A new stochastic Fubini-type theorem: On interchanging expectations and Itō integrals*,†

Michael Heinrich Baumann‡,§

January 8, 2019

Abstract

When a stochastic process is given through a stochastic integral or a stochastic differential equation (SDE), an analytical solution does not have to exist—and even if there is a closed-form solution, the derivation of this solution can be very complex. When the solution of the stochastic process is not needed but only the expected value as a function of time, the question arises whether it is possible to use the expectation operator directly on the stochastic integral or on the SDE and to somehow calculate the expectation of the process as a Riemann integral over the expectation of the integrands and integrators. In this paper, we show that if the integrator is linear in expectation, the expectation operator and an Itō integral can be interchanged. Additionally, we state how this can be used on SDEs and provide an application from the field of mathematical finance.

MSC (2010): 60H05, 60H10
Keywords: Stochastic Analysis, Itō integral, Expectations, Fubini Theorem, Semimartingale, Stochastic Process

1 Introduction and Motivation

In this paper, we present a new finding in stochastic calculus, namely, that—under specific conditions—it is allowed to interchange an expectation operator $\mathbb{E}$ and a stochastic integral, actually an Itō integral $X \bullet Z_t = \int_0^t X_{s-} dZ_s$. In case the integrator is a Brownian motion, this result is easy to show. However, when we allow all semimartingales as integrators Z (cf. the theorem of Bichteler-Dellacherie: the set of "good integrators" is exactly the set of semimartingales) the result is not trivial. Since both the expectation operator as

*The work of the author was supported by a scholarship of Hanns-Seidel-Stiftung e. V. (HSS), funded by Bundesministerium für Bildung und Forschung (BMBF)

†Parts of this work also appeared in the doctoral thesis of the author [1].

‡Department of Mathematics, University of Bayreuth, 95447 Bayreuth, Germany

§michael.baumann@uni-bayreuth.de

well as the Itō integral are integrals, the result is a kind of stochastic Fubini-type theorem, although different to the stochastic Fubini or Fubini-type theorems in the literature [2, 3, 4, 5, 6] and Thm. IV.64, Thm. IV.65 in [7]. In some of these papers/results different probability spaces for the expectation and for the stochastic process are used. In others the measure is time dependent and the integration is performed in time over that measure. That means, the assumptions for these stochastic Fubini-theorems are very different to those in the work at hand.

Our result was inspired by the analysis of trading rules [8], to which we will come back in the example, Section 4. There are market models, e.g., the Cox-Ingersoll-Ross model, with price processes modeled via stochastic differential equations (SDEs) that cannot be solved analytically. However, often an analytical solution of an SDE describing a price process or the gain function of a trading rule is not needed at all since the expected value of the solution as a function of time is enough for applications. In this paper it is shown that it is allowed to swap the expectation operator and the Itō integral under specific conditions: loosely spoken, the integrator has to be "linear in expectation." With this, we then apply the expectation operator on both sides of an SDE to get a deterministic ordinary differential equation (ODE) for the expectation of the solution.

2 The Setting: Adapted, Cádlág Integrand and Semimartingale Integrator

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete and filtered probability space that fulfills the usual hypotheses (cf. part I., p. 3 of [7]) where $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration and $T > 0$ is the model's time horizon. We denote the space of all adapted, càdlàg processes with $\mathbb{D}$ and the space of all adapted, càglàd processes with $\mathbb{L}$. Furthermore, we define $\mathbb{D}_0 := \{Y \in \mathbb{D} \mid Y_0 = 0\}$ and $\mathbb{L}_0 := \{Y \in \mathbb{L} \mid Y_0 = 0\}$. We need some basic definitions and lemmas to set up the stochastic Itō integral in the form it is used in the main part of this paper, Section 3, and how it is common in the field of stochastic analysis (cf. [7]).

Definition 2.1 (*up* Convergence)**.** Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes and H be a stochastic process. We say that H_n converges uniformly in probability (*up*) to H if $\sup\limits_{t\in[0,T]} |H_{n,t} - H_t| \to 0$ for $n \to \infty$ in probability.

Definition 2.2 (Simple Predictable Process)**.** A process H is called simple predictable if it can be written as

$$H_t(\omega) = \sum_{i=1,\dots,n} Z_{i-1}(\omega)\mathbb{I}_{]\!]T_{i-1},T_i]\!]}(\omega, t)$$

with $n \in \mathbb{N}$, $(T_i)_{i=1,\dots,n}$ stopping times with $0 = T_0 \le T_1 \le \ldots \le T_n = T$, and Z_i $\mathcal{F}_{T_i}$-measurable random variables ($i = 0, \dots, n$) with $|Z_i| < \infty$. With $\mathcal{S}$ we denote the set of all simple predictable processes.

Definition 2.3 (Stochastic Integral for Simple Predictable Processes)**.** Let H be a simple predictable process and $X \in \mathbb{D}$. We define the stochastic integral of H over X as the linear function $H \bullet X : \mathcal{S} \to \mathbb{D}, H \mapsto \int_0^t H_s dX_s$ with

$$H \bullet X_t := \sum_{i=1,\dots,n} Z_{i-1}(X_{T_i \wedge t} - X_{T_{i-1} \wedge t}).$$

Lemma 2.4. *The closure of $\mathcal{S}$ with respect to a metric d_{up} induced by the up convergence, called $\overline{\mathcal{S}_{up}}$, i.e.,*

$$Y_n \to Y \ up, n \to \infty \Leftrightarrow: d_{up}(Y_n, Y) \to 0, n \to \infty,$$

equals $\mathbb{L}_0$.

A proof can be found in [9] Thm. 3.55. In this setting, the integral $H \bullet X$ is defined for all $H \in \mathbb{L}_0$ and all semimartingales X:

Definition 2.5 (Stochastic Integral on $\overline{\mathcal{S}_{up}}$)**.** Let X be a semimartingale, $H \in \overline{\mathcal{S}_{up}} = \mathbb{L}_0$, and $(H_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ be a sequence with $H_n \to H$ *up*. The stochastic integral of H over X is $\lim_{n\to\infty}(H_n \bullet X) =: H \bullet X \ \left(= \int H_t dX_t\right)$.

This definition is well defined (see [9] Bem. 3.53). Next, we recapitulate some findings concerning random grids before stating our new findings in Section 3.

Definition 2.6 (Random Grid Tends to Identity)**.** Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of random grids given through $\sigma_n = (T_0^n, T_1^n, \dots, T_{k_n}^n)$ with $0 = T_0^n \leq T_1^n \leq \dots \leq T_{k_n}^n = T$ stopping times, $k_n \in \mathbb{N}$. We say that $(\sigma_n)_{n\in\mathbb{N}}$ tends to identity if

$$\|\sigma_n\| = \max_{i=1,\dots,k_n} |T_i^n - T_{i-1}^n| \to 0 \ a.s., \ n \to \infty.$$

Let Y be a stochastic process and $\sigma = (T_0, T_1, \dots, T_k)$ be a random grid. Via

$$Y^{(\sigma)}(\omega, t) = \sum_{i=1,\dots,k} Y_{T_{i-1}}(\omega, t)\mathbb{I}_{]\!]T_{i-1},T_i]\!]}(\omega, t)$$

we define the simple predictable process $Y^{(\sigma)}$. And for an integrator X we define the integral

$$Y^{(\sigma)} \bullet X := \sum_{i=1,\dots,k} Y_{T_{i-1}}(X^{T_{i-1}} - X^{T_i}).$$

Lemma 2.7. *If X is a semimartingale, Y is an element of $\mathbb{D}_0$ or $\mathbb{L}_0$, and $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of random grids tending to identity, it holds $Y^{(\sigma_n)} \bullet X \to Y_- \bullet X$ up, $n \to \infty$.*

This lemma is proven in [7] Thm. II.21. Note that $Y_- = (Y_{t-})_t$ is the process Y made left-continuous. Analogously, we define $H_+ = (H_{t+})_t$ as the process H made right-continuous. Now, we recapitulated everything needed for our new stochastic Fubini-type theorem.

3 The stochastic Fubini-type Theorem

The main contribution of this work is the next theorem: It states that it is allowed to interchange the two integrals of interest, namely the Itō integral and $\mathbb{E}$ if the "integrator is linear in expectation."

Theorem 3.1. *Let Z be a semimartingale with $\mathbb{E}[Z_t - Z_s] = \zeta(t-s)$ $\forall 0 \le s \le t \le T$ and $\zeta \in \mathbb{R}$. Further let $X \in \mathbb{D}_0$ be integrable (i.e., $\mathbb{E}[|X_t|] < \infty$ $\forall t$) and $Z_t - Z_s$ independent of X_s for all $0 \le s \le t \le T$. Let $Y_t = X \bullet Z_t = \int_0^t X_{s-} dZ_s$ be integrable, too, and $\mathbb{E}[X_t]$ continuous. Then it holds that*

$$\mathbb{E}[Y_t] = \int_0^t \mathbb{E}[X_s]\zeta ds.$$

Proof. If $X \in \mathbb{D}_0$ it follows that $X_- \in \mathbb{L}_0$. We define a sequence of random grids through $\sigma_n = (0, \frac{T}{n}, \frac{2T}{n}, \ldots, T)$, cf. Def. 2.6. Note that $(\sigma_n)_n$ tends to identity and that all σ_n are deterministic. We define the sequence of simple predictable processes X^n via

$$X^n(\omega, t) = \sum_{i=1,\ldots,2^n} X_{\frac{(i-1)T}{2^n}}(\omega, t) \mathbb{I}_{]\frac{(i-1)T}{2^n}, \frac{iT}{2^n}]}(t).$$

With Lemma 2.7 it follows that $X^n \bullet Z \to X \bullet Z$ *up*.

We choose a subsequence of X^n s.t. the convergence $X^n \bullet Z \to X \bullet Z$ is uniformly in time and *a.s.* in ω (which is possible since the limit is in probability). Additionally, we set all $X^n(\omega) \equiv 0$ where either the convergence does not hold (because it is just *a.s.*) or where the distance (as the supremum over t) between $(X^n \bullet Z)(\omega)$ and $(X \bullet Z)(\omega)$ is ≥ 1. We rename this new sequence to X^n and note that nothing changes concerning the convergence, besides that the convergence is dominated by the integrable function $|Y_t| + 1$ (a).

The convergence $X^n \to X$ is pointwise in t for *a.a.* ω. For each t (and *a.a.* ω), we can find an n^* so that $|X_t^n(\omega) - X_t(\omega)| < 1$ for all $n \ge n^*$. That means, we can treat the convergence like it was bounded (with boundary $|X_t| + 1$) (e).

Furthermore, we define for all X^n a sequence $X^{n,m}$ $(m \ge n)$ of representations via

$$X^{n,m}(\omega, t) = \sum_{i=1,\ldots,2^n} \sum_{j=1,\ldots,2^{m-n}} X_{\frac{(i-1)T}{2^n}}(\omega, t) \mathbb{I}_{]\frac{(i-1)T}{2^n} + \frac{(j-1)T}{2^m}, \frac{(i-1)T}{2^n} + \frac{jT}{2^m}]}(t).$$

Note that $X^{n,n} = X^n$ and that all $X^{n,m}$ are just representations of X^n for all $m \ge n$ (i.e., all $X^{n,m}$ and X^n are exactly the same function; convergences are monotonous) (b). It holds that X_u and X_u^n are independent of $Z_w - Z_v$ for all $0 \le u \le v \le w \le T$ (c). For shortening the notation, we insert a subscript t at the end of the formulae instead of subscript $\wedge t$ in each random variable. Further, note that $\mathbb{E}[X_t]$ is bounded on $[0, T]$, thus, for a sequence that converges to $\mathbb{E}[X_t]$ this convergence can assumed to be bounded (with the same argument as above) (d).

This leads to:

$$
\begin{aligned}
\mathbb{E}[Y_t] =& \mathbb{E}[X \bullet Z_t] \\
=& \mathbb{E}\left[\lim_{n\to\infty} X^n \bullet Z_t\right] \\
\overset{(a)}{=}& \lim_{n\to\infty} \mathbb{E}[X^n \bullet Z_t] \\
=& \lim_{n\to\infty} \mathbb{E}\left[\lim_{m\geq n,\ m\to\infty} X^{n,m} \bullet Z_t\right] \\
\overset{(b)}{=}& \lim_{n\to\infty} \lim_{m\geq n,\ m\to\infty} \mathbb{E}[X^{n,m} \bullet Z_t] \\
=& \lim_{n\to\infty} \lim_{m\geq n,\ m\to\infty} \mathbb{E}\left[\sum_{j=1,\dots,2^m} X^n_{\frac{(j-1)T}{2^m}} \left(Z_{\frac{jT}{2^m}} - Z_{\frac{(j-1)T}{2^m}}\right)\right]_t \\
\overset{(c)}{=}& \lim_{n\to\infty} \lim_{m\geq n,\ m\to\infty} \left(\sum_{j=1,\dots,2^m} \mathbb{E}\left[X^n_{\frac{(j-1)T}{2^m}}\right] \mathbb{E}\left[Z_{\frac{jT}{2^m}} - Z_{\frac{(j-1)T}{2^m}}\right]\right)_t \\
=& \lim_{n\to\infty} \left(\lim_{m\geq n,\ m\to\infty} \sum_{j=1,\dots,2^m} \mathbb{E}\left[X^n_{\frac{(j-1)T}{2^m}}\right] \cdot \frac{\zeta}{2^m}\right)_t \\
=& \lim_{n\to\infty} \int_0^t \mathbb{E}[X^n_s]\zeta ds \\
\overset{(d)}{=}& \int_0^t \lim_{n\to\infty} \mathbb{E}[X^n_s]\zeta ds \\
\overset{(e)}{=}& \int_0^t \mathbb{E}\left[\lim_{n\to\infty} X^n_s\right]\zeta ds \\
=& \int_0^t \mathbb{E}[X_s]\zeta ds
\end{aligned}
$$

□

Next, we present an alternative proof, which is more constructive and does not use Lemma 2.7.

Proof. Since X_{t-} is predictable we find a sequence of simple predictable processes $(H^n_t)_t$ so that $H^n \to X,\ n \to \infty,\ up$ and $H^n \bullet Z \to X \bullet Z, n \to \infty,\ up$. In the next step we choose a subsequence so that $H_n \to X$ is uniformly in time and $a.s.$ in ω (which is possible since the limit is in probability). Note that the limit for the integral is still *up*. For shortening the notation, we rename it to H^n again.

Now, H^n_t is of the form $H^n_t(\omega) = \sum_{i=1,\dots,k_n} M^n_{i-1}(\omega)\mathbb{I}_{]\!]T^n_{i-1},T^n_i]\!]}(\omega,t)$ with $k_n \in \mathbb{N}$, $(T^n_i)_{i=1,\dots,k_n}$ stopping times with $0 = T^n_0 \leq T^n_1 \leq \ldots \leq T^n_{k_n} = T$ and M^n_i $\mathcal{F}_{T^n_i}$-measurable random variables $(i = 0, \dots, k_n - 1)$ with $|M^n_i| < \infty$.

We note that we can replace M_{i-1}^n by $X_{T_{i-1}^n}$. This is true since $M_{i-1}^n = H^n_{T_{i-1}^n+}$ and on the interval $]\!]T_{i-1}^n, T_i^n]\!]$ the distance between H^n and X goes to zero (for $a.a.\ \omega$ and $n \to \infty$) and the distance between $X_{T_{i-1}^n}$ and $H^n_{T_{i-1}^n+}$ goes to zero (for $a.a.\ \omega$ and $n \to \infty$). For clear, $X_{T_i^n}$ are $\mathcal{F}_{T_i^n}$-measurable random variables ($i = 0, \dots, k_n - 1$) with $\left|X_{T_i^n}\right| < \infty$.

As a consequence, the new sequence of processes with $X_{T_{i-1}^n}$ instead of M_{i-1}^n is a sequence of simple predictable processes and still converges uniformly in time and $a.s.$ in ω to X_{t-} and still $H^n \bullet Z \to X \bullet Z,\ n \to \infty,\ up$ holds (with another rename for H^n).

Now we choose another subsequence (of that subsequence) so that the latter limit is $a.s.$ in ω, too. And we set all $H^n(\omega) \equiv 0$ if the distances (as the supremum over t) between $H^n(\omega)$ and $X(\omega)$ or between $(H^n \bullet Z)(\omega)$ and $(X \bullet Z)(\omega)$ is ≥ 1. And we set $H^n \equiv 0$ for all ω where H^n does not converge to X or where $H^n \bullet Z$ does not converge to $X \bullet Z$. These definitions do not change anything on the convergences, save that the convergences are dominated now.

Since $H^n \bullet Z$ converges uniformly in t to Y $a.s.$ and Y is integrable, we can use the dominated convergence theorem (e.g., with boundary $|Y_t| + 1$) to obtain $\lim_{n\to\infty} \mathbb{E}[(H^n \bullet Z)_t] = \mathbb{E}[Y_t]$. Now we have to calculate $\mathbb{E}[(H^n \bullet Z)_t]$. For each H_t^n, which is a simple predictable process, we define a sequence $(H^{n,m})_m$ of simple predictable processes via $H^{n,m} = \sum_{j=1,\dots,2^m} H^n_{\frac{(j-1)T}{2^m}} \mathbb{I}_{\left(\frac{(j-1)T}{2^m}, \frac{jT}{2^m}\right]}$ ($m \geq 0$).

With $\vartheta_i^{n,m-}(\omega)$ we denote the largest point of the grid $\{\frac{0}{2^m}, \frac{T}{2^m}, \dots, T\}$ with $\vartheta_i^{n,m-}(\omega) \leq T_i^n(\omega)$ and with $\vartheta_i^{n,m+}(\omega)$ we denote the smallest point of the grid $\{\frac{0}{2^m}, \frac{T}{2^m}, \dots, T\}$ with $\vartheta_i^{n,m+}(\omega) \geq T_i^n(\omega)$. Without loss of generality, we choose m big enough s.t. all jumps (that are not at the same point of time) of $H^n(\omega)$ are separated by the dyadic grid (ω-by-ω). It holds, since Z is càdlàg:

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left|(H^{n,m} \bullet Z)(\omega)_t - (H^n \bullet Z)(\omega)_t)\right| \\
&\leq \sup_{t\in[0,T]} \left(\sum_{i=1,\dots,n_k} \left| \left(H^n_{\vartheta_i^{n,m+}}(\omega) - H^n_{\vartheta_i^{n,m-}}(\omega) \right) \left(Z_{\vartheta_i^{n,m+}} - Z_{T_i^n} \right) \right| \right)_t \\
&\to 0,\ m \to \infty
\end{aligned}
$$

For all ω, $H^{n,m} \bullet Z$ converges to $H^n \bullet Z$ ($m \to \infty$), especially there exists an m^* so that the distance is smaller than 1 for all $m \geq m^*$ (for all t). That means, again, we can use the dominated convergence theorem (with boundary $|H^n \bullet Z_t| + 1$) to get

$$
\begin{aligned}
\mathbb{E}\left[(H^n \bullet Z)_t\right] =& \mathbb{E}\left[\left(\lim_{m\to\infty} H^{n,m}\right) \bullet Z_t\right] \\
=& \mathbb{E}\left[\lim_{m\to\infty} (H^{n,m} \bullet Z)_t\right] \\
=& \lim_{m\to\infty} \mathbb{E}\left[H^{n,m} \bullet Z_t\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \mathbb{E}\left[\left(\sum_{j=1,\ldots,2^m} H^n_{\frac{(j-1)T}{2^m}} \left(Z_{\frac{(j-1)T}{2^m}} - Z_{\frac{jT}{2^m}}\right)\right)_t\right] \\
&= \lim_{m\to\infty} \left(\sum_{j=1,\ldots,2^m} \mathbb{E}\left[H^n_{\frac{(j-1)T}{2^m}}\right] \mathbb{E}\left[Z_{\frac{(j-1)T}{2^m}} - Z_{\frac{jT}{2^m}}\right]\right)_t \\
&= \lim_{m\to\infty} \left(\sum_{j=1,\ldots,2^m} \mathbb{E}\left[H^n_{\frac{(j-1)T}{2^m}}\right] \zeta \frac{1}{2^m}\right)_t \\
&= \int_0^t \mathbb{E}\left[H^n_s\right] \zeta ds.
\end{aligned}
$$

Here, we used that X is independent of the increments of Z and thus H^n are also independent. Putting these results together and using a third and a fourth time the dominated convergence theorem (but these times for X with boundary $|X_t| + 1$ and for $\mathbb{E}[X]$ which is bounded on $[0, T]$) completes the proof:

$$
\begin{aligned}
\mathbb{E}[Y_t] &= \lim_{n\to\infty} \mathbb{E}[(H^n \bullet Z)_t] \\
&= \lim_{n\to\infty} \int_0^t \mathbb{E}[H^n_s]\zeta ds \\
&= \int_0^t \lim_{n\to\infty} \mathbb{E}[H^n_s]\zeta ds \\
&= \int_0^t \mathbb{E}\left[\lim_{n\to\infty} H^n_s\right] \zeta ds \\
&= \int_0^t \mathbb{E}[X_s]\zeta ds
\end{aligned}
$$

(since X is càglàd).

□

Now, we apply this theorem to SDEs.

Theorem 3.2. *Let Z be a d-dimensional vector of semimartingales with stochastically independent and stationary increments, which implies that there are $\zeta^j \in \mathbb{R}$ s.t. $\mathbb{E}[Z^j_t - Z^j_s] = \zeta^j(t-s)$, and $Z^i_0 = 0\ \forall i = 1, \ldots, d$. Let $F^i_j : \mathbb{D}^n \to \mathbb{D}$ be linear operators ($i = 1, \ldots, n,\ j = 1, \ldots, d$). Let $J \in \mathbb{D}^n$ be a vector of processes with $\mathbb{E}[J_t] = \delta_t \in \mathbb{R}^n$. We define*

$$
X^i_t = J^i_t + \sum_{j=1,\ldots,d} \int_0^t F^i_j(X)_{s-} dZ^j_s.
$$

Let all Z^j be independent of each other. If X_t is an integrable process, it holds *with* $\mathbb{E}[X_t^i] = \xi_t^i$ *that*

$$\xi_t^i = \delta_t^i + \sum_{j=1,\dots,d} \int_0^t F_j^i(\xi_s)\zeta^j dt,$$

which is an ordinary differential equation (ODE).

Proof. First, note that if $F_j^i : \mathbb{D}^n \to \mathbb{D}$ are linear operators ($i = 1, \dots, n,\ j = 1, \dots, d$), F_j^i are functional Lipschitz. Note that $\mathbb{E}[Z_t^j - Z_s^j] = \zeta^j(t-s)$ is justified since Z has stationary increments. We note that X_t^i is independent of the increments $Z_{t+h}^j - Z_t^j$ (due to the SDE and the independent increments of Z). Further, note that ξ^i is continuous since Z has stationary increments and due to the construction of the SDE (otherwise $sup_{t\in[0,T]}|\xi_t^i| = \infty$). So we can apply the expectation operator on both sides of the SDE and use Thm. 3.1.

□

Theorem 3.2 is very helpful in the case of SDEs (which fulfill the conditions of the theorem) that cannot be solved analytically (or only with very high effort). When we are interested only in the expectation of the solution, we do not need to solve the SDE, instead we can apply the theorem.

Before coming to the next section, we mention that there exist several stochastic Fubini theorems or Fubini-type theorems in the literature, e.g., the works of [2, 3, 4, 5, 6] and Thm. IV.64, Thm. IV.65 [7]. To the best of the author's knowledge these settings are different to our assumptions. For example, the authors of these papers use different probability spaces for the expectation and for the stochastic process or the measure is time dependent and the integration is performed in time over that measure.

4 Example: Linear Long Feedback Trading on Merton's Jump Diffusion Model

In this section we show a useful application of the results shown above, i.e. Thm. 3.1 and Thm. 3.2. The example provided in this section is from the field of mathematical finance, also known as stochastic finance. We investigate the performance of an asset trader (who is identified with his or her trading strategy). The trading strategy tells the trader how much money should be invested in a specific asset at time t. (This is related to portfolio selection, however, we investigate a one asset market with a risk-less bond.) The market, i.e. the price, is exogenously given and not influenced by the trader. One question that arises is how much gain (or loss) a trader can expect. In our example, we analyze a so-called feedback-based trading rule, i.e., we assume that the trader calculates the amount to be invested solely via his or her own gain. The price model under investigation in this section is Merton's jump diffusion model. This model has a known analytical solution and even the expected gain of the trading strategy under analysis is known. We chose this model to show that our results are in line with the literature. As we will show, we do neither need a solution for the price model nor for the trader's investment to calculate the expected gain (which is different to the literature). That means, our method can also be used when the price model does not have a known solution or even when the price model is not solvable—as long as the conditions of Thm. 3.1 are fulfilled.

Let us have a look at a linear long feedback trading rule L, where a trader in every point in time t invests, i.e., holds the net asset position,

$$I_t^L = I_0^* + K g_t^L$$

of a specific asset. That means, the trader invests some initial investment $I_0^* > 0$ plus $0 < K$ times his or her own gain g_t^L at time t. When we denote the price process of the asset with p_t and assume this process to be a semimartingale, i.e. esp. cádlág, we can calculate the traders gain (or loss) at time t via

$$g_t^L := \int_0^t \left(\frac{I^L}{p}\right)_{t-} dp_t.$$

For further information about linear feedback trading see [1, 8, 10].

When we assume the price to be governed by Merton's jump diffusion model (see [8, 11]), which is given via the SDE

$$dp_t = (\mu - \lambda\kappa)p_t dt + \sigma p_t dW_t + p_t dN_t,$$

there are several ways to calculate the expected feedback trading gain. Note that we use the purely formal d-notation to shorten the integral notation of the SDE. That does not mean that we deal with "real differential equations;" as long as there are stochastic parts in the differential equations we always have to translate them into integral equations.

Before coming to the target of this section, the calculation of the expected feedback trading gain, we further explain Merton's price model. The "jump-less" trend is given through the parameter $\mu > -1$ though the trend part of the SDE is $\mu - \lambda\kappa$ and the volatility of the diffusion part via $\sigma > 0$, where W_t is a standard Brownian motion (also known as Wiener process). So far, the model is similar to a geometric Brownian motion. However, additionally, there is the jump-part modeled via the Poisson-driven process N_t with jump intensity $\lambda > 0$ (i.e., the time interval between two consecutive jumps is $Exp(\lambda)$-distributed and the number of jumps $N(t)$ up to time t is $\mathcal{P}oi(\lambda t)$-distributed), i.i.d. jumps $(Y_i - 1)_{i\in\mathbb{N}} \geq -1$ and expected jump height $\kappa := \mathbb{E}[Y_1 - 1] > -1$. When the start price is $p_0 > 0$ the SDE can be solved:

$$p_t = p_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)\prod_{k=1}^{N(t)} Y_i,$$

see [11]. In the limits $\lambda \to 0$ or $Y_1 \to \delta_1$ the model converges to a geometric Brownian motion with trend $\mu > -1$ and volatility $\sigma > 0$. Note that the conditions "> -1" are not always needed for mathematical reasons but rather for economical reasons to avoid negative prices. More detailed information about this market model can be found in [8].

When calculating the gain of the linear long feedback trading rule in Merton's model, we can work directly with the SDEs. The integral equation describing the trading strategy can be written as the SDE:

$$dI_t^L = K dg_t^L,$$

and the integral equation describing the gain as

$$dg_t^L = I_t^L \frac{dp_t}{p_t}.$$

Putting in the price process $\frac{dp_t}{p_t} = (\mu - \lambda\kappa)dt + \sigma dW_t + dN_t$ leads to an SDE for the investment I^L:

$$dI_t^L = KI_t^L(\mu - \lambda\kappa)dt + KI_t^L \sigma dW_t + KI_t^L dN_t,$$

which again is a process described by Merton's jump diffusion model. However, the trend is $K(\mu - \lambda\kappa)$, the volatility is $K\sigma$, and the jumps are specified through the intensity λ and the *i.i.d.* jumps $X_i - 1 = K(Y_i - 1)$ with expected jump height $K\kappa$.

To calculate the expected gain $\mathbb{E}[g_t^L]$ we have at least three possibilities: For the first and the second possibility we can solve the SDE for the investment an get the stochastic process I_t^L (see [11]). Using the converted formula for the trading rule $g_t^L = \frac{I_t^L - I_0^*}{K}$ leads to the gain/loss process. Next, the expected gain $\mathbb{E}[g_t^L]$ can be calculated: First, directly via the theorem of Fubini-Tonelli as done in [8] or, second, by use of an extension of Wald's lemma as done in [1] Sections 4.2 and 9.1.1 (cf. [12]).

The third possibility is to use our Theorem 3.2 on the SDE governing the investment I_t^L. The advantage of latter way is that no SDE has to be solved, neither for the price process p_t nor for the investment I_t^L.

We have to note that $\mathbb{E}[t - s] = 1 \cdot (t - s)$, $\mathbb{E}[W_t - W_s] = 0 \cdot (t - s)$, and $\mathbb{E}[N_t - N_s] = \lambda\kappa \cdot (t - s)\ \forall T \geq t > s \geq 0$ as well as that the initial investment I_0^* is not stochastic. Shortened and purely formal one could write $\mathbb{E}[dt] = dt$, $\mathbb{E}[dW_t] = 0$, and $\mathbb{E}[dN_t] = \lambda\kappa dt$. It follows:

$$d\mathbb{E}[I_t^L] = K(\mu - \lambda\kappa) \cdot 1 \cdot \mathbb{E}[I_t^L] + K\sigma \cdot 0 \cdot \mathbb{E}[I_t^L] + 1 \cdot K\lambda\kappa \cdot \mathbb{E}[I_t^L] = K\mu\mathbb{E}[I_t^L],$$

which is an ODE with solution $\mathbb{E}[I_t^L] = I_0^* e^{K\mu t}$. Via $\mathbb{E}[g_t^L] = \frac{\mathbb{E}[I_t^L] - I_0^*}{K}$ it follows

$$\mathbb{E}[g_t^L] = \frac{I_0^*}{K}\left(e^{K\mu t} - 1\right),$$

which is in line with the first and the second possibility [1, 8].

Since this section is to show an application of our stochastic Fubini-type theorem, we do not discuss market requirements in detail. For that and for economics interpretations of this result the reader may consult [8, 10]. Note that third method, which uses Thm. 3.2, is also possible for market models where no analytical solution is known or even for models that are not solvable.

5 Conclusion

In this work, we showed that under specific assumptions—i.a., that the integrator is linear in expectation—it is allowed to swap an expectation operator and an Itō integral, i.e., the expectation as a function of time of a stochastic process that is given via a stochastic integral can be calculated via a Riemann integral of the expectations of the integrand and the integrator. The result that it is allowed to interchange the expectation operator, which is an integral, and an Itō integral is a stochastic Fubini-type theorem. It is extended to stochastic differential equations (SDEs) and an application from the field of mathematical finance is provided.

Acknowledgment

The author thanks Michaela Baumann. Furthermore, the author wishes to thank Lars Grüne, Melanie Birke, and Bernhard Herz.

References

[1] M. H. Baumann, *Performance and effects of linear feedback stock trading strategies*, doctoral thesis, Universität Bayreuth, 2018.

[2] T. Hille, *On a weak stochastic Fubini theorem*, ETH Zürich, 2014.

[3] J. Van Neerven, M. C. Veraar, *On the stochastic Fubini theorem in infinite dimensions*, in *Stochastic Partial Differential Equations and Applications-VII*, CRC Press, 2005, chapter 26, pp. 323–336.

[4] M. C. Veraar, *The stochastic Fubini theorem revisited*, Stochastics An International Journal of Probability and Stochastic Processes, **84**(4), Taylor & Francis, 2012, pp. 543–551.

[5] T. Kailath, A. Segall, M. Zakai, *Fubini-type theorems for stochastic integrals*, Sankhyā: The Indian Journal of Statistics, Series A, **40**(2), JSTOR, 1978, pp. 138–143.

[6] M. A. Berger, J. V. Mizel, *Theorems of Fubini type for iterated stochastic integrals*, Trans. Amer. Math. Soc., **252**, 1979, pp. 249–274.

[7] P. E. Protter, *Stochastic integration and differential equations*, Springer, 2005.

[8] M. H. Baumann, *On stock trading via feedback control when underlying stock returns are discontinuous*, IEEE Trans. Automat. Control, **62**(6), IEEE, 2017, pp. 2987–2992.

[9] C. Kühn, *Stochastische Analysis mit Finanzmathematik WS 2004/05*, Lecture notes on "Stochastische Analysis mit Finanzmathematik," Goethe Universität Frankfurt, December 2016.

[10] B. R. Barmish, J. A. Primbs, *On arbitrage possibilities via linear feedback in an idealized Brownian motion stock market*, IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), 2011, pp. 2889–2894.

[11] R. C. Merton, *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economics, **3**(1-2), Elsevier, 1976, pp. 125–144.

[12] A. Wald, *On cumulative sums of random variables*, The Annals of Mathematical Statistics, **15**(3), The Institute of Mathematical Statistics, 09.1944, pp. 283–296.