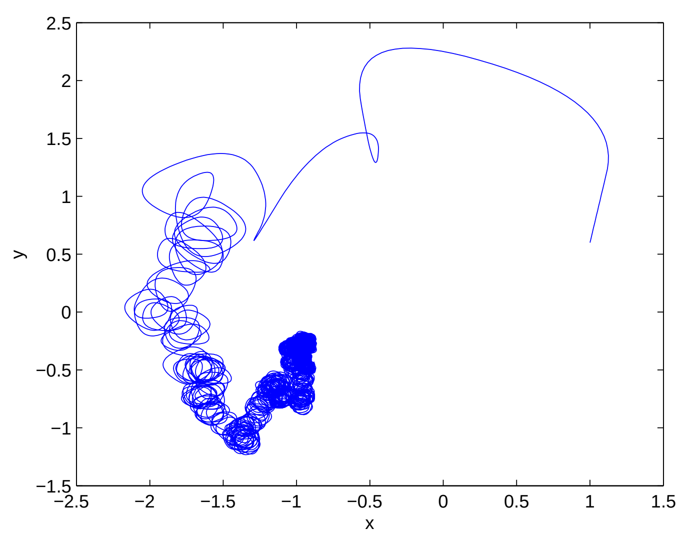

Figure 4.8: Force actuated vehicle trajectory

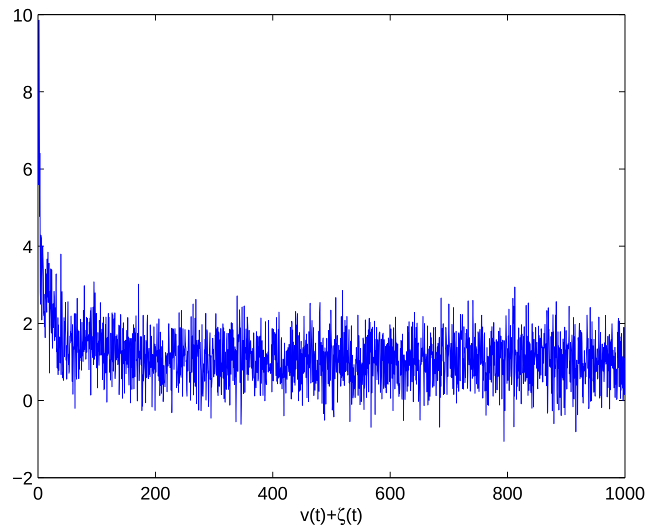

Figure 4.9: Noisy measurements of the criterion function

vehicle/sensor are

$$\dot{x} = v\cos\theta, \qquad \dot{y} = v\sin\theta, \qquad \dot{\theta} = \Omega_0, \tag{4.103}$$

where (x, y) are the coordinates of the center of the vehicle, θ its orientation and v, Ω_0 are the forward and angular velocity inputs. Our ES algorithm will be tuning only the forward velocity input v, keeping the angular velocity Ω_0 constant. The whole scheme containing both the vehicle and the discrete-time control algorithm is represented in Fig. 4.10. Our

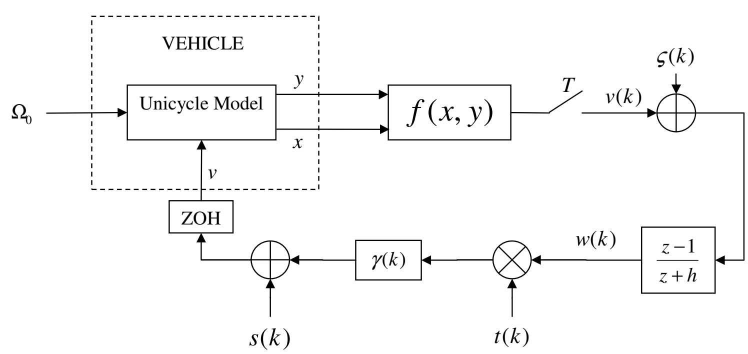

Figure 4.10: Extremum seeking scheme for the unicycle vehicle model

immediate concern is the mapping of the system variables induced by the unicycle in discrete time. It is straightforward to show that we have

$$x(k) = 2\frac{\sin\frac{\omega_0}{2}}{\Omega_0}\sum_{j=0}^{k-1} v(j)\cos\omega_0(j+\frac{1}{2})$$

$$y(k) = 2\frac{\sin\frac{\omega_0}{2}}{\Omega_0}\sum_{j=0}^{k-1} v(j)\sin\omega_0(j+\frac{1}{2}) \tag{4.104}$$

where $\omega_0 = T\Omega_0$. Assuming that the additive signal is given by $s(k) = \alpha(k)\sin\omega(k+\frac{1}{2})$, we obtain that its maps to the nonlinearity inputs are

$$s_x^*(k) = \frac{\sin\frac{\omega_0}{2}}{\Omega_0}\sum_{j=0}^{k-1}\alpha(j)[\cos(\omega+\omega_0)(j+\frac{1}{2}) + \cos(\omega-\omega_0)(j+\frac{1}{2})] \tag{4.105}$$

$$s_y^*(k) = \frac{\sin\frac{\omega_0}{2}}{\Omega_0}\sum_{j=0}^{k-1}\alpha(j)[\sin(\omega+\omega_0)(j+\frac{1}{2}) + \sin(\omega-\omega_0)(j+\frac{1}{2})] \tag{4.106}$$

Assuming that k is large enough we obtain, after convenient trigonometric transformations, that

$$s_x^*(k) \approx k_x\alpha(k)\cos\omega k\cos\omega_0 k \tag{4.107}$$

$$s_y^*(k) \approx k_y\alpha(k)\cos\omega k\sin\omega_0 k \tag{4.108}$$

where k_x and k_y are appropriately defined constants. Furthermore, we define

$$\tilde{x}(k) = x^* - x(k) + s_x^*(k) \tag{4.109}$$

$$\tilde{y}(k) = y^* - y(k) + s_y^*(k) \tag{4.110}$$

and obtain

$$\tilde{Z}(k+1) = \tilde{Z}(k) + 2\frac{\sin\frac{\omega_0}{2}}{\Omega_0}\cos\omega_0(k+\frac{1}{2})\varepsilon(k)c(\omega k)w(k) \tag{4.111}$$

where $c(\omega k) = \cos(\omega k - \varphi)$.

Using the same methodology as in Section 4.4 we can analyze the scheme represented by the equations (4.107)-(4.111). First, we extract the linear part of the second term on the right hand side of (4.111). We obtain, similarly as in Section 4.4, that

$$L(k) = k_3 \cos \omega_0 (k + \frac{1}{2}) \cos(\omega k - \varphi) S(z,k)[\tilde{Z}(k)] \tag{4.112}$$

where $S(z,k)[\tilde{Z}(k)] = q_x M(z)[s_x^*(k)\tilde{x}(k)] + q_y M(z)[s_y^*(k)\tilde{y}(k)]$ and k_3 is an appropriate constant. In order to apply the methodology used in Section 4.3 and Section 4.4, we transform the products in (4.112) and (4.107) into sums using standard trigonometric transformations, and we apply Lemma 4.1 or (4.77) in the following way

$$\begin{aligned} & 2\cos((\omega \pm \omega_0)k - \varphi + \frac{\omega_0}{2}) M(z)[\sin((\omega \pm \omega_0)k)\alpha(k)\tilde{y}(k)] = \\ = \; & \sin(2(\omega \pm \omega_0)k - \varphi + \frac{\omega_0}{2}) \mathrm{Re}\{M(e^{j\omega}z)[\alpha(k)\tilde{y}(k)]\} \\ + \; & \cos(2(\omega \pm \omega_0)k - \varphi + \frac{\omega_0}{2}) \mathrm{Im}\{M(e^{j\omega}z)[\alpha(k)\tilde{y}(k)]\} \\ - \; & \mathrm{Im}\{e^{j(\varphi - \frac{\omega_0}{2})} M(e^{j\omega}z)[\alpha(k)\tilde{y}(k)]\}, \end{aligned} \tag{4.113}$$

Notice that the above relation shows the influence of the y-channel to the x-channel, *i.e.* it defines the weight of $\tilde{y}$ in the linear part of the relation for $\tilde{x}$, similarly as in the single integrator case. After a similar treatment of all the terms appearing in $L(k)$ defined by (4.112), we obtain an expression analogous to (4.78), consisting of four terms containing sine and cosine functions with the frequencies $2(\omega \pm \omega_0)$ as multipliers, four additional terms containing sine and cosine functions with the frequencies $2\omega k$ and $2\omega_0 k$ as multipliers, together with the main term analogous to $A(k)$ in (4.78) not containing any sinusoidal component. The last term is again crucial for stability of the scheme. Following (4.113), one can derive that in the unicycle case we have

$$A''(k) = \begin{bmatrix} \mathrm{Re}\{e^{j\varphi'} m_\alpha(k)\} & \mathrm{Im}\{e^{j\varphi'} n_\alpha(k)\} \\ -\mathrm{Im}\{e^{j\varphi'} m_\alpha(k)\} & \mathrm{Re}\{e^{j\varphi'} n_\alpha(k)\} \end{bmatrix}$$

where $\varphi' = \varphi - \frac{\omega_0}{2}$. For the matrix K'' analogous to K we get

$$K'' = \begin{bmatrix} \cos(\varphi' + \psi) & \sin(\varphi' + \psi) \\ -\sin(\varphi' + \psi) & \cos(\varphi' + \psi) \end{bmatrix} |M(e^{j\omega}\}| \tag{4.114}$$

having in mind that the scheme in Fig. 4.10 contains the same processing blocks as the scheme in Fig. 4.1. The above matrix is positive definite for

$$-\frac{\pi}{2} < \varphi' + \psi < \frac{\pi}{2} \tag{4.115}$$

which does not impose any additional problem in *a priori* selection of φ in the multiplying signal. Therefore, the scheme in Fig. 4.10 is stable under the conditions (A.4.1-6) plus condition (4.115).

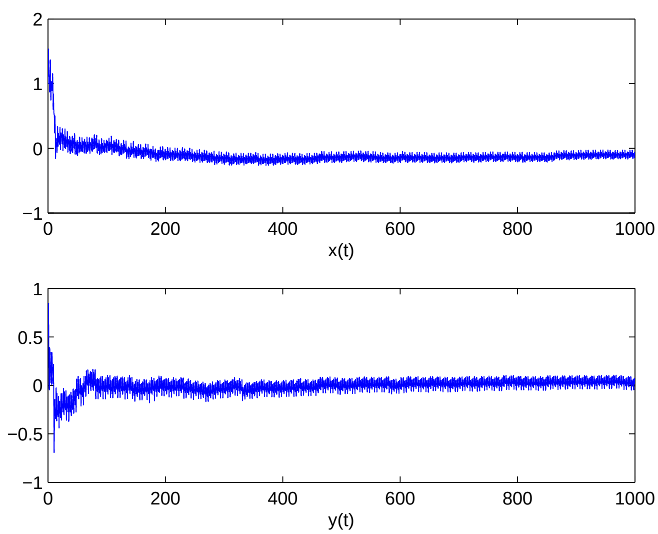

Figure 4.11: Nonholonomic vehicle coordinates

Example 4.3 In this example we illustrate the simulation results for optimal positioning of the Kalman estimator, as described in the Example 4.1, with the single integrators replaced by the unicycle. The Kalman estimator parameters are assumed to be the same as in the Example 4.1. We apply the scheme in Fig. 4.10 the same way as in the case of velocity actuated vehicle (by taking the estimator's squared residuals as the criterion function to be minimized). The parameters of the scheme are set to be $h = 0.07$, $\omega = 0.6\pi$, $\alpha(k) = \frac{1}{k^{0.25}}$, $\epsilon(k) = \frac{1}{k^{0.75}}$, $\varphi = 0$, $\Omega_0 = \omega/5$, which satisfy the conditions (A.4.2-6), as well as the condition (4.115). The convergence of the coordinates $x(t)$ and $y(t)$ to the optimal point is illustrated in Fig. 4.11. The trajectory of the unicycle is shown in Fig. 4.12.

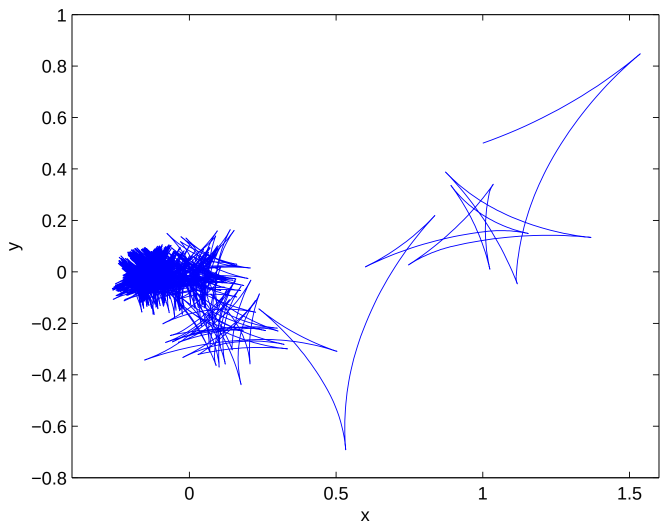

Figure 4.12: Nonholonomic vehicle trajectory

4.7 Multi-Target Extremum Seeking Using Global Utility Functions

This section is devoted to the problem of target assignment in multi-agent systems using multi-variable extremum seeking algorithm with specially designed global utility functions which capture the dependance among different, possibly conflicting, agents' objectives.

Assume we are faced with the problem of assigning N targets, defined by N minima of N unknown functions (assuming each function has exactly one minimum), to N agents, so that each agent is assigned with a different target. In particular, assume that each, out of N agents can measure N different (unknown) functions $f_i(x_i, y_i)$, $i = 1, ..., N$, x_i and y_i are agents coordinates, and the goal is to design an algorithm that will automatically lead the agents to a configuration in which they will cover the minima of all the functions f_i, $i = 1, ..., N$. Towards this goal, we can define global utility functions that depend on all the measured functions by all the agents, that would have exactly $N!$ minima corresponding to all the configurations in which each target is covered by exactly one agent. If by f_{ij} we denote the function f_i measured by the j-th agent, the simplest utility function satisfying the formulated condition is:

$$F(f_{11}, f_{12}, ..., f_{NN}) = m_1 f_{11} f_{12} ... f_{1N} + m_2 f_{21} f_{22} ... f_{2N} + ... + m_N f_{N1} f_{N2} ... f_{NN} \quad (4.116)$$

where $m_i > 0$, $i = 1, ..., N$ are weighting parameters which determine the target significance (the larger m_i is, the more important the target i is). Assuming that the values of the function f_i at minimum points are zero, the function (4.116) will have exactly $N!$ minima corresponding to all the configurations in which the agents has covered all the targets (minima of all the functions f_i). Hence, we can apply the multi-variable extremum seeking algorithm ([6]) in order to find the local extremum of the function (4.116). This extremum will correspond to the best configuration of the agents in which they cover all the targets. This final configuration is the closest one to the initial positions of all the agents, taking into account the weights of the targets m_i. The proposed algorithm, involving multi-variable extremum seeking, is shown in Fig. 4.13. Obviously, this scheme is centralized, since we

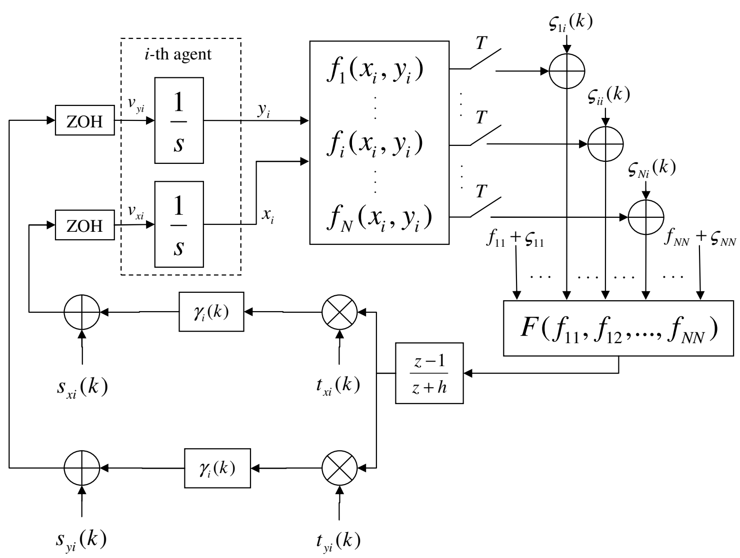

Figure 4.13: Multi-target extremum seeking using utility function F

assume that all the agents have access to all the measurements of the other agents. The signals $s_{xi}(k)$, $s_{yi}(k)$, $t_{xi}(k)$ and $t_{yi}(k)$ are in the same form as in the Section 4.4. Each agent applies a 2D ES scheme (such as the one shown in Fig. 4.2, assuming velocity driven vehicles), but with different frequency sinusoidal perturbation, which has to satisfy

$\omega_i + \omega_j \neq \omega_k$, for all $i, j, k = 1, ..., N$ ([6]). Therefore, the resulting scheme is a $2N$ dimensional ES scheme, which solves the problem of seeking the closest local extremum of the given utility function F, having in mind the assumption that all the agents have the access to this function. Similar schemes as in Sections 4.5 and 4.6 can be applied in the cases of force actuated vehicles or unicycles, respectively.

Another utility function that can be applied is in the following form ([2]):

$$F(f_{11}, f_{12}, ..., f_{NN}) = m_1(1 - e^{-\delta f_{11}})...(1 - e^{-\delta f_{1N}}) + ... + m_N(1 - e^{-\delta f_{N1}})...(1 - e^{-\delta f_{NN}}) \tag{4.117}$$

where $\delta \neq 0$ is a parameter which determines the level of utility dependence. This function involves the normalized deviations from the targets and, hence, is less sensitive to very large deviations ($1 - e^{-\delta f_{ij}} \to 1$ when $f_{ij} \to \infty$,) at the expense of slower convergence. This is desirable property in the proposed ES based algorithm, having in mind that for large deviations from the minima the algorithm easily blows up.

Remark 4.3 The important assumption in the proposed multi-agent scheme is that the values of the functions f_i at the minimum points are zero. If this is not the case and the minimal values are unknown, an adaptive strategy can be applied, in which the agents would, in each step, subtract a percentage of the final values of the functions f_i (which should be zero) and then initialize the proposed algorithm again until these values are close enough to zero.

Example 4.4 In this example we illustrate the proposed multi target extremum seeking algorithm from Fig. 4.13 for the case of two agents. Functions measured by agents, whose minima we are seeking are $f_{1i} = x_i^2 + y_i^2$, $i = 1, 2$ and $f_{2i} = (x_i - 1)^2 + (y_i - 1)^2$, $i = 1, 2$. The utility function (4.116) is applied, with $m_1 = 1$ and $m_2 = 1$. In Fig. 4.14, the trajectories of the two vehicles (modeled as single integrators) are shown, for the initial condition $x_1(0) = 0.6$, $y_1(0) = 0.6$, $x_2(0) = 0.4$ and $y_1(0) = 0.4$. It can be seen that both minima are covered by the agents, each one going to the closer one. In Fig 4.15, trajectories are shown for the initial conditions $x_1(0) = 1.2$, $y_1(0) = 1.1$, $x_2(0) = 0.7$ and $y_1(0) = 0.7$. In this case, minimum of f_1 is closer to both agents; hence both, agents aim at this target at beginning. The second one changes its target to the minimum of f_2 when the first one

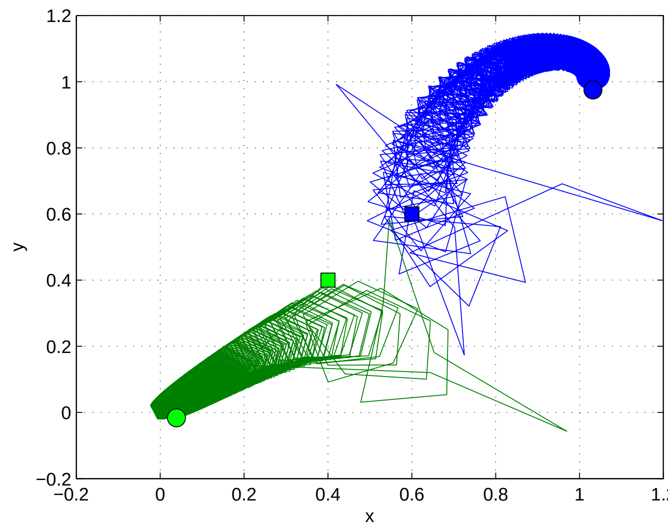

Figure 4.14: Two targets ES: trajectories of the vehicles

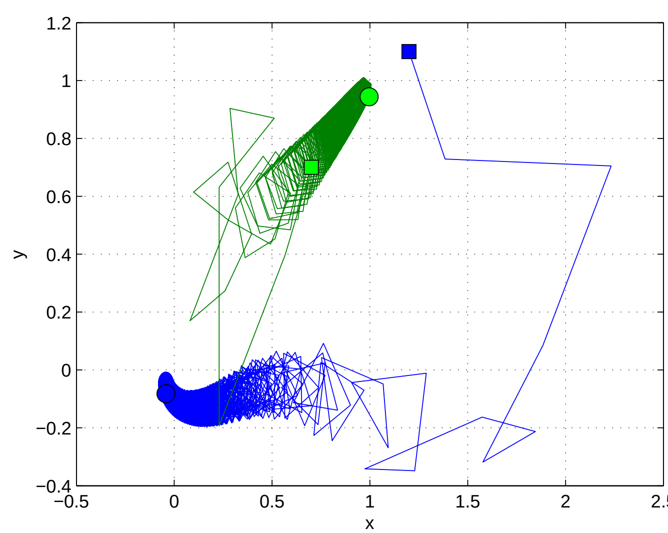

Figure 4.15: Two targets ES: trajectories of the vehicles

is much closer to the minimum of f_1.

Example 4.5 In this example the proposed multi-target algorithm is applied for the case of three agents. The measured functions are $f_{1i} = x_i^2 + y_i^2$, $i = 1, 2, 3$, $f_{2i} = (x_i - 1)^2 + (y_i - 1)^2$, $i = 1, 2, 3$ and $f_{3i} = (x_i + 1)^2 + (y_i + 1)^2$, $i = 1, 2, 3$. Hence, the targets are $(0, 0)$, $(1, 1)$ and $(-1, -1)$. The utility function (4.116) is applied, with $m_1 = 1$, $m_2 = 1$ and $m_3 = 1$. In this example, the additive measurement noise of small variance (0.1) is added to the agents' measurements. The trajectories of the agents are shown in Fig 4.16, for the initial conditions $x_1(0) = 1.4$, $y_1(0) = 0.8$, $x_2(0) = 0.25$, $y_2(0) = 0.2$, $x_3(0) = -0.5$ and $y_3(0) = -0.5$. Note

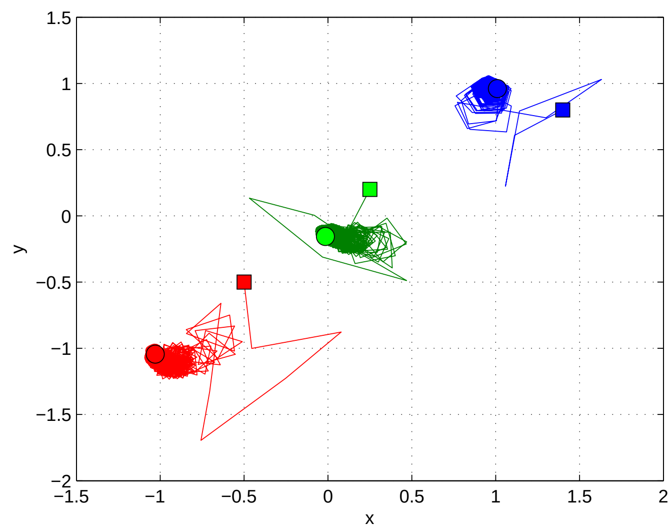

Figure 4.16: Three targets ES: trajectories of the vehicles

here that depending on the noise realization different agents can end up in different targets. In Fig 4.17, the trajectories are shown for the case of constant, non-vanishing integrator gains and amplitudes of the sinusoidal perturbations in the ES scheme. The benefit of the proposed time-varying scheme, capable of eliminating the measurement noise is obvious.

Example 4.6 Finally, this example illustrates the application of the utility function (4.117) for two agents and for the two cases of the weighting coefficients m_1 and m_2. Measured functions are the same as in the Example 4.4. In Fig 4.18 trajectories are shown for the initial conditions $x_1(0) = 0.5$, $y_1(0) = 0.5$, $x_2(0) = 1.2$ and $y_1(0) = 0.7$, and for the weighting coefficients $m_1 = 1$, $m_2 = 1$. In Fig 4.19 the trajectories are shown for the same

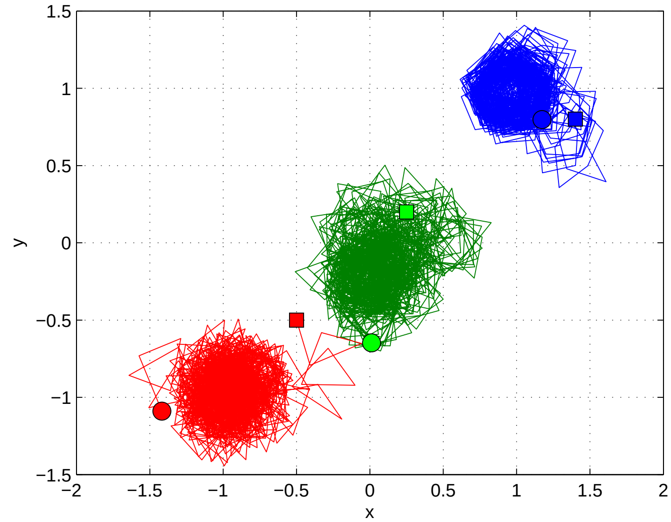

Figure 4.17: Three targets ES: trajectories of the vehicles

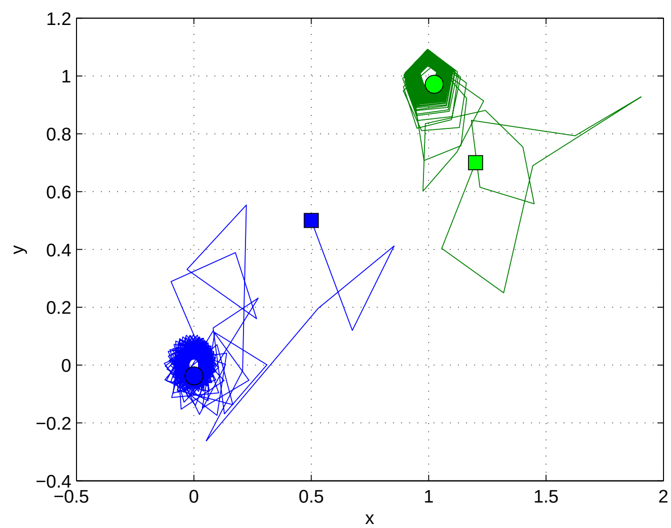

Figure 4.18: Two targets ES: trajectories of the vehicles

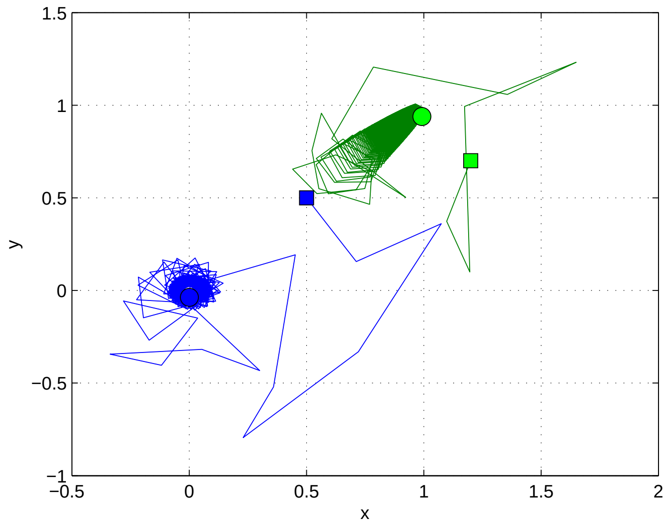

Figure 4.19: Two targets ES: trajectories of the vehicles

initial conditions but for different weights: $m_1 = 6$ and $m_2 = 1$. In this case, the first target has greater priority, so that the second agent (starting at the point $(1.2, 0.7)$)) also aims at the first target at the beginning, changing the trajectory towards the second target when the first one is close enough to the more important one.

Chapter 5

Conclusions and Future Directions

5.1 Thesis Summary

The primary focus of this thesis is on two crucial problems in multiple agent, networked control systems and mobile sensor networks. The first one is the problem of decomposition of complex/large-scale, networked systems into smaller, overlapping subsystems, formulating their local estimation and/or control laws and defining communication schemes (over unreliable communication channels) which would ensure stability, acceptable performance, robustness and scalability of the overall system. The second problem addressed in this thesis, which is the critical problem within mobile sensor networks, is the problem of searching positions for mobile nodes on which optimal sensing capabilities can be achieved.

Novel, consensus based state and parameter estimation schemes have been proposed, in both continuous-time and discrete-time. The algorithms are based on: a) overlapping system decomposition, b) implementation of local state or parameter estimators according to local resources, c) formulation of the inter-agent communication scheme based on the consensus algorithm, which provides the global state or parameter estimates to all the agents in the network. Stability and the asymptotic properties of the proposed algorithms have been analyzed. Also, conditions concerning network complexity have been derived for achieving asymptotic elimination of the measurement noise (when the number of agents go to infinity). For the state estimation scheme, a strategy for obtaining consensus gains based on the minimization of the total mean-square error is proposed. Properties and performance of the proposed schemes have been illustrated in several examples.

Several structures for multi-agent control based on a dynamic consensus strategy have been proposed. After formally defining the problem of multi-agent control with information structure constraints, two novel classes of overlapping decentralized control algorithms based on consensus are presented. In the first class, an agreement between the agents is required at the level of control inputs, while for the second class of algorithms, the agreement is required at the state estimation level. In this case, a control scheme based on state estimation with consensus, coupled with a globally optimal state feedback, is presented and analyzed. The proposed control algorithms have been illustrated by several examples which demonstrate their effectiveness. Also, the proposed consensus based control scheme has been applied to decentralized overlapping tracking control of a planar formation of UAVs. A comparison with the design methodology based on the expansion/contraction paradigm and the inclusion principle is given.

In order to address the problem of searching the optimal sensing positions for mobile sensors, new assumptions have been introduced into the extremum seeking algorithm with sinusoidal perturbation. It has been assumed that the integrator gain and the perturbation amplitude are time varying (decreasing in time with a proper rate) and that the output is corrupted with measurement noise. The convergence of the algorithm, with probability one and in the mean square sense, has been proved. The proposed one dimensional algorithm has been extended to two dimensional, hybrid schemes and directly applied to the optimal mobile sensor positioning, where the vehicles are modeled as single integrators, double integrators or unicycles. Also, a multi-target assignment problem, where multiple objectives need to be fulfilled by a number of agents has been addressed. An algorithm based on multivariable local extremum seeking of a suitably constructed global utility function has been proposed and analyzed. It has been shown how the utility function parameters and agents' initial conditions impact the trajectories and destinations of the agents. Several simulation studies illustrate the proposed algorithms.

5.2 Future Directions

One natural direction for future work is to extend the proposed extremum seeking based algorithms for the optimal sensor placement to the multi-objective, decentralized optimization scenarios, which are more desirable having in mind local constraints and communication unreliability of the mobile sensor networks. In particular, local utility functions can be designed for each agent whose local optimization can lead to an overall goal. A natural approach to these problems is the game theoretic approach where we treat each agent as a player in a dynamic game, so that we can use game theoretic (cooperative or non-cooperative) concepts to find particular strategies which correspond to global equilibriums.

In order to apply the proposed consensus based stochastic approximation algorithm to the problem of system identification and adaptive control (*e.g.* for the identifications of ARMA processes) the assumptions regarding local regression models need to be relaxed. An analyzes of the asymptotic behavior of the proposed scheme with this relaxation, which will lead to colored noise models, can be considered as future work. Also, in order to achieve faster convergence, matrix gains in local recursive algorithms (e.g. local least square algorithms) can be considered. Furthermore, a promising direction is to apply the proposed consensus based scheme to a case when the local agents are using errors-in-variables models ([18]) for the local identification. In this case, the agents would calculate input-output correlations according to their local models, and the consensus scheme can be applied in order to achieve agreement upon the correlation functions.

Another future direction is to analyze more rigorously connective stability of the proposed multi-agent control structures. In particular, a methodology based on vector Lyapunov functions can be used in order to find subsystem interconnection gains (gains in the consensus scheme) which would guarantee stability of the overall system.

Furthermore, the interaction between control and communication factors for the proposed networked large scale systems can be explored in more details. In particular, one can analyze other communication scenarios and architectures and introduce the parameters such as delays, channel capacity, quantization errors or particular communication protocols in the analysis of the proposed networked estimation or control algorithms.

Applications of the proposed decentralized control and estimation algorithms are numerous. Besides the proposed application to formations of UAVs, a promising one would be to extend it to the control of formations of aircraft in deep space. This deserves a particular attention since the absolute positions of the aircraft can not be measured; hence the agreement on the local estimates of the positions and the velocities among the surrounding aircraft (through the proposed consensus-based algorithms) is of crucial importance for accurate positioning.

Bibliography

[1] A. E. Abbas, *Multiattribute utility copulas*, To appear in *Operations Research.*

[2] A. E. Abbas and R. A. Howard, *Attribute dominance utility*, Decision Analysis **2** (2005), 185–206.

[3] B. Acikmese, F. Y. Hadaegh, D. P. Scharf, and S. R. Ploen, *Decentralized control and overlapping decomposition of mechanical systems*, IET Control Theory Appl. **1** (2007), 461–474.

[4] B. D. O. Anderson and J. B. Moore, *Optimal filtering*, Prentice Hall, New York, 1979.

[5] P. Antsaklis and J. Baillieul (eds.), *Proceedings of the ieee: Special issue on technology of networked control systems*, vol. 95, January 2007.

[6] K. B. Ariyur and M. Krstic, *Analysis and design of multivariable extremum seeking*, Proc. American Control Conference, 2002, pp. 2903–2853.

[7] K. B. Ariyur and M. Krstić, *Real time optimization by extremum seeking*, Wiley, Hoboken, NJ, 2003.

[8] Y. Bar-Shalom and X.R. Li, *Multitarget-multisensor tracking: Principles and techniques*, YBS Publishing, 1995.

[9] B. Baran, E. Kaszkurewicz, and A. Bhaya, *Parallel asynchronous team algorithms: convergence and performance analysis*, IEEE Trans. Parallel and Distributed Syst. **7** (1996), 677–688.

[10] R. Bellman, *Introduction to matrix analysis*, McGraw Hill, New York, 1960.

[11] C. Belta and V. Kumar, *Abstraction and control for groups of robots*, IEEE Transactions on Robotics **20** (2004), 865–875.

[12] A. Berman and R. J. Plemmons, *Nonnegative matrices in mathematical sciences*, Academic Press, New York, 1979.

[13] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and distributed computation: Numerical methods*, Prentice Hall, 1989.

[14] C. M. Bishop, *Neural networks for pattern recognition*, Clarendon Press, Oxford, 1995.

[15] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, *Convergence in multiagent coordination, consensus and flocking*, Proc. IEEE Conf. Decision and Control, 2005.

[16] V. Borkar and P. Varaiya, *Asymptotic agreement in distibuted estimation*, IEEE Trans. Autom. Control **27** (1982), 650–655.

[17] C. G. Cassandras and W. Li, *Sensor networks and cooperative control*, European Journal of Control **11** (2005), 436–463.

[18] H. F. Chen, *Stochastic approximation and its applications*, Kluwer Academic Publisher, 2003.

[19] X. B. Chen and S. S. Stanković, *Decomposition and decentralized control of systems with multi-overlapping structure*, Automatica **41** (2005), 1765–1772.

[20] J. Y. Choi, M. Krstić, K. B. Ariyur, and J. S. Lee, *Extremum seeking control for discrete-time systems*, IEEE Trans. Autom. Control **47** (2002), 318– 323.

[21] K. L. Chung, *On stochastic approximation method*, Annals of Math. Statistics **25** (1954), 463–483.

[22] D. Cvetković, M. Doob, and H. Sachs, *Spectra of graphs*, Academic Press, 1979.

[23] A. Fax and R.M Murray, *Information flow and cooperative control of vehicle formations*, IEEE Trans. Automat. Contr. **49** (2004), 1465–1476.

[24] F. R. Gantmacher, *Theory of matrices*, Nauka, Moscow, 1966.

[25] V. Gazi and K. M. Pasino, *Stability analysis of swarms*, IEEE Trans. Automat. Contr. **48** (2003), 692–697.

[26] H. Gharavi and S. Kumar (eds.), *Proceedings of the ieee: Special isssue on sensor networks and applications*, vol. 91, August 2003.

[27] C. Godsil and G. Royle, *Algebraic graph theory*, Springer Verlag, New York, 2001.

[28] R. M. Gray, *Toeplitz and circulant matrices: a review*, Now Publishers, Boston-Delft, 2005.

[29] H. R. Hashimpour, S. Roy, and A. J. Laub, *Decentralized structures for parallel Kalman filtering*, IEEE Trans. Autom. Control **33** (1988), 88–93.

[30] R. A. Horn and C. A. Johnson, *Matrix analysis*, Cambridge Univ. Press, 1985.

[31] M. Huang and J. H. Manton, *Stochastic approximation for consensus seeking: mean square and almost sure convergence*, Proc. 46th IEEE Conf. Decision and Control, 2007, pp. 306–311.

[32] ———, *Stochastic Lyapunov analysis for consensus algorithms with noisy measurements*, Proc. 2007 American Control Conference, 2007, pp. 1419–1424.

[33] A. Iftar, *Decentralized estimation and control with overlapping input, state and output decomposition*, Automatica **29** (1993), 511–516.

[34] A. Iftar and Ü. Özgüner, *Contractible controller design and optimal control with state and input inclusion*, Automatica **26** (1990), 593–597.

[35] M. Ikeda and D. D. Šiljak, *Decentralized control with overlapping information sets*, J. Optimization Theory and Applications **34** (1981), 279–310.

[36] ———, *Overlapping decentralized control with input, state and output inclusion*, Control Theory and Advanced Technology **2** (1986), 155–172.

[37] M. Ikeda, D. D. Šiljak, and D.E. White, *Decentralized control with overlapping information sets*, J. Optimization Theory and Applications **34** (1981), 279–310.

[38] ______, *An inclusion principle for dynamic systems*, IEEE Trans. Autom. Control **29** (1984), 244–249.

[39] A. Jadbabaie, J. Lin, and A. Morse, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Trans. Automat. Contr. **48** (2003), 988–1001.

[40] N. J. Killingsworth and M. Krstić, *PID tuning using extremum seeking*, IEEE Contr. Systems Magaz. (2006), no. 2, 70–79.

[41] A. A. Krasovsky, *Dynamic of continuous self tuning systems*, Fizmatgiz, Moscow, 1963.

[42] M. Krstić and H. H. Wang, *Stability of extremum seeking fefedback for general nonlinear dynamic systems*, Automatica **36** (2000), 595–601.

[43] H. J. Kushner and D. S. Clark, *Stochastic approximation methods for contrained and unconstrained systems*, Springer, New York, 1978.

[44] H. J. Kushner and G. Yin, *asymptotic properties of distributed and communicating stochastic approximation algorithms*, SIAM J. Control Optim. **25** (1987), 1266–1290.

[45] ______, *stochastic approximation algorithms for parallel and distributed processing*, Stochastics **22** (1987), 219–250.

[46] G. Ladde and D. D. Šiljak, *Convergence and stability of distributed stochastic iterative processes*, IEEE Trans. Automat. Contr. **35** (1990), 665–672.

[47] G. Laferriere, A. Williams, J. Caughman, and J. J. P. Veerman, *Decentralized control of vehicle formation*, Systems and Control Letters **54** (2005), 899–910.

[48] Z. Lin, *Coupled dynamic systems: from structure towards stability and stabilizability*, Ph.D. thesis, University of Toronto, 2006.

[49] Z. Lin, B. Francis, and M. Maggiore, *Necessary and sufficient conditions for formation control of unicycles*, IEEE Trans. Automat. Contr. **50** (2005), 121–127.

[50] C. Manzie and M. Krstić, *Extremum seeking with stochastic perturbations*, IEEE Trans. Autom. Control **54** (2009), 580–585.

[51] L. Moreau, *Stability of multiagent systems with time-dependent communication links*, IEEE Trans. Automat. Contr. **50** (2005), 169–182.

[52] I. S. Morosanov, *Method of extremum control*, Automat. Remote Control **18** (1957).

[53] M. B. Nevel'son and R. Z. Khasminsky, *Stochasic approximation and recursive estimation (in Russian)*, Nauka, Moscow, 1972.

[54] J. Nilsson, *Analysis and design of real-time systems with random delays*, Ph.D. thesis, Lund Institute of Technology, 1996.

[55] J. Nilsson, B. Benhardsson, and B. Wittenmark, *Stochastic analysis and control of real-time systems with random time-delays*, Automatica **34** (1998), 57–64.

[56] R. Olfati-Saber, *Distributed Kalman filter with embedded consensus filters*, Proc. IEEE Conf. Decision and Control, 2005.

[57] R. Olfati-Saber, A. Fax, and R. Murray, *Consensus and cooperation in networked multi-agent systems*, Proceedings of the IEEE **95** (2007), 215–233.

[58] R. Olfati-Saber and R. Murray, *Consensus problems in networks of agents with switching topology and time-delays*, IEEE Trans. Automat. Contr. **49** (2004), 1520–1533.

[59] Ü. Özgüner and W.R. Perkins, *Optimal control of multilevel large-scale systems*, Intern. Journal of Control **28** (1978), 967–980.

[60] I. F. Pierce, *Matrices with dominating diagonal blocks*, Journ. of Economic Theory **9** (1974), 159–170.

[61] B. T. Polyak, *Convergence and rate of convergence of iterative stochastic algorithms, I general case*, Automation and Remote Control (1976), no. 12, 83–94.

[62] ———, *Convergence and rate of convergence of iterative stochastic algorithms, II linear case*, Automation and Remote Control (1977), no. 4, 101–107.

[63] ———, *New stochastic approximation type procedures*, Automation and Remote Control (1990), no. 7, 98–107.

[64] B. T. Polyak and Ya. Z. Tsypkin, *Pseudogradient algorithms of adaptation and learning*, Automation and Remote Control (1973), no. 3, 45–68.

[65] ———, *Adaptive estimation algorithms (convergence, optimality, stability)*, Automation and Remote Control (1979), no. 3, 71–84.

[66] ———, *Criterial algorithms of stochastic optimization*, Automation and Remote Control (1984), no. 6, 95–104.

[67] B. T. Poznyak and D. O. Chikin, *Asymptotic properties of the stochastic approximation procedure with dependent noise*, Automation and Remote Control (1984), no. 12, 78–93.

[68] W. Ren, *Consensus strategies for cooperative control of vehicle formations*, IET Control Theory Appl. **1** (2007), 505–512.

[69] ———, *On consensus algorithms for double integrator dynamics*, IEEE Trans. Autom. Control **53** (2008), 1503–1509.

[70] W. Ren and E. Atkins, *Distributed multi-vehicle coordinated control via local information exchange*, Int. Journal Robust Nonlinear Contr. **17** (2007), 1002–1033.

[71] W. Ren, R. W. Beard, and E. M. Atkins, *A survey of consensus problems in multi-agent coordination*, Proc. American Control Conference, 2005, pp. 1859–1864.

[72] W. Ren, R. W. Beard, and D. B. Kingston, *Multi-agent Kalman consensus with relative uncertainty*, Proc. American Control Conference, 2005.

[73] W. Ren and R.W. Beard, *Consensus seeking in multi-agent systems using dynamically changing interaction topologies*, IEEE Trans. Autom. Control **50** (2005), 655–661.

[74] C. W. Sanders, E. C. Tacker, and T. D. Linton, *A new class of decentralized filters for interconnected systems*, IEEE Trans. Autom. Control **19** (1974), 259–262.

[75] ______, *Specific structures for large scale state estimation algorithms having information exchange*, IEEE Trans. Autom. Control **23** (1978), 255–260.

[76] G. Saridis, *Stochastic approximation methods for identification and control - a survey*, IEEE Trans. Autom. Control **19** (1974), 798–809.

[77] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poola, and S. S. Sastry, *Foundations of control and estimation over lossy networks*, Proceedings of the IEEE **95** (2007), 163–187.

[78] E. Y. Shapiro, *On the lyapunov matrix equation*, IEEE Trans. Autom. Control **19** (1974), 594–596.

[79] D. D. Šiljak, *Large scale dynamic systems: Stability and structure*, North-Holland, 1978.

[80] ______, *Decentralized control of complex systems*, Academic Press, New York, 1991.

[81] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poola, M. Jordan, and S. S. Sastry, *Kalman filtering with intermittent observations*, IEEE Trans. Autom. Control **49** (2004), 1453–1464.

[82] R. S. Smith and F. Y. Hadaegh, *Closed-loop dynamics of cooperative vehicle formations with parallel estimators and communication*, IEEE Trans. Autom. Control **52** (2007), 1404–1414.

[83] V. Solo, *Stochastic approximation with dependent noise*, Stoch. Processes and Appl. **13** (1982), 157–170.

[84] J. L. Speyer, *Computation and transmission requirements for a decentralized linear quadratic gaussian control problem*, IEEE Trans. Autom. Control **24** (2004), 266–269.

[85] J.A. Stankovic, T.E. Abdelzaher, Chenyang Lu, Lui Sha, and J.C. Hou, *Real-time communication and coordination in embedded sensor networks*, Proceedings of the IEEE **91** (2003), 1002 – 1022.

[86] M. S. Stanković, S. S. Stanković, and D. M. Stipanović, *Consensus based multi-agent control structures*, Proc. IEEE Conf. Decision and Control, 2008.

[87] M. S. Stanković and D. M. Stipanović, *Stochastic extremum seeking with applications to mobile sensors*, Submitted to *Automatica*.

[88] ———, *Discrete time extremum seeking for autonomous vehicle target tracking in stochastic environment*, Submitted to *IEEE Conference on Decision and Control*, 2009.

[89] ———, *Stochastic extremum seeking with applications to mobile sensor networks*, Accepted for presentation at *American Control Conference*, 2009.

[90] S. S. Stanković, X. B. Chen, M. R. Mataušek, and D. D. Šiljak, *Stochastic inclusion principle applied to decentralized automatic generation control*, Int. J. Control **72** (1999), 276–288.

[91] S. S. Stanković, S. M. Mladenović, and D. D. Šiljak, *Headway control of a platoon of vehicles based on the inclusion principle*, Complex Dynamic Systems with Incomplete Information (E. Reithmeier and G. Leitmann, eds.), vol. 1, Shaker Verlag, Aachen, 1999, pp. 153–167.

[92] S. S. Stanković and D. D. Šiljak, *Sequential LQG optimization of hierarchically structured systems*, Automatica **25** (1989), 545–559.

[93] ———, *Contractibility of overlapping decentralized control*, Systems and Control Letters **44** (2001), 189–199.

[94] S. S. Stanković, M. S. Stanković, and D. M. Stipanović, *Decentralized parameter estimation by consensus based stochastic approximation*, Accepted for publication in *IEEE Trans. Autom. Control*.

[95] ———, *Decentralized parameter estimation by consensus based stochastic approximation*, Proc. IEEE Conference on Decision and Control, 2007.

[96] ———, *A consensus based overlapping decentralized estimator in lossy networks: Stability and denoising effects*, Proc. American Control Conference, 2008.

[97] ———, *Consensus based overlapping decentralized estimation with missing observations and communication faults*, Automatica **45** (2009), 1397–1406.

[98] ———, *Consensus based overlapping decentralized estimator*, IEEE Trans. Autom. Control **54** (2009), 410–415.

[99] S. S. Stanković, M. J. Stanojević, and D. D. Šiljak, *Decentralized overlapping control of a platoon of vehicles*, IEEE Trans. Control Syst. Techn. **8** (2000), 816–832.

[100] S. S. Stanković, D. M. Stipanović, and M. S. Stanković, *Decentralized overlapping tracking control of a formation of autonomous unmanned vehicles*, Accepted for presentation at *American Control Conference*, 2009.

[101] D. M. Stipanović, G. İnhalan, R. Teo, and C. Tomlin, *Decentralized overlapping control of a formation of unmanned aerial vehicles*, Automatica **40** (2004), 1285–1296.

[102] D. M. Stipanović, A. Melikyan, and N. Hovakimyan, *Guaranteed strategies for nonlinear multi-player pursuit-evasion games*, To appear in *International Game Theory Review.*

[103] E. C. Tacker and C. W. Sanders, *Decentralized structures for state estimation in large scale systems*, Large Scale Systems **40** (1980), 39–49.

[104] H. G. Tanner, G. J. Pappas, and V. Kumar, *Leader-to-formation stability*, IEEE Trans. Robotics Autom. **20** (2004), 443–455.

[105] J. N. Tsitsiklis, *Problems in decentralized decision making and computation*, Ph.D. thesis, Dep. Electrical Eng. Comput. Sci., M.I.T., Cambridge, MA, 1984.

[106] J. N. Tsitsiklis and M. Athans, *Convergence and asymptotic agreement in distibuted decision problems*, IEEE Trans. Autom. Control **29** (1984), 42–50.

[107] J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans, *Distributed asynchronous deterministic and stochastic gradient optimization algorithms*, IEEE Trans. Autom. Control **31** (1986), 803–812.

[108] Ya. Z. Tsypkin, *Adaptive optimization algorithms with a priori uncertainty*, Automation and Remote Control (1979), no. 6, 94–108.

[109] H. H. Wang, M. Krstić, and G. Bastin, *Optimizing bioreactors by extremum seeking*, Int. Journal Adapt. Contr. Sign. Proc. **13** (1999), 651–669.

[110] L. Xiao and S. Boyd, *Fast linear iterations for distributed averaging*, Systems and Control Letters **53** (2004), 65–78.

[111] L. Xiao, S. Boyd, and S. Lall, *A scheme for robust distributed sensor fusion based on average consensus*, Proc. Int. Conf. Inf. Proc. in Sensor Networks, 2005.

[112] P. Yang, R.A. Freeman, and K.M. Lynch, *Multi-agent coordination by decentralized estimation and control*, - submitted to *I*EEE Trans. Autom. Control, 2006.

[113] G. Yin, *Recent progress in parallel stochastic approximation*, IMA Preprint Series, 1989.

[114] C. Zhang, D. Arnold, N. Ghods, A. Siranosian, and M. Krstić, *Source seeking with nonholonomic unicycle without position measurement and with tuning of forward velocity*, Systems and Control Letters **56** (2007), 245–252.

[115] C. Zhang, A. Siranosian, and M. Krstić, *Extremum seeking for moderately unstable systems and for autonomous vehicle target tracking without position measurements*, Automatica **43** (2007), 1832–1839.

[116] Y. Zhu, Z. You, J. Zhao, K. Zhang, and X.R. Li, *The optimality for the distributed kalman filtering fusion with feedback*, Automatica **37** (2001), 1489–1493.

Curriculum Vitae

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EDUCATION:

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  - Course work: Nonlinear and Adaptive Control, Simulation of Dynamic Systems, Control of Stochastic Systems, Robot Dynamics and Control, Control and Optimization of Complex Systems, Topics in Wireless Communications, Sensing, Actuation, and Computation, Game Theory
  - Adviser: Prof. Dušan Stipanović
  - Dissertation title: *Control and Estimation Algorithms for Multiple-Agent Systems*
- 2002-2006: School of Electrical Engineering, University of Belgrade, Graduate studies
  - GPA 10/10
  - Degree: *Master of Science*
  - Master's thesis title: *Fast Bayesian Learning Methods with an Application to Text Categorization*
  - Adviser: Prof. Milan Milosavljević
- 1997-2002: School of Electrical Engineering, University of Belgrade, Undergraduate studies
  - Major: Electronics, Telecommunications and Control
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  - Degree: *Dipl. Ing.* (5 years Bachelor)

- Diploma thesis: *Support Vector Machines in Pattern Recognition*
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- 1993-2007: Mathematical High School, Belgrade

EMPLOYMENT:

- 2006-2009: Research Assistant, University of Illinois, Urbana-Champaign, Coordinated Science Laboratory
  - Research in decentralized estimation and control over lossy networks with applications to control of the formations of UAVs and mobile sensor networks
- 2007: Teaching Assisant, University of Illinois, Urbana-Champaign, Department of Industrial and Enterprise Systems Engineering
  - Courses: Introduction to Control Systems, Digital Control Systems
- Dec 2002 - Aug 2006: Employed full time by the Institute for Telecommunications and Electronics "IRITEL", Belgrade, Serbia. Professional work:
  - Integrated circuits design (FPGA - XILINX Spartan 3, using VHDL) of Multiple Service Access Node that supports V5 signaling protocol, TDM cross connect, POTS, ISDN, IP/Ethernet over TDM, TDM over IP/Ethernet, Voice over IP, ADSL, SHDSL
  - Realization of software drivers and applications for Embedded Real-Time Linux (Timesys) for PowerQUICC 2 processor (C, C++ programming languages)

PROFESSIONAL ACTIVITIES:

- Reviewed papers for the *IEEE Transactions on Automatic Control*, *IEEE Transactions on Circuits and Systems I and II*, *American Control Conference* 2008, *IEEE Conference on Decision and Control* 2007,2008 and 2009.
- Chair of the session "Stochastic Adaptive Control" at the *American Control Conference* 2009.

AWARDS:

1. Scholarship for the best undergraduate students, awarded by the Ministry of Education, Government of Serbia.
2. Scholarship for the best graduate students, awarded by the Ministry of Science and Technology, Government of Serbia.
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4. First on the entrance examination for the School of Electrical Engineering in Belgrade (out of 500 students)

JOURNAL PAPERS:

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2. S. S. Stanković, M. S. Stanković and D. M. Stipanović. Consensus Based Overlapping Decentralized Estimation With Missing Observations and Communication Faults, *Automatica*, Vol. 45(6), pp. 1397-1406, 2009.

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4. S. S. Stanković, M. S. Stanković and D. M. Stipanović. Consensus Based Overlapping Decentralized Estimator, *IEEE Trans. Automatic Control*, Vol. 54(2), pp. 410-415, 2009.

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1. M. S. Stanković and D. M. Stipanović. Discrete Time Extremum Seeking by Autonomous Vehicles in a Stochastic Environment, accepted for publication, *IEEE Control and Decision Conference*, 2009.

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3. M. S. Stanković, S. S. Stanković and D. M. Stipanović. Consensus Based Multi-Agent Control Structures, Proc. *IEEE Control and Decision Conference*, 2008, pp. 4364-4369

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5. S. S. Stanković, M. S. Stanković and D. M. Stipanović. Consensus Based Overlapping Decentralized Estimation With Missing Observations and Communication Faults, Proc. *17th IFAC World Congress*, 2008.

6. S. S. Stanković, M. S. Stanković and D. M. Stipanović. Decentralized Parameter Estimation by Consensus Based Stochastic Approximation, Proc. *CDC*, 2007.

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