Nonminimally Coupling Matter to Curvature: from Inflation to Gravitational Waves and Non-Metricity

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2019

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UNIVERSIDADE DO PORTO

DOCTORAL THESIS

NONMINIMALLY COUPLING MATTER TO CURVATURE: FROM INFLATION TO GRAVITATIONAL WAVES AND NON-METRICITY

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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Centro de Física do Porto
Departamento de Física e Astronomia

January 2019

Declaration of Authorship

I, Cláudio Filipe Vieira Gomes, declare that this thesis titled, "NONMINIMALLY COUPLING MATTER TO CURVATURE: FROM INFLATION TO GRAVITATIONAL WAVES AND NON-METRICITY", is based on original work done in collaboration with other researchers during the course of my Ph.D., from January 2015 to December 2018. The content of the Chapters of the present work is based on the released publications [1–4], which are listed below for convenience:

- **Inflation in non-minimal matter-curvature coupling theories**
  C. Gomes, J. G. Rosa and O. Bertolami,
  JCAP **06**, 021 (2017)
  arXiv:1611.02124 [gr-qc]

- **Inflation with Planck data: A survey of some exotic inflationary models**
  C. Gomes, O. Bertolami and J. G. Rosa,
  Phys. Rev. D **97**, 104061 (2018)
  arXiv:1803.08084 [hep-th]

- **Gravitational waves in theories with a non-minimal curvature-matter coupling**
  O. Bertolami, C. Gomes and F. S. N. Lobo,
  Eur. Phys. J. C **78**, 303 (2018)
  arXiv:1706.06826 [gr-qc]

- **Nonminimally coupled Weyl Gravity**
  C. Gomes and O. Bertolami,
  arXiv:1812.04976 [gr-qc]

I have also co-authored the following articles [5, 6], which have not been included in the present thesis:

- **Cosmology with SKA**
  O. Bertolami and C. Gomes,
  arXiv:1809.08663 [astro-ph.CO]

- **Sliding Vectors**
  J. M. M. Moreira and C. F. V. Gomes,
  to appear

Cláudio Gomes
Porto, December 2018

"Space: the final frontier. These are the voyages of a young physicist. His four-year mission: to explore strange new physics and phenomena. To seek out new explanations and new understanding of the Cosmos. To boldly go where no man has gone before!"

inspired by Star Trek

Acknowledgements

I would like to thank:

- my supervisor, Prof. Orfeu Bertolami, for his guidance and support throughout my work. I thank him every discussion on exciting topics we had developed and the opportunity for listening to his vision of Physics and society;
- to Fundação para a Ciência e Tecnologia for the research grant SFRH/BD/102820/2014 and to Centro de Física do Porto for travel expenses to Conferences, under the project UID/FIS/04650/2013;
- my mother for her unconditional support for all these years;
- my friends for their kindness and for keeping reminding me to have some off-work life;
- to my collaborators, Dr. João G. Rosa and Dr. Francisco S. N. Lobo, for fruitful discussions and jointly work.

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Abstract

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Doctor of Philosophy

NONMINIMALLY COUPLING MATTER TO CURVATURE: FROM INFLATION TO GRAVITATIONAL WAVES AND NON-METRICITY

by Cláudio Filipe Vieira Gomes

In this work, we study alternative theories of gravity with a non-minimal coupling between matter and curvature. In particular, the inflationary paradigm is analysed in the context of such theories, providing some observational predictions which are compared with the most recent data. This discussion motivated the study of other alternative or "exotic" inflationary models in the light of the observational data. During inflation, gravitational waves can be produced, therefore a more general but detailed discussion on this subject is mandatory, particularly since gravitational wave signals from massive astrophysical objects mergers have been recently detected. Finally, given that the metric compatibility postulate for the connection is not the most general mathematical scenario, the case of Weyl gravity in the presence of these theories of gravity is developed.

This thesis is based on work developed in Refs. [1–4].

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Resumo

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Doctor of Philosophy

NONMINIMALLY COUPLING MATTER TO CURVATURE: FROM INFLATION TO GRAVITATIONAL WAVES AND NON-METRICITY

by Cláudio Filipe Vieira Gomes

Neste trabalho, as teorias alternativas da gravitação Einsteiniana que incluem um acoplamento não mínimo entre matéria e geometria são estudadas. Em particular, o paradigma inflacionário no contexto destes modelos é analisado e as previsões comparadas com os dados observacionais mais recentes. Esta discussão motiva o estudo de modelos alternativos ou "exóticos" que ainda não haviam sido testados à luz dos dados recentes. Durante a inflação podem ser geradas ondas gravitacionais, pelo que a exploração mais geral e detalhada deste tópico no contexto das teorias com acoplamento não-mínimo é realizada em seguida, além da recente descoberta de sinais destas ondas advindas da colisão de objetos astrofísicos maciços. Finalmente, atendendo a que o postulado da compatibilidade métrica da conexão não constitui o cenário mais abrangente matematicamente, o caso particular da gravidade de Weyl na presença destas teorias alternativas é desenvolvido.

Esta tese baseia-se no trabalho desenvolvido nas Refs. [1–4].

Contents

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List of Abbreviations

NMC	**N**on-**M**inimal **C**oupling
GR	**G**eneral **R**elativity
d.o.f.	**d**egrees **o**f **f**reedom
GW	**G**ravitational **W**ave
lhs	**l**eft **h**and **s**ide
rhs	**r**ight **h**and **s**ide
CMB	**C**osmic **M**icrowave **B**ackground
RW	**R**obertson-**W**alker

Part I

Introduction

Chapter 1

The non-minimal matter-curvature coupling: an extension of General Relativity

1.1 General Relativity

GENERAL Relativity (GR) was proposed by Albert Einstein in 1915 as a theory of spacetime geometry and it is one of the most extraordinary theories ever conceived by the human mind. His work consisted of four fundamental papers published in November of 1915 [7–10]. Although his theory has been derived mainly through elegance, aesthetics and simplicity criteria, GR is extremely successful in accounting for the weak-field experimental regime of gravitation where it agrees with data with an impressive precision (we refer the reader to Refs. [11,12] and references therein).

In GR the concept of spacetime geometry is crucial. In fact, the main features of the curvature are encoded in the Riemann tensor:

$$\bar{R}^{\lambda}{}_{\mu\sigma\nu} := \partial_\sigma \bar{\Gamma}^{\lambda}{}_{\mu\nu} - \partial_\nu \bar{\Gamma}^{\lambda}{}_{\mu\sigma} + \bar{\Gamma}^{\rho}{}_{\mu\nu}\bar{\Gamma}^{\lambda}{}_{\rho\sigma} - \bar{\Gamma}^{\rho}{}_{\mu\sigma}\bar{\Gamma}^{\lambda}{}_{\rho\nu} \,, \tag{1.1}$$

where $\bar{\Gamma}^{\lambda}{}_{\mu\nu}$ is the affine connection, which can be decomposed into three parts:

$$\bar{\Gamma}^{\lambda}{}_{\mu\nu} = \{^{\lambda}{}_{\mu\nu}\} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} \,, \tag{1.2}$$

where $\{^{\lambda}{}_{\mu\nu}\} := \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$ are the Christoffel symbols, $K^{\lambda}{}_{\mu\nu} := \frac{1}{2}T^{\lambda}{}_{\mu\nu} + T_{(\mu}{}^{\lambda}{}_{\nu)}$ is the contortion built from the torsion tensor, $T^{\lambda}{}_{\mu\nu} := 2\bar{\Gamma}^{\lambda}{}_{[\mu\nu]}$, and $L^{\lambda}{}_{\mu\nu} := \frac{1}{2}g^{\lambda\rho}\left(-Q_{\mu\rho\nu} - Q_{\nu\rho\mu} + Q_{\rho\mu\nu}\right)$ is the disformation built from the non-metricity tensor, $Q_{\lambda\mu\nu} := D_\lambda g_{\mu\nu}$, where D_λ is the generalised covariant derivative built from the affine connection, $\overline{\Gamma}^{\lambda}{}_{\mu\nu}$ [13]. In General Relativity, the connection is torsionfree and the metric is preserved, hence it corresponds to the Christoffel symbols and is called Levi-Civita connection, which we denote by $\Gamma^{\lambda}{}_{\mu\nu}$. But there are other metric-affine theories based on combinations of how many of the three aforementioned parts vanish. This can be illustrated in Fig. (1.1).

Henceforth, and otherwise stated, we employ the Levi Civita connection. Then, the action functional for General Relativity reads:

$$S = \int \left[\kappa R + \mathcal{L}\right]\sqrt{-g}\,d^4x \,, \tag{1.3}$$

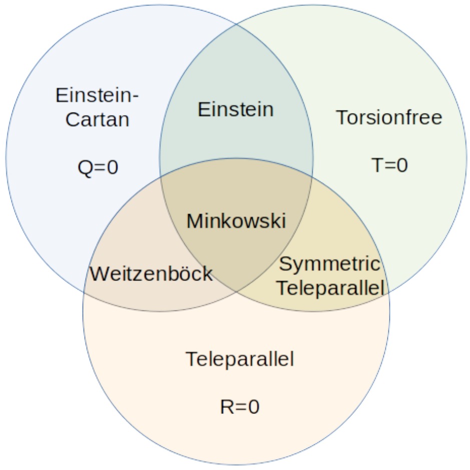


FIGURE 1.1: General picture of combinations of non-vanishingness of the terms in the affine connection decomposition. Based on Ref. [14].

where $\kappa := \frac{c^4 M_P^2}{2}$ with c being the light speed, $M_P^2 = (8\pi G)^{-1}$ being the reduced Planck mass, and G being the Newton's gravitation constant, $R = g^{\mu\nu} R^{\lambda}{}_{\mu\lambda\nu}$ is the Ricci scalar (or scalar curvature), $\mathcal{L}$ is the Lagrangian of the matter fields, and g is the determinant of the metric, $g_{\mu\nu}$.

1.1.1 Metric formalism

If we vary the action with respect to the metric, we get the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2\kappa} T_{\mu\nu} \, , \tag{1.4}$$

where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\mathcal{L}\sqrt{-g})}{\delta g^{\mu\nu}}$ is the energy-momentum tensor. The left hand side (lhs) of the field equations is called the Einstein tensor and is denoted by $G_{\mu\nu}$. From the Bianchi identities, the Einstein tensor is covariantly conserved, $\nabla_\mu G^{\mu\nu} = 0$, and due to the field equations, the energy-momentum tensor is also covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$.

1.1.2 Palatini formalism

There is a subtle caveat in the metric formalism: we postulate that the affine connection depends on the metric and its derivatives. Nevertheless, this is not necessarily the most general case. In fact, we can recall the basics of Differential Geometry and Topology: the metric field gives the notion of inner product and defines distances in the manifold and angles between vectors in the tangent space; whilst the affine connection is related to the parallel transport along a curve of tangent vectors to a manifold from one point to another. Therefore, GR or any other theory can be analysed in the Palatini formalism, where the action functional can be varied with respect to the metric and to the connection independently.

However, the matter Lagrangian does not feel this independent connection, but only the metric field and its derivatives. In this formalism, the variation of action 1.3, keeping in mind that $R(\Gamma) = g^{\mu\nu}R_{\mu\nu}(\Gamma)$, with respect to the metric reads:

$$R_{(\mu\nu)}(\Gamma) - \frac{1}{2}g_{\mu\nu}R(\Gamma) = \frac{1}{2\kappa}T_{\mu\nu}\,, \tag{1.5}$$

and the variation with respect to the connection gives:

$$\nabla_\mu\left(\sqrt{-g}g^{\mu\nu}\right) = 0\,, \tag{1.6}$$

where the latter is simply the relation for the Levi-Civita connection. Hence, we get the metric formalism equations, albeit the fact that in the Palatini formalism the Levi-Civita connection is a dynamic consequence of the theory and not an *a priori* assumption.

Although this approach is known as the Palatini formalism, it seems that this procedure was firstly considered by Einstein [15]. In fact, Atillio Palatini derived the so-called Palatini identity:

$$\delta R_{\mu\nu} = \nabla_\rho(\delta\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho_{\rho\mu})\,, \tag{1.7}$$

which is remarkably similar to the Lie derivative, $\mathcal{L}_\xi$, which measures in a coordinate independent way the change of a tensor field along the flow of a given vector field, $\xi = \xi^\rho\partial_\rho$:

$$\mathcal{L}_\xi R_{\mu\nu} = \nabla_\rho(\mathcal{L}_\xi\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\mathcal{L}_\xi\Gamma^\rho_{\rho\mu})\,. \tag{1.8}$$

There are also other formalisms: the metric-affine one, which regards the metric and the connection as two independent variables, but additionally the matter sector also feels the connection; the vierbein formalism, which writes the theory in the tangent space; among others (we refer the reader to the review in Ref. [16]).

1.1.3 Some concerning issues with GR

Despite GR's impressive agreement with most observational data, there are some conundrums: the so-called ultraviolet (UV) and infrared (IR) behaviours; in other words, the small scales (high energies) and the large scales (small energies) limits. In the first case, we refer to the lack of a fully consistent quantum version of GR and in the other limit to the large scale enigmatic behaviour of the Universe, namely the so-called dark matter and dark energy problems [17].

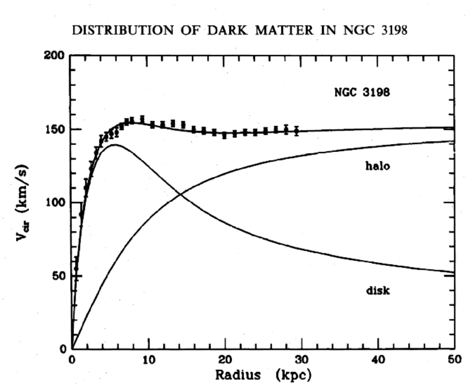
Attempts to find a quantised version of GR have been pursued. However, it is shown that the theory is not renormalisable at one-loop [18]. In addition, quantum corrections to GR in string theory or in the Noether symmetry approach [19–25] show that higher-order curvature invariant terms have to be included in the action. These terms would affect the UV limit of gravity. Actually, this reasoning lead to important models in the early Universe, as it was the case of the cosmic inflation, which we shall discuss in Section 1.4. In fact, in the 1980's, Stephen Hawking and James Hartle started a new branch of research, the quantum cosmology. In this approach the spacetime is described by the Universe wavefunction, which obeys to the Wheeler-De Witt equation, an equation analogous to the Schroedinger's one [26–30].

A different approach to a quantum gravitational theory was performed in the spirit of effective field theory. The latter was well known in the context of Particle Physics and Statistical Mechanics, and relies on the identification of the relevant degrees of freedom (d.o.f.) which describe phenomena at a given scale, while it ignores substructure and degrees of freedom at lower scales by integrating out these heavy d.o.f. Thus, the method has been employed to General Relativity [31–34] and also in a covariant way using the background field method [35–38]. This approach leads to measurable quantum corrections to the Newtonian potential [39–41].

At the other end of length scales, there have been strong evidences that GR fails. Namely, observational data requires either a different gravitational theory or the presence of two dark components: dark matter and dark energy; which have not been directly observed so far. Assuming they do exist, data requires that 26.8% of the Universe energy content is due to dark matter and 68.3% of dark energy [42]. This means that just the remaining 4.9% is of matter we know.

The issue of dark matter [43] started to be discussed at the early 20th century by Lord Kelvin when he computed the mass of the Milky Way from the observation of the stars' velocity dispersion, which was in disagreement with the value obtained from the visible stars. Thus, the idea of the existence of some "dark bodies" in the galaxy was born. In 1906, Henri Poincaré analysed the work of Kelvin and coined the term "dark matter" (or "matière escure" in French) [44]. Further attention was drawn to this topic from several authors. In particular, Fritz Zwicky estimated the mass of the Coma Cluster using the virial theorem, which states that in a relaxed gravitational system the kinetic, K, and potential, U, energies obey the relation $2K + U = 0$, obtaining a value 400 times higher than the one computed from the brightness and number of galaxies in the cluster [45]. Although, this value was overestimated due to the less rigorous value of the Hubble constant, he correctly inferred the need of "dunkle materie" (dark matter). In 1970, two studies were published concerning the missing matter in galactic rotation curves in the optical waveband [46, 47]. Vera Rubin and Kent Ford showed that the velocity rotation curve of Andromeda Nebula stayed roughly constant for large radii, and Kenneth Freeman showed that for the two galaxies NGC 300 and M33 the maximum velocity happened for larger radius than the one predicted from photometry. This introduced a new motivation for the understanding on the nature of dark matter. A characteristic rotation curve is shown in Fig 1.8b. More recently, the observation of clusters of galaxies which collided a long time ago indicated that two main patterns could be detected: X-ray emission of hot gas, which indicates that stars from the two clusters passed right through each other and the gravitational lensing effect due to dark matter [48–50] as shown in Fig. 1.2b.

A vast list of particle physics motivated candidates for dark matter have been proposed, which can be classified into three main categories: hot, warm and cold dark matter, in increasing order of mass. Notwithstanding, hot dark matter consists of ultrarelativistic particles such as neutrinos. However, they show great tension with structure formation simulations as they provide much less structures than the ones we observe in the Universe. In its turn, warm dark matter particles would have masses of the order of keV as the sterile neutrinos, although initially they have been proposed from an extension of the Standard Model of Particle Physics. These neutrinos have right-handed chirality, and only interact through gravity. Warm dark matter models show good agreement with observations [52–55]. The last scenario: the cold dark matter is the most studied case as it relies on heavy particles,



(A) Galactic rotation curve of the galaxy NGC 3198 showing the plateau behaviour for large radius, which cannot be explained from the observed ordinary matter within General Relativity [51].

(B) The "bullet cluster" 1E 0657-56 with X-ray hot gas in red and gravitational lensing effect due to "dark matter" in blue [48–50]

FIGURE 1.2: Two observational evidences for the existence of dark matter or a gravitational behaviour different from General Relativity.

with masses from GeV to TeV, and assumes that they interact very weakly. Some candidates of this scenario are the Weakly Interacting Massive Particles (WIMP), which although thermally produced in the early Universe went out of the equilibrium as the expansion rate of spacetime overdominated the production/annihilation rate of WIMP and whose abundance became constant [56, 57]; and the Feeble Interaction Massive Particles, which were not very abundant in the early Universe but were created by out of equilibrium decaying processes of Standard Model particles [58, 59]. However, this type of dark matter fails on small scales as it leads to the satellite galaxies problem: more small structures, such as small galaxies orbiting bigger ones, than the ones observed. Furthermore, none of the three types of candidates for dark matter were directly observed so far (see e.g. Ref. [60] for a review on dark matter searches).

Some of the searches for dark matter occur in Particle Physics colliders such as the Large Hadron Collider (LHC) in Switzerland, mainly due to the possibility of the decaying of the superrenormalisable Higgs boson into new particles [61]. In particular, $SU(3) \times SU(2) \times U(1)$ singlet fields can have renormalisable coupling to the Higgs, which motivates the so called "Higgs portal" dark matter [62–64]. Non-renormalisable terms may however appear provided that the dark matter field lives in a hidden sector which is sequestered by the visible sector, where Standard Model particles live, including the Higgs boson itself [65]. On the other hand, the dark matter scalar field can be coupled to both the Higgs field and the scalar curvature, thus having some bearings on inflation and electroweak symmetry breaking [66–68]. Moreover, unifying scenarios of dark matter and dark energy to dark energy can also be found through this procedure [69].

In addition to the astronomical observations on the dark matter effect, there were other ones which showed another intriguing feature: the Universe is undergoing an accelerated expansion phase. Indeed, in 1998, two teams reported that conclusion from the observation of Supernovae Type Ia, used as standard candles to accurately measure distances, the High-Z Supernova Search Team [70] and the Supernova Cosmology Project [71]. These results led to the Nobel Prize in Physics in 2011 awarded to Saul Perlmutter, Brian Schmidt and Adam G. Riess. This acceleration was postulated to be due to some form of smoothly distributed dark energy.

The simplest mechanism for the dark energy is the cosmological constant: a constant in Einstein's field equations, which was introduced by Einstein himself but in a rather different context. At that time, it was believed that the Universe was static, and the way to reconciliate the theory with such Universe was to include a cosmological constant. Just after the Hubble's observations on the redshift of galaxies, which indicated that the farthest galaxies moved faster away from us than the local ones, Einstein decided to consider the introduction of the mentioned constant as a huge mistake. However, much later it was realised that a positive cosmological constant could account for the dark energy effect. This leads to an equation of state parameter $w := p/\rho = -1$, where p is the pressure and ρ denotes the energy density.

There are also other possibilities for dark energy. One of such models relies on a scalar field, ϕ, whose dynamics at late times become potential dominated, and consequently $w \approx -1$. However, the scalar field can evolve in time, and its equation of state "freezes" towards or "thaws" away from the nowadays value, $w = -1$ [72–74].

There are also models which attempt to explain dark matter and dark energy through a single unifying mechanism. This is the case of an exotic fluid with equation of state $p = -A/\rho^{\alpha}$, where A and α are real constants, with $0 < \alpha \leq 1$ [75,76]. This model will be further developed in Chapter 4 in the context of Inflation. We refer the reader to Refs. [77,78] for review on models of dark energy.

As we have seen, despite all the beauty of the mathematics and the impressive agreement with some observational data, General Relativity may fail at very low and very high scales. Hence, one may think that GR is not the ultimate theory. Thus, several modified gravity models have been proposed in the literature. We shall discuss two of them: $f(R)$ theories and the non-minimal matter-curvature coupling theories. The latter will be a central point of the present thesis.

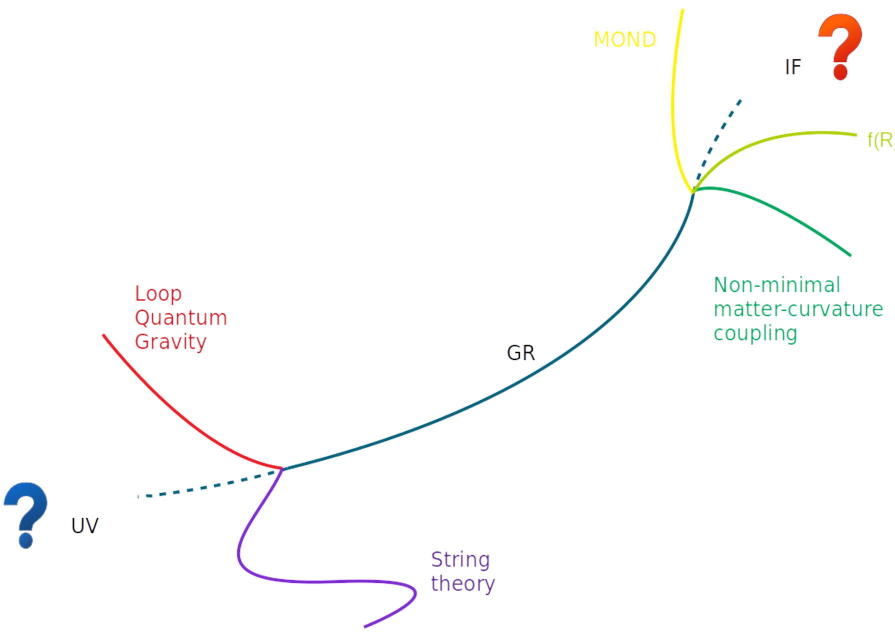


FIGURE 1.3: General Relativity may not be the final theory. At small length scales, we need a fully consistent quantum theory of gravity, whilst at large length scales we need to provide an explanation for the behaviour dark matter and dark energy in the Universe.

1.2 f(R) theories

We have pointed out that attempts to build an UV completion of GR relied on higher-order curvature terms in the action. Thus, at least from a phenomenological point of view, we can replace the Ricci scalar in the action by a generic function of it, $f(R)$ [79–81]:

$$S = \int \left[\kappa f(R) + \mathcal{L}\right] \sqrt{-g} d^4 x \,. \tag{1.9}$$

The function can be expanded in a power series:

$$f(R) = \sum_{n=-\infty}^{+\infty} a_n \left(\frac{R}{M_n^2}\right)^n \,, \tag{1.10}$$

where a_n are the coefficient of the terms in the series, and M_n are some mass scales to be determined. In the metric formalism, the variation of the action relatively to the metric gives:

$$F(R) R_{\mu\nu} - \frac{f(R)}{2} g_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu} + \Delta_{\mu\nu} F \,, \tag{1.11}$$

where we have defined the operator $\Delta_{\mu\nu} := \nabla_\mu \nabla_\nu - g_{\mu\nu}\Box$, with $\Box := \nabla_\mu \nabla^\mu$ being the D'Alambertian, and $F(R) := df/dR$. The trace of the field equations yield a dynamical equation for the Ricci scalar, thus instabilities associated to a negative sign on the *mass* term for R should be avoided. The Dolgov-Kawasaki criterion provides a necessary and sufficient condition for that: $F'(R) \geq 0$ [82].

However, in the Palatini formalism, the metric and the connection variations yield, respectively:

$$F(R) R_{(\mu\nu)}(\Gamma) - \frac{f(R)}{2} g_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu} \,, \tag{1.12}$$

$$\bar{\nabla}_\mu \left(\sqrt{-g} F(R) g^{\mu\nu}\right) = 0 \,, \tag{1.13}$$

where $\bar{\nabla}$ is the covariant derivative built from the connection and we should bear in mind that $f(R) = f\left(g^{\mu\nu} R_{\mu\nu}(\Gamma)\right)$.

In fact, for $f(R)$ theories, the Palatini formalism reduces the order of the field differential equations. In particular, the trace of the field equations yields an algebraic equation for R, hence a "mass" term for the Ricci scalar does not appear and instabilities related with negative masses are not a problem. This is the so called Dolgov-Kawasaki criterion [82].

Other formalisms in $f(R)$ theories are also possible, in particular the metric-affine version was analysed in Ref. [83].

In what concerns to the observational issues with GR we pointed out earlier, $f(R)$ theories were considered to have a power-law form, as Eq. (1.10), with a positive exponent for galaxies. In fact, it was found that the Newtonian potential has a logarithmic correction, if one looks for a constant tangential velocity of test particle in a stable circular orbit. However, this approach exhibits no galaxy-dependent parameters and yields an universal asymptotic velocity. This is in disagreement with the Tully-Fisher and Faber-Jackson laws, which are empirical power-law relations between the asymptotic rotation velocity of a galaxy and its total visible mass, $M \propto v^m$, yielding an exponent $m = 4$ for spiral and $m = 6$ for elliptical galaxies, respectively. On the other hand, if the metric is perturbed, this leads to a correction on the Newtonian potential of the form $\Phi_N = -Gm/r(1 + (r/r_0)^\beta)$, where the exponent β is a parameter characteristic of each galaxy [79,84,85].

As for the dark energy problem, $f(R)$ theories can provide a natural framework for an accelerated expansion, depending on a suitable choice for the exponent n in $f(R) \sim R^n$. Actually, let us consider, for simplicity, the absence of matter, $\rho = p = 0$, and a homogeneous and isotropic Universe given by a Robertson-Walker (RW) metric:

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij} dx^i dx^j \,, \tag{1.14}$$

where γ_{ij} is the Friedmann metric and $a(t)$ is the scale factor.

The equation of state parameter is defined as $w := p/\rho = \hat{p}/\hat{\rho}$, where the effective pressure is written as $\hat{p} = p - \frac{1}{2}(f - FR) + (\ddot{R} + 2H\dot{R})\,F' + F''\dot{R}^2$ and the effective energy density is given by $\hat{\rho} = \rho + \frac{1}{2}(f - FR) - 3HF'\dot{R}$, with $F' := dF/dR$ and $F'' := d^2F/dR^2$. A condition to have an acceleration phase is that the equation of state parameter is $w < -1/3$ (this will be further developed in Section 1.4), thus for $f(R) \sim R^n$ it is possible to achieve such regime depending on suitable choices for n [79–81,86].

It is worth mentioning that $f(R)$ theories in the Jordan frame can be conformally transformed into a scalar-tensor theory in the Einstein frame:

$$S_E = \int \left[\kappa\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi) + \mathcal{L}\left(F^{-1}\tilde{g}_{\mu\nu}, \Psi\right)\right]\sqrt{-\tilde{g}}\,d^4x \tag{1.15}$$

where the conformal metric reads $\tilde{g}_{\mu\nu} = F^2 g_{\mu\nu}$, the auxiliary scalar field is $\Phi := \sqrt{3\kappa}\ln F(R)$ and the potential has the form $V(\Phi) := \kappa\frac{FR-f}{F^2}$, and Ψ are the matter fields.

Actually, $f(R)$ theories are a relevant modification since they allow for a successful model of inflation, namely the Starobinsky's model [87]. Furthermore, they are also quite useful at late times where they account for dark matter and dark energy unsolved problems without postulating some exotic particles or fluids yet to be discovered (we refer the reader to Refs. [17,88] and references therein for a more detailed discussion).

A further generalisation of $f(R)$ was proposed in Ref. [89] by including a non-minimal coupling between matter and curvature.

1.3 The non-minimal matter-curvature coupling model

GENERAL Relativity is built upon the principle of a minimal coupling between the metric and matter. This means that the partial derivatives are locally promoted to covariant ones, which, in particular, means that $\partial_\mu T^{\mu\nu} \to \nabla_\mu T^{\mu\nu}$ and we get a covariant conservation equation for the energy-momentum tensor. This was necessary to obtain the field equations and to get the action functional. However, this is not the most general case. In fact, the energy-momentum tensor can be no longer covariantly conserved, which results in an extra force in the geodesics equation that can account for the dark matter effect on galaxies. This was the idea behind the birth of the extension of $f(R)$ theories with a non-minimal coupling between matter and curvature [89].

The action functional for these models read:

$$S = \int d^4x\sqrt{-g}\left[\kappa f_1(R) + f_2(R)\,\mathcal{L}\right] \,, \tag{1.16}$$

where $f_1(R), f_2(R)$ are arbitrary functions of the Ricci scalar R.

Varying the action with respect to the metric, $g_{\mu\nu}$, leads to the following field equations:

$$\Theta R_{\mu\nu} - \frac{f_1(R)}{2} g_{\mu\nu} = \frac{f_2}{2\kappa} T_{\mu\nu} + \Delta_{\mu\nu}\Theta \,, \tag{1.17}$$

where we defined $\Theta := F_1 + F_2\mathcal{L}/\kappa$. It is straightforward to see that choosing $f_1(R) = R$ and $f_2(R) = 1$, one recovers General Relativity.

Taking the trace of the previous equations, one finds:

$$\Theta R - 2f_1 = \frac{f_2}{2\kappa} T - 3\Box\Theta \,. \tag{1.18}$$

Using the Bianchi identities in the metric field equations, one finds that the matter energy-momentum tensor is not covariantly conserved in this model:

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{f_2}\left(g^{\mu\nu}\mathcal{L} - T^{\mu\nu}\right)\nabla_\mu R \,. \tag{1.19}$$

This results in an extra force in the geodesic equation, which for a perfect fluid is given by [89]:

$$f^\mu = \frac{1}{\rho + p}\left[\frac{F_2}{f_2}\left(\mathcal{L} - p\right)\nabla_\nu R + \nabla_\nu p\right] h^{\mu\nu} \,, \tag{1.20}$$

where $h^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu$ is the projection operator, U^μ denotes the particle's 4-velocity and we have used the energy-momentum tensor of a perfect fluid $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$. As we can see, the extra force explicitly depends on the matter Lagrangian choice.

In the context of GR, it is well known that $\mathcal{L} = -\rho$, $\mathcal{L} = p$ and $\mathcal{L} = -nA$, where n is the particle number density and A is the physical free energy, defined as $A := \rho/n - Ts$, with T being the fluid temperature and s the entropy per particle, can be derived from the same action by adding specific surface integrals to it [90–92]. This yields the same energy-momentum tensor for a perfect fluid, hence these on-shell descriptions are degenerated.

However, in the non-minimal matter-curvature coupling model, these choices give the same energy-momentum tensor, but different gravitational equations. In particular, the geodesic motion is quite different from one choice to another. Let us note that if $\mathcal{L} = -\rho$ the extra force becomes $f^\mu = -\left(\nabla_\nu\Phi_c + \frac{\nabla_\nu p}{\rho+p}\right)h^{\mu\nu}$, with the non-minimal coupling potential $\Phi_c := \ln f_2$, whilst for $\mathcal{L} = p$, the extra force reduces to the pressure-gradient force in GR, and for the third option, $\mathcal{L} = -nA$, it is simply $f^\mu = \frac{1}{\rho+p}\left(-(nA+p)\nabla_\nu\Phi_c + \nabla_\nu p\right)h^{\mu\nu}$. This is a rather surprising result, since even in the null dust matter distribution, $p = 0$, the parallel transport is not conserved as it still depends on the matter Lagrangian density, hence these extra forces may be physically distinguishable [93].

On the other hand, this may hint the existence of an underlying principle or symmetry which gives the correct Lagrangian for a perfect fluid and the usual general relativistic formulation is incomplete [93]. This case was analysed in Ref. [94] for the case of a barotropic fluid, just assuming that the matter Lagrangian does not depend on metric derivatives and that the particle number density is conserved.

Furthermore, the existence of the extra force hints an explicit violation of the equivalence principle, which states that bodies in free fall have the same acceleration regardless of their inner structures. This principle is one of the basics assumptions of GR, and it is very constrained at the solar system level [95]. However, this is not a fundamental principle of physics, hence long range deviations may be possible and lead to interesting observational features [96]. In fact, some higher-dimensional theories have a low-energy version which breaks the equivalence principle, such as Kaluza-Klein theories [97] or string theory whose dilaton and moduli scalar fields couple to matter [98]. Even in the context of dark matter and dark energy, it was shown that there are cosmological evidences for a putative violation of the equivalence principle [99–102]. Therefore, observational tests of the equivalence principle are very important to constrain parameters of a theory at solar system level, although it should not be taken as granted for larger scales [103, 104].

The viability and physical meaningfulness of any gravity model is fundamental and can be scrutinised by studying its stability. One of such criteria is to study the energy conditions and the stability of the model under the Dolgov-Kawasaki analysis [82]. To do so, we first note that the field equations of the NMC models, Eq. (1.17), can be recast in another form:

$$G_{\mu\nu} = \frac{f_2}{\left(F_1 + \frac{F_2\mathcal{L}}{\kappa}\right)} \frac{1}{2\kappa} \left[T_{\mu\nu} + \hat{T}_{\mu\nu}\right] \,, \tag{1.21}$$

with the effective energy-momentum tensor defined as

$$f_2\hat{T}_{\mu\nu} := \Delta_{\mu\nu}\left(F_1 + \frac{F_2\mathcal{L}}{\kappa}\right) + \frac{1}{2}g_{\mu\nu}\left(f_1 - F_1 R - \frac{F_2 R\mathcal{L}}{\kappa}\right) \,. \tag{1.22}$$

From Eq.(1.21) one can define an effective gravitational constant according to the similarity with Einstein's field equations: $\hat{G}_{eff} := \frac{f_2}{\left(F_1+\frac{F_2\mathcal{L}}{\kappa}\right)}\frac{1}{2\kappa}$. In order to have attractive gravity, we have to require that $\hat{G}_{eff} > 0$, which is a condition arising from the model.

There are four energy conditions which need to be satisfied. The strong (SEC) and null (NEC) energy conditions are expressed by the purely geometric and theory independent Raychaudhury equation, which provides the time evolution of the expansion of a congruence of timelike or null geodesics, respectively, together with the requirement that gravity is attractive. The weak energy condition (WEC) arises from the requirement that the energy density measured by any observer is non-negative, whilst the dominant energy condition (DEC) states that in any orthonormal basis the energy dominates the other components of the energy-momentum tensor [105]. For the non-minimal coupling between matter and curvature models, and for a Friedmann-Robertson-Walker metric, these energy conditions yield, for a perfect fluid [106]:

- SEC

$$\begin{cases} \hat{\rho} + \hat{p} \geq 0 \,, \\ \hat{\rho} + 3\hat{p} \geq 0 \end{cases} \tag{1.23}$$

- NEC

$$\hat{\rho} + \hat{p} \geq 0 \,, \tag{1.24}$$

- WEC

$$\begin{cases} \hat{\rho} \geq 0 \,, \\ \hat{\rho} + \hat{p} \geq 0 \end{cases} \tag{1.25}$$

- DEC

$$\hat{\rho} \geq |\hat{p}| \tag{1.26}$$

where $\hat{\rho} = \rho + \frac{1}{2}\left(\frac{f_1}{f_2} - \frac{F_1+2\mathcal{L}F_2}{f_2}R\right) - 3H\frac{F_1'+2\mathcal{L}F_2'}{f_2}\dot{R}$ is the effective energy density and $\hat{p} := p - \frac{1}{2}\left(\frac{f_1}{f_2} - \frac{F_1+2\mathcal{L}F_2}{f_2}R\right) + (\ddot{R}+2H\dot{R})\frac{F_1'+2\mathcal{L}F_2'}{f_2} + \frac{F_1''+2\mathcal{L}F_2''}{f_2}\dot{R}^2$ is the effective pressure. When setting $f_1 = R$ and $f_2 = 1$, we recover GR's results.

The other criterion is the Dolgov-Kawasaki's one, which is related with the value of the "mass" term for the Ricci scalar. In GR, the trace of the field equation gives an algebraic equation for R, thus GR does not suffer from such instabilities. However, modified gravity models may lead to a dynamical equation for the scalar curvature as it is the case of the NMC model. Therefore, requiring the "mass" of the scalar curvature to be positive defined implies [106,107]:

$$F_1' + \frac{1}{\kappa}F_2'\mathcal{L} \geq 0\,. \tag{1.27}$$

It was shown in Ref. [106] that for power-law functions, f_1 and f_2, the positiveness of the effective coupling, the energy conditions and the Dolgov-Kawasaki criterion are degenerate. However, in other gravity theories this needs not to hold.

Provided the NMC model satisfy the above conditions, there are many physical situations which need to be addressed. The first one is the dark matter problem, where these theories can mimic the profiles at galaxies [108,109] and clusters [110]. In fact, by requiring the Tully-Fisher law and that the energy-momentum tensor is roughly conserved, $\nabla_\mu T^{\mu\nu} \approx 0$, but the metric itself is perturbed, thus providing the extra force effect, then the mimicked dark matter density is given in terms of $R/2\kappa - \rho$. If we assume that the non-minimal coupling function has a power-law behaviour, $f_2(R) = 1 + (R/M_n^2)^n$, where M_n is a typical mass scale for each galaxy, it is found that the dark matter profile is obtained at low curvatures and long range for a negative exponent, n [108]. One concrete example of this procedure is shown in Fig. 1.4.

However, the dark matter behaviour on clusters of galaxies is slightly different. Actually, clusters are the largest astronomical relaxed objects that have entered hydrostatic and virial equilibrium, hence allowing for a reliable measurement of the total mass [111]. Furthermore, observations on X-rays and gravitational lensing show that the clusters' mass is around $7-10$ times larger than the visible mass from stars and the intracluster medium, and this effect is higher in the innermost regions of the structure. This feature contrasts with the one observed in galaxies, thus leading to a positive exponent n in a power-law function f_2 together with the requirement that the extra-force is small relatively to the Newtonian one [110].

Those studies [108,110] found that the exponents were non-integers, which may come from the simplification on the specific form of the non-minimal coupling function f_2 relying on a monomial behaviour. If this function was polynomial on R, with more exponents and mass scales, its functional form could be analysed in more detail and be more adequate for describing deviations from spherical symmetry, for instance. However, this is a highly non-trivial task and it demands further detailed observations with smaller uncertainties in the future.

As for the dark energy problem, the same approach performed in $f(R)$ theories can be made by requiring $w > -1/3$ and taking into account the generalised effective energy density and pressure: $\hat{\rho} = \rho + \frac{1}{2}\left(\frac{f_1}{f_2} - \frac{F_1+2\mathcal{L}F_2}{f_2}R\right) - 3H\frac{F_1'+2\mathcal{L}F_2'}{f_2}\dot{R}$ and $\hat{p} = p - \frac{1}{2}\left(\frac{f_1}{f_2} - \frac{F_1+2\mathcal{L}F_2}{f_2}R\right) +$

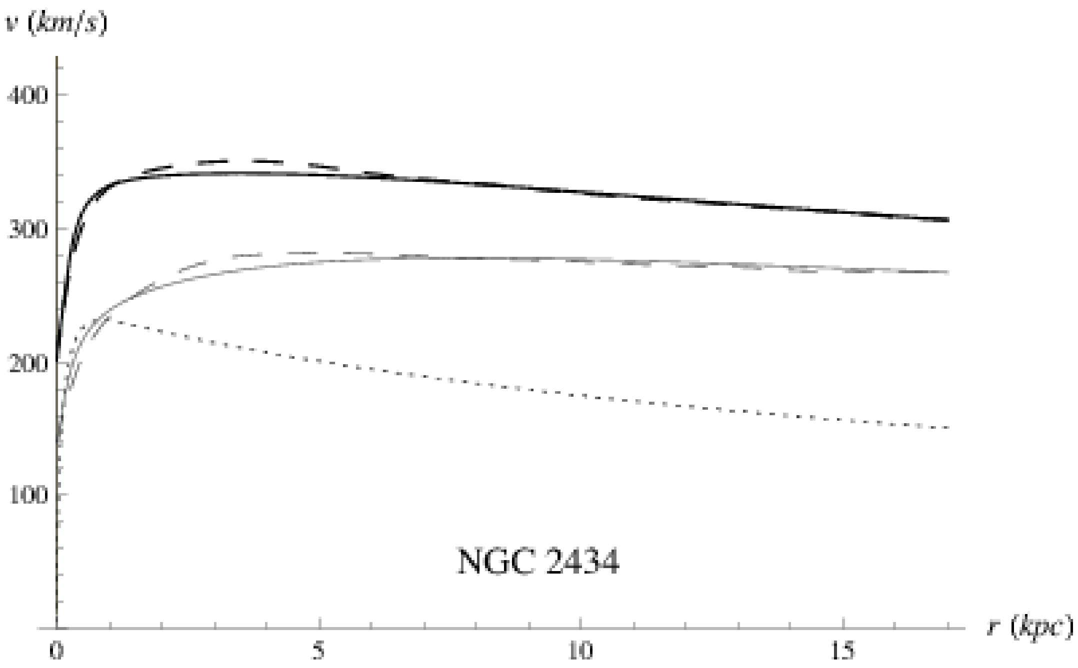


FIGURE 1.4: Galactic rotation curve: dark matter (dashed) vs. non-minimal coupling (full) with $f_2(R) \sim R^{-1/3}$ for the NGC 2434 galaxy. The dotted curve comes from visible matter, whilst the dashed grey is from the dark matter profile. [108].

$\left(\ddot{R}+2H\dot{R}\right)\frac{F_1'+2\mathcal{L}F_2'}{f_2}+\frac{F_1''+2\mathcal{L}F_2''}{f_2}\dot{R}^2$, as before. It was also shown that for $f_1(R) = R$ and $f_2(R) = 1+(R/R_n)^n$ or $f_2(R) = e^{(R/R_n)^n}$, we can find inhomogenenous spherical mass distributions with constant curvature solutions. These distributions exhibit admissible values for density and size of the interstellar medium. It was argued that the averaging of spherical distributions throughout the Universe can lead to an universal value for the cosmological constant [112, 113]. Furthermore, it is quite relevant to stress that the same equation of state parameter found for the dark matter scenario is recovered in the dark energy description, $w = \frac{n}{1-n}$, which may hint for an unified model of the dark sector [112, 113].

The identification of the NMC model with a scalar-tensor theory was derived in Ref. [114] based on tensor-multi-scalar field theories [115]. Contrarily to the pure $f(R)$ case, the NMC in the Einstein frame is equivalent to two scalar field theory, where although these are two distinct degrees of freedom, only one of them is dynamical. Additionally the potential term cannot be decomposed into a sum of potentials of each scalar field [114].

Furthermore, solar system observations impose tight constraints on gravity models [11, 12]. Unlike to pure $f(R)$ theories where there is an evident disagreement with data [116], the NMC model was shown to have a time-time metric component of the form of Newtonian plus Yukawa term whose effect can be negligible when the characteristic mass scales of the two functions, f_1 and f_2, are similar [117–119]. In fact, the Newtonian approximations need to be carefully addressed, since the background has to be the cosmological one. This poses very feeble constrains in some specific models of NMC theories [117–119].

The NMC model induces an extra potential term in the cosmic version of the virial theorem, also known as Layzer-Irvine equation, which can be identified with the missing mass problem on clusters [120]. These theories are stable under cosmological perturbations [121] and have some consequences for black hole solutions [122]. They also provide a natural framework for preheating [123] and has some bearings on stellar stability [124].

Some wormholes solutions in the NMC model are possible provided that normal matter satisfies the energy conditions at the throat, and the higher order terms in curvature derivatives of the NMC are responsible for the null energy condition violation [125–127].

1.3.1 Palatini formulation

Until now, we have explored some new physics coming from this non-minimal coupling between matter and curvature in the metric formalism. However, it is worth mentioning that these theories were also studied in the Palatini formulation [128]. In fact, the variation of the action relatively to the metric yields:

$$\Theta R_{(\mu\nu)}(\Gamma) - \frac{1}{2} f_1 g_{\mu\nu} = \frac{f_2}{2\kappa} T_{\mu\nu} \,, \tag{1.28}$$

where the functions $f_i(g^{\mu\nu} R_{\mu\nu}(\Gamma))$ are now written in terms of the connection. In its turn, the variation with respect to the connection gives:

$$\bar{\nabla}_\mu \left[\sqrt{-g}\, \Theta\, g^{\mu\nu} \right] = 0 \tag{1.29}$$

where $\bar{\nabla}_\mu$ is the covariant derivative built from the connection.

In this formalism, it was shown that it is possible to express the connection with a Levi-Civita form in terms of a metric, $h_{\mu\nu} = \Theta g_{\mu\nu}$, conformal to the physical one. Furthermore, the energy-momentum tensor is still not conserved and the extra force is also present [128]. It was also discussed an action with a more general dependence on the matter Lagrangian, which results in two self-interactions of matter in the geodesic equation [128].

We further point out that more general matter-curvature couplings were proposed in the literature. For instance, we can assume that the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar and of the matter Lagrangian, $f(R, \mathcal{L})$ [129]. This model has some bearings on the geodesic deviation, Raychaudhuri equation, and tidal forces [130]. Furthermore, extensions of $f(R, \mathcal{L})$ gravity were considered by including a generalised scalar field and kinetic term dependences [131] or by a NMC between the scalar curvature and the trace of the energy-momentum tensor, $f(R, T)$ [132]. The latter includes $f(R, \mathcal{L})$ gravity when $T \propto \mathcal{L}$. Of course, we can also add further mixed terms, such as $R_{\mu\nu} T^{\mu\nu}$ [133, 134] and so on. We refer the reader to Refs. [135, 136] for a more exhaustive and detailed review of modified gravity based on matter-curvature couplings.

Although modified gravity models may explain dark matter and dark energy at large scales, they must pass several other observational tests and need to be consistent with the Universe evolution. In particular, they need to be compatible with inflation, either by providing the mechanism for this accelerated expansion or by not preventing the inflationary expansion due to another mechanism. In the next Section, we will discuss the main ideas behind inflation.

1.4 Inflation

IN 1929, the astronomer Edwin Hubble found that the nearby galaxies were moving away from each other at a rate known as the Hubble constant, although the proposal of an expanding Universe had been firstly derived from the Einstein's equations by Alexander Friedmann in 1922 and by George Lemaître in 1927.

The most interesting fact of Hubble's discovery was that the farther the galaxies are, the faster they move away. Straightforwardly, physicists concluded that the Universe was expanding, hence somewhere in the past it should have been smaller, hotter and denser. From this, it came the idea of a Big Bang, that is, there was an extremely hot and dense singularity that remained expanding until now, giving birth to the Universe. This simple model was very successful in explaining the Big Bang Nucleosynthesis (predicting the correct abundance of all light elements), the history of the Universe itself and the existence of an universal background temperature (predicted by Ralph Alpher, Robert Herman and George Gamow in 1948, in order to explain why the heavy elements were only produced in stars).

After the accidental discovery of the Cosmic Microwave Background Radiation by Arno Penzias and Robert Wilson in 1964, who were awarded the Nobel Prize in 1978, one striking conclusion came across: there was a low temperature (around $2,7$K) that permeated the entire Universe (in fact the work of Alpher, Herman and Gamow predicted a temperature of about 5K). But the only possible explanation for this feature was that at some point in the past all regions in space were in causal contact, otherwise, today two distant regions of the Universe should not exhibit the same temperature.

Several theoretical issues arose later on, namely the monopoles, the flatness and the horizon problems; in other words, why there was no observed topological defects like magnetic monopoles putatively generated in the context of Grand Unified Theories; why the geometry of the Universe is so flat nowadays; and why do we observe such a large scale homogeneity and isotropy? The only possible answer to these questions was the cosmic inflation hypothesis [87, 137–139], which implied that the Universe had undergone an accelerated expansion at very early times, between the Planck and the Electroweak epochs. This also provides a mechanism to understand the origin of the observable large scale structures due to quantum fluctuations of the inflaton field, ϕ.

An inflationary epoch can arise from a modified type of gravity, as in $f(R)$ theories with $f(R) = R + \alpha R^2$ [87], or from the dynamics of a scalar field [137–139], the inflaton. Features of the scalar field potential are now quite constrained by the recent measurements of the Cosmic Microwave Background (CMB) [140, 141]. The first realisations of inflation were excluded on various grounds, whilst Starobisky's model is compatible with Planck data at 1σ level. In fact, given the invariance of a scalar field under Lorentz transformations, homogeneity and isotropic are not spoiled, as they would if inflation was driven by a vector field [142], unless higher curvature terms are taken into consideration and the vector field obeys to a $SO(3)$ symmetry as shown in Refs. [143–145]. However, further analysis needs to be done in order to fully compare these vector field inflationary models with existing data.

It is worth mentioning that most of modified gravity models can be formulated in the Einstein frame as General Relativity and a scalar field with a given potential. Thus, we shall now explore the basic ideas of the inflation paradigm and discuss scalar field inflation.

1.4.1 General inflationary dynamics

Let us consider a homogeneous and isotropic Universe described by the Robertson-Walker (RW) line element, Eq. 1.14.

The standard Lagrangian density of a homogeneous and isotropic scalar field, the inflaton, is given by:

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \ , \tag{1.30}$$

which can be identified with a perfect fluid with energy density $\rho_\phi := \dot{\phi}^2/2 + V(\phi)$ and pressure $p_\phi = \dot{\phi}^2/2 - V(\phi)$.

In fact, the Einstein's field equations give the following set of equations for a perfect fluid in the RW metric:

$$H^2 = \frac{\rho}{3M_P^2}\,, \tag{1.31}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}\left(\rho + 3p\right)\,, \tag{1.32}$$

where the first equation is called Friedmann equation [146, 147] whilst the second is Raychaudhury equation [148] (although different from the geometric one named in the same way which gives the expansion of a congruence). We further note that the Hubble parameter is given by $H = \dot{a}/a$.

Before continuing, let us discuss three crucial concepts in inflation: the particle and event horizons, and the Hubble radius. The first corresponds to the maximal comoving distance from which an observer at time, t, is able to receive signals traveling at the speed of light from an early time, t_0:

$$d_p = c\int_{t_0}^{t}\frac{dt}{a} = c\int_{a_0}^{a}(aH)^{-1}d\ln a\,, \tag{1.33}$$

where $(aH)^{-1}$ is the comoving Hubble radius, which is the maximal distance between two communicating particles at the present time.

In its turn, the event horizon is related to the maximal distance a particle now can influence another one at a later time, t_f:

$$d_h = c\int_{t}^{t_f}\frac{dt}{a}\,. \tag{1.34}$$

Therefore, the inflationary epoch is characterised by a particle horizon bigger than a Hubble horizon, such that although two particles which were in causal contact early on, they cannot communicate with each other at the present time, or at the CMB era. This can be expressed by a set of equivalent formulations (for more details on Inflation, see, for instance, Refs. [149, 150]):

- Accelerated expansion arising from a shrinking comoving Hubble radius:

$$\frac{d}{dt}\left(aH\right)^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0 \implies \ddot{a} > 0\,, \tag{1.35}$$

- Slowly-varying Hubble parameter from a shrinking comoving Hubble radius:

$$\frac{d}{dt}\left(aH\right)^{-1} = -\frac{1}{a}(1-\epsilon) < 0\,, \tag{1.36}$$

  where $\epsilon := -\dot{H}/H^2$.

- Quasi-de Sitter spacetime with $\epsilon \ll 1$:

$$ds^2 \approx -dt^2 + e^{2Ht}d\vec{r}^2\,. \tag{1.37}$$

- Negative pressure or violation of the strong energy condition, which can be seen by combining the Friedmann equation and the continuity equation, $\nabla_\mu T^{\mu 0} = 0 \iff$

$\dot{\rho} + 3H(\rho + p) = 0$, in the definition of ϵ:

$$\epsilon := -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{p}{\rho}\right) < 1 \iff w := \frac{p}{\rho} < -\frac{1}{3} . \tag{1.38}$$

- Constant density from combining the Friedmann equation and the continuity equation:

$$\left|\frac{d \ln \rho}{d \ln a}\right| = 2\epsilon < 1 . \tag{1.39}$$

In fact, all of these definitions for an inflationary period can be expressed in terms of two main Hubble slow-roll parameters. The first one was already defined above and corresponds to the fractional change of the Hubble parameter in a Hubble time:

$$\epsilon := -\frac{\dot{H}}{H^2} = -\frac{d \ln H}{d \ln N} \ll 1 , \tag{1.40}$$

where $N := \int H dt$ is the number of e-folds of inflation, which measures the time for the spacetime to increase by a factor of e. In order to have an homogeneous Universe after inflation, this process has to last a sufficiently long time. Thus, a second slow-roll parameter can be defined as the fractional change of ϵ per Hubble time:

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} = -\frac{\ddot{H}}{\dot{H} H} , \tag{1.41}$$

whose absolute value has to be small, $|\eta| < 1$.

Inflation can be achieved from a scalar field as already pointed out. Therefore, we shall now explore inflation driven by a scalar field, for which two main realisations are possible: cold and warm inflation.

1.4.2 Cold Inflation

We start with the Lagrangian, Eq. (1.30), for which we require the inflaton to slow-roll along its potential. During this process, the energy density of the scalar field is potential dominated. Thus, the Friedmann's equation becomes:

$$H^2 = \frac{\rho_\phi}{3M_P^2} = \frac{\dot{\phi}^2/2 + V(\phi)}{3M_P^2} \approx \frac{V(\phi)}{3M_P^2} \approx const , \tag{1.42}$$

which coincides with the definitions of a inflationary period.

The equation of motion for ϕ is a Klein-Gordon equation and it is given by:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \implies \dot{\phi} \approx -\frac{V'}{3H} . \tag{1.43}$$

Hence, the Hubble parameter can now be expressed as:

$$H = \frac{\partial \ln a}{\partial t} = \dot{\phi}\frac{\partial \ln a}{\partial \phi} \approx -\frac{V'}{3H}\frac{\partial \ln a}{\partial \phi} , \tag{1.44}$$

from which, and resorting to previous relations and then integrating over ϕ, we get the dependence of the scale factor in terms of the inflaton:

$$a = a_0 e^{-M_P^{-2} \int d\phi V/V'} . \tag{1.45}$$

The inflaton behaviour can be extracted from the dynamics of two slow-roll potential parameters. The first one measures the slope of the potential

$$\epsilon_\phi := \frac{M_P^2}{2} \left(\frac{V'}{V} \right)^2 , \tag{1.46}$$

where flat potentials lead to $\epsilon_\phi < 1$, thus ensuring inflation; whilst the second which is related with the curvature of the potential

$$\eta_\phi := M_P^2 \frac{V''}{V} . \tag{1.47}$$

These slow-roll parameters are different from the Hubble's ones. However, resorting to the Friedmann equation and during slow-roll we have the following identities: $\epsilon_\phi \approx \epsilon$ and $\eta_\phi \approx 2\epsilon - \frac{1}{2}\eta$.

Quantum fluctuations

In fact, scalar field inflation also provides the seeds for the formation of structures in the Universe, such as galaxies, clusters, filaments, voids and ensued substructures. This is achieved due to the quantum fluctuations of the inflaton field. Actually, Eq. 1.43 was derived bearing in mind homogeneity and isotropy of the Universe, but quantum fluctuations are affected by the neglected terms, namely the Laplacian operator of the inflaton. Thus, considering small quantum fluctuations around the classical background value of the scalar field:

$$\phi = \bar{\phi} + \delta\phi , \tag{1.48}$$

together with the equation of motion of the scalar field, $\Box\phi = V'$, and in the slow-roll regime, we may consistently neglect the effective scalar field mass, i.e. $V''(\bar{\phi})$, such that we obtain:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{1}{a^2}\nabla^2\delta\phi \simeq 0 . \tag{1.49}$$

From this equation, we are able to compute the power spectrum of inflaton perturbations in the Bunch-Davies vacuum, which yields [149, 150]:

$$P_\phi(k) = \frac{H^2}{2k^3} . \tag{1.50}$$

The relation between the inflaton and the gauge-invariant comoving curvature perturbation lead to a dimensionless power spectrum of inflaton perturbations [149, 150]:

$$\Delta^2_{\mathcal{R}}(k) = \left(\frac{H}{2\pi} \right)^2 \left(\frac{H}{\dot{\phi}} \right)^2 . \tag{1.51}$$

The associated scalar spectral index, which measures deviations from scale invariance of the spectrum, is then given by:

$$n_s - 1 := \frac{d \ln \Delta^2_{\mathcal{R}}}{d \ln k} = \frac{\dot{\phi}}{H} \frac{d \ln \Delta^2_{\mathcal{R}}}{d\phi} \approx 2\eta_\phi - 6\epsilon_\phi \,, \tag{1.52}$$

Additionally, we are able to compute the spectrum of tensor perturbations [149, 150]:

$$\Delta^2_t(k) = \frac{8}{M_P^2} \Delta^2_\phi(k) = \frac{2}{\pi^2} \frac{H^2}{M_P^2} \,, \tag{1.53}$$

from which we may compute the tensor spectral index [149, 150]:

$$n_t := \frac{d \ln \Delta^2_t}{d \ln k} = \frac{\dot{\phi}}{H} \frac{d \ln \Delta^2_t}{d\phi} \,, \tag{1.54}$$

and the tensor-to-scalar ratio, which measures the amount of gravitational waves produced during inflation, is written as:

$$r := \frac{\Delta^2_t}{\Delta^2_{\mathcal{R}}} \approx 16\epsilon_\phi \,. \tag{1.55}$$

Some of these quantities have been constrained by the Planck collaboration [141]:

$$n_s = 0.9603 \quad , \qquad A_s = \Delta^2_{\mathcal{R}}(k = k_*) = 2.2 \times 10^{-9} \quad , \qquad r < 0.11 \,. \tag{1.56}$$

After cold inflation, there is an epoch called reheating which allows the Universe getting hotter due to oscillations of the scalar field around its potential's minimum (where the inflaton may also decay into other massive particles), thus transforming the vacuum energy density via its oscillations into radiation energy density. Moreover, the inflaton field typically rapidly decays into bosons due a broad parametric resonance, which means that these particles are far from equilibrium and can decay into other particles that will become thermalised. This is called the preheating [151, 152].

However, there is a different realisation of inflation called warm inflation, which will shall discuss next.

1.4.3 Warm Inflation

In the warm inflation scenario, both inflaton and radiation play a relevant role and are coupled to the scalar curvature by an *ad hoc* term, $\Upsilon\dot{\phi}^2$. The first time that a term like that appeared in the literature was in Ref. [153] in a discussion about the reheating phase of the Universe just after the cold inflation scenario. Later on, a similar term was added to the inflaton and radiation equations providing a different dynamical realisation of inflation [154, 155]. Thus, the inflaton obeys:

$$\ddot{\phi} + (3H + \Upsilon)\,\dot{\phi} + V'(\phi) = \zeta \,, \tag{1.57}$$

and the radiation follows:

$$\dot{\rho}_r + 4H\rho_r = \Upsilon\dot{\phi}^2 \,, \tag{1.58}$$

where $\Upsilon\dot{\phi}^2$ is a phenomenological friction term and ζ is a fluctuating force related to the Υ factor by a fluctuation-dissipation theorem and which does not affect the overall dynamics of the inflaton since $\langle\zeta\rangle = 0$, meaning that it will only be relevant on the inflaton fluctuation dynamics (which provides the seeds for the structure formation in the Universe). We also point out that the Υ is more general than the decay width of particle interaction, Γ, although in some specific situations it can become the same.

During the slow-roll regime, we may disregard the $\ddot{\phi}$ and the $\dot{\rho}_r$ terms in the previous equations, so that:

$$\dot{\phi} \approx -\frac{V'(\phi)}{3H+\Upsilon}, \tag{1.59}$$

and

$$\rho_r \approx \frac{\Upsilon}{4H}\dot{\phi}. \tag{1.60}$$

In fact, by defining a new parameter $Q = \Upsilon/3H$, we can systematically classify this inflationary paradigm [155–157]. On one hand, cold inflation can be seen as the limit of warm inflation when both Q and $T \ll H$ are extremely small. On the other hand, two other regimes can be obtained, namely:

- Weak dissipative regime ($\Upsilon < H$):

$$3H\dot{\phi} \approx -V'(\phi) \quad , \quad 4H\rho_r \approx \Upsilon\dot{\phi}^2 ; \tag{1.61}$$

- Strong dissipative regime ($\Upsilon > H$):

$$3(1+Q)H\dot{\phi} \approx -V'(\phi, T) \quad , \quad 4H\rho_r \approx \Upsilon\dot{\phi}^2 . \tag{1.62}$$

These scenarios lead to some dependencies on the temperature for some observables. Furthermore, the slow-roll approximation for warm inflation is characterised by four parameters:

$$\epsilon_\phi < 1+Q \quad , \quad |\eta_\phi| < 1+Q \quad , \quad \beta_\Upsilon < 1 \quad , \quad \delta_T < 1 , \tag{1.63}$$

where: $\beta_\Upsilon := M_P^2 \frac{\Upsilon' V'}{\Upsilon V}$ and $\delta_T := T\frac{\partial V'/\partial T}{V'}$.

Thus, warm inflation has some distinct implications for the inflaton potential. In the case of cold inflation, we have the inflaton rolling down its sufficiently flat potential, driving the Universe to an accelerated expansion. During this stage, the vacuum energy density, corresponding to the inflaton's energy density, remains almost the same and manifestly dominant over the radiation component; after the slow-roll period the inflaton decays into other particles, thus reheating the Universe. However, in the warm picture, the inflaton rolls down its potential (which does not need to be extremely flat) and since it is coupled to radiation, it continuously converts some of its energy into radiation, thus heating the Universe without the need of a reheating stage after the slow-roll period. In Fig. (1.5), we present a pictoric comparison between the two realisations of inflation.

Thus, inflation is a fundamental piece for understanding the Universe. We have referred that gravitational waves could be produced due to tensor perturbations of the metric. However, this is not the unique case: colliding massive objects, such as black holes or neutrons stars, among other sources can also produce them. In the next Section, we shall examine this topic.

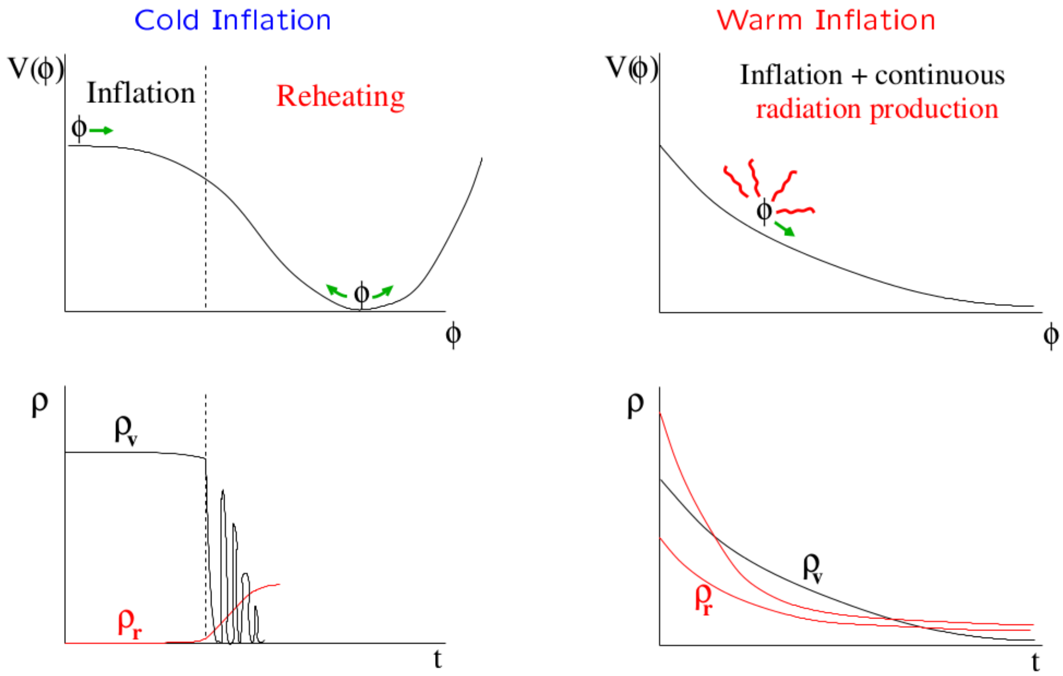


FIGURE 1.5: Comparison between warm and cold inflation as far as the potential and energy density evolutions is concerned [158, 159]. The strong and the weak dissipative regimes in warm inflation are shown in the right-bottom picture and correspond to higher or lower values for the radiation energy density component relatively to the vacuum energy density.

1.5 Gravitational Waves

One of the most surprising predictions of GR is the existence of ripples in the fabric of spacetime, the gravitational waves (GW). In fact, they have discovered as a weak field solution to GR in November of 1916 [160]. However, the first conclusions were not completely correct since Einstein only realised the quadrupole nature of such waves in 1918 [161]. Later on, in 1937 Einstein and Nathan Rosen showed that gravitational waves not only existed in the linear regime of GR, but also in the full nonlinear theory [162].

Nonetheless, the first indirect observational evidence of the existence of gravitational waves was obtained from the energy loss of the binary pulsar PSR 1913+16 discovered by Russell Hulse and Joseph Taylor in 1974 [163]. This observation awarded them the Nobel Prize in Physics in 1993. And only very recently, in 2015, GW were directly detected from black holes binaries mergers by the Laser Interferometer Gravitational wave Observatory (LIGO) collaboration [164], and this led to the Nobel Prize in Physics in 2017 being awarded to Rainer Weiss, Barry C. Barish and Kip S. Thorne. It is worth mentioning that this discovery was not only about the observation of a direct imprint of a GW, but also about the first time a signal of a merger of black holes and the subsequent formation of a more massive black hole was observed. This observed signal is shown in Fig. 1.6.

In fact, the detection was possible due to the two interferometers of the LIGO project located at Hanford, in Washington, and at Livingstone, in Louisiana, in the USA. Both instruments are based on Michelson interferometry, as shown in Fig. 1.7.

Just after the first signal, others were detected from various black hole binaries [166–168]. Furthermore, another amazing feature was found: all the mergers happened at frequencies

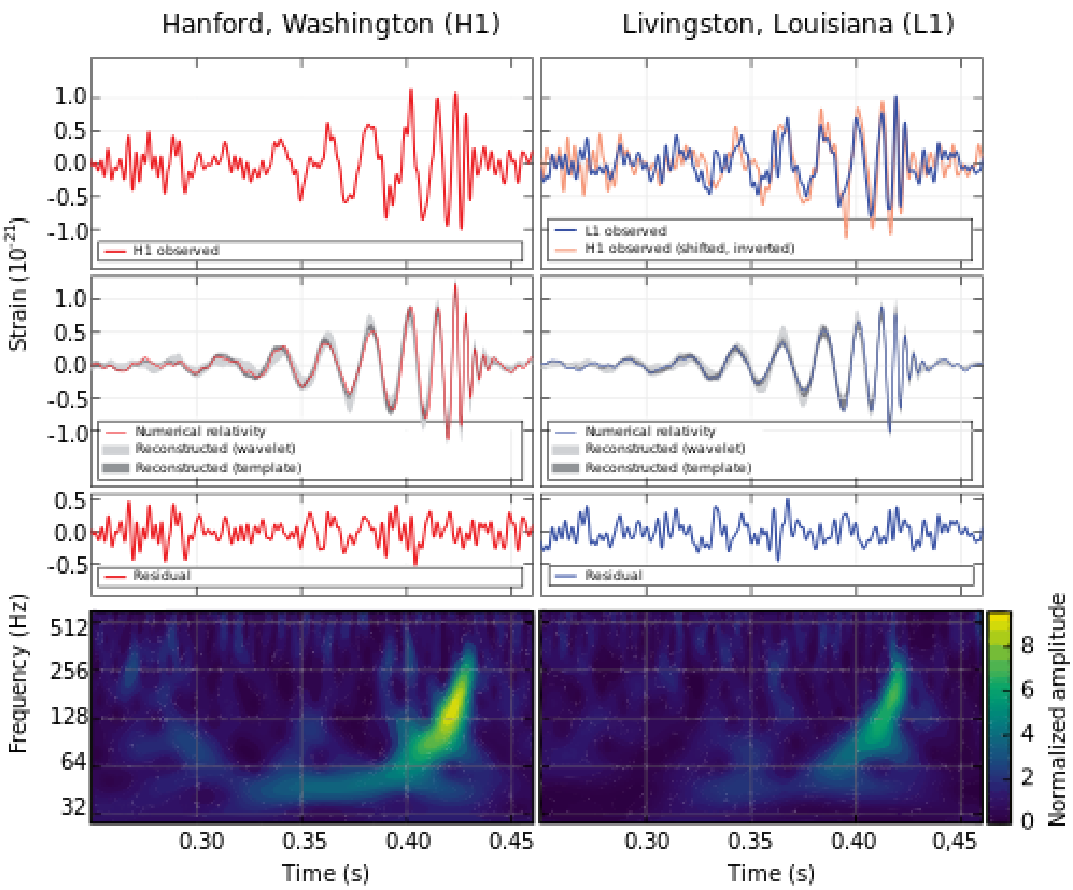


FIGURE 1.6: The first observed signal of a gravitational wave. The strain, $\Delta l/l$, increases during the ring down of the two black holes, has a peak in the merging and decreases after the formation of the final black hole. The frequency increases over time and reaches the maximum at the merger. This is called the chirp, in analogy to the chirp of a bird [164].

which lie in the human audition range. Afterwards, another important observation was achieved: the merge of binary neutron stars. This resulted in both a gravitational wave signal and an electromagnetic counterpart [169–172]. Recently, the conclusions from the two observing runs of Advanced LIGO and Advanced Virgo interferometers combined with four new ones were addressed in Refs. [173, 174]. These astonishing discoveries have paved the way for a new era in astronomy and physics, and opened a new window to assess the ultimate nature of gravity.

In the two interferometers from LIGO collaboration there is a phase shift of half wavelength, so that when the two beams are summed in the beam splitter, after being reflected by the mirrors, are expected to produce a null signal. Therefore, when this is not the case, a signal of some distortion was detected. As one may expect, there are several interference signals that directly affect the observations: seismic activity, vehicles passing by, trains, thermal noise, among many others. This is why both interferometers have a lot of supplementary detectors to fully characterise this background noise and exclude it from an eventual gravitational wave measurement.

Actually, both LIGO interferometers have two 4 km arms, which can detect length variations of the order of 10^{-19} m, due to a precision of $\frac{\Delta l}{l} = 5 \times 10^{-22}$ $(at\ 500Hz)$. This means that they are able to detect length variations 10^5 smaller than the proton size. This is the

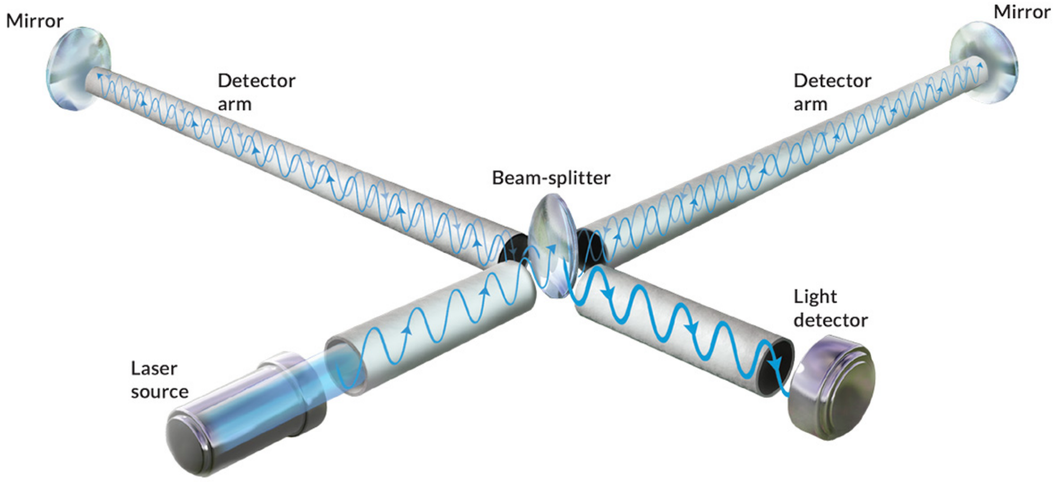


FIGURE 1.7: Schematic representation of a Michelson interferometer. A light beam is sent to a "beam splitter", where it is split in two beams: one which goes straight and another which is reflected by 90º. Both beams reach a mirror and are reflected back to the beam splitter that sums the signals and send them to a detector. Figure extracted from Ref. [165].

most precise instrument ever built in History!

In fact, a gravitational wave corresponds to the linear excitation of the metric:

$$h_{\mu\nu} := g_{\mu\nu} - g^{(0)}_{\mu\nu} \,, \tag{1.64}$$

where $g^{(0)}_{\mu\nu}$ is the background metric.

In linearised GR, this can be found by plugging this ansatz into the metric field equations, Eq. 1.4, and assuming that the background is Minskowskian, $g^{(0)}_{\mu\nu} = \eta_{\mu\nu} = diag(-1, 1, 1, 1)$:

$$-\frac{1}{2}\left[\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} - \partial_\nu\partial_\rho\bar{h}^\rho_\mu - \partial_\mu\partial_\rho\bar{h}^\rho_\nu\right] = \frac{1}{2\kappa}T_{\mu\nu} \,, \tag{1.65}$$

where $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ with $h := h^\mu_\mu$, and we have dropped terms quadratic in such quantity since we are assuming that $|h_{\mu\nu}| \ll 1$. This is a wave equation for $\bar{h}_{\mu\nu}$ with a matter source and it is valid outside the source, where the nonlinear corrections are negligible. Furthermore, there is a useful gauge which greatly simplifies the wave equation, the Lorenz gauge:

$$\partial_\mu\bar{h}^{\mu\nu} = 0 \,. \tag{1.66}$$

This gauge condition has a residual gauge transformation invariance of the form $\bar{h}^{\mu\nu} \to \bar{h}^{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$ provided that the extra functions ξ^μ satisfy $\Box\xi^\mu = 0$.

In this gauge, similar to the electromagnetic case, the equation of motion for the metric perturbation is:

$$\Box\bar{h}_{\mu\nu} = -\frac{1}{\kappa}T_{\mu\nu} \,, \tag{1.67}$$

In this linearised version of GR, the conservation law is $\partial_\mu T^{\mu\nu} = 0$. However, linearisation ignores that "gravity gravitates", i.e., the non-linearity of GR includes the fact that the energy–momentum tensor associated with the gravitational field acts also as a source for the gravitational field. A way to circumvent this issue is to compute the energy-momentum tensor associated with the perturbation $h_{\mu\nu}$ and to introduce it as a source. This leads to a correction to the metric perturbation, $h^{(1)}_{\mu\nu}$, which in turn generates a correction to the energy-momentum tensor. This process can continue iteratively and leads to the full non-linear theory. This approach motivates the validity of the linearisation process since it provides a self-consistent analysis of gravity in a fixed Minkowskian geometry [175,176].

Thus, we can proceed to analyse the solutions of Eq. (1.67) in vacuum, which are straightforwardly found:

$$\bar{h}^{\mu\nu} = A^{\mu\nu} e^{ik_\alpha x^\alpha} \,, \tag{1.68}$$

where $A^{\mu\nu}$ is a symmetric tensor whose entries are complex constants, $k_\alpha = (k_0, \vec{k})$ is the 4-wavenumber, a vector with real entries which satisfy $k_\alpha k^\alpha = 0$. These solutions are called plane waves. The gauge condition, Eq. (1.66), provides the constraint:

$$A^{\mu\nu} k_\nu = 0 \,. \tag{1.69}$$

The residual gauge freedom on ξ^μ can be used to further simplify the analysis. Without loss of generality, we can choose a referential where the propagation of the gravitational wave coincides with the $z-$direction, $k = (k, 0, 0, k)$ with $k = \omega / c$ given in terms of the angular frequency ω. By doing this, it is found that only two components of $A^{\mu\nu}$ are independent, hence:

$$A^{\mu\nu}_{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = a e_1^{\mu\nu} + b e_2^{\mu\nu} \,, \tag{1.70}$$

where $e_1^{\mu\nu} := A^{\mu\nu}|_{a=1,b=0}$ and $e_2^{\mu\nu} := A^{\mu\nu}|_{a=0,b=1}$ are the linear polarisation tensors, which correspond to the $+$ and $\times$ polarisation states of a gravitational wave, respectively (see Fig. 1.8). This choice of ξ^μ is the so called Transverse-Traceless gauge (TT). Furthermore, these predicted modes in GR are massless and travel at the speed of light, and their amplitude is inversely proportional to the distance from the source.

Nonetheless, other metric gravitational theories are shown to exhibit up to six polarisation states: two tensor, two vector and two scalar modes. These six modes are shown in Fig. 1.9.

Actually, we have to bear in mind that the linear superposition of plane waves is also a solution for the linear equations of motion, hence the full solution is:

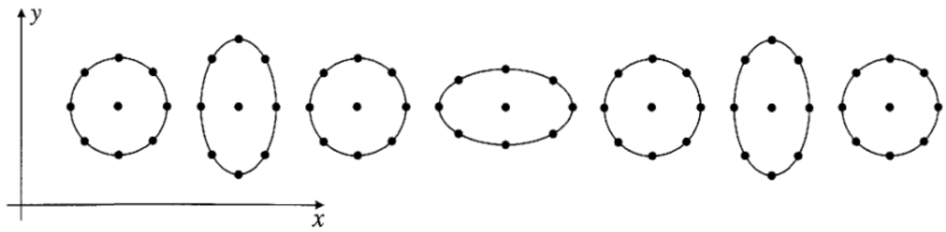
$$\bar{h}^{\mu\nu}(x) = \int A^{\mu\nu}(\vec{k}) e^{ik_\alpha x^\alpha} d\vec{k} \,. \tag{1.71}$$

However, we have to bear in mind that the physical solutions correspond to the real part of the previous equation.

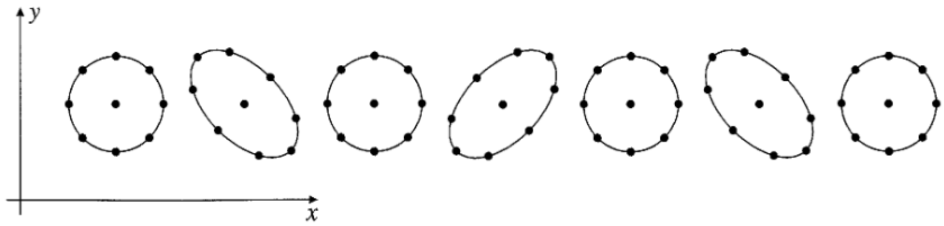
When we consider a non-zero energy-momentum tensor, the solution can be obtained from the Green's function method, yielding:

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(ct_r, \vec{y})}{|\vec{x} - \vec{y}|} d\vec{y} \,, \tag{1.72}$$

where $\vec{x}$ are the spatial coordinates of the field at the point where $\bar{h}^{\mu\nu}$ is computed, $\vec{y}$ are the spatial coordinated of a point in the source, $ct_r := ct - |\vec{x} - \vec{y}|$ is the distance at the retarded times. This comes from the fact that the perturbation of the gravitational field at the event



(A) Gravitational wave with "+" polarisation passing perpendicularly to ring of test particles. The GW emerges out of the plane of the paper. Extracted from Ref. [177].



(B) Gravitational wave with "×" polarisation passing perpendicularly to ring of test particles. The GW emerges out of the plane of the paper. Extracted from Ref. [177].

FIGURE 1.8: The two polarisation states in GR.

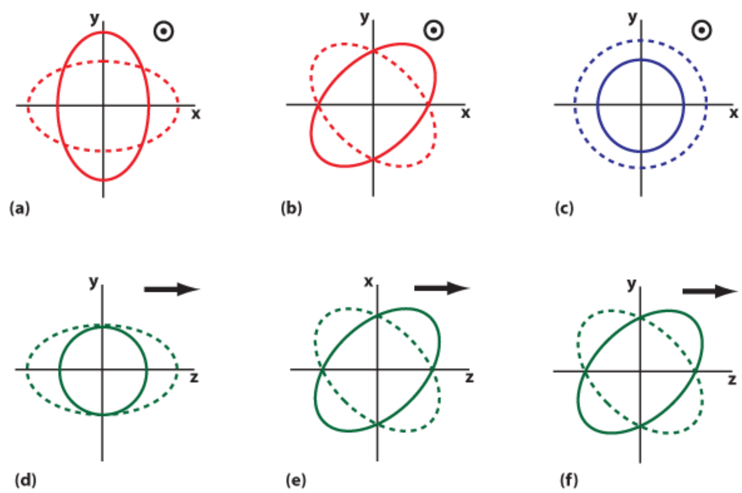


FIGURE 1.9: The polarisation states in a generic metric theory, when a gravitational wave passes through a ring of test particles. The GW travels in the $+z$ direction. In GR only the modes a) and b) exist. Extracted from Ref. [12].

$(ct, \vec{x})$ is the result of the influences of the energy and momentum sources at the points $(ct, \vec{y})$ which lie in the past lightcone.

In fact, the previous solutions can be Taylor expanded in terms of multipoles (following the notation of Ref. [176]):

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!} M^{\mu\nu i_1 \dots i_\ell}(ct_r) \partial_{i_1} \dots \partial_{i_\ell} \, , \tag{1.73}$$

where $M^{\mu\nu i_1 \dots i_\ell}(ct_r) := \int T^{\mu\nu}(ct_r, \vec{y}) y^{i_1} \dots y^{i_\ell} d\vec{y}$. Given that the ℓ-term goes as $1/r^{\ell+1}$ only the first terms will dominate. If we consider only the first term, which gives the asymptotic behaviour far from the source (compact source approximation), we find:

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4 r} \int T^{\mu\nu}(ct_r, \vec{y}) d\vec{y} \,, \tag{1.74}$$

where we note that the time-time component of this equation is the total energy/mass of the particles, and the time-spatial components correspond to the 4-momentum of the particles along the direction of each axis. These two quantities are conserved given the conservation law $\partial_\mu T^{\mu\nu} = 0$. However the spatial-spatial components correspond to integrated internal stresses in the source and can be rewritten, taking into account the conservation law, as the quadrupole formula:

$$\bar{h}^{ij}(ct, \vec{x}) = -\frac{2G}{c^6 r} \frac{d^2 I^{ij}(ct_r)}{dt_r^2} \,, \tag{1.75}$$

where $I^{ij} := \int T^{00} y^i y^j d\vec{y}$ is the quadrupole-moment tensor, which is constant on each hypersurface of constant time.

However, extensions of GR predict different observational signatures. For instance, the existence of additional polarisations states which propagate with different velocities, or have effective masses. Some of these observables have been already highly constrained: an upper bound on the mass of the graviton [167] and a lower and an upper bounds on the speed and the group velocity of the gravitational wave [172, 178]. In fact, alternative gravity models equivalent to cosmological scalar fields in scalar-tensor theories of gravity are in strong disagreement with recent data [179–182].

As a matter of fact, the observation of gravitational waves signal have paved a new prime arena for gravitation. It is now possible to test GR in the strong regime, when the non-linear effects are computed, as well as several other gravitational theories.

Part II

Results

Chapter 2

Inflation in NMC

As discussed in Chapter 1, inflation is the paradigm which solves the initial condition problems of the standard Hot Big Bang model. Most of the inflationary models are based on General Relativity (GR). However, there are many reasons why we should seek inflationary models beyond GR. Therefore, we explore in this Chapter the inflationary dynamics within alternative theories with a non-minimal coupling between matter and curvature.

In fact, the idea that non-minimal couplings can affect inflation is not new. For example, in the Bezrukov-Shaposhnikov model, the Higgs boson acts as the inflaton non-minimally coupled to the scalar curvature [183]. However, in this model the full scalar Lagrangian is not coupled with gravity, but only a specific function of the inflaton. This scenario is compatible with data [184]. Additionally, other models with curvature-matter couplings in the context of inflation were studied [185, 186].

The goal of this Chapter is to explore the modifications to inflationary scenarios arising from a slowly rolling scalar field non-minimally coupled to an arbitrary function of the Ricci scalar. Afterwards attention to the observational data will be given, and some of the most commonly used inflationary potentials will be analysed in these scenarios.

This Chapter is based on Ref. [1].

2.1 Inflation in the NMC model

We start by considering a homogeneous and isotropic Universe, which is described by a RW metric, Eq. (1.14). We further consider a flat Euclidean metric $\gamma_{ij} = \delta_{ij}$, since inflation will exponentially smooth the effects of spatial curvature, hence addressing the flatness problem.

As for the matter content in the Universe, we assume that it is well described by a set of perfect fluids which dominate the energy balance at different epochs. For each fluid, the energy-momentum tensor takes the form:

$$T^{\mu\nu} = (\rho + p)\, U^\mu U^\nu + p g^{\mu\nu} \,, \tag{2.1}$$

where U^μ is the fluid's 4-velocity. The time component of the non-conservation law, Eq. (1.19), gives:

$$\nabla_\mu T^{\mu 0} = \dot{\rho} + 3H\,(\rho + p) = \begin{cases} 0 & \text{if } \mathcal{L} = -\rho \\ -\frac{F_2}{f_2}\,(\rho + p)\,\dot{R} & \text{if } \mathcal{L} = p \end{cases} \,, \tag{2.2}$$

where the dots denote time derivatives.

In the case of a homogeneous scalar field whose Lagrangian density is given by Eq. (1.30), it is possible to make the following identifications with a perfect fluid energy density and pressure: $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ and $p_\phi = \dot{\phi}^2/2 - V(\phi) = \mathcal{L}_\phi$. The inflaton's dynamics for the NMC theories, Eq. (1.16), is governed by the field equation:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = -\frac{F_2}{f_2}\dot{R}\dot{\phi} \equiv -\Gamma\dot{\phi}\,, \tag{2.3}$$

where $V'(\phi) = dV/d\phi$. It is straightforward to verify that the non-minimal matter-curvature coupling will induce a friction term in the inflaton's equation of motion, which seems to be analogous to the case of warm inflation scenario [154, 155] we have discussed in the Introduction.

In addition, a similar term also appears in the continuity equation for a radiation component, whose Lagrangian density is defined as $\mathcal{L}_r = p$. Thus:

$$\dot{\rho}_r + \frac{4}{3}\left(3H + \frac{F_2}{f_2}\dot{R}\right)\rho_r = 0\,. \tag{2.4}$$

This expression can be integrated, yielding [123]:

$$\rho_r(t) = \rho_{ri}\left(\frac{a_i}{a(t)}\right)^4\left[\frac{f_2(R_i)}{f_2(R)}\right]^{4/3}\,. \tag{2.5}$$

When $\Gamma := \frac{F_2}{f_2}\dot{R} > 0$, we may expect this friction term to facilitate the slow-roll of the inflaton by damping its motion. However, contrarily to warm inflation, $\Gamma > 0$ yields a sink and not a source for the radiation energy density. Hence, the NMC would provide a way to dilute radiation faster than the expansion rate instead of counter-acting this effect. The other possibility, $\Gamma < 0$, has a negligible effect (proportional to the slow-roll parameter ϵ) on the inflaton's motion for strong curvature/density regimes, where inflation takes place. Therefore, this term cannot sustain a thermal bath during this period, and inflation will occur in a supercooled regime unless we consider interactions between the inflaton and other degrees of freedom, as e.g. in Refs. [187, 188]. Therefore, we will henceforth consistently neglect the Γ term.

2.1.1 The slow-roll approximation

Let us now study the slow-roll regime during inflation. To do so, we firstly note that the metric field equations, Eq. 1.17, can be rewritten as:

$$\left(F_1 + \frac{F_2\mathcal{L}}{\kappa}\right)G_{\mu\nu} = \frac{1}{2\kappa}f_2 T_{\mu\nu} + \Delta_{\mu\nu}\left(F_1 + \frac{F_2\mathcal{L}}{\kappa}\right) + \frac{1}{2}g_{\mu\nu}\left(f_1 - F_1 R - \frac{F_2 R\mathcal{L}}{\kappa}\right)\,, \tag{2.6}$$

where $G_{\mu\nu}$ is the Einstein tensor.

In the context of inflation, it is very useful to express several parameters with an explicit dependence on the reduced Planck mass. Therefore, we shall introduce henceforth the notation $M_P^{-2} := 8\pi G = 2\kappa$ for the reduced Planck mass in units such that $c = 1$.

From the 00 and the ii components of the field equations, we get [121]:

$$f_2\rho = 3FH^2 + \frac{1}{2}\left(M_P^2 f_1 - FR\right) + 3H\dot{F}\,, \tag{2.7}$$

$$f_2 p = -3FH^2 - \frac{1}{2}\left(M_P^2 f_1 - FR\right) - 2H\dot{F} - \ddot{F} - 2\dot{H}F\,, \tag{2.8}$$

where $F = M_P^2 F_1 - F_2\rho$. We thus see that $p = -\rho$ up to terms that depend on time derivatives of the scalar curvature R and matter density ρ, which vanish in the exact de Sitter limit.

The time-time component can be written is a more elegant way, resulting in a modified Friedmann equation [189]:

$$H^2 = \frac{1}{6}\frac{1}{F_1 + \frac{F_2\mathcal{L}}{\kappa}}\left[\frac{f_2\rho}{\kappa} - 6H\partial_t\left(F_1 + \frac{F_2\mathcal{L}}{\kappa}\right) + \left(F_1 + \frac{F_2\mathcal{L}}{\kappa}\right)R - f_1\right]. \tag{2.9}$$

Resorting to the Hubble slow-roll parameters, ϵ and η, and noting that $R = 6\left(\dot{H} + 2H^2\right)$, then we can write:

$$R = 6H^2\left(2 - \epsilon\right) \implies \dot{R} \approx -24H^3\epsilon\,, \tag{2.10}$$

which means that a slow-roll regime is achieved by consistently neglecting the time variation of ρ and H requires, as in the GR case, that the conditions $\epsilon, |\eta| \ll 1$ are satisfied.

We shall henceforth consider the case of a pure non-minimal coupling, i.e. $f_1(R) = R$, so to isolate its effects. Taking the slow-roll limit of the Friedmann equation in these theories (2.9), for $\mathcal{L} = p \simeq -\rho$, we then obtain:

$$H^2 \simeq \left(\frac{f_2}{1 + \frac{2F_2\rho}{M_P^2}}\right)\frac{\rho}{3M_P^2}\,. \tag{2.11}$$

It is thus clear that this Friedmann equation may have a different behaviour from its GR version, depending on the inflaton energy density and on the explicit form of the non-minimal matter-curvature coupling function. In particular, a different behaviour in the high and low densities regimes is expected, which will be discussed in the next sections. However, we have to ensure that the effective gravitational constant does not change sign during inflation so that small anisotropies are stable. We refer the reader to the Appendix A for a discussion on this topic.

In addition, the ij-component of the field equations, for the flat FRW model in the pure NMC case, with a Lagrangian $\mathcal{L} = p$ and noting that $R = 12H^2 + 6\dot{H} = 6H^2 + 6\frac{\ddot{a}}{a}$, yields:

$$-\left[2\frac{\ddot{a}}{a} + H^2\right]\left(1 - \frac{F_2 p}{M_P^2}\right) + 3H^2\frac{F_2 p}{M_P^2} = \frac{f_2 p}{M_P^2} + \frac{2}{M_P^2 a^2}\Delta_{ij}\left(F_2 p\right)\,. \tag{2.12}$$

Substituting the Friedmann equation, Eq. (2.11), in the above equation and consistently neglecting $\Delta_{ij}(F_2 p)$, one obtains:

$$2\frac{\ddot{a}}{a} = \frac{f_2\bar{\rho}}{1 + F_2\bar{\rho}}\left[1 - \frac{1}{3}\left(\frac{1 + 4F_2\bar{\rho}}{1 + 2F_2\bar{\rho}}\right)\right]\,, \tag{2.13}$$

with $\bar{\rho} = \rho/M_P^2$. Once again, we obtain an accelerated period in the slow-roll regime for $p \simeq -\rho$, as in GR.

2.1.2 Quantum fluctuations

As we have seen in the Introduction, the inflaton's quantum fluctuations provide a natural mechanism for generating structure formation in the Universe. Thus, we aim now to determine the properties of the spectrum of such quantum fluctuations by considering the full Klein-Gordon equation, although the friction term in the slow-roll approximation can be neglected, yielding:

$$\Box\phi = -V'(\phi)\,, \tag{2.14}$$

where $\Box$ denotes the D'Alembertian operator. Since this is the same equation as in the GR case, the equation of motion for the small quantum fluctuations around the classical background value of the inflaton field, $\bar{\phi}$, is straightforwardly found:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{1}{a^2}\nabla^2\delta\phi \simeq 0\,, \tag{2.15}$$

where we note that the effective scalar field mass term, $V''(\bar{\phi})$, was consistently neglected.

Thus the power spectrum of inflaton perturbations in the Bunch-Davies vacuum is also immediate:

$$P_\phi(k) = \frac{H^2}{2k^3}\,, \tag{2.16}$$

although we stress that this has not the GR behaviour due to the modifications to the Friedmann equation introduced by the non-minimal matter-curvature coupling. The relation between the scalar field and the gauge-invariant comoving curvature perturbation is also unaffected in the slow-roll regime, so the dimensionless power spectrum of inflaton perturbations is:

$$\Delta^2_{\mathcal{R}}(k) = \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}}\right)^2 . \tag{2.17}$$

Given that the Ricci scalar, R, is quadratic in curvature and tensor perturbations, the NMC model will not modify the action functional for these perturbations at the leading quadratic order. Hence, the spectrum of tensor perturbations retains its standard form:

$$\Delta^2_t(k) = \frac{8}{M_P^2}\Delta^2_\phi(k) = \frac{2}{\pi^2}\frac{H^2}{M_P^2}\,. \tag{2.18}$$

As abovementioned, we are interested in studying deviations from GR observational signatures in the high curvature/density regime, but whose low-density behaviour reduces to the GR' one. In particular, we seek for non-minimal coupling functions of the form $f_2(R) = 1 + (R/M^2)^n$ for $n > 1$. The deviations produced by such functions will be discussed next.

2.1.3 The high density limit

Let us consider now the high density regime of inflation in the NMC models of gravity indicated above. In the literature, exponential and power-law non-minimal coupling functions, $f_2(R)$, have been employed to tackle cosmological and astrophysical problem such as dark matter, dark energy, reheating after inflation, and black holes [108,110,112,113,122,123]. Hence, we shall explore these two possibilities within inflation.

If we a have a non-minimal coupling function as $f_2(R) = e^{R/M^2}$, then the modified Friedmann Eq. (2.11) simplifies to $\frac{R}{M^2} \simeq 1$ for $\rho/(M_P^2 M^2) \gtrsim 0.07$, which means that $H \simeq$

$M/\sqrt{6}$. For mass scales $M \ll M_P$, this solution is achieved for energy densities far below the Planck mass.

The remaining class of functions of interest is the power-law type, $f_2(R) = 1 + \left(\frac{R}{M^2}\right)^n$, where we find an asymptotic solution:

$$\frac{R}{M^2} = \left(\frac{2}{n-2}\right)^{1/n} + \mathcal{O}\left(\frac{M_P^2 M^2}{\rho}\right) \quad , \qquad n > 2\,. \tag{2.19}$$

We point out that for the special case where $n = 2$ the modified Friedmann equation yields $H^2 = \rho/3M_P^2$, which is the well known solution in GR that is now retrieved in addition to the trivial case $f_2(R) = 1$. For $n < 2$ there are no real solutions for R/M^2, hence it lies outside the present physical discussion. It is worth noting that for $n > 2$, the solution yields $R/M^2 \sim \mathcal{O}(1)$, which was also found in the above exponential case, so that $H \sim M$ up to numerical factors. This means that this behaviour is found for energy densities $\rho \gtrsim M_P^2 M^2$, which are sub-Planckian provided that $M \ll M_P$. For instance, the case $n = 3$ with $\rho \gtrsim M_P^2 M^2$ lealds to a Friedmann equation with the asymptotic behaviour:

$$H^2 \simeq \frac{M^2}{12}\left(2^{1/3} - \frac{1}{6 \times 2^{1/3}} \frac{M_P^2 M^2}{\rho}\right)\,. \tag{2.20}$$

In Fig. 2.1 we show the comportment of the modified Friedmann equation $H^2(\rho)$ for different power-law functions. In fact, for $n > 2$ the asymptotic regime is quickly achieved and the low density regime coincides with Friedmann equation in GR. Since there is no strong deviation between power-law functions with indices $n > 2$, we shall explore the cubic case in the next section, although the analysis for other powers is similar. Actually, the cubic power corresponds to the simplest monomial which yields a non-trivial correction to GR in the inflationary regime.

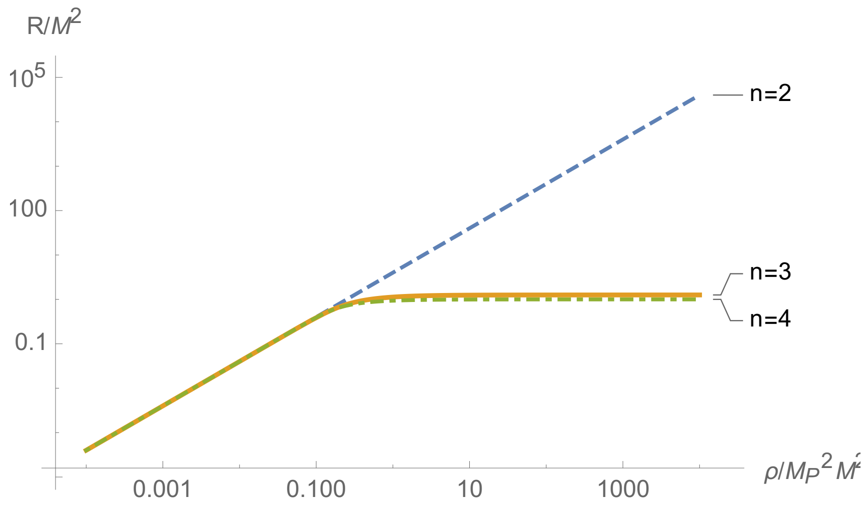


FIGURE 2.1: Solutions of the Friedmann equation for power-law non-minimal coupling functions, $f_2(R) = 1 + \left(\frac{R}{M^2}\right)^n$. The special case $n = 2$ corresponds exactly to the GR solution. For $n > 2$ the behaviour is quite similar for different powers. For $n < 2$ there are no real solutions, hence a plot in the real plane is not possible.

2.2 Cubic non-minimal coupling: $f_2(R) = 1 + \left(\frac{R}{R_0}\right)^3$

For a cubic non-minimal coupling, $f_2(R) = 1 + \left(\frac{R}{M^2}\right)^3$, and defining $H_0^2 := 2^{1/3}M^2/12$ and $\tilde{\rho} := \rho/M_P^2 H_0^2$, the modified Friedmann equation is recast as:

$$H^2 = H_0^2 \frac{-2^{2/3}\tilde{\rho} + \left(\tilde{\rho}^3 + \sqrt{\tilde{\rho}^3(4+\tilde{\rho}^3)}\right)^{2/3}}{2^{1/3}\tilde{\rho}\left(\tilde{\rho}^3 + \sqrt{\tilde{\rho}^3(4+\tilde{\rho}^3)}\right)^{1/3}} . \tag{2.21}$$

However, it is very hard to find numerical and graphical solutions for this cumbersome function, therefore, we can approximate to the following expression, with the advantage of being faster in getting the numerical results, but still being highly accurate in comparison with the full expression:

$$H^2 = H_0^2 \frac{\tilde{\rho} + \tilde{\rho}^2}{3 + 2\tilde{\rho} + \tilde{\rho}^2} . \tag{2.22}$$

This approximated expression gives the correct low and high density limits and is very close to the exact result in the intermediate density region.

Bearing this approximation in mind and taking into account the discussion for a generic Friedmann equation of the form $H^2 = H^2(\rho)$ in the Appendix B, the scalar index and the tensor-to-scalar ratio read:

$$n_s = 1 - 2\epsilon_\phi \left(\frac{\tilde{\rho}^2 + 6\tilde{\rho} + 3}{\tilde{\rho}^2 + 2\tilde{\rho} + 1}\right) + \frac{2}{3}\eta_\phi \left(\frac{\tilde{\rho}^2 + 2\tilde{\rho} + 3}{\tilde{\rho} + 1}\right) , \tag{2.23}$$

$$r = \frac{16}{9}\epsilon_\phi \left(\frac{\tilde{\rho}^2 + 2\tilde{\rho} + 3}{\tilde{\rho} + 1}\right)^2 . \tag{2.24}$$

It is straighforward to recover the GR results in the low-density limit, $\tilde{\rho} \ll 1$, i.e., $n_s - 1 = -6\epsilon_\phi + 2\eta_\phi$ and $r = 16\epsilon_\phi$, while at high densities, $\tilde{\rho} \gg 1$, we obtain:

$$n_s \simeq 1 - 2\epsilon_\phi + \frac{2}{3}\eta_\phi \frac{V}{H_0^2 M_P^2} , \tag{2.25}$$

$$r \simeq \frac{16}{9}\epsilon_\phi \left(\frac{V}{H_0^2 M_P^2}\right)^2 . \tag{2.26}$$

These are the most relevant observables which have been constrained by the Planck collaboration [141]. Therefore, we are able to compare in the $n_s - r$ plane the predictions of the NMC model with the common inflaton's potentials used in the literature. To do so, we have developed a numerical code to compute the field value at which the CMB fluctuations at the pivot scale become super-horizon giving a number of e-folds between 50 and 60, needed to solve the flatness and horizon problems. Furthermore, the end of inflation was given in each case via the strongest of the conditions (B.2) and (B.3), so that we could compute n_s and r as a function of the potential parameters.

For each scalar potential, we define the dimensionless parameter $x := V_0/M_P^2 H_0^2$, where the constant V_0 sets the scale of the inflationary potential, such that when $x \ll 1$ we retrieve the GR limit. On one hand, monomial models are defined uniquely in terms of V_0, thus we show the results as a function of x. On the other hand, the hilltop and the Higgs-like potentials are parametrised by more than one parameter, hence we chose to plot the results for a fixed x and allow the γ parameter from the potential to vary, as it will be discussed in the next subsections.

2.2.1 Monomial models

Monomial models correspond to potentials of the form:

$$V = V_0 \left(\frac{\phi}{M_P}\right)^n , \tag{2.27}$$

where the constant V_0 sets the scale of inflation and $n \in \mathbb{Z}^+$. The observational predictions of such potentials depend exclusively on the ratio $x = V_0/H_0^2 M_P^2$ due to the non-minimal matter-curvature coupling.

In GR, only monomial potentials with $n \geq 2$ are ruled out by the observational constraints imposed by Planck data [141]. We find that these conclusions hold for the NMC model, although a further constraint on the ratio x was considered. For $n = 1$ and $n = 2$ powers, the observational predictions are shown in Fig. 2.2, where it is clear that $n = 2$ is ruled out, while the linear potential with $n = 1$ is still allowed at the 2σ confidence level for $x \lesssim 0.2$.

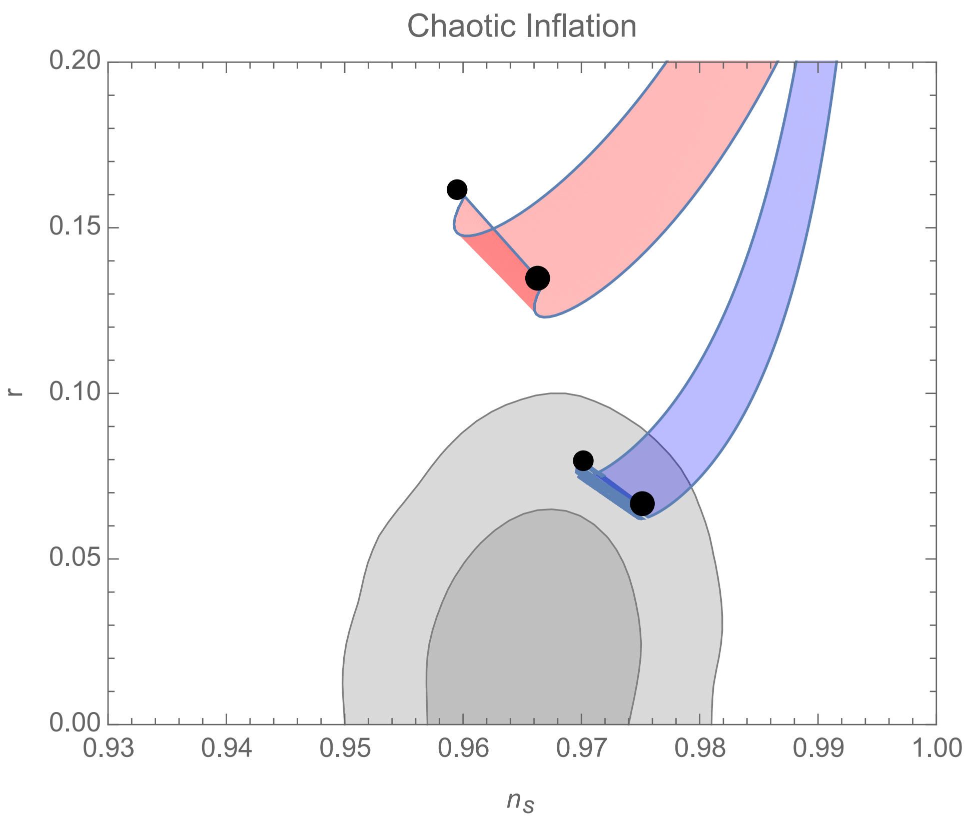


FIGURE 2.2: Observational predictions for monomial potentials in the NMC model with $n = 1$ shown in blue and $n = 2$ shown in red. The parameter $x = V_0/M_P^2 H_0^2$ decreases towards zero from top to bottom. Upper and lower bounds correspond to $N_e = 50$ and $N_e = 60$, respectively. The black circles correspond to the GR predictions.

2.2.2 Hilltop Potentials

The hilltop potentials are the ones which can be cast in the following form:

$$V = V_0 \left[1 - \frac{\gamma}{n} \left(\frac{\phi}{M_P} \right)^n \right] , \tag{2.28}$$

where γ is a running parameter to be constrained with data. Since for this potential we have two degrees of freedom, we fix the ratio $x = 5$ and allow $\gamma \in]0, 1[$. The observational predictions for the quadratic and quartic hilltop potentials, $n = 2$ and $n = 4$, respectively, as a function of the parameter γ are shown in Fig. 2.3. Despite these results, it is worth to notice that lower values for x lead to observational predictions progressively closer to the GR behaviour for the hilltop potentials, whilst for larger values of x the scalar-to-tensor ratio increases.

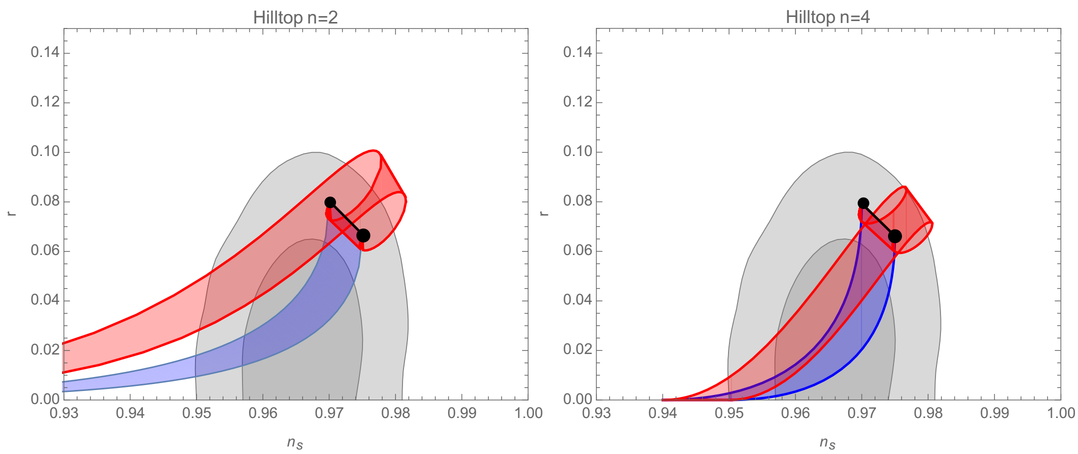


FIGURE 2.3: Observational predictions for the quadratic (left) and quartic (right) hilltop potentials with $x = 5$ are shown in red, and are compared with the GR predictions which are shown in blue. The region between upper and lower bounds correspond to the number of e-folds ranging from $N_e = 50$ to $N_e = 60$, respectively. The black circles correspond to the GR prediction for a linear potential. The light and dark grey regions correspond, respectively, to the 68% and 95% C.L. contours obtained by the Planck collaboration [141].

It is found that the non-minimal coupling generically increases the value of the tensor-to-scalar ratio, which means that it predicts slightly higher values for the tensor modes comparatively to GR, particularly for the quadratic exponent. In both plots, γ decreases from left to right and there is a turning point (which we coin "piparote") which results from the fact that the potential changes from high density regime, which occurs for smaller values of γ, to the low density regime. Interestingly, in the low density regime, which corresponds to $\gamma \to 0$, we recover the GR limit in both cases, which coincides with a linear scalar potential $V(\phi) \propto \phi$. The GR limit is also retrieved for $x \to 0$ regardless the value of γ.

For a number of e-folds between 50 and 60, the hilltop models are always compatible with Planck's data [141] at the 2σ level, independently of the value of the ratio $V_0/M_P^2 H_0^2$. This happens for the branch near the GR behaviour, i.e. for smaller values of γ.

2.2.3 Higgs-like Potential

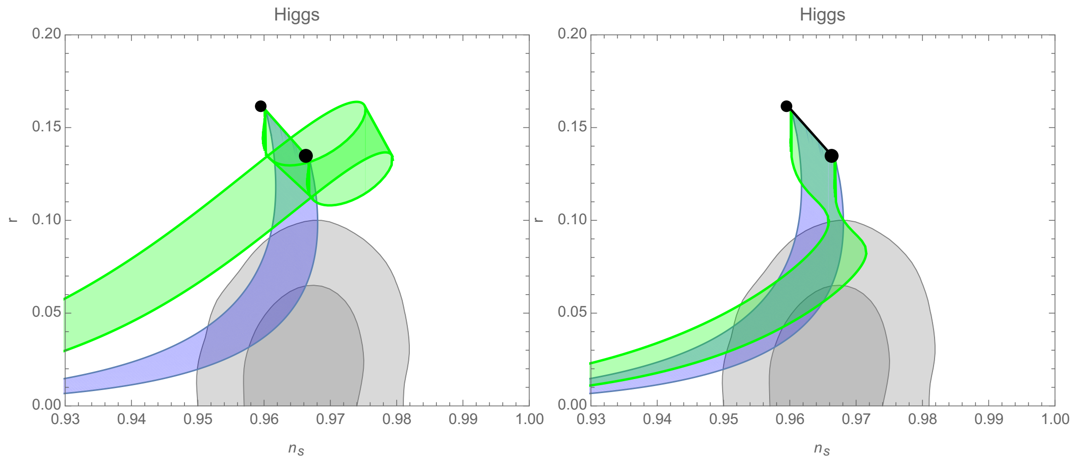
A Higgs-like potential is of the form:

$$V(\phi) = \lambda(\phi^2 - v^2)^2 , \tag{2.29}$$

which can be rewritten as

$$V = V_0 \left[1 - \frac{\gamma}{2} \left(\frac{\phi}{M_P} \right)^2 + \frac{\gamma^2}{16} \left(\frac{\phi}{M_P} \right)^4 \right] , \tag{2.30}$$

with the following identifications: $V_0 := \lambda v^4$ and $\gamma := 4M_P^2/v^2$. This expression resembles a hilltop model, and the previous analysis is straightforwardly applied. In fact, this potential can be seen as a completion of the quadratic hilltop model that is bounded from below. Once again, we fix the ratio x by choosing two values: $x = 5$ and $x = 2$. The resulting plots are shown in Fig. 2.4.



(A) Higgs potential for $x = V_0/M_P^2H_0^2 = 5$.

(B) Higgs potential for $x = V_0/M_P^2H_0^2 = 2$.

FIGURE 2.4: Observational predictions for the Higgs-like potential in the NMC model with the ratios $x = 5$ (left) and $x = 2$ (right) are shown in green, and the corresponding GR predictions are shown in blue. Upper and lower bounds correspond to $N_e = 50$ and $N_e = 60$, respectively. The black circles correspond to the GR prediction for a quadratic potential. The light and dark grey regions are the 68% and 95% C.L. contours obtained by the Planck collaboration, respectively [141]. The running parameter γ decreases from nearly 1 to 0 from left to right in both plots.

Analogously to the hilltop potentials, the value of the tensor-to-scalar ratio is also boosted. In the NMC scenario, for $x = V_0/M_P^2H_0^2 \gtrsim 4.8$, the Higgs model is ruled out by the Planck results [141], as shown in Fig. 2.4a. For smaller ratios, the behaviour is similar to GR; as an example, if we set the ratio to be $V_0/M_P^2H_0^2 = 2$, we get the results shown in Fig. 2.4b. For $x \lesssim 2.6$ the Higgs potential is in agreement with data mostly at the 2σ level, albeit there exist some small parametric regions where the potential is favoured at 1σ.

2.2.4 Consistency relation

We have stated earlier that the standard consistency relation found in GR between the tensor-to-scalar ratio and the tensor spectral index is modified in the context of the non-minimal coupling between the inflaton and curvature. Hence in Fig. 2.5, we plot the variation of the ratio $|r/n_t|$ in the cubic power-law coupling as a function of the normalised energy density at horizon-crossing, $\rho/(M_P^2 H_0^2)$.

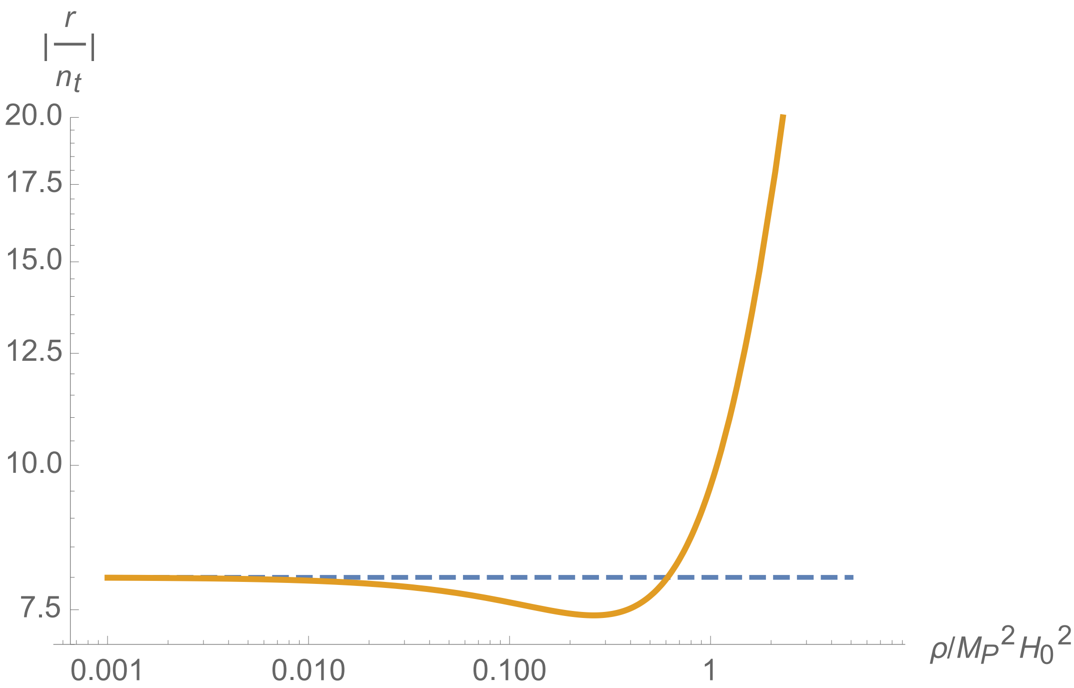


FIGURE 2.5: Variation of the ratio $|r/n_t|$ in the cubic NMC model as a function of the energy density (solid orange curve), in contrast to the constant prediction in GR, $r = 8|n_t|$, given by the dashed blue line.

As seen in Fig. 2.5, the GR limit is obtained for small energy density, $\rho \leq 0.01 H_0^2 M_P^2$, as we would expected since the Friedmann equation reduces to the standard GR form in this regime. However, for intermediate ranges, $\rho/(M_P^2 H_0^2) \in [0.01, 0.6]$, a ratio smaller than the GR value is found. This behaviour resembles the characteristic ones from the warm inflation scenario [187], or from multifield models of inflation [190–192].

On the other hand, energy densities higher than $0.6 H_0^2 M_P^2$ exhibit a larger $|r/n_t|$ ratio, meaning that for $r \lesssim 0.11$ tensor modes should exhibit an almost scale invariant spectrum. This comportment as found in models where the initial inflaton state is not the Bunch-Davies vacuum [193–197]. Such large deviations in the high density limit imply that future observational constraints, namely for the tensor-to-scalar ratio and the tensor index, may allow for a way to distinguish between General Relativity and the NMC matter-curvature models.

Additionally, we note that in the high-density limit the amplitude of the scalar power spectrum is given by:

$$A_s \simeq \frac{2}{\pi^2 M_P^2} \frac{H_0^2}{r} \, . \tag{2.31}$$

Bearing in mind that the potentials, which were analysed above, meet the observations for scenarios where $r \sim 10^{-2}$, we can infer the magnitude of the non-minimal coupling mass scale from the amplitude of the scalar power spectrum which is constrained to be

$A_s \sim 2.2 \times 10^{-9}$ [141]. Thus, the mass scale is found to be $M \sim 10^{-5} M_P \sim 10^{13}$ GeV, which perhaps not surprisingly, is comparable to the magnitude of the Hubble parameter in most inflationary models in GR.

After inflation, as in GR, the reheating process takes place. This period is characterised by the transference of energy from the inflaton into other light particles. Since the radiation energy density has a quartic dependence on the temperature, $\rho_R \propto T^4$, for $T < \sqrt{M M_P}$, the effects of the non-minimal coupling function should be negligible in the scenario that we are considering. In fact, for $M \sim 10^{13}$ GeV, which means that for temperatures below 10^{15} GeV the non-minimal coupling plays no relevant role.

Furthermore, in Ref. [123], it was shown that the non-minimal coupling could not produce inflation by itself, however it was important in the reheating and preheating epochs. In the present discussion, these conclusions do not hold, since inflation is driven by a real homogeneous scalar field, and the non-minimal coupling function exists in addition. Nonetheless, afterwards it plays a negligible role since the curvature/energy density rapidly decreases after inflation. Thus it is expected that reheating follows the GR framework (see e.g. Ref. [198]).

Chapter 3

Exotic Inflation

In the previous Chapter, we have discussed inflation in the context of alternative theories of gravity with a non-minimal coupling between matter and curvature. In fact, other formulations of non-minimal inflaton-curvature couplings were also analysed, being compatible with data for some inflationary potentials [185, 186].

Furthermore, there are other inflationary models based on scalar fields arising from superstring and supergravity theories [199]. One example of such scenarios is to the so called D-brane inflation [200], where the inflaton is a geometric modulus and represents the distance between two stacks of 3D branes and anti-branes. In these theories, the particles from the Standard Model are confined to the branes that survive brane-antibrane annihilation at the end of inflation. Two special cases of D-brane inflationary models have drawn some attention and are compatible with Planck's data [141]: the quadratic [201] and quartic [202,203] models.

Moreover, there are inflationary models exhibiting more than one scalar field, as in the so-called hybrid inflation [204, 205], where a rapid rolling, sometimes called waterfall, of a scalar field, σ, is prompted by a second scalar field, ϕ. Despite this pleasant feature, a blue tilted power spectrum is predicted which is therefore ruled out.

Notwithstanding, these models are just a small sample of the vast literature on this subject. In fact, there are many non-standard inflationary models which have not been compared with data. An attempt to fulfil this gap for some of those models is made in this Chapter. Taking into consideration the previous discussion on general Friedmann equations of the form $H^2 = H^2(\rho)$, and the developed numerical code, we explore the generalised Chaplygin gas inflationary model, as well as some inflationary models within supergravity and braneworld scenarios. We further analyse the degeneracy between some modified gravity theories and plateau-like potentials in GR which yield a primordial spectrum of fluctuations similar to the one from Starobinsky model.

This Chapter is based on Ref. [2].

3.1 Chaplygin-inspired inflation

The generalised Chaplygin gas (GCG) model is a proposal to unify dark matter and dark energy by resorting to an exotic fluid with equation of state [75,76]:

$$p_{GCG} = -\frac{A}{\rho_{GCG}^{\alpha}} , \tag{3.1}$$

where p_{GCG} and ρ_{GCG} are the pressure and the energy density of the gas, respectively, and A and α are constants, with the exponent ranging between $0 < \alpha \leq 1$. The original Chaplygin gas model corresponds to $\alpha = 1$.

This unified model admits a complex scalar field construction [75,206]. Under the Thomas-Fermi approximation, which introduces a functional relation between the kinetic energy and the potential of a field, this construction gives rise to a formulation built upon an underlying scalar field with a suitable potential [75].

The GCG has an equation of state that leads to a fluid energy density that behaves as $\rho_{GCG} = \left(A + \rho_m^{1+\alpha}\right)^{\frac{1}{1+\alpha}}$, where $\rho_m \propto a^{-3}$ resembles the energy density of non-relativistic matter. Such behaviour can also be achieved from a modification of gravity, particularly from a generalised Born-Infeld action for a scalar field, θ, (we refer the reader to Fig. 3.1 for a visual scheme) yielding a Friedmann equation of the form [207]:

$$H^2 = \frac{1}{3M_P^2}\left[A + \rho_\phi^{1+\alpha}\right]^{\frac{1}{1+\alpha}} . \tag{3.2}$$

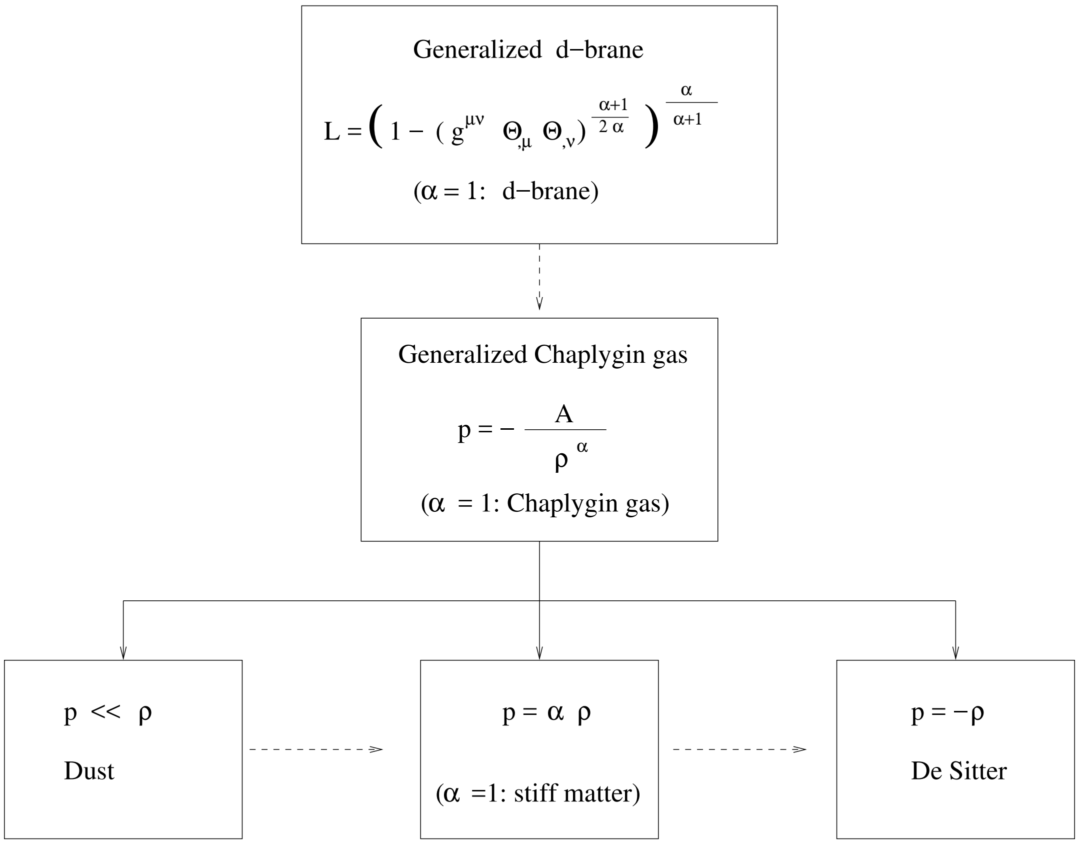


FIGURE 3.1: The Generalised Chaplygin scenario arising from a generalised d-Brane action and leading to three different behaviours of the Universe evolution. Extracted from Ref. [75].

One can immediately identify such a scalar field with the inflaton, provided it obeys the standard classical equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 . \tag{3.3}$$

We shall now consider two main cases in the slow-roll regime, $\rho_{GCG} = \rho_\phi \approx V(\phi)$, namely $\alpha = 1$ and $\alpha = 0.5$ for linear and quadratic monomial potentials. Moreover, we shall also analyse the quartic and quadratic hilltop models, but only in the $\alpha = 1$ case given the numeric limitations arisen when computing the observables.

For monomial potentials, Eq. (2.27), the slow-roll parameters can be expressed as:

$$\begin{aligned} \epsilon_\phi &= \frac{n^2}{2\tilde{\phi}^2}\,, \\ \eta_\phi &= \frac{n(n-1)}{\tilde{\phi}^2}\,, \end{aligned} \tag{3.4}$$

where we used the notation $\tilde{\phi} := \phi/M_P$.

In its turn, slow-roll parameters for the hilltop potentials, Eq. (2.28), read:

$$\begin{aligned} \epsilon_\phi &= \frac{\gamma^2\tilde{\phi}^{2n-2}}{2\left(1-\frac{\gamma}{n}\tilde{\phi}^n\right)^2}\,, \\ \eta_\phi &= -\frac{(n-1)\tilde{\phi}^{n-2}}{1-\frac{\gamma}{n}\tilde{\phi}^n}\,. \end{aligned} \tag{3.5}$$

The number of e-folds of inflation after horizon-crossing of the relevant CMB scales is straightforwardly computed:

$$N_e = -\frac{1}{M_P^2}\int_{\phi_*}^{\phi_e}\frac{\left(A+V(\phi)^{1+\alpha}\right)^{\frac{1}{1+\alpha}}}{V'(\phi)}d\phi\,, \tag{3.6}$$

where the value of the inflaton at the end of inflation, ϕ_e, for a modified Friedmann equation of the form $H^2 = H^2(\rho)$, is given, for each potential, by the strongest conditions of the following set, as shown in Appendix B:

$$\epsilon_\phi \sim 3\left(\frac{M_P^2H^2}{V}\right)^2\left(M_P^2\frac{dH^2}{d\rho}\right)^{-1}\,, \tag{3.7}$$

$$\epsilon_\phi \sim 9\frac{H^2M_P^2}{V}\,. \tag{3.8}$$

Then, the observables, namely the scalar spectral index and the tensor-to-scalar ratio, are given by:

$$\begin{aligned} n_s &= 1-6\epsilon_{\phi_*}\frac{V(\phi_*)^{2+\alpha}}{\left(A+V(\phi_*)^{1+\alpha}\right)^{\frac{\alpha+2}{\alpha+1}}}+2\eta_{\phi_*}\frac{V(\phi_*)}{\left(A+V(\phi_*)^{1+\alpha}\right)^{\frac{1}{1+\alpha}}}\,, \\ r &= 16\epsilon_{\phi_*}\frac{V(\phi_*)^2}{\left(A+V(\phi_*)^{1+\alpha}\right)^{\frac{2}{1+\alpha}}}\,. \end{aligned} \tag{3.9}$$

Analogously to the inflationary scenario in the NMC model discussed in the previous Chapter, the monomial potentials have a running free parameter due to the modification on the Friedmann equation, namely $x := \frac{A}{V_0^{1+\alpha}}$. In Fig. 3.2, we plot the observational predictions for the linear and quadratic potentials in the Chaplygin inspired inflation. Imposing the obvious fact that inflation has to end, some constraints arise for the ratio x in all cases, except in the linear potential case for $\alpha = 1$, where it that requirement always holds and an attractor point is found at $(n_s, r) = (1, 0)$ for $x \to \infty$.

In Fig. 3.3 we show the observational predictions for the quadratic and quartic hilltop

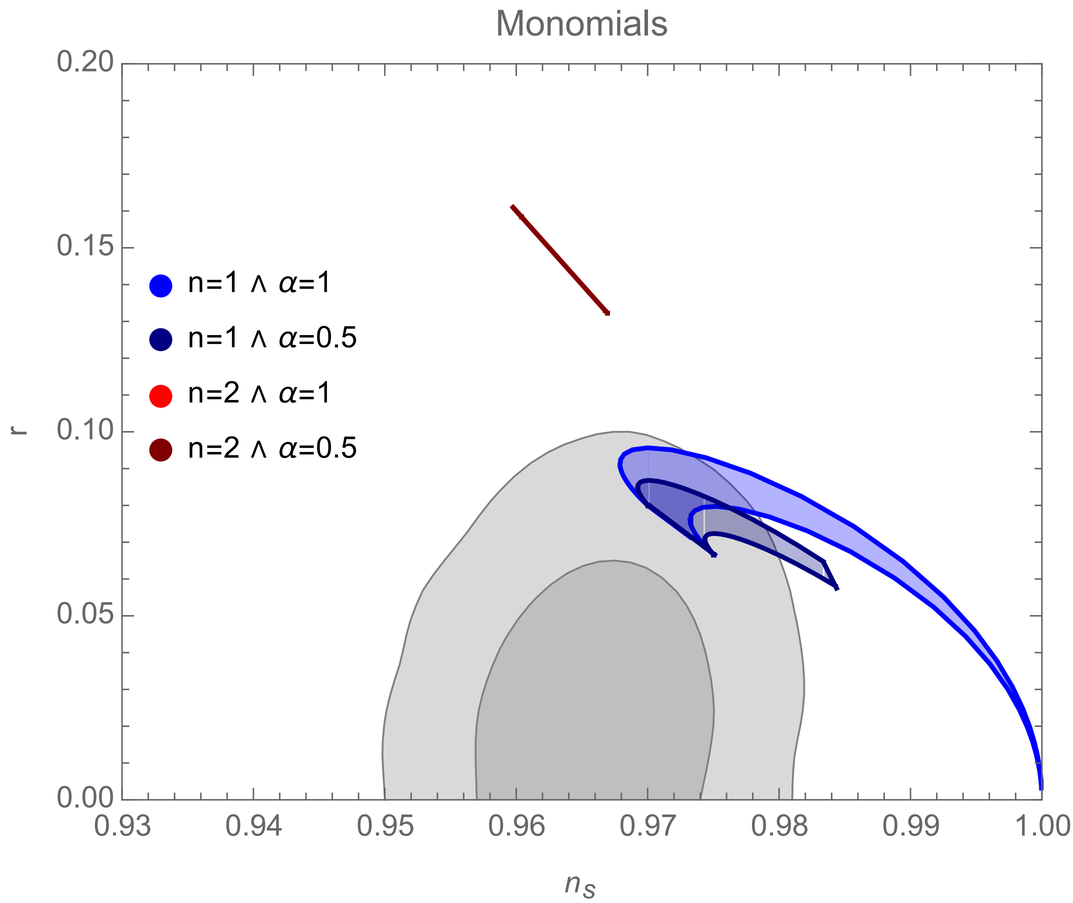


FIGURE 3.2: Predictions for monomial potentials in the Chaplygin inspired inflation in comparison with Planck data in grey for 1σ and 2σ confidence levels. For $n = 1$, if $\alpha = 0.5$, then $x := \frac{A}{V_0^{1+\alpha}} < 0.53$. For $n = 2$, if $\alpha = 1$, then $x < 0.59$, and for $\alpha = 0.5$ we must have $x < 0.53$. Upper and lower bounds correspond to $N_e = 50$ and $N_e = 60$, respectively.

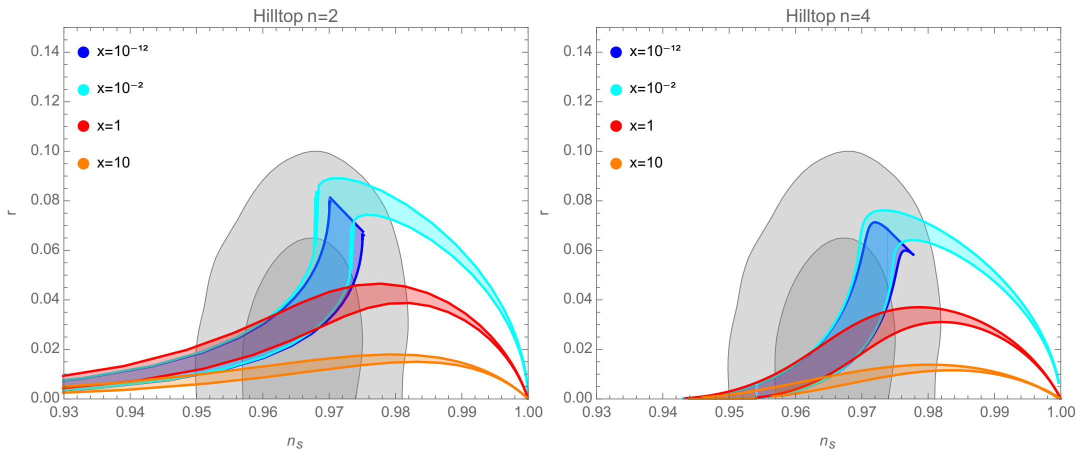


FIGURE 3.3: Predictions for the quadratic (left) and quartic (right) hilltop potential in a generalised Chaplygin model with $\alpha = 1$ in comparison with Planck data in grey. For $x \ll 1$, one recovers the standard Friedmann equation prediction, whilst for larger values of x the potential behaviour is quite different. The region between the upper and lower bounds correspond to a number of e-folds between $N_e = 50$ and $N_e = 60$, respectively.

potentials for $\alpha = 1$, as a function of x. In fact, as $x \to 0$, we find the predictions of such potentials in GR for both cases, as expected. However, for larger values of x the deviations are more evident: the tensor-to-scalar ratio diminishes and an attractor point is found at $(n_s, r) = (1, 0)$ for $\gamma \to 0$. Nonetheless, there is a considerable range of values for x which are observably favoured.

3.2 $N = 1$ Supergravity Inflation

We shall now study a particular case of $N = 1$ supergravity (SUGRA), which is an attempt to quantise GR by imposing local supersymmetry through a super-Poincaré algebra. The potential which will be analysed describes the interaction of chiral superfields, and is specified by the Kähler potential, $K(\Phi, \Phi^\dagger)$. Thus, the scalar potential reads [208]:

$$V = \frac{e^K}{4}\left(G_a (K^{-1})^a_b G^b - 3|W|^2\right) , \tag{3.10}$$

where $G_a = K_a W + W_a$ denotes the Kähler function, the indices a, b label the derivatives with respect to the chiral superfields, Φ, and $W(\Phi)$ corresponds to the superpotential that describes the Yukawa and the scalar couplings of the supersymmetric theory.

We shall assume that the superpotential can be split into three sectors:

$$W = P + G + I\,, \tag{3.11}$$

where P represents the supersymmetry-breaking part, G is a gauge term, and I is an inflationary sector.

In the following discussion, we shall discuss the supersymmetric model constructed in Refs. [209, 210] with the minimal choice for the Kähler potential, $K = \Phi\Phi^\dagger$, and the inflaton superpotential, $I = \Delta^2 M_P f(\frac{\Phi}{M_P})$, where Δ sets the inflation scale, and $f(\Phi/M_P)$ is a function unconstrained by the underlying R-symmetry of the model (this symmetry is responsible for transforming different supercharges into each other). Hence, the scalar potential as a function of the inflaton field, ϕ, reads [208]:

$$V_I(\phi) = e^{|\phi|^2/M_P^2}\left(\left|\frac{\partial I}{\partial\phi} + \frac{\phi^* I}{M_P^2}\right|^2 - \frac{3|I|^2}{M_P^2}\right)_{\Phi=\phi} . \tag{3.12}$$

We can further impose that SUSY remains unbroken in the global minimum, $\left|\frac{\partial I}{\partial\Phi} + \frac{\Phi^* I}{M_P^2}\right|_{\Phi=\phi_0} = 0$, and that we have a vanishing cosmological constant at present times, $V_I(\phi_0) = 0$. Therefore, the simplest superpotential I that satisfies these conditions is:

$$I(\phi) = V_0(\phi - \phi_0)^2\,, \tag{3.13}$$

where $V_0 = \Delta^2/M_P$. As we can see from this expression, there is a flat region in the vicinity to the origin for $\phi_0 = M_P$, which yields a vanishing inflaton mass. Therefore, we can Taylor expand the potential around its origin [209, 210]:

$$V_I(\phi) = \Delta^4\left(1 - 4\tilde{\phi}^3 + \frac{13}{2}\tilde{\phi}^4 - 8\tilde{\phi}^5 + \frac{23}{3}\tilde{\phi}^6 + \ldots\right) , \tag{3.14}$$

where $\tilde{\phi} := \phi/M_P$ as before.

Now, we can proceed in order to compute the slow-roll parameters:

$$\begin{aligned} \epsilon_\phi &\simeq 2\left(\frac{-6\tilde{\phi}^2 + 13\tilde{\phi}^3 - 20\tilde{\phi}^4 + 23\tilde{\phi}^5}{1 - 4\tilde{\phi}^3 + \frac{13}{2}\tilde{\phi}^4 - 8\tilde{\phi}^5 + \frac{23}{3}\tilde{\phi}^6}\right)^2 , \\ \eta_\phi &\simeq \frac{-24\tilde{\phi} + 78\tilde{\phi}^2 - 160\tilde{\phi}^3 + 230\tilde{\phi}^4}{1 - 4\tilde{\phi}^3 + \frac{13}{2}\tilde{\phi}^4 - 8\tilde{\phi}^5 + \frac{23}{3}\tilde{\phi}^6} . \end{aligned} \tag{3.15}$$

In addition, the number of inflationary e-folds follows from:

$$N_e = -\frac{1}{M_P^2}\int_{\phi_*}^{\phi_e} \frac{V(\phi)}{V'(\phi)} d\phi . \tag{3.16}$$

Most surprisingly, both the equation of motion for the inflaton and the Friedmann equation coincide with the well known ones in general relativity. Thus, it is immediate to find the scalar index and the tensor-to-scalar ratio at horizon crossing:

$$n_s = 1 - 6\epsilon_{\phi_*} + 2\eta_{\phi_*} , \tag{3.17}$$

$$r = 16\epsilon_{\phi_*} . \tag{3.18}$$

However, when compared with data, this model predicts a scalar index in the range $n_s \in [0.92, 0.94]$ which is ruled out by the Planck data. Additionally, it leads to very small tensor-to-scalar ratio, actually of the order of $10^{-9} - 10^{-8}$, which, although viable, in practical terms poses difficulties for the observation. Furthermore, other small field realisations in minimal $N = 1$ SUGRA with only one chiral superfield also show some tension with data [211]. However, it is important to stress that Planck data favours slightly lower values of the scalar spectral index when taking into account as free parameters the following quantities: the total neutrino mass, M_ν, and the effective number of relativistic degrees of freedom, N_{eff}. This usually happens in scenarios with M_ν larger than the minimal value allowed by neutrino oscillation data and in scenarios with a low-reheating temperature for which $N_{eff} < 3.046$ [212]. Hence, we cannot fully rule out the SUGRA model.

However, for standard observational data [141], this SUGRA model is excluded. Therefore, we shall consider in the next sections some supergravity models on the brane. But first, we shall analyse some features from braneworld scenario.

3.3 Brane Inflation

The braneworld scenario is a model to explain the weakness of the gravitational force when compared the other three fundamental interactions: electromagnetic, weak and strong nuclear forces. In the 5-dimensional brane scenario, the standard model matter fields are localised in a 3-brane and only gravity propagates in the fifth dimension (bulk). In fact, we can find a 4-dimensional version of the Einstein field equations [213]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \frac{1}{M_P^2}T_{\mu\nu} + \frac{1}{M_5^6}S_{\mu\nu} - E_{\mu\nu} , \tag{3.19}$$

where $M_5 := (8\pi G_5)^{-2}$ is the 5-dimensional reduced Planck mass associated to the 5-dimensional gravitational constant G_5, $T_{\mu\nu}$ is the energy-momentum on the brane, $S_{\mu\nu}$ is a tensor which depends only on quadratic powers of $T_{\mu\nu}$, and $E_{\mu\nu}$ is the projection of the 5-dimensional Weyl tensor on the 3-brane. We can assume that the spacetime on the brane is homogeneous and isotropic, i.e., is well described by the RW metric, such that the Friedmann equation look as [213–215]:

$$H^2 = \frac{\Lambda}{3} + \frac{\rho}{3M_P^2} + \frac{\rho^2}{9M_5^6} + \frac{\epsilon}{a^4}\,, \tag{3.20}$$

where ϵ is an integration constant and a is the scale factor. Given its smallness, the cosmological constant has a negligible contribution and the last term on the previous equation quickly vanishes when inflation sets in. Hence, we can further simplify the Friedmann equation [216,217]:

$$H^2 = \frac{\rho}{3M_P^2}\left(1 + \frac{\rho}{2\lambda}\right)\,, \tag{3.21}$$

where $\lambda := 3M_5^6/2M_P^2$ is the 3-brane tension.

It is instructive to make the following comment: there are other models which render a similar Friedmann equation. One of such models is as the Cardassian one [218], where we have an *ad hoc* correction of the form $\left(1 + \frac{\rho^n}{2\lambda}\right)$ in the Friedmann equation. This model has already been compared with observational data, and the results showed that both the ΛCDM and the Cardassian models were equally favoured within WMAP constraints [219]. Furthermore, Gamma Ray Burst data further narrowed the allowed values for the exponent of such model or even a modified version of it - the modified polytropic Cardassian model [220]. Another striking similar model is found in the context of loop quantum cosmology [221,222], where the corrections to the standard Friedman equation have a minus sign, $H^2 \sim \rho\left(1 - \frac{\rho}{2\lambda}\right)$. This model is compatible with the observations [223,224].

Looking back at the brane model under consideration, the equation of motion for the fields are not changed relatively to GR, namely Eq. (3.3). However if there existes some vector field defined on the bulk and coupled to the Ricci tensor, a spontaneous Lorentz symmetry breaking would be possible and the equations of motion would change [225].

As in the previous discussion, we must require the smallness of the slow-roll parameters, in order to achieve a successful inflationary period. The number of inflationary e-folds is given by:

$$N_e = -\int \frac{1}{\sqrt{2\epsilon_{\tilde{\phi}}}}\left(1 + \frac{1}{2\lambda}V(\tilde{\phi})\right)d\tilde{\phi}\,. \tag{3.22}$$

At last, the scalar index and the tensor-to-scalar ratio follow from:

$$\begin{aligned} n_s &= 1 - 6\epsilon_{\phi_*}\frac{1 + \frac{1}{\lambda}V(\phi_*)}{\left(1 + \frac{1}{2\lambda}V(\phi_*)\right)^2} + 2\eta_{\phi_*}\frac{1}{\left(1 + \frac{1}{2\lambda}V(\phi_*)\right)^2}\,. \\ r &= 16\epsilon_{\phi_*}\frac{1}{\left(1 + \frac{1}{2\lambda}V(\phi_*)\right)^2}\,. \end{aligned} \tag{3.23}$$

In Fig. 3.4, the predictions of this brane inflationary model for the linear and quadratic monomial potentials are plotted in comparison with the allowed observational regions from Planck data [141]. The quadratic monomial model is excluded by the observations. In contrast, the linear model is consistent with data. The running parameter for these potentials is

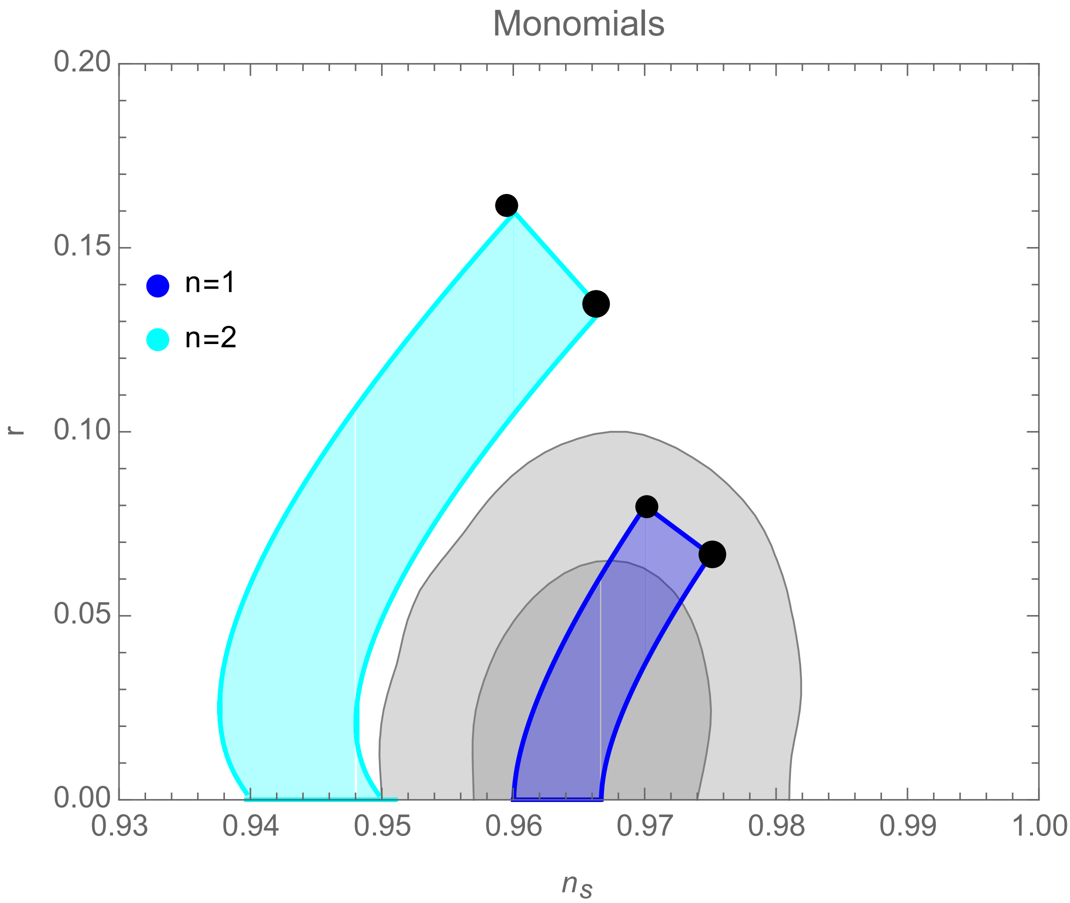


FIGURE 3.4: Predictions for monomial potentials in a braneworld inflation scenario. As the factor $y := \frac{V_0}{\lambda} \sim 0$, the linear or the quadratic behaviour in the standard Friedmann equation, represented by the black dots, is obtained. On the other hand, if y goes to larger values, tensor-to-scalar ratios become smaller. The quadratic model is disfavoured by Planck data (in grey), but the linear potential is always compatible with data, and, in particular, for $y \simeq 0.05$ it becomes 1σ compatible with Planck observations. Upper and lower bounds for each potential correspond to $N_e = 50$ and $N_e = 60$, respectively.

the ratio $y := \frac{V_0}{\lambda}$, which when it becomes larger, leads to a smaller tensor-to-scalar ratio. In particular, when the ratio $y \geq 0.05$, the linear monomial becomes 1σ compatible with data.

We can also study the quadratic and hilltop potentials, whose predictions are shown in Fig. 3.5. For these cases, we fix some values for y and allow the potential parameter γ to run. Thus, when the y factor grows, the scalar spectral index becomes larger. Nonetheless, in the limit $\gamma \to 0$ the general relativity predictions for a linear potential are recovered, independently of the value of the free parameter y.

3.4 $N = 1$ Supergravity inflation in the braneworld scenario

Let us now implement a $N = 1$ SUGRA model in the braneworld framework introduced in the previous section. To to so, let us look again at the $N = 1$ SUGRA scenario discussed in Sec. 3.2. If we aim to extract new predictions, then we must look at possible significant deviations from the standard Friedmann equation. These are attained when we consider the case where $\phi_0 = 0$, which renders a large-field inflationary model, rather that the studied case

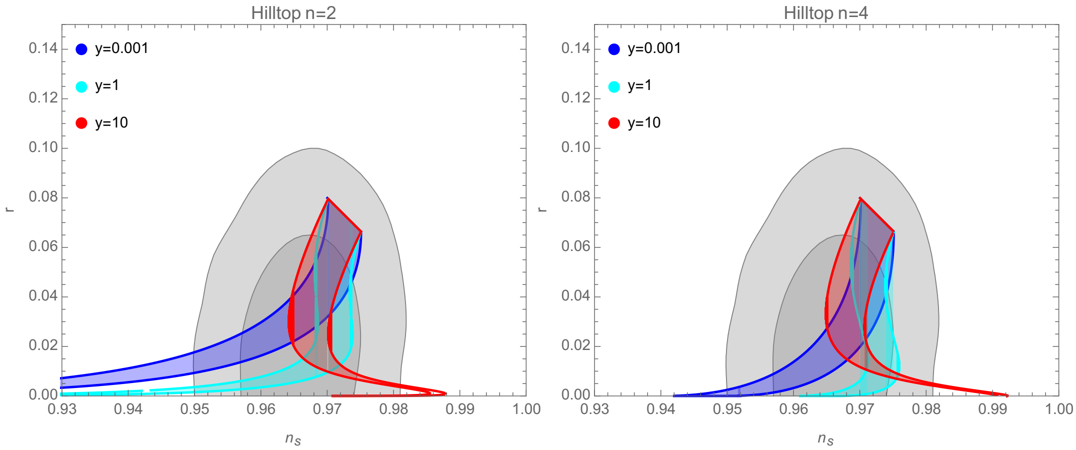


FIGURE 3.5: Predictions for the quadratic (left) and quadratic (right) hilltop potential in a braneworld inflationary scenario in comparison with Planck data in grey. When the factor $y := \frac{V_0}{\lambda} \sim 0$, one gets the linear behaviour in GR, but as y grows, one predicts larger values of n_s for $\gamma \to 1$. Nevertheless, they are compatible with data for a large subset of the parameter space. Upper and lower bounds for each case correspond to $N_e = 50$ and $N_e = 60$, respectively.

$\phi_0 = M_P$ that leads to a hilltop-like potential, which now would not yield new measurable effects.

Thus, the relevant part of the inflaton potential, along the real ϕ direction, reads [217]:

$$V(\phi) = V_0 e^{\tilde{\phi}^2} \left(4\tilde{\phi}^2 + \tilde{\phi}^4 + \tilde{\phi}^6\right) , \tag{3.24}$$

where, as before, $\tilde{\phi} := \frac{\phi}{M_P}$. The slow-roll parameters for this potential are written as:

$$\begin{aligned} \epsilon_\phi &= 2\left(\frac{4+6\tilde{\phi}^2+4\tilde{\phi}^4+\tilde{\phi}^6}{4\tilde{\phi}+\tilde{\phi}^3+\tilde{\phi}^5}\right)^2 , \\ \eta_\phi &= \frac{8+52\tilde{\phi}^2+64\tilde{\phi}^4+30\tilde{\phi}^6+4\tilde{\phi}^8}{4\tilde{\phi}^2+\tilde{\phi}^4+\tilde{\phi}^6} , \end{aligned} \tag{3.25}$$

and the number of inflationary e-folds after horizon-crossing is:

$$N_e = -\int \frac{1}{\sqrt{2\epsilon_{\tilde{\phi}}}} \left(1 + \frac{1}{2\lambda} V(\tilde{\phi})\right) d\tilde{\phi} \, . \tag{3.26}$$

For this potential, we allow the parameter $y := \frac{V_0}{\lambda}$ to run, and we find that a very small tensor-to-scalar ratio is found as depicted in Fig. 3.6. Nonetheless, this model is compatible with the observational data within the 2σ region for $y := \frac{V_0}{\lambda} \gtrsim 10^6$ with $N_e = 50$, and $y \gtrsim 2 \times 10^3$ with $N_e = 60$.

3.5 Plateau potentials in general relativity

A very popular inflationary model is the Starobinsky one, where a de Sitter phase is driven by a particular function of the scalar curvature: $f(R) = R + \alpha R^2$. In addition to this model, other similar scenarios, such as the Higgs-inflation with a non-minimal gravitational

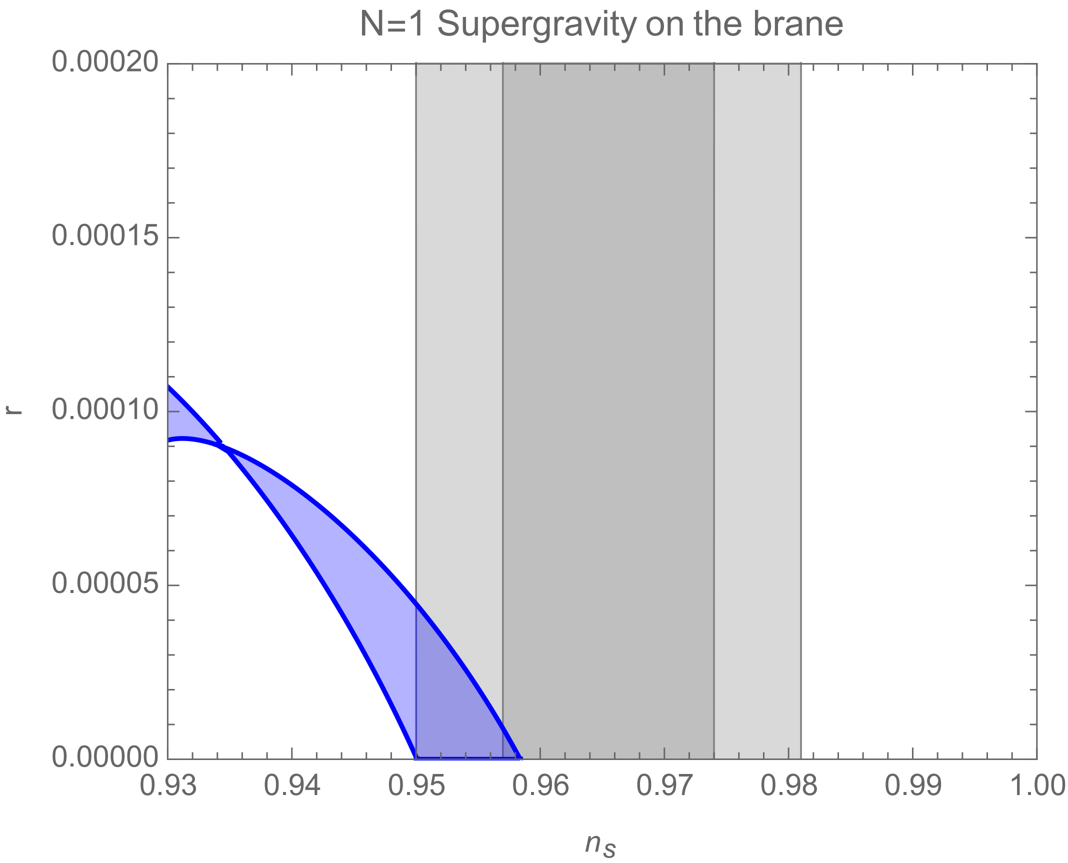


FIGURE 3.6: Predictions for the $N = 1$ supergravity inflation model on the brane world scenario. When the factor $y := \frac{V_0}{\lambda}$ increases, predictions go from the left to the right. The upper line at $n_s = 0.93$ corresponds to 50 e-folds of inflation, while the lower one to 60. For 50 e-folds and $y \gtrsim 10^6$ and for 60 e-folds and $y \gtrsim 10^3$ the model is in agreement with Planck data at 2-σ confidence level (in light grey).

coupling, share a common property, namely a very flat and negatively curved effective scalar potential in the Einstein frame. In fact, the data release by the Planck collaboration made it very clear that observations favoured potentials with such features. This is also the case of the hilltop or the natural inflationary potentials, or some modified gravity theories, at least in the context of single field cold inflation. However, warm inflation allows further monomial potentials, such as the quartic one $V(\phi) = \lambda\phi^4$, rendering predictions in agreement with Planck data [187,188].

To complement the discussion of exotic inflationary scenarios, we shall pursue a different approach in such a way that most modifications of gravity are degenerate with GR plus a scalar field, although in the Einstein frame. However, we note that in the previous Chapter we have examined inflation driven by a scalar field whose Lagrangian density was non-minimally coupled to the curvature. Such theories, as it has been discussed earlier, are equivalent to a theory with two scalar fields with distinct properties, so the analysis in this Section do not apply directly to that case.

In fact, the predictions of the Starobinsky model are degenerated to the ones from generic plateau-like potentials within GR with a scalar inflaton.

We start by defining a p-plateau, $p > 2$, as a potential function for which $p-1$ derivatives vanish at some field value ϕ_0, with $V^{(p)}(\phi_0) < 0$. In the vicinity of the central plateau value,

ϕ_0, the potential may be written as:

$$V(\phi) = V_0 \left[1 - \frac{\mu}{p} \left(\frac{\phi - \phi_0}{M_P} \right)^p + \ldots \right] , \tag{3.27}$$

where $\mu > 0$. The slow-roll parameters are found to be:

$$\begin{aligned} \epsilon_\phi &\simeq \frac{\mu^2}{2} \Delta\tilde{\phi}^{2p-2} , \\ \eta_\phi &\simeq -\mu(p-1)\Delta\tilde{\phi}^{p-2} , \end{aligned} \tag{3.28}$$

where we have defined $\Delta\tilde{\phi} := (\phi - \phi_0)/M_P \ll 1$. We further note that $\epsilon_\phi \ll |\eta_\phi|$ when $\mu \lesssim 1$. The number of inflationary e-folds of inflation after horizon-crossing is thus:

$$N_e = -\frac{1}{M_P^2} \int_{\phi_*}^{\phi_e} \frac{V(\phi)}{V'(\phi)} d\phi \simeq \frac{\Delta\tilde{\phi}_*^{2-p}}{(p-2)\mu} , \tag{3.29}$$

provided that $\Delta\tilde{\phi}_e \gg \Delta\tilde{\phi}_*$ and $p > 2$. As for the spectral index and the tensor-to-scalar ratio, we get:

$$\begin{aligned} n_s - 1 &= -6\epsilon_{\phi_*} + 2\eta_{\phi_*} \simeq -2\mu(p-1)\Delta\tilde{\phi}_*^{p-2} \\ &\simeq -\frac{p-1}{p-2}\frac{2}{N_e} , \\ r &= 16\epsilon_{\phi_*} \simeq 8\mu^2\Delta\tilde{\phi}_*^{2p-2} \simeq 8\mu^s(p-2)^{s-2}N_e^{s-2} , \end{aligned} \tag{3.30}$$

where $s := 2/(2-p)$. A very flat plateau is characterised by the condition $p \gg 1$, then:

$$\begin{aligned} n_s &\simeq 1 - \frac{2}{N_e} , \\ r &\simeq \frac{8}{p^2 N_e^2} . \end{aligned} \tag{3.31}$$

It is straightforward to see that the scalar spectral index prediction coincides with that of the Starobinsky potential and Higgs-inflation non-minimally coupled to the curvature, rendering a scalar index of $n_s \simeq 0.96 - 0.967$ for 50-60 e-folds of inflation. In Fig. 3.7, the predictions for different plateaux with integer $p \geq 3$ and $\mu = 10^{-3} - 1$ for 60 e-folds of inflation after horizon-crossing are shown. Thus, it is clearly evident that for large powers p, the observational predictions become insensitive to the curvature of the potential, μ. We further notice that a flat plateau with integer power $p \geq 6$ meets Planck data at 1σ confidence level.

In addition to the identical scalar spectral index, plateau potentials are degenerate with the Starobinksy model, for which $r \simeq 12/N_e^2$, in the sense that the tensor-to-scalar ratio can be arbitrarily large for any finite value of p provided $\mu \to 0$. The effective scalar potential for the Starobinsky model in the Einstein frame is $V(\phi) = V_0 \times \left(1 - e^{-\sqrt{2/3}\phi/M_P}\right)^2$, which may be seen as an infinite degree plateau at $\phi_0 \to +\infty$. However, our analysis does not immediately apply. This result shows that Planck data is not strictly favouring modifications of GR, but rather to a model with a sufficiently flat plateau in the scalar effective potential, within a general relativistic description.

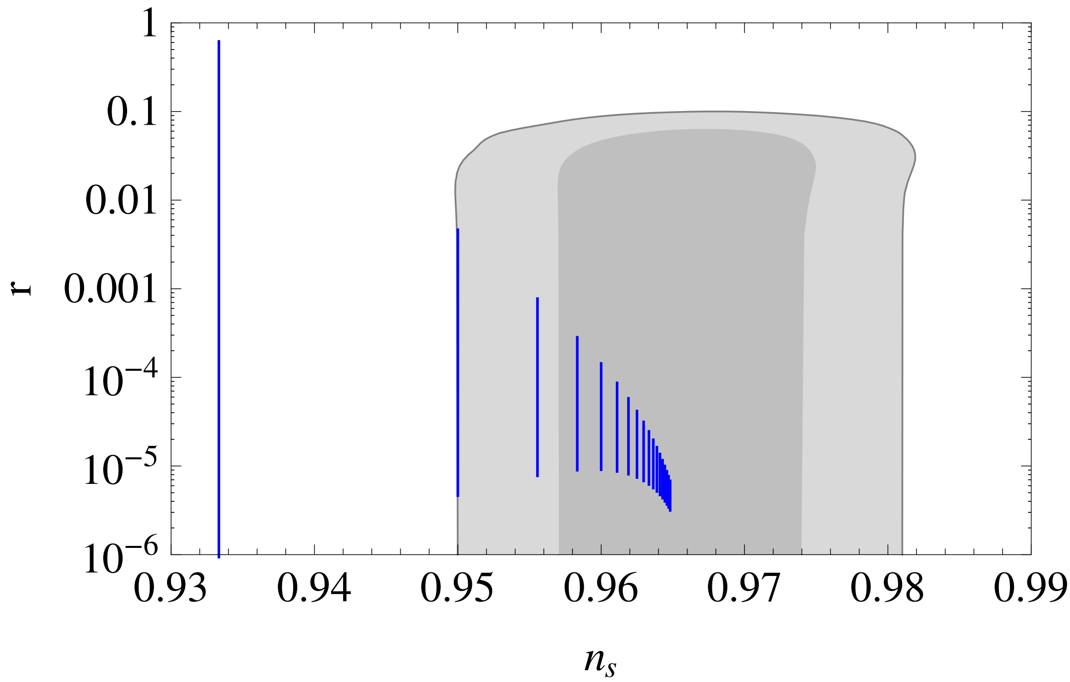


FIGURE 3.7: Observational predictions for plateau potentials of order $p = 3, \ldots, 20$ (increasing integers from left to right), for 60 e-folds of inflation and $\mu \in [10^{-3}, 1]$. Planck data is shown in grey.

Chapter 4

Gravitational waves in NMC

A striking property of General Relativity is the prediction of gravitational waves, which have been directly observed in recent years. However, this solution of the field equations also appear in other models of gravity, usually with new features which may be observable and, therefore, provide a way to distinguish between models of gravity [226].

As an example, we note that further scalar and vectorial modes or massive modes are expected in most of extensions of GR. For instance, in higher order extended theories of gravity [227], where scalar invariants other than the Ricci scalar are included, spin-0 and spin-2 massive modes are expected besides the massless spin-2 field (graviton), and even some ghost modes might arise. Therefore, much attention should be taken in finding consistent physical solutions both theoretically and observationally. However, there are many alternative theories whose predictions are degenerate to the GR's ones [228], thus further techniques should be devised in order to distinguish these models [229].

An interesting tool to discriminate these theories is the application of the Newman-Penrose formalism in addition to the usual perturbation theory [230–232]. This prescription can be also used in the Petrov classification for the Weyl tensor of the theory [233]. Interestingly, in $f(R)$ theories, the aforementioned formalisms seemed to render inconsistencies, as the scalar breathing mode that appeared in the Newman-Penrose formalism [234] was not present in the usual perturbative approach [235]. However, later on, it was realised that the issue resided in the assumption of the two traceless and transverse gauge conditions, which are no longer incompatible with each other [236]. Nevertheless, this issue does not appear in the Palatini formalism, where only two tensor modes as in GR are found [234]. However, this should not be surprising since the Palatini formulation of $f(R)$ theories is equivalent to a Brans-Dicke theory with the parameter $\omega_{BD} = -3/2$. In fact, this particular value corresponds to a vanishing kinetic term for the scalar field, rendering a nondynamical contribution, and therefore no additional scalar degree of freedom is expected. Despite these apparent ambiguities between different formalisms, gravitational waves are still a very robust tool to test theories of gravity.

In most cases, the gravitational wave solution of a given theory is considered in vacuum. However, this can be extended to account for matter source terms in GR and other alternative theories of gravity by resorting to the Green functions' method. Nonetheless, there are further approaches, namely the Campbell-Morgan formalism [237], or taking into account the interaction of gravitational waves with matter [238] or even the cyclotron damping and Faraday rotation of GW in the presence of collisionless plasmas [239,240]. A further relevant topic was considered: the presence of a cosmological constant, which induces the metric field equations to lose their residual gauge freedom [241], and has some bearings on the physical metric [242].

Thus, we are motivated to explore in this Chapter the main properties of the NMC models in what concerns gravitational waves. We shall analyse these solutions in both the linearised version and the Newman-Penrose formalism. We further assess the implications of a GW propagating in a background dominated by a cosmological constant and briefly discuss the non-trivial case of a dark energy-like fluid.

This Chapter is based on Ref. [3], which we will follow closely.

4.1 Linearised NMC theories

We start by considering the action functional of Eq. (1.16), and for convenience we shall adopt units such that $\kappa = 1/2$, which mean $M_P^2 = c = 1$.

The trace of the metric field equations, Eq. (1.17), can be cast in the following form:

$$\Box \left(F_1 + 2F_2 \mathcal{L}_m \right) = \frac{2f_1 - \left(F_1 + 2F_2 \mathcal{L}_m \right) R + f_2 T}{3} , \tag{4.1}$$

from which it is clear that the trace equation can be seen as a Klein-Gordon equation, but for two fields $\Phi_1 := F_1$ and $\Phi_2 := 2F_2\mathcal{L}_m$, rather than just one as in pure $f(R)$ theories [235], with an effective potential given by:

$$\frac{dV}{d\Phi_1} := \frac{2f_1 - F_1 R}{3}, \qquad \frac{dV}{d\Phi_2} := \frac{-2F_2 \mathcal{L}_m R + f_2 T}{3} , \tag{4.2}$$

where $\mathcal{L}_m$ is the matter Lagrangian density.

This identification is valid since the NMC theories are equivalent to a two-scalar field model in the Einstein frame, as referred in the Introduction. In fact, one of the scalar fields is not dynamical, but mixes with the dynamical one [114].

Before proceeding any further, let us define the following perturbative fields:

$$\delta f := \left(F_1 - 2F_2 \mathcal{L}_m + F_2 T \right) \delta R , \tag{4.3}$$

$$\delta f' := \left(F_1' + 2F_2' \mathcal{L}_m \right) \delta R , \tag{4.4}$$

$$\delta h := f_2 \delta T , \tag{4.5}$$

$$\delta h' := 2F_2 \delta \mathcal{L}_m . \tag{4.6}$$

Now the trace equation can be written as:

$$3\Box \left(\delta f' + \delta h' \right) = -R_0 \left(\delta f' + \delta h' \right) + \delta f + \delta h , \tag{4.7}$$

where we have computed such perturbative fields at a constant curvature background R_0.

Linearising the field equations with the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} \ll 1$:

$$\begin{aligned}
&\left(F_1' \delta R + 2F_2' \mathcal{L}_m \delta R + 2F_2 \delta L \right) R_{\mu\nu} + \left(F_1 + 2F_2 \mathcal{L}_m \right) \delta R_{\mu\nu} \\
&- \frac{1}{2} \eta_{\mu\nu} F_1 \delta R - \frac{1}{2} h_{\mu\nu} f_1 - \left[\nabla_\mu \nabla_\nu - h_{\mu\nu} \Box \right] \left(F_1 + 2F_2 \mathcal{L}_m \right) \\
&- \left[\nabla_\mu \nabla_\nu - \eta_{\mu\nu} \Box \right] \left(F_1' \delta R + 2F_2' \mathcal{L}_m \delta R + 2F_2 \delta \mathcal{L}_m \right) \\
&= f_2 \delta T_{\mu\nu} + F_2 T_{\mu\nu} \delta R .
\end{aligned} \tag{4.8}$$

We shall consider that the background is Minkowskian, which means that the scalar curvature vanishes at lowest order, $R_{\mu\nu} = R = 0$. In fact, for constant curvature we have $\nabla_\mu F_i = F_i' \nabla_\mu R = 0$. Hence:

$$\begin{aligned} &(F_1 + 2F_2\mathcal{L}_m)\,\delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}F_1\delta R - \frac{1}{2}h_{\mu\nu}f_1 \\ &- \left[\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right]\left(F_1'\delta R + 2F_2'\mathcal{L}_m\delta R + 2F_2\delta\mathcal{L}_m\right) \\ &+ \left[\partial_\mu\partial_\nu - h_{\mu\nu}\Box\right](F_1 + 2F_2\mathcal{L}_m) = f_2\delta T_{\mu\nu} + F_2 T_{\mu\nu}\delta R\,. \end{aligned} \tag{4.9}$$

We further require that the Lagrangian density can be written as $\mathcal{L}_m \to \mathcal{L}_m + \delta\mathcal{L}_m(x^\mu)$, such that $\nabla_\mu\mathcal{L}_m = 0$. This is the case of a cosmological constant, for instance. Actually, these assumptions further simplify the linearised field equations:

$$\begin{aligned} &(F_1 + 2F_2\mathcal{L}_m)\,\delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}F_1\delta R - \frac{1}{2}h_{\mu\nu}f_1 \\ &- \left[\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right]\left(\delta f' + \delta h'\right) = f_2\delta T_{\mu\nu} + F_2 T_{\mu\nu}\delta R\,. \end{aligned} \tag{4.10}$$

However, we should stress that the NMC theories reduce to pure $f(R)$ theories in the far-field and in the absence of matter. Therefore, we expect to have six polarisation states [234]. On the other hand, if matter is present then the polarisation states can be highly non-trivial. In particular, the longitudinal mode, which can be found in the trace of the field equations, can be decoupled into two scalar modes under some conditions, which is motivated by the fact that the NMC model is equivalent to two scalar fields in the Einstein frame.

A particular matter Lagrangian density, which although simple, renders non-trivial solutions is the cosmological constant as a source, which will be analysed and the resulting polarisation states found in the next section.

4.2 GWs in the presence of a cosmological constant

Let us consider a geometry determined by a cosmological constant term:

$$\mathcal{L}_m = -\Lambda \Rightarrow T_{\mu\nu} = -\Lambda g_{\mu\nu} \Rightarrow T = -4\Lambda\,. \tag{4.11}$$

Then, the field equations can be written as:

$$\begin{aligned} &(F_1 - 2F_2\Lambda)\,\delta G_{\mu\nu} - \frac{1}{2}(f_1 - 2f_2\Lambda)\,h_{\mu\nu} \\ &= \left[\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right]\left(\delta f' + \delta h'\right)\,, \end{aligned} \tag{4.12}$$

where the perturbed Einstein's tensor is defined as $\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta R$. We should bear in mind that $R_0 = 0$, which means that the fluctuations $\delta f' = \left(F_1' - 2F_2'\Lambda\right)\delta R$ and $\delta h' = 0$, since $\delta\Lambda = 0$.

It is worth mentioning that the above perturbations can be expressed in terms of the metric perturbations. In fact, the perturbed Ricci tensor and scalar curvature are given by:

$$\delta R_{\mu\nu} = \frac{1}{2}\left[-\Box h_{\mu\nu} + h_{\nu\ ,\mu\alpha}^{\ \alpha} + h_{\mu\alpha,\ \nu}^{\ \ \ \alpha} - h_{,\mu\nu}\right]\,, \tag{4.13}$$

$$\delta R = h^{\alpha\beta}{}_{,\alpha\beta} - \Box h\,, \tag{4.14}$$

respectively.

It is important to choose a gauge such that the equations become simpler. The most obvious choice is the following gauge:

$$\partial^{\mu}\left[h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h-\eta_{\mu\nu}\left(\frac{\delta f'+\delta h'}{F_1-2F_2\Lambda}\right)\right]=0\,,\tag{4.15}$$

where we should bear in mind that $F_1 - 2F_2\Lambda := F_1(R_0) - 2F_2(R_0)\Lambda$ is computed at the curvature $R_0 = 0$.

Thus, the field equations, in this gauge, read:

$$\Box\left(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h-\eta_{\mu\nu}\Omega\right)=\frac{f_1-2f_2\Lambda}{F_1-2F_2\Lambda}h_{\mu\nu}\,,\tag{4.16}$$

with

$$\Omega:=\frac{\delta f'+\delta h'}{F_1-2F_2\Lambda}\,.\tag{4.17}$$

An interesting conclusion can be immediately drawn: in the adopted gauge, a massive degree of freedom (scalar mode) is found, in addition to a dressed graviton that breaks residual gauge invariance $h'_{\mu\nu} = h_{\mu\nu} - \partial_{(\mu}\epsilon_{\nu)}$, as expected [241]. It is important to notice that the presence of a cosmological constant regarded as an integration constant in the pure gravity sector [243–245], that is $f_1(R) = R - 2\Lambda$, leads to rather different results. However, this has been done in the context of $f(R)$ theories and do not correspond to the present analysis in the matter sector.

Equation (4.16) can be rearranged in the following form:

$$\Box\left(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\right)=\frac{f_1-2f_2\Lambda}{F_1-2F_2\Lambda}h_{\mu\nu}+\frac{\eta_{\mu\nu}}{3}\left(h^{\alpha\beta}{}_{,\alpha\beta}-\Box h\right)\,.\tag{4.18}$$

However, we note that $\Box\Omega = \frac{1}{3}\delta R = \frac{1}{3}\left[-\frac{1}{2}\Box h + \Box\Omega\right] \Rightarrow \Box\Omega = -\frac{1}{4}\Box h$. Thus, the previous equation can be recast as:

$$\Box\left(h_{\mu\nu}-\frac{1}{4}\eta_{\mu\nu}h\right)=\frac{f_1-2f_2\Lambda}{F_1-2F_2\Lambda}h_{\mu\nu}\,.\tag{4.19}$$

As we can see, the scalar mode can be completely absorbed in the form of a scaling of the trace of the graviton, although it is still present in the gauge choice, Eq. (4.15). The solution to the wave equation is straightforwardly obtained:

$$h_{\mu\nu}=A^{+}e^{ik_\alpha x^\alpha}e^{+}_{\mu\nu}+A^{\times}e^{ik_\alpha x^\alpha}e^{\times}_{\mu\nu}\,,\tag{4.20}$$

where A^+, $A^\times$ are the amplitudes of the "plus" and "cross" polarisations, and $e^+_{\mu\nu}$, $e^\times_{\mu\nu}$ are the usual polarisation tensors, respectively, as discussed in the Introduction. However, a massive dispersion relation for the tensor modes arises:

$$k_\alpha k^\alpha:=\omega^2-k^2=\frac{f_1-2f_2\Lambda}{F_1-2F_2\Lambda}\,.\tag{4.21}$$

This implies that we have a traceless metric perturbation, $h = 0$, which naturally occurs when considering a cosmological constant. Furthemore, the squared mass of the graviton, which corresponds to the coefficient of the $h_{\mu\nu}$ term in rhs of Eq. (4.19), has an upper bound equal to $m_g < 7.7\times10^{-23}\ eV/c^2$ [167]. Taking this bound into account and the two functions, $f_1(R)$ and $f_2(R)$, and their first derivatives evaluated at $R = 0$, the restrictions for the

NMC model imply that the denominator of the mass term has to be much larger than the numerator, and that both of them need to have the same sign. In the case of a pure non-minimal matter-curvature coupling ($f_1(R) = R$), and using the observational value of the cosmological constant $\Lambda = 4.33 \times 10^{-66}\, eV^2$ [42] the following constraint is found: $f_2(0) > -6.8 \times 10^{20}$ in units such that $c = 1$. This poses a very weak restriction on the value of the non-minimal coupling function evaluated at vanishing curvature. In addition, we avoid tachyonic instabilities by requiring that $-f_2(0)\Lambda > 0 \Rightarrow f_2(0) < 0$. Therefore, in what concerns to these bounds a non-minimal coupling to the cosmological constant model is a quite viable model.

An additional observable has been restricted by data: the group velocity, v_g, of the gravitational wave [172]. This quantity follows from the dispersion relation:

$$v_g := \frac{\partial \omega}{\partial k} = \frac{1}{\sqrt{1 + \frac{f_1 - 2f_2\Lambda}{F_1 - 2F_2\Lambda}\frac{1}{k^2}}} \approx 1 - \frac{m_{gw}^2}{2k^2}\,, \tag{4.22}$$

where $m_{gw}^2 := \frac{f_1 - 2f_2(0)\Lambda}{F_1 - 2F_2(0)\Lambda} \ll 1$. Therefore the NMC model predicts a group velocity of the gravitational wave slightly smaller, but very close to the speed of light, thus avoiding Cerenkov radiation. This is consistent with the most recent data from the neutron star merger $-3 \times 10^{-15} < \frac{v_g - c}{c} < 7 \times 10^{-16}$ [172]. Given the dependence on k, or equivalently, the energy dependence on the group velocity, if $k \ll 1$ occurs then some problems can arise, this is known as the soft-graviton case. Nevertheless, for the NMC model we have $\omega \sim k$, thus leading to $v_g = 1 - \frac{m_{gw}}{2k^2} \approx 1 - 7.0 \times 10^{-23} \to 1^-$, which is compatible with the observed frequencies of gravitational waves $f \sim 250\ Hz$, and the smallness of the mass of the graviton.

There is also a further observable: the so-called "speed" of a gravitational wave, c_{gw}, which follows from a modified dispersion relation in models which exhibit rotation invariance of the form [178, 246]

$$\omega^2 = m_g^2 + c_{gw}^2 k^2 + a\frac{k^4}{\Delta}\,, \tag{4.23}$$

where Δ is a high-energy scale cut-off that has been constrained to be large [247], and a is an operator that is model dependent. This quantity has been constrained to be in the range $0.55 < c_{gw} < 1.42$ [178]. By comparing Eq. (4.23) with the non-minimal coupling dispersion relation, we find that $c_{gw} = 1$, hence being observationally viable.

4.2.1 Longitudinal scalar mode

We have seen that the longitudinal scalar mode Ω could be absorbed in the solution of the wave equation through a scaling effect. However, we shall further explore its nature and properties. Taking into account that the perturbation $\delta h'$ vanishes for the case of a cosmological constant as a source term, the solution to the wave equation for Ω is straightforwardly found. In fact, linearising the trace equation gives:

$$\Box\Omega = m_\Omega^2 \Omega\,, \tag{4.24}$$

with

$$\Omega := \frac{\delta f'}{F_1 - 2F_2\Lambda} = \frac{F_1' - 2F_2'\Lambda}{F_1 - 2F_2\Lambda}\delta R\,, \tag{4.25}$$

and the mass term

$$m_{\Omega}^2 := \frac{1}{3}\left[\frac{F_1 - 2F_2\Lambda}{F_1' - 2F_2'\Lambda}\right] . \tag{4.26}$$

Furthermore, for a gravitational wave passing through a galaxy in a curved background due to a cosmological constant, for instance, the mass of the longitudinal mode changes as in the $f(R)$ case [248]. In fact, if a constant matter Lagrangian of the form $\mathcal{L} = -\rho_0$ is considered, the underlying physics would remain the same and it would only be necessary to replace Λ by ρ_0. More elaborated forms for the matter Lagrangian would render highly non-trivial technical problems to be handled. However, our conclusions would not change drastically given the weakness of the coupling of gravity to matter.

4.2.2 Newman-Penrose analysis

In addition to the linearisation procedure to characterise the gravitational wave solution of the NMC model, there is a very useful formalism due to Ezra T. Newman and Roger Penrose which accounts for the full non-linear theory with a cosmological constant background [232]. This Newman-Penrose (NP) formalism is presented in the Appendix C. The NMC model with a cosmological constant background is equal to the $f(R)$ case upon the identification $f(R) := f_1(R) - 2f_2(R)\Lambda$.

Let us start by considering the following forms for the two functions $f_1(R) = R - \alpha R^{-\beta}$ and $f_2(R) = 1 + \gamma R^n$, which have been popular in the literature for other contexts. The trace of the metric field equations yields:

$$\begin{aligned} R - 4\Lambda = 3\Box &\left(\alpha\beta R^{-\beta-1} - 2\gamma n\Lambda R^{n-1}\right) \\ &+\alpha(\beta+2)R^{-\beta} + 2\Lambda\gamma(2-n)R^n . \end{aligned} \tag{4.27}$$

A special case is worth to be noted: homogeneous and static scalar curvatures ($\nabla_\mu R = 0$) and exponents of the functions of the scalar curvature obeying to $n = -\beta = 2$ yield the trivial solution, $R = 0$. This is the case of a flat spacetime as the one discussed in the previous subsection.

However, if $n = -\beta = 2$ but the curvatures are not homogeneous and/or static, then:

$$\Box R = -\frac{R - 4\Lambda}{6(\alpha + 2\gamma\Lambda)} , \tag{4.28}$$

whose solution is:

$$R(z,t) = 4\Lambda + R_0 e^{ik_\alpha x^\alpha} , \tag{4.29}$$

where R_0 is an integration constant and $\vec{k} = (\omega, 0, 0, k)$ which obeys to $k_\alpha k^\alpha = 1/(6\alpha + 12\gamma\Lambda)$. The z and t dependences of the Ricci scalar should not look surprising given that we have not specified the spacetime yet.

Substituting this solution into Eq. (1.17), we find that the nonvanishing terms of the Ricci tensor are:

$$R_{tt} = -\Lambda - \frac{R_0 e^{ik_\alpha x^\alpha}}{2}\left[1 - 4\left(\alpha + 2\gamma\Lambda\right)k^2\right] , \tag{4.30}$$

$$R_{zz} = \Lambda + \frac{R_0 e^{ik_\alpha x^\alpha}}{6}\left[1 + 12\left(\alpha + 2\gamma\Lambda\right)k^2\right] , \tag{4.31}$$

$$R_{tz} = -\left(\alpha + 2\gamma\Lambda\right)k\omega R_0 e^{ik_\alpha x^\alpha} , \tag{4.32}$$

$$R_{xx} = R_{yy} = \Lambda + \frac{1}{6}R_0 e^{ik_\alpha x^\alpha} . \tag{4.33}$$

It is clear that we get $R = -R_{tt} + R_{xx} + R_{yy} + R_{zz} = 4\Lambda + R_0 e^{ik_\alpha x^\alpha}$ as expected. In fact, the term R_{tz} was missing in Ref. [234], thus leading to some incomplete conclusions.

In fact, the $n = -\beta = 2$ case renders a Starobinsky-like model with a quadratic non-minimal coupling (let us recall that the quadratic term yields a GR behaviour with respect to the inflationary paradigm in Chapter 2 or in Ref. [1]).

The only nonvanishing NP-Ricci quantities, which characterise the existence of different polarisation modes, are:

$$\Phi_{00} = \frac{\alpha + 2\gamma\Lambda}{2}\left(\omega - k\right)^2 R_0 e^{ik_\alpha x^\alpha} , \tag{4.34}$$

$$\Phi_{22} = \frac{\alpha + 2\gamma\Lambda}{2}\left(\omega + k\right)^2 R_0 e^{ik_\alpha x^\alpha} , \tag{4.35}$$

$$\Phi_{11} = \tilde{\Lambda} - \frac{\Lambda}{6} . \tag{4.36}$$

These quantities can be used to study the Ricci tensor, or alternatively its traceless version, the Plebański tensor, $S_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/4$, where a classification likewise the Petrov one for the Weyl tensor can be performed. Relatively to the NP-Weyl quantities, we must solve the full non-linear theory to extract the metric field and to compute the Riemann tensor. It is relevant to note that expansions of the metric in perturbation theory may lead to completely different Petrov classifications, and therefore the full metric solution is mandatory to correctly characterise the theory.

For $n = -\beta$, and defining $\phi := R^{-\beta-1}$ the Ricci scalar has the same form as in Ref. [234] but with a rescaling:

$$\frac{\phi^{-\frac{1}{\beta+1}} - 4\Lambda}{\alpha + 2\gamma\Lambda} = 3\beta\Box\phi + (\beta+2)\phi^{\frac{\beta}{1+\beta}} . \tag{4.37}$$

We can see that for $\beta \geq 1$, at late times we have $R^{-\beta} \gg R, \Lambda$, which leads to the Klein-Gordon equation:

$$\Box\phi \approx -\frac{\beta+2}{3\beta}\phi^{\frac{\beta}{1+\beta}} \quad \vee \quad \alpha + 2\gamma\Lambda = 0. \tag{4.38}$$

Provided that $\alpha + 2\gamma\Lambda \neq 0$, the solution to the above equation is:

$$R(z,t) = \left[i\xi\frac{(z - z_0) - vt}{\sqrt{1 - v^2}} + C^{-1/2}\right]^{-2} , \tag{4.39}$$

where

$$\xi := \frac{1}{2(\beta+1)}\left[\frac{2(\beta+2)(\beta+1)}{3\beta(2\beta+1)}\right]^{1/2} , \tag{4.40}$$

wit $\mathcal{C}$ being an integration constant and v the wave propagation velocity which arises from the Lorentz transformation. Then the only nonvanishing components of the Ricci tensor are:

$$R_{tt} = \frac{R}{6\beta}\left[3 - 2\frac{\beta+2}{1-v^2}\right] , \tag{4.41}$$

$$R_{xx} = R_{yy} = \frac{R}{6\beta}\left[-3 + 2(\beta+2)\right] , \tag{4.42}$$

$$R_{zz} = \frac{R}{6\beta}\left[-3 - 2\frac{\beta+2}{1-v^2}v^2\right] , \tag{4.43}$$

$$R_{tz} = \frac{R}{6\beta}\frac{\beta+2}{1-v^2}2v . \tag{4.44}$$

These results lead to the following non-null and independent NP-Ricci quantities:

$$\Phi_{00} = -\frac{R}{12\beta}(\beta+2)\frac{1-v}{1+v} , \tag{4.45}$$

$$\Phi_{22} = -\frac{R}{12\beta}(\beta+2)\frac{1+v}{1-v} , \tag{4.46}$$

$$\Phi_{11} = \frac{R}{12\beta}(\beta+2) = 2\frac{\beta+2}{\beta}\tilde{\Lambda} . \tag{4.47}$$

The NP-Weyl quantities follows a discussion similar to the previous case, and therefore a full analysis lie beyond the scope of the present work.

Another situation can be studied: an exponent $\beta < -2$, and the same conditions as above hold. In this case, we have $R^{-\beta} \ll R, \Lambda$ and the Ricci scalar is defined in an implicit way through:

$$\frac{(z-z_0) - vt}{\sqrt{1-v^2}} = \sqrt{6\beta(\alpha + 2\gamma\Lambda)}\frac{(\beta+1)}{(\beta+2)}\sqrt{\frac{\beta R^{-1}}{\beta+1}} \times$$
$$\times\, {}_2F_1\left(\tfrac{1}{2}, 1+\tfrac{\beta}{2}; 2+\tfrac{\beta}{2}; \tfrac{4\beta\Lambda R^{-1}}{\beta+1}\right) , \tag{4.48}$$

where ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function [249]. This special function can be evaluated for each specific value β. In order to get some insight on the physical implications, let us consider the case of $\beta = -3$, where the Ricci scalar is given by:

$$R(t,z) = 6\Lambda - \frac{\left((z-z_0) - vt\right)^2}{108(\alpha + 2\gamma\Lambda)(1-v^2)} . \tag{4.49}$$

The main conclusion from the study of the NMC theories with a cosmological constant, or with a constant matter Lagrangian density of the form $\mathcal{L} = -\rho_0 = const.$, reduces to the analysis of a $f(R)$ theory. However, the analysis of the polarisation states can only be complete if the full metric of the theory is found [250]. Even though, the nonvanishing components of the Ricci tensor, seen throughout the present discussion, suggest the presence of extra polarisation modes of the gravitational waves of this alternative model. Moreover, as we get more observational data from mergers of black holes or neutron stars, the detection of such modes could provide a direct way of testing alternative theories of gravity, such as $f(R)$ and the NMC.

4.3 GWs in the presence of a dark energy-like fluid

The next simple, although non-trivial example which can be studied is a perfect fluid matter source having an equation of state parameter of the form $w = -1$, which implies $p = -\rho$:

$$\mathcal{L} = -\rho \approx \text{const.} \Rightarrow T_{\mu\nu} = -\rho g_{\mu\nu} \Rightarrow T = -4\rho \,. \tag{4.50}$$

Although this case resembles the previous one with a cosmological constant, there is a strong difference, namely that, in general, $\delta\rho \neq 0$. This implies that the linearised field equations now read:

$$\begin{gathered}\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \eta_{\mu\nu}\Omega\right) = \\ \frac{f_1 - 2f_2\rho}{F_1 - 2F_2\rho}h_{\mu\nu} + \frac{2f_2\delta\rho}{F_1 - 2F_2\rho}\eta_{\mu\nu} \,,\end{gathered} \tag{4.51}$$

where

$$\Omega := \frac{\delta f' + \delta h'}{F_1 - 2F_2\rho} \,. \tag{4.52}$$

We shall note that

$$3\Box\Omega = \delta R - \frac{4}{F_1 - 2F_2\rho}\delta\rho \,, \tag{4.53}$$

which means $\Box\Omega = -\frac{1}{4}\Box h - 2\frac{\delta\rho}{F_1 - 2F_2\rho}$, thus:

$$\Box\left(h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}h\right) = \frac{f_1 - 2f_2\rho}{F_1 - 2F_2\rho}h_{\mu\nu} \,. \tag{4.54}$$

This result is similar to the cosmological constant case, since the massive longitudinal mode is absorbed into the scaling of the trace of the graviton term. Furthermore, in this rescaling the matter perturbation term also disappears due to the fact one required $\rho = \rho_0 + \delta\rho$, and the fluctuation is a subleading term, and therefore should not manifest itself in the equation for the tensor mode in the gauge Eq. (4.15).

In what concerns the nature and properties of the longitudinal modes, if these two modes in the Klein-Gordon equation can be decoupled, which occurs when matter and curvature perturbations evolve separately at linear order, we have:

$$\Box\omega_f = m^2_{\omega_f}\omega_f \,, \tag{4.55}$$

$$\Box\omega_h = m^2_{\omega_h}\omega_h \,, \tag{4.56}$$

with

$$\omega_f := \frac{\delta f'}{F_1 - 2F_2\rho} = \frac{F_1' - 2F_2'\rho}{F_1 - 2F_2\rho}\delta R \,, \tag{4.57}$$

and

$$\omega_h := \frac{\delta h'}{F_1 - 2F_2\rho} = \frac{-2F_2}{F_1 - 2F_2\rho}\delta\rho \,. \tag{4.58}$$

The masses of the two modes are defined as:

$$m^2_{\omega_f} := \frac{1}{3}\left[\frac{F_1 - 2F_2\rho}{F_1' - 2F_2'\rho} - R_0\right] \,, \tag{4.59}$$

$$m^2_{\omega_h} := \frac{1}{3}\left[\frac{2f_2}{F_2} - R_0\right] \,, \tag{4.60}$$

respectively. Let us note that in our case $R_0 = 0$. By requiring that both $\frac{F_1 - 2F_2\rho}{F_1' - 2F_2'\rho} \geq 0$ and $f_2/F_2 \geq 0$, tachyonic instabilities are avoided.

Notwithstanding, this is not the most general solution, as the evolution of matter and curvature perturbations could depend on each other, thus being coupled.

In addition, a dark energy-like fluid, whose matter Lagrangian $\mathcal{L} = -\rho \approx const.$, has some evident similarities with the case of a cosmological constant in what concerns to the tensor modes, but an extra longitudinal mode related to $\delta\rho$ comes into play.

Let us note that since at linear order the graviton is propagating with a velocity $v \lesssim c$, then the classification system developed in Refs. [230, 231] for metric theories do not necessarily hold [251], and we have to be able to find the full non-linear solution of the field equations. After the mathematical issues properly addressed, there are some engineering difficulties which need to be overcome.

4.4 Detection of NMC gravitational waves

The detectability of the extra polarisation states of the NMC theories may become possible if one resorts to the theory-independent method proposed in Ref. [252]: which relies on the antenna angular pattern functions of the detector of gravitational waves. In its turn, the antenna pattern provides measurements on the linear combination of the polarisation states. Therefore, the response signal from combined Gravitational Wave detectors, such as the Advanced-LIGO, the Advanced-Virgo or the Einstein GW Telescope, could be used to discriminate between models of gravity, and in particular between GR and the NMC.

As far as the strong gravity regime is concerned, a previous work on black hole solutions for the NMC models [122] brought some insight on the issue: once the Newtonian limit was ensured and the null energy condition was satisfied, the Schwarzschild and Reissner–Nordstrom solutions of GR are recovered in the non-minimally coupled curvature-matter theories provided quantities such as mass, charge, and cosmological constant are suitably “dressed". Consequently, further detailed analyses on gravitational waves generation by black holes collisions will result in a modification on the relevant parameters so to account for the effects of the non-minimal coupling.

In concluding, future work requires more studies in the new scalar and vectorial modes by extending the formalism of cross-correlation analysis. Hence, the energy density of the spectrum of a stochastic background of gravitational waves would then be possible [253].

Chapter 5

Nonminimally coupling Weyl Gravity

UNTIL now, our discussion focussed on a metric compatible and torsionless connection. However, as it was pointed out at the Introduction, this is not the most general case and when the other parts are taken into account different theories of gravity are found and new observational features emerge. For instance, some extensions of GR assume either just torsion [254] or only non-metricity [14,255,256]. In fact, there is a vast literature on theories with torsion (we refer the reader to Refs. [257,258] for reviews on the topic). As far as non-metricity is concerned, it was, for instance, constrained by the Bhabha scattering in the metric-affine formulation of Ricci-based gravity models in vacuum [259], and a lower bound on the scale of non-metricity was found to be greater than 1 TeV.

Furthermore, modified gravity models can also be built from more general connections. In fact, extensions with torsion, such as in $f(T)$ theories [260,261], or of non-metricity [262], were considered, leading to significantly different results from $f(R)$ theories. However, the differences if $f(R) = R$ are not so sharp, thus it is said that they provide equivalent formulations of GR [263]. This inspired the analysis for non-minimal coupled versions with curvature and torsion [264] and with non-metricity [265].

Thus, we shall pursue this line of research in the present Chapter, by considering a particular interesting case of non-metricity which is the Weyl Gravity [266]. Originally, this idea was proposed to unify gravity and electromagnetism through an auxiliary vector field. Afterwards, a reformulation of this scenario was due to Dirac leading to a simpler action, but a new scalar field was introduced to describe spacetime in addition to the metric [267]. More recently, Weyl gravity-like theories have regained attention in order to address issues such as dark matter and dark energy problems or inflation [268].

As a matter of fact, the interest of vector fields in cosmology is not new [142–145]. However, a vector field is, in general, incompatible with the homogeneity and isotropy of the Universe, leading to a preferred direction. This can be circumvent provided that it is embedded in a $SO(3)$ symmetry [143–145].

Then, we shall discuss the main features of Weyl gravity with a non-minimal coupling between matter and curvature, as well as some implications for cosmology, namely the contribution from the theory to the cosmological constant.

This Chapter is based on Ref. [4].

5.1 Weyl Gravity

Hermann Weyl proposed a model of gravity relying on non-metricity. This proposal is motivated by a generalised covariant derivative defined from the expression:

$$D_\lambda g_{\mu\nu} = A_\lambda g_{\mu\nu} \,, \tag{5.1}$$

where,

$$D_\lambda g_{\mu\nu} = \nabla_\lambda g_{\mu\nu} - \bar{\bar{\Gamma}}^\rho_{\mu\lambda} g_{\rho\nu} - \bar{\bar{\Gamma}}^\rho_{\nu\lambda} g_{\rho\mu} \,, \tag{5.2}$$

∇_λ is the usual covariant derivative built from the Levi-Civita connection and $\bar{\bar{\Gamma}}^\rho_{\mu\nu} = -\frac{1}{2}\delta^\rho_\mu A_\nu - \frac{1}{2}\delta^\rho_\nu A_\mu + \frac{1}{2} g_{\mu\nu} A^\rho$ is the Weyl connection, constructed from the Weyl vector, A^μ. We further notice that the covariant derivative of the inverse metric is given by: $D_\lambda g^{\mu\nu} = -A_\lambda g^{\mu\nu}$.

In the context of the Weyl connection, we follow the convention that the contraction of first index of the Riemann tensor with its third one yields the Ricci tensor, $\bar{R}_{\mu\nu} := \bar{R}^\lambda_{\mu\lambda\nu}$, which can be expressed as:

$$\bar{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} A_\mu A_\nu + \frac{1}{2} g_{\mu\nu} \left(\nabla_\lambda - A_\lambda\right) A^\lambda + F_{\mu\nu} + \frac{1}{2}\left(\nabla_\mu A_\nu + \nabla_\nu A_\mu\right) := R_{\mu\nu} + \bar{\bar{R}}_{\mu\nu} \,, \tag{5.3}$$

where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the strength tensor of the Weyl field. We can further contract the Ricci tensor giving the Ricci scalar or scalar curvature:

$$\bar{R} = R + 3\nabla_\lambda A^\lambda - \frac{3}{2} A_\lambda A^\lambda := R + \bar{\bar{R}} \,. \tag{5.4}$$

As we can see, the spacetime is no longer specified just by the metric, but also by the Weyl vector field. Therefore, the constant Riemann curvature solutions found in the context of GR, namely the de Sitter, anti-de Sitter and Minkowski spaces, do not follow immediately. In fact, they now also depend on the values of the vector field and only coincide with the usual spaces of GR for a vanishing Weyl vector.

The contracted Bianchi identities can be generalised taking into account the Weyl covariant derivative:

$$\mathcal{D}_\mu \bar{G}^{\mu\nu} = -\frac{1}{2}\mathcal{D}_\mu F^{\mu\nu} \,, \tag{5.5}$$

which give:

$$\mathcal{D}_\mu \bar{G}^{(\mu\nu)} = 0 \,, \tag{5.6}$$

$$\mathcal{D}_\mu F^{\mu\nu} = 0 \,, \tag{5.7}$$

where $\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \bar{R}$ is the Einstein-like tensor for the $\bar{R}^{(\mu\nu)}$ curvature and we have defined for convenience $\mathcal{D}_\mu := \left(D_\mu + 2A_\mu\right)$.

Now, we can proceed to accommodate Weyl gravity within the non-minimal matter-curvature coupling scenario of Ref. [89].

5.2 Weyl Gravity with a non-minimal matter-curvature coupling

We shall now consider the non-minimal coupling model, Eq. (1.16), with the Weyl gravity's scalar curvature:

$$S = \int \left(\kappa f_1(\bar{R}) + f_2(\bar{R})\mathcal{L}\right) \sqrt{-g} d^4x \,. \tag{5.8}$$

If we vary the action functional relatively to the vector field, we obtain, up to boundary terms, a constraint equation:

$$\nabla_\lambda \bar{\Theta} = -A_\lambda \bar{\Theta} \,, \tag{5.9}$$

where $\bar{\Theta} := F_1(\bar{R}) + (F_2(\bar{R})/\kappa)\mathcal{L}$, and $F_i := df_i/d\bar{R}$. On the other hand, varying the action with respect to the metric and taking into account the previous constraint, we get:

$$\left[R_{\mu\nu} + \bar{\bar{R}}_{(\mu\nu)}\right] \bar{\Theta} - \frac{1}{2} g_{\mu\nu} f_1 = \frac{f_2}{2\kappa} T_{\mu\nu} \,, \tag{5.10}$$

where $\bar{\bar{R}}_{(\mu\nu)} = \frac{1}{2} A_\mu A_\nu + \frac{1}{2} g_{\mu\nu} \left(\nabla_\lambda - A_\lambda\right) A^\lambda + \nabla_{(\mu} A_{\nu)}$.

The constraint equation for the vector field reduces a fourth order theory, as the usual NMC, into a second order version.

The trace of the metric field equations reads:

$$\bar{R}\bar{\Theta} - 2f_1 = \frac{f_2}{2\kappa} T \,. \tag{5.11}$$

If we take the trace of the metric field equations and plug it into Eq. (5.10), we have the trace-free equations:

$$\bar{\Theta} \left[R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\right] + \bar{\Theta} \left(\bar{\bar{R}}_{(\mu\nu)} - \frac{1}{4} g_{\mu\nu} \bar{\bar{R}}\right) = \frac{f_2}{2\kappa} \left[T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T\right] \,. \tag{5.12}$$

In its turn, the divergence of the field equations leads to a generalised version of the covariant non-conservation law for the energy-momentum tensor:

$$\nabla_\mu T^{\mu\nu} = \frac{2}{f_2} \left[\frac{F_2}{2} \left(g^{\mu\nu} \mathcal{L} - T^{\mu\nu}\right) \nabla_\mu R + \nabla_\mu \left(\Theta \bar{B}^{\mu\nu}\right) - \frac{1}{2} \left(F_1 g^{\mu\nu} + F_2 T^{\mu\nu}\right) \nabla_\mu \bar{\bar{R}}\right] \,, \tag{5.13}$$

where we have defined the tensor: $B^{\mu\nu} := \frac{3}{2} A^\mu A^\nu + \frac{3}{2} g^{\mu\nu} \left(\nabla_\lambda - A_\lambda\right) A^\lambda$.

On its hand, we can resort to the Weyl divergence of the energy-momentum tensor, yielding:

$$\mathcal{D}_\mu T^{\mu\nu} = \frac{2\bar{\Theta}}{f_2} \left[\mathcal{D}_\mu \bar{R}^{(\mu\nu)} - \frac{1}{2} \mathcal{D}_\mu \left(g^{\mu\nu} \frac{f_1}{\bar{\Theta}}\right) - \mathcal{D}_\mu \left(\frac{f_2}{2\bar{\Theta}}\right) T^{\mu\nu}\right] \,, \tag{5.14}$$

where we opt to keep this form for the equation since it will be useful in the next section when considering the contracted Bianchi identities.

5.2.1 The sequestering of the cosmological constant

An interesting idea firstly proposed by Einstein and further explored by other authors is unimodular gravity [269–273]. This scenario restricts GR with the condition on the determinant of the metric, $\sqrt{-g}=1$, which leads to a traceless version of the field equations:

$$R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R=\frac{1}{2\kappa}\left(T_{\mu\nu}-\frac{1}{4}g_{\mu\nu}T\right)\ . \tag{5.15}$$

It is worth to note that these equations are formally equivalent to substitute the trace equation into the field equations. However, in unimodular gravity, the field equations lost one degree of freedom when the condition for g was taken. Bearing in mind the contracted Bianchi identities $\nabla_\mu G^{\mu\nu}=0$ together with the covariant conservation of the energy-momentum tensor, we can take the divergence of Eq. (5.15) leading to a conserved quantity $\nabla^\nu\left(R+\frac{1}{2\kappa}T\right)=0=\nabla^\nu(4\lambda)$ which can be identified as a cosmological constant.

This procedure motivated further techniques to "sequester" the cosmological constant from a gravity theory [274, 275]. In fact, adding specific Lagrangian multipliers to the action functional lead to a similar version of the previous scenario:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{1}{2\kappa}\left(T_{\mu\nu}-\frac{1}{4}g_{\mu\nu}\langle T\rangle\right)\ , \tag{5.16}$$

where $\langle T\rangle := \int T\sqrt{-g}d^4x/\int\sqrt{-g}d^4x$ is the cosmic average of the trace of the energy-momentum tensor, T. Since the matter Lagrangian can be decomposed as a sum of a vacuum contribution and the matter species part, $\mathcal{L}=-2\kappa\lambda_0+\mathcal{L}_m$, and noting that the cosmic average of a constant is the constant itself, then, it is immediate to get:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda=\frac{1}{2\kappa}\tau_{\mu\nu}\ , \tag{5.17}$$

where $\Lambda := \langle\tau\rangle/8\kappa$ is the cosmological constant term, $\tau_{\mu\nu}$ is the energy-momentum tensor built from $\mathcal{L}_m$ and τ its trace.

This scheme was applied in the context of the relaxed regime for the non-minimal coupling model [276], and some conditions for the functions $f_1(R)$ and $f_2(R)$ were found in order to overcome the discrepancy between the vacuum energy Λ_0 and the integration constant Λ.

Thus, we now aim to get an integration constant from the contracted Bianchi identities for the non-minimally coupled Weyl model. Let us make $\kappa=1$ for the sake of simplicity and considering the identity $(\Box\nabla^\nu-\nabla^\nu\Box)\,H=R^{\mu\nu}\nabla_\mu H$, for some scalar function H, we get:

$$\begin{aligned}
\frac{1}{4}\nabla^\nu R &= \nabla_\mu\left(R^{\mu\nu}-\frac{1}{4}g^{\mu\nu}R\right)\\
&=\frac{1}{4}\nabla^\nu R+\nabla_\mu\left(\frac{1}{\bar{\Theta}}\right)\left[-(R^{\mu\nu}+B^{\mu\nu})\bar{\Theta}-\frac{1}{2}g^{\mu\nu}f_1+\frac{f_2}{2}T^{\mu\nu}+\Box^{\mu\nu}\bar{\Theta}\right]+\\
&\quad+\frac{1}{4}\nabla^\nu\left[R+B-2f_1-\frac{f_2}{2\bar{\Theta}}+\frac{3}{\bar{\Theta}}\Box\bar{\Theta}\right]
\end{aligned} \tag{5.18}$$

$$\iff \nabla^\nu\left[R+B-2f_1-\frac{f_2}{2\bar{\Theta}}+\frac{3}{\bar{\Theta}}\Box\bar{\Theta}\right]=0\ , \tag{5.19}$$

where the last equality trivially vanishes since it corresponds to the covariant derivative of the trace of the field equations. As it was found in Ref. [276], by applying the Bianchi identities no cosmological constant arises from the model.

We can proceed in a similar way for the generalised contracted Bianchi identities:

$$\frac{1}{4}\mathcal{D}_\mu\left(g^{\mu\nu}\bar{R}\right) = \mathcal{D}_\mu\left[\bar{R}^{(\mu\nu)} - \frac{1}{4}g^{\mu\nu}\bar{R}\right] = \mathcal{D}_\mu\left[\frac{f_2}{2\bar{\Theta}}\left(T^{\mu\nu} - \frac{1}{4}g^{\mu\nu}T\right)\right] = \mathcal{D}_\mu\left(\frac{f_2}{2\bar{\Theta}}\right)T^{\mu\nu} +$$

$$+\left[\mathcal{D}_\mu\bar{R}^{(\mu\nu)} - \frac{1}{2}\mathcal{D}_\mu\left(g^{\mu\nu}\frac{f_1}{\bar{\Theta}}\right) - \mathcal{D}_\mu\left(\frac{f_2}{2\bar{\Theta}}\right)T^{\mu\nu}\right] - \frac{1}{4}\mathcal{D}_\mu\left(g^{\mu\nu}\frac{f_2}{2\bar{\Theta}}T\right) \tag{5.20}$$

$$\iff \mathcal{D}_\mu\left(\bar{R}^{(\mu\nu)} - \frac{1}{4}g^{\mu\nu}\bar{R}\right) + \frac{1}{4}\mathcal{D}_\mu\left[g^{\mu\nu}\left(-\frac{f_2}{2\bar{\Theta}}T - 2\frac{f_1}{\bar{\Theta}}\right)\right] = 0 \tag{5.21}$$

$$\iff \frac{1}{4}\mathcal{D}_\mu\left[g^{\mu\nu}\left(\bar{R} - \frac{f_2}{2\bar{\Theta}}T - 2\frac{f_1}{\bar{\Theta}}\right)\right] = 0\,, \tag{5.22}$$

where as in the previous scheme, it yields a trivial solution from which a cosmological constant does not follow.

Notwithstanding, there is another discussion which may hint for a cosmological constant construction, which resorts to the constant sectional curvature solutions of a theory. This possibility shall be analysed in the next subsection.

5.2.2 The space form behaviour

In order to assess whether the vacuum of the NMC Weyl gravity is well defined and admits a constant generalised curvature solution, we resort to the analysis of its space form behaviour. To do so, let us recall that a pseudo-Riemannian manifold is said to be a space form if and only if the following equation holds [277]:

$$\bar{R}_{abcd} = K(g_{ac}g_{db} - g_{ad}g_{cb}) \implies \bar{R}_{bd} = 3Kg_{db} \implies \bar{R} = 12K\,, \tag{5.23}$$

where K is some real constant, which in GR is directly related to the cosmological constant, $K = \Lambda/3$. Given that in our case, we have a vector field affecting the vacuum state, we generalise the space form property so to allow for homogeneous and isotropic evolving spaces where $K = K(t)$. Furthermore, requiring an homogeneous and isotropic scalar curvature, but allowing it to evolve in time, leads to a cosmological term with a time dependence which might account for the dark energy behaviour of the Universe. Actually, a time varying cosmological term was found in the context of the Brans-Dick theory [278].

Even in the absence of matter, the vacuum also contributes to the Lagrangian density, $\mathcal{L} = -2\kappa\Lambda_0$, so that the metric field equations and their trace are given by:

$$\left[R_{\mu\nu} + \bar{\bar{R}}_{(\mu\nu)}\right]\bar{\Theta} = \frac{1}{2}g_{\mu\nu}f_1 - g_{\mu\nu}f_2\Lambda_0\,, \tag{5.24}$$

$$\left[R + \bar{\bar{R}}\right]\bar{\Theta} = 2f_1 - 4f_2\Lambda_0\,. \tag{5.25}$$

The trace of the metric field equations, Eq. (5.25), can be recast in the following form:

$$\frac{2f_1 - \bar{R}F_1}{2f_2 - \bar{R}F_2} = 2\Lambda_0\,, \tag{5.26}$$

which may provide a way to sequester the cosmological constant. Let us consider the case of a pure non-minimal coupling, $f_1 = \bar{R}$, then:

$$f_2(\bar{R}) = \frac{\bar{R}}{2\Lambda_0} + C_2\bar{R}^2 \iff f_2(\bar{R} = 12K(t)) = \frac{6K(t)}{\Lambda_0} + C_2'K(t)^2\,, \tag{5.27}$$

where $\mathcal{C}_2$ is an integration constant and $\mathcal{C}'_2 := 144\mathcal{C}_2$.

On the other hand, we can consider a feeble non-minimal coupling, $f_2(\bar{R}) \approx 1$, along with a general function for the pure gravity, which does not need to reduce to GR, then:

$$f_1(\bar{R}) = 2\Lambda_0 + \mathcal{C}_1\bar{R}^2 \iff f_1(\bar{R} = 12K(t)) = 2\Lambda_0 + \mathcal{C}'_1 K(t)^2 \,, \tag{5.28}$$

where $\mathcal{C}_1$ is an integration constant and $\mathcal{C}'_1 := 144\mathcal{C}_1$.

A conclusion can be immediately drawn: the contribution of the cosmological term, $K(t)$, in general differs from the one arising from the vacuum energy, Λ_0.

A further issue need to be tackled: the homogeneity and isotropy of the Universe cannot be spoiled, hence some conditions for the vector field have to be found. Thus, if we combine equations (5.24) and (5.25) into a single one, we get:

$$4\left[R_{\mu\nu} + \bar{\bar{R}}_{(\mu\nu)}\right]\bar{\Theta} = g_{\mu\nu}\left[R + \bar{\bar{R}}\right]\bar{\Theta} \,, \tag{5.29}$$

where the Riemann tensor with a Levi-Civita connection can itself generate a space form manifold, hence its contractions can be written as $R_{\mu\nu} = 3\Lambda g_{\mu\nu}$ and $R = 12\Lambda$. These terms cancel each other in the previous equation, leading to an equation for the vector field for the non-trivial case, $\bar{\Theta} \neq 0$:

$$g_{\mu\nu}\nabla_\lambda A^\lambda + g_{\mu\nu}\frac{1}{2}A_\lambda A^\lambda = 2A_\mu A_\nu + 4\nabla_{(\mu}A_{\nu)} \,. \tag{5.30}$$

Hence, an ansatz for the Weyl vector need to be assumed. In particular, the following ansatz:

$$A_0 = \xi(t) \,,\; A_i = \chi(t)\delta_i^a L_a \,, \tag{5.31}$$

admits invariance under spatial rotations, $SO(3)$ transformations, with generators L_a, which exhibits the homogeneity and isotropy of space. The time-time component and each diagonal space-space components of Eq. (5.30) for this ansatz lead to:

$$\dot{\xi} + \frac{1}{2}\xi^2 + \frac{1}{2}\chi^2 = 0 \,, \tag{5.32}$$

whilst the time-spatial components give:

$$\dot{\chi} + 2\xi\chi = 0 \,. \tag{5.33}$$

Two non-trivial cases can be studied: $\xi = \xi_0 = const.$ and $\xi = \chi$. The first leads to the solution:

$$\chi(t) = \chi_0 e^{-2\xi_0(t-t_0)} \,, \tag{5.34}$$

where χ_0 and t_0 are integration constants. As we can see, an exponentially decreasing contribution from the vector field is found, which for the space form manifold condition, and resorting to the definition of $\bar{R}_{\mu\nu}$, gives:

$$12K(t) = \left(12\Lambda + \frac{3}{2}\xi_0\right) - 6\xi_0\chi(t) - \frac{9}{2}\chi(t)^2 \,, \tag{5.35}$$

which at late times is expected to be constant, $12K(t) \to 12K = 12\Lambda + \frac{3}{2}\xi_0$.

In its turn, the other case of interest, $\xi = \chi$, leads to:

$$\chi(t) = \frac{\chi_0}{1 + \chi_0(t - t_0)} , \tag{5.36}$$

which for the space form condition yields:

$$12K(t) = 12\Lambda + 3\partial^0\chi(t) - \frac{3}{2} \times 2\chi^2(t) = 12\Lambda . \tag{5.37}$$

In this case, however the vector field does not contribute to the vacuum, and the curvature of the space form coincides with the curvature of the Levi-Civita connection.

Chapter 6

Conclusions

THERE are several reasons, both theoretical and observational, to seek for a theory of gravity beyond General Relativity. The lack of a fully consistent UV completion of GR, and the existence of the dark matter and dark energy problems at the IV regime point towards additional curvature terms in the Einstein-Hilbert action. One of the vast plethora of models of gravity is an extension of the so-called $f(R)$ theories including a non-minimal coupling between curvature and the matter Lagrangian density [89]. This model has some interesting properties, such as the covariant non-conservation of the energy-momentum tensor of matter fields, which gives rise to an extra force term in the geodesics for a perfect fluid. This feature allows for mimicking the dark matter and dark energy behaviours at galactic and cosmological scales. This model has survived a vast set of observational constraints and some stability conditions have been found, hence motivating further studies. Therefore, this work presented some implications of such theories in different contexts.

In Chapter 2, we have studied the implications of a scalar field inflaton non-minimally coupled to curvature. It was found that the NMC model exhibited two main regimes during the slow-roll of the inflaton along its potential: the low density regime where the observational predictions coincided, in the limit, with the ones from GR; and the high density regime whose asymptotic behaviour for the Friedmann equation resemble the contribution from Starobinsky model, and whose predictions differed significantly from GR. In order to have a complete analysis, some of the most commonly used inflationary potentials, namely monomials, hilltop and Higgs-like potentials, were compared with Planck data. In general, the NMC in the high density regime led to a higher scalar-to-tensor ratio, which means that a higher amount of gravitational waves are expected. However, the inflationary potentials were in agreement with data for large ranges on the running parameters. Furthermore, we have shown that the consistency relation $|r/n_t|$ may deviate strongly from the GR's prediction, particularly at high densities, where we have found $|r/n_t| \gtrsim 8$. This implies that future observational constraints in both the tensor-to-scalar ratio and the tensor index may help distinguishing GR from the NMC curvature-matter models of gravity.

The discussion of generic Friedmann equations of the form $H^2 = H^2(\rho)$ and the numerical code developed in the previous Chapter motivated the discussion of some "exotic" inflationary models relying on the Generalised Chaplygin gas, on Supergravity and on Brane world scenarios in the light of recent data. This analysis was done in Chapter 3. The results showed that there were some regions on the running parameters which met the data. However, a model of $N = 1$ supergravity with an R-symmetry, yielding a hilltop-like potential with vanishing first and second derivatives at the origin, which was observationally disfavoured at more than 2σ. A similar model could, however, be made compatible with the Planck data by considering a brane-localised scenario. On the other hand, we have also found that scalar potential with a very flat plateau section led to predictions which are degenerated to modified gravity models such as the popular Starobinsky model, which is essentially likewise a special plateau-potential in the Einstein frame. Thus, data may be pointing towards to potentials with these properties rather than to modified gravity.

Gravitational waves are predicted in the context of inflation, but also from the merger of compact object, such as black holes and neutron stars. Therefore, in Chapter 4, we explored the gravitational wave solutions of the NMC model resorting to both perturbation theory and the Newman-Penrose formalism. In particular, the perturbation of the trace of the field equations exhibited a behaviour that could be interpreted as the dynamics of the effective scalar field decoupled into two scalar modes: one arising from perturbations on the Ricci scalar and the other from perturbations on the matter Lagrangian density. When considering a matter Lagrangian from a cosmological constant term, the analysis was similar to the $f(R)$ case. A further technical difficult need to be overcome in the future: a way to compute the full metric solution of the field equations in order to classify the Weyl tensor for the theory. However, the mass of the graviton and on the speed and group velocity of the gravitational waves at linear level are in agreement with the most recent bounds.

Finally, in Chapter 5, we have relaxed the metric compatibility assumption on the connection in a particular case: the Weyl gravity. In this scenario, the spacetime is not uniquely determined by the metric, but also by a vector field. Therefore, we allowed the NMC model to account for the non-metricity. It was found that this property reduced the order of the field equations, hence given a second order well defined theory. Furthermore, space form manifolds are also admitted in these theories, and that the non-metricity affects the vacuum. In order to have a homogeneous and isotropic Universe, the Weyl vector, A_μ, had to obey a $SO(3)$ symmetry. Thus, two time decaying solutions were found: one leading to a constant curvature at late times with a correction, and the other degenerated into the well known constant curvature of GR. However, these solutions, likewise GR, can be made compatible with observations by fine tuning.

As a final remark we would say that in this thesis we have shown that the non-minimal coupling theories: i) are compatible with inflation driven by a scalar field; ii) have gravitational wave solutions, which predict the existence of extra polarisations modes although the tensor ones are quite similar to the GR case; iii) with the non-metricty property may allow for new features in a gravity theory.

Part III

Appendices

Appendix A

Avoidance of singularities

In fact, as seen in the Introduction, it is common to express the field equations of alternative theories of gravity in an effective Einstein-like field equations, Eq. (1.21). From these equations we can define an effective gravitational constant: $\hat{G}_{eff} \equiv \frac{f_2}{\left(F_1+\frac{F_2\mathcal{L}}{\kappa}\right)}\frac{1}{2\kappa}$. Although extremely convenient, this kind of identification may present singularities, particularly if it changes sign and one considers homogeneous, but anisotropic models [279, 280]. This may pose a serious problem in the NMC model, given that inflation had to work beyond the assumptions of homogeneity and isotropy. Therefore, we aim to properly address this issue in this appendix.

We have explored inflation in the presence of a pure non-minimal coupling, $f_1 = R$. In this scenario, when deriving the Friedmann equation during slow-roll, Eq. (2.11), we found a natural effective gravitational constant: $G_{eff} \sim \frac{f_2}{1+\frac{2F_2\rho}{M_P^2}}G$. This expression resulted from the balance between the convenient constant $\hat{G}_{eff}$ and the effective energy-momentum tensor. In fact, G_{eff} remains positive throughout the entire slow-roll regime. But after slow-roll, that expression no longer holds and we need to look back at the full $\hat{G}_{eff}$ which will be positive given that the inflaton will be kinetically dominated.

Moreover, if field and metric perturbations cannot avoid a zero appearing in the expression $1-2\frac{F_2V(\phi)}{M_P^2}$ (related to $\hat{G}_{eff}$) this poses further restrictions on the free parameters for each inflationary potential: (x,γ). Therefore, in order to circumvent the problem, we must require that the effective gravitational constant remains positive during inflation. For the slow-roll, this occurs for the hypersurface $\frac{V(\phi)}{V_0} \approx \frac{2.57653}{x}$, for $f_1 = R$ and $f_2 = 1+\left(\frac{R}{M^2}\right)^3$.

Hence, we are able to find regions defined by a range of the parameters which are compatible with Planck's data. In particular, for the linear monomial model, $n = 1$, this requirement implies that $x \lesssim 0.24$ for 50 e-folds and $x \lesssim 0.22$ for $N_e = 60$, which gives $r < 0.11$, in perfect agreement with data. The quadratic monomial potential, $n = 2$, requires $x \lesssim 0.012$ for $N_e = 50$ and $x \lesssim 0.01$ for $N_e = 60$, even though it is still excluded by data. As for the quadratic hilltop potential, and for $x = 5$ the coupling parameter is constrained to be $\gamma \lesssim 0.0018$ when $N_e = 50$ and $\gamma \lesssim 0.0015$ for $N_e = 60$. Graphically, this means that only the region from the turning around point ("piparote") till $\gamma \to 0$ is allowed. However, as the ratio x drops, larger values for the parameter are allowed; as an example, for $x = 2$ all values in the range $\gamma \in]0,1[$ are allowed. The case of the quartic hilltop potential follows in a similar way: for the ratio $x = 5$, the free parameter $\gamma \lesssim 3.1\times10^{-7}$ for 50 e-folds and $\gamma \lesssim 2.1\times10^{-7}$ for 60 e-folds. Once more, this results in that only the region ranging from the turn around point towards the $\gamma \to 0$ limit is admissible. Furthermore, as we lower x, higher values for γ are allowed which turns into a larger graphical area. As for the Higgs-like potential, by setting $x = 5$, the following constraints are found: $\gamma \lesssim 0.0049$ for $N_e = 50$

and $\gamma \lesssim 0.0041$ for $N_e = 60$. For $x = 2$, the free parameter has to be $\gamma \lesssim 0.04$ for the range $N_e \in [50, 60]$.

Now, we have to ensure that after inflation, a change on the sign if the effective gravitational constant does not occur. In fact, the pressure becomes kinetically dominated $p \approx K$, as the inflaton oscillates around the minimum of its potential, so that $F_1 + \frac{F_2 \mathcal{L}}{\kappa}$ stays positive throughout the whole history of the Universe. Hence, we expect the behaviour of the Universe evolution to be identical to GR scenario since one is in the low curvature regime, and the characteristic mass scale of the non-minimal coupling function, which is coupled to the inflaton, to be of the order of 10^{13} GeV as it was found at the end of Subsection (2.2.4). Additionally, neither the curvature nor the inflaton Lagrangian, i.e. its pressure, blow up since $F_1 + \frac{F_2 \mathcal{L}}{\kappa} < \infty$. Thus, the homogeneity and isotropy of the Universe depicted in the FRW metric are not spoiled.

It is worth mentioning, that a singularity could appear between inflation and radiation epochs, mainly due to the high complexity of the full field equations and the reheating model-dependency. However, this issue can be removed when considering the effects of both matter and tensor perturbations, in the same fashion as in inflationary scenarios in the context of other modified gravity models. Of course, the aforementioned constraints no longer hold and a highly non-trivial analysis of such issues is required for reaching a general conclusion.

Appendix B

Inflation with a general modified Friedmann equation

THE non-minimal matter-curvature coupling model induces a modification on the Friedmann equation, which in the context of slow-roll inflation, has a generic dependence on the energy density of the scalar field. Therefore, this motivates a thorough analysis of inflationary dynamics arising from a Friedmann equation of the form $H^2 = H^2(\rho)$, with the requirement that inflaton's equation of motion during slow-roll, $3H\dot{\phi} \simeq -V'(\phi)$, is unaffected.

Thus, the acceleration equation is given by:

$$\frac{\ddot{a}}{a} = H^2\left[1 - 3\frac{dH/d\rho}{H(\rho)}\dot{\phi}^2\right] = H^2\left[1-\epsilon\right] \ , \tag{B.1}$$

where we may write the slow-roll parameter in the form $\epsilon = 3\frac{dH/d\rho}{H(\rho)}\dot{\phi}^2$. In order to have an accelerated period characterised by $\ddot{a} > 0$, the condition $\epsilon < 1$ has to be satisfied. The latter we can recast in terms of the field slow-roll parameter $\epsilon_\phi = M_P^2(V'/V)^2/2$ as:

$$\epsilon_\phi < 3\left(\frac{M_P^2H^2}{V}\right)^2\left(\frac{dH^2}{d\bar{\rho}}\right)^{-1} \ , \tag{B.2}$$

for $\rho \simeq V(\phi)$. It is straightforwardly found that in the GR limit, this reduces to $\epsilon_\phi < 1$. In addition, the slow-roll also requires $\rho \simeq -p$, i.e. $\dot{\phi}^2/2 < V$, which is equivalent to:

$$\epsilon_\phi < 9\frac{H^2M_P^2}{V} \ . \tag{B.3}$$

While in GR the two conditions on the field slow-roll parameter, ϵ_ϕ, are essentially the same (up to $\mathcal{O}(1)$ factors), in general they yield different constraints, and the end of inflation is set by the strongest of these conditions. In fact, for the non-minimal inflaton-curvature coupling model considered in the high density regime, $V \gg H^2M_P^2$, the second condition, Eq. (B.3), yields a much stronger constraint than Eq. (B.2).

The general expression for the number of inflationary e-folds reads:

$$N_e = \int H dt = \int \frac{H}{\dot{\phi}}d\phi \approx -3\int_{\phi_i}^{\phi_e}\frac{H^2(\rho)}{V'}d\phi \ . \tag{B.4}$$

Now, we can proceed to compute some of the relevant quantities which can be constrained by the CMB observational data. We start by the amplitude of the dimensionless scalar curvature power spectrum, which is given by:

$$\Delta_{\mathcal{R}}^2 = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 = \frac{9H^6(\rho)}{4\pi^2 V'^2} = \frac{1}{24\pi^2}\frac{V}{M_P^4}\epsilon_\phi^{-1}\left(\frac{3H^2M_P^2}{V}\right)^3 , \tag{B.5}$$

where it easy to see that the amplitude of curvature perturbations is suppressed for $V \gg H^2M_P^2$. The associated scalar spectral index is now given by:

$$\begin{aligned} n_s - 1 &\equiv \frac{d\ln\Delta_{\mathcal{R}}^2}{d\ln k} = \frac{\dot{\phi}}{H}\frac{d\ln\Delta_{\mathcal{R}}^2}{d\phi} = -2\frac{dH}{d\rho}\frac{V'^2}{H^3} + \frac{2}{3}\frac{V''}{H^2} \\ &\simeq -6\left(3\frac{dH^2}{d\bar{\rho}}\right)\left(\frac{V}{3H^2M_P^2}\right)^2\epsilon_\phi + 2\left(\frac{V}{3H^2M_P^2}\right)\eta_\phi , \end{aligned} \tag{B.6}$$

where $\eta_\phi = M_P^2 V''/V$ satisfies the slow-roll condition $|\eta_\phi| \ll H^2M_P^2/V$ similarly to Eq. (B.3).

Another relevant quantity is the amplitude of the tensor spectrum:

$$\Delta_t^2 = \frac{8}{M_P^2}\left(\frac{H}{2\pi}\right)^2 = \frac{2}{\pi^2 M_P^2}H^2(\rho) , \tag{B.7}$$

from which the tensor spectral index is found:

$$n_t \equiv \frac{d\ln\Delta_t^2}{d\ln k} \simeq -\frac{2V'^2}{3H^3(\rho)}\frac{dH}{d\rho} \simeq -2\left(3\frac{dH^2}{d\bar{\rho}}\right)\left(\frac{V}{3H^2M_P^2}\right)^2\epsilon_\phi , \tag{B.8}$$

as well as the tensor-to-scalar ratio:

$$r \equiv \frac{\Delta_t^2}{\Delta_s^2} = \frac{8}{9}\frac{V'^2}{M_P^2}\frac{1}{H^4(\rho)} \simeq 16\left(\frac{V}{3H^2M_P^2}\right)^2\epsilon_\phi . \tag{B.9}$$

Furthermore, the non-minimal coupling between matter and curvature model does not change the Lyth bound obtained in GR, which related the distance travelled by the inflaton field in the field space during inflation and the scalar-to-tensor ratio. In fact, for any modified Friedmann equation of the form $H^2 = H^2(\rho)$ we get:

$$\frac{\Delta\phi}{M_P^2} = \mathcal{O}(1)\sqrt{\frac{r}{0.01}} . \tag{B.10}$$

The last comment is dedicated to the consistency relation between the tensor-to-scalar ratio and the tensor spectral index, which turns out to be:

$$r = -8\left(3\frac{dH^2}{d\bar{\rho}}\right)^{-1} n_t . \tag{B.11}$$

This expression is in general considerably different from the GR case, $r = -8n_t$, possibly yielding a ‘smoking gun’ for a non-minimally coupled inflationary regime.

Appendix C

The Newman-Penrose formalism

A way to properly analyse the polarisation modes of a gravitational theory is obtained from the Newman-Penrose formalism, named after Ezra T. Newman and Roger Penrose [232]. Originally, it was used treat to GR in terms of spinor notation, but later on it started to be employed for other theories of gravity. In this formalism, we decompose the Riemann tensor into its irreducible parts: the Weyl tensor, the Ricci tensor and the scalar curvature. In fact, the independent components of the Weyl tensor are encoded in ten Ψ scalar functions, while the independent components of the Ricci tensor (and its trace itself) are encoded in nine Φ scalar functions plus a single $\tilde{\Lambda}$ scalar. The classification on the algebric symmetries of the Weyl scalars of a given theory was performed by by Aleksei Zinovyevich Petrov in 1954 [233] and, independently, by Felix Pirani in 1957 [281]. An alternative classification relies on the Plebanski tensor, built from the trace free Ricci tensor, which shares the symmetries of the Weyl tensor [250, 282].

In fact, the tensors of a given gravity theory are projected onto a complete vector basis at each point in spacetime, in a so called tetrad formalism. The vector basis is chosen to reflect a given symmetry depending on the issue which needs to be addressed. In the case of gravitational wave solutions of a gravity theory, it is useful to chose the gravitational wave oriented in the $+z$ direction, so that one can use the null-complex tetrads [231]:

$$\mathbf{k} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_t + \mathbf{e}_z\right) \ , \qquad \mathbf{l} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_t - \mathbf{e}_z\right) \ , \tag{C.1}$$

$$\mathbf{m} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_x + i\mathbf{e}_y\right) \ , \quad \bar{\mathbf{m}} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_x - i\mathbf{e}_y\right) \ , \tag{C.2}$$

which obey $-\mathbf{k}\cdot\mathbf{l} = \mathbf{m}\cdot\bar{\mathbf{m}} = 1$ and $\mathbf{k}\cdot\mathbf{m} = \mathbf{k}\cdot\bar{\mathbf{m}} = \mathbf{l}\cdot\mathbf{m} = \mathbf{l}\cdot\bar{\mathbf{m}} = 0$, respectively.

Therefore, the Newman-Penrose quantities in this tetrad basis read:

$$\Psi_0 \quad := C_{kmkm} = R_{kmkm} \tag{C.3}$$

$$\Psi_1 \quad := C_{klkm} = R_{klkm} - \frac{R_{km}}{2} \tag{C.4}$$

$$\Psi_2 \quad := C_{km\bar{m}l} = R_{km\bar{m}l} + \frac{R}{12} \tag{C.5}$$

$$\Psi_3 \quad := C_{kl\bar{m}l} = R_{kl\bar{m}l} + \frac{R_{l\bar{m}}}{2} \tag{C.6}$$

$$\Psi_4 \quad := C_{l\bar{m}l\bar{m}} = R_{l\bar{m}l\bar{m}} \tag{C.7}$$

$$\Phi_{00} \quad := \frac{R_{kk}}{2} \tag{C.8}$$

$$\Phi_{11} \quad := \frac{R_{kl} + R_{m\bar{m}}}{4} \tag{C.9}$$

$$\Phi_{22} := \frac{R_{ll}}{2} \tag{C.10}$$
$$\Phi_{01} := \frac{R_{km}}{2} = \Phi_{10}^* \equiv \left(\frac{R_{k\bar{m}}}{2}\right)^* \tag{C.11}$$
$$\Phi_{02} := \frac{R_{mm}}{2} = \Phi_{20}^* \equiv \left(\frac{R_{\bar{m}\bar{m}}}{2}\right)^* \tag{C.12}$$
$$\Phi_{12} := \frac{R_{lm}}{2} = \Phi_{21}^* \equiv \left(\frac{R_{l\bar{m}}}{2}\right)^* \tag{C.13}$$
$$\tilde{\Lambda} := \frac{R}{24}, \tag{C.14}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. In order to better understand the previous notation, let us recall that any tensor T can be written as $T_{abc...} = T_{\mu\nu\lambda...}a^{\mu}b^{\nu}c^{\lambda\cdots}$, where $a, b, c, \ldots$ are vectors of the null-complex tetrad basis $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$, whilst $\mu, \nu, \ldots$ run over the spacetime indices.

Actually, in a metric theory that admits plane null wave solutions, the Newman-Penrose quantities reduce to only six real independent components in a given null frame: $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$. Thus a complete classification system for this kind of theories can be built from the "little group" $E(2)$, which is a subgroup of the Lorentz group that leaves the wavevector invariant [230, 231, 233]. This set, under the action of the rotation group, yields the following helicities $\{0, \pm 1, \pm 2, 0\}$. A closer look at both Ψ_3 and Ψ_4 scalars show that they are complex quantities, hence for each of them two polarisations modes are found, that correspond to the real and to the imaginary parts of these functions. In particular, the Ψ_2 scalar denotes a longitudinal mode, the real and imaginary parts of Ψ_3 account for the mixed vectorial $x-$ and $y-$modes, the Ψ_4 corresponds to the two transverse tensor polarisations $(+, \times)$, and the transverse scalar breathing mode is accounted for Φ_{22}. This is the motivation for Fig. 1.9 in the Introduction.

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