Numerical Calculation of Nonsmooth Control Lyapunov Functions via Piecewise Affine Approximation ⋆

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Abstract: We review the numerical computation of control Lyapunov functions for switched systems in the Filippov sense and in the sample-and-hold sense. The computed (control) Lyapunov functions are piecewise affine and the decrease conditions involve either generalized gradients or the Dini subderivate. Further conditions for stabilization or stabilizability yield a (mixed) linear integer optimization problem.

Keywords: (control) Lyapunov function, asymptotic stability, controllability, switched systems, piecewise linear analysis, linear programming, mixed-integer problems

1. INTRODUCTION

In recent years, algorithms for the computation of piecewise affine (PWA) Lyapunov functions have been developed for various different settings. Starting from ordinary differential equations in Hafstein (2007), via differential inclusions in Baier et al. (2012) to control Lyapunov functions in Baier and Hafstein (2014) and Baier et al. (2018), to mention only those results that are most important for this paper. The construction in all these references is similar: the conditions ensuring that a PWA function defined on a partition of the state space is a Lyapunov function are formulated as inequalities for a (typically very large) set of variables describing the function. These inequalties are then used as constraints in an optimization problem, whose solution yields the desired Lyapunov function, provided the problem is feasible.

The contribution of this paper is to revisit the algorithms developed in these references for switched systems with different settings and to show how the structure of the resulting inequalities depends on the problem setting. From the references mentioned above, it is already known that there is a fundamental difference between a Lyapunov function that ensures asymptotic stability for all possible solutions — a so called strong or robust Lyapunov function — and a Lyapunov function that certifies asymptotic stability only for at least one solution for each initial value — a so called weak or control Lyapunov function (CLF). While the optimization problem to be solved is a linear problem in the case of robust Lyapunov functions, it becomes nonlinear in the case of CLFs. While the resulting problem can still be solved using a big-M linearization resulting in a mixed integer linear problem, the computational effort is significantly higher. In fact, such a difficulty cannot be avoided as the problem of finding a CLF (which is equivalent to jointly finding a control law and a corresponding Lyapunov function) is known to be a non-convex problem, see Prajna et al. (2004). Another difficulty related to finding CLFs is that for certain problems only nonsmooth Lyapunov functions may exist, as explained in Section 2 below. This significantly affects the formulation of the inequalities needed for ensuring the Lyapunov function property.

In this paper we formulate the piecewise affine Lyapunov function conditions for four different cases. We do this in the context of switched systems where the switching signal is the control input, as the proposed algorithm is naturally able to handle this discrete structure. We consider the case in which a Lyapunov function is sought for a pre-designed state dependent switching law and the case of a CLF, i.e., where a switching law and the corresponding Lyapunov function are sought for at the same time. For both settings, we consider algorithms looking for PWA Lyapunov functions, with and without the assumption that a smooth Lypunov function exists. The main result of the analysis of these four cases is that only in the case where the switching law is pre-designed and a PWA Lyapunov function is sought are the resulting inequalities linear. In all other cases, the problem must be converted into a mixed integer linear problem in order to find a solution.

Notation: We denote by $\mathcal{B}_r(m)$ the *closed Euclidean ball* with center $m \in \mathbb{R}^n$ and radius $r > 0$, therefore $\mathcal{B}_1(0)$ is the *closed unit ball* in $\mathbb{R}^n$. $\mathbb{N}_0$ denotes the set of non-negative integer numbers.

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The *interior* and *closure* of a set $A \subset \mathbb{R}^n$ are denoted by $\operatorname{int} A$, $\operatorname{cl} A$ and the *(closed) convex hull* as $\overline{\operatorname{co}}A$ and $\operatorname{co} A$. A *set-valued map* F with images $F(x) \subset \mathbb{R}^n$ for $x \in X \subset \mathbb{R}^m$ is denoted as $F : X \rightrightarrows \mathbb{R}^n$, the *space of compact, nonempty sets* is denoted by $\mathcal{K}(\mathbb{R}^n)$.

2. PROBLEM

2.1 Setting

For a compact subset $G \subseteq \mathbb{R}^n$, Lipschitz-continuous *vector fields* $f_\mu : G \to \mathbb{R}^n$, $\mu = 1, \dots, M$, and a state dependent *switching strategy* $\sigma : G \to \{1, \dots, M\}$, we consider the differential equation

$$\dot{x}(t) = f_{\sigma(x(t))}(x(t)). \tag{1}$$

The equation contains a state dependent selection $\sigma(x)$ from M different right-hand sides f_μ, $\mu = 1, \dots, M$. Since such a selection will in general lead to a discontinuous map $x \mapsto f_{\sigma(x)}(x)$, we must be careful when defining an appropriate solution concept for (1).

One possible solution concept relies on using the Filippov regularization, cf. (Filippov, 1988, § 2.7) and (Aubin and Cellina, 1984, Sec. 2.1), in order to turn the discontinuous differential equation (1) into a differential inclusion.

Definition 1. The *Filippov regularization* for the system (1) is given by

$$F(x) := \bigcap_{\delta > 0} \overline{\operatorname{co}}\{g(\mathcal{B}_\delta(x) \cap G)\}, \tag{2}$$

where $g(x) = f_{\sigma(x)}(x)$.

Now we can define solutions of (1) via solutions of the differential inclusion $\dot{x}(t) \in F(x(t))$ with F from (2).

Definition 2. Consider a set-valued map $F : G \rightrightarrows \mathbb{R}^n$ with images in $\mathcal{K}(\mathbb{R}^n)$. An absolutely continuous function $x : I \to G$ is a *solution of the differential inclusion*, if

$$\dot{x}(t) \in F(x(t)) \quad \text{for a.a. } t \in I. \tag{3}$$

For the existence of a solution of Problem (3), upper semi-continuity of the set-valued map F and convexity of the images are important tools, see (Aubin and Cellina, 1984, Sec. 2.1, Theorem 3), (Aubin and Frankowska, 1990, Theorem 10.1.3). As stated in (Aubin and Cellina, 1984, p. 101), these properties hold for (2), since the f_μ are assumed to be Lipschitz and thus locally bounded.

Remark 3. If for a given switching law $\sigma : G \to \{1, \dots, M\}$ and all $\mu \in \{1, \dots, M\}$ we define

$$H_\mu := \{x \in G \,|\, \sigma(x) = \mu\} \text{ and } G_\mu := \operatorname{cl} H_\mu, \tag{4}$$

then (1) can be restated as

$$\dot{x}(t) = f_\mu(x(t)) \quad \text{for } x(t) \in H_\mu \tag{5}$$

and the Filippov regularization of (1) is given by

$$F(x) = \operatorname{co}\{f_\mu(x) \mid \mu \in I_{\mathcal{G}}(x)\}. \tag{6}$$

Here,

$$I_{\mathcal{G}}(x) := \{\mu \in \{1, \dots, M\} \mid x \in G_\mu\} \tag{7}$$

is the *set of active indices of the subregions* G_μ.

As we will discuss at the end of Section 2.2, Filippov solutions can be too large a class of solutions for stability considerations. As an alternative, we define the following concept of sample-and-hold solutions.

Definition 4. Consider a *partition* $\pi = \{t_i\}_{i \in \mathbb{N}_0}$ with $t_0 = 0$, $t_{i+1} > t_i$ for $i \in \mathbb{N}_0$ and $t_i \xrightarrow[i \to \infty]{} \infty$. The *diameter* of π is defined as $\operatorname{diam}(\pi) := \sup_{i \in \mathbb{N}_0}(t_{i+1} - t_i)$. A *sample-and-hold solution* $x_\pi : [0, \infty) \to \mathbb{R}^n$ of (1) for the partition π and initial value x_0, is an absolutely continuous function satisfying $x_\pi(0) = x_0$ and, for all $i \in \mathbb{N}_0$

$$\dot{x}_\pi(t) = f_{\sigma(x_\pi(t_i))}(x_\pi(t)) \text{ for a.a. } t \in (t_i, t_{i+1}]. \tag{8}$$

A *limiting sample-and-hold solution* is a function x which is obtained as the limit of sample-and-hold solutions $\{x_{\pi_j} \mid j \in \mathbb{N}\}$ with $\operatorname{diam}(\pi_j) \xrightarrow[j \to \infty]{} 0$ converging uniformly on bounded intervals.

2.2 Stability notions

We consider system (1) and assume that $f_{\sigma(0)}(0) = 0$. Our goal now is to study the stabiliy of this equilibrium point. To this end, we introduce two stability notions related to the two solution concepts. The first notion applies to the Filippov regularization of (1).

Definition 5. System (1) is *asymptotically stable* at the *equilibrium* 0 in the *Filippov sense*, if the following conditions hold:

(i) For each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_0 \in \mathcal{B}_\delta(0) \cap G$ and all solutions $x(t)$ of (2), (3) with $x(0) = x_0$ the inequality $\|x(t)\| \le \varepsilon$ holds for all $t \ge 0$.

(ii) For all solutions $x(t)$ of (2), (3) the convergence $x(t) \to 0$ holds as $t \to \infty$.

The second stability notion is adapted to the sample-and-hold solution concept.

Definition 6. System (1) is *asymptotically stable* at the *equilibrium* 0 in the *sample-and-hold sense*, if the following holds:

(i) For each $\varepsilon > 0$ there exists $\delta > 0$ and $\rho > 0$ such that for all $x_0 \in \mathcal{B}_\delta(0) \cap G$ and all partitions π with $\operatorname{diam}(\pi) < \rho$ the sample-and-hold solution x_π with $x_\pi(0) = x_0$ satisfies $\|x_\pi(t)\| \le \varepsilon$ for all $t \ge 0$.

(ii) For each $\Delta > 0$ and $\varepsilon > 0$ there exists $\rho > 0$ and $T > 0$ such that for all $x_0 \in \mathcal{B}_\Delta(0) \cap G$ and all partitions π with $\operatorname{diam}(\pi) < \rho$ the solution x_π with $x_\pi(0) = x_0$ satisfies $\|x_\pi(t)\| \le \varepsilon$ for all $t \ge T$.

We note that both definitions demand that the solutions under consideration exist and stay in G for all times $t \ge 0$. This is an idealized assumption which simplifies the presentation. It is conceptually easy but notationally cumbersome to weaken this assumption by restricting the stability definitions to an appropriate subset of initial conditions in G. However, since it does not affect the key features of the subsequent considerations, we use this simplification.

Definition 7. A switched system with vector fields f_μ, $\mu \in \{1, \dots, M\}$ is called *stabilizable in the Filippov sense* or in the *sample-and-hold sense*, respectively, if there exists a switching strategy $\sigma : G \to \{1, \dots, M\}$ such that (1) is asymptotically stable at 0 in the Filippov sense or in the sample-and-hold sense, respectively.

It was shown in Clarke et al. (1997) that asymptotic controllability (i.e., asymptotic stability using time-dependent instead of state-dependent switching signals) implies stabilizability in the sample-and-hold sense but not necessarily in the Filippov sense. The difference between the two stability notions is also reflected in their Lyapunov function characterization, which we discuss next.

2.3 Lyapunov function characterization

Let us first fix a given switching law $\sigma : G \to \{1, \dots, M\}$. We start by defining a smooth Lyapunov function for the switched system (1), see (Clarke et al., 1998, Definition 1.1).

Definition 8. A function $V : G \to \mathbb{R}$ is a *smooth Lyapunov function* for (1), if

(i) V is *positive definite*, i.e., $V(0) = 0$ and $V(x) > 0$ for all $x \in G$, $x \neq 0$
(ii) V is continuously differentiable
(iii) the *decrease condition* holds, i.e.,

$$\langle \nabla V(x), f_{\sigma(x)}(x) \rangle \le -\gamma(\|x\|) \quad \forall x \in G \tag{9}$$

for a positive definite function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$.

By looking at (9) for all Filippov solutions of (2), (3) associated to the given switching law σ and by exploiting continuity of ∇V, it is straightforward to check that a smooth Lyapunov function for (1) satisfies

(iii') $$\max_{w \in F(x)} \langle \nabla V(x), w \rangle \le -\gamma(\|x\|) \quad \forall x \in G$$

for the Filippov regularization (2) of (1). Conversely, since $f_{\sigma(x)}(x) \in F(x)$, (iii') implies (iii). A function satisfying (i), (ii) and (iii') is called a smooth *strong Lyapunov function* for F from (2).

In summary, in the smooth case the two forms of the decrease condition (iii) and (iii') are equivalent. The following proposition then follows from (Clarke et al., 1998, Theorem 1.2).

Proposition 9. A smooth Lyapunov function in the form of Definition 8 exists if and only if system (1) is asymptotically stable at 0 in the Filippov sense.

Since in general this is a stronger property than asymptotic stability in the sample-and-hold sense, we need to weaken Definition 8 in order to obtain an equivalent Lyapunov function condition for asymptotic stability in the sample-and-hold sense. To this end, we introduce nonsmooth Lyapunov functions, for which we need the following definition.

Definition 10. The *Dini subderivate* of a Lipschitz function $V : X \to \mathbb{R}$ in direction $w \in \mathbb{R}^n$ at $x \in X$ is defined by

$$DV(x; w) := \liminf_{t \downarrow 0} \frac{V(x + tw) - V(x)}{t}.$$

Now, we can formulate the nonsmooth decrease condition.

Definition 11. A function $V : G \to \mathbb{R}$ is a *nonsmooth Lyapunov function in the Dini sense* for (1), if

(i) V is *positive definite*
(ii) V is Lipschitz continuous
(iii) the *decrease condition in the Dini sense* holds, i.e.,

$$DV(x; f_{\sigma(x)}(x)) \le -\gamma(\|x\|) \quad \forall x \in G \tag{10}$$

for a positive definite function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$.

It follows from Proposition 13, below, that the existence of a nonsmooth Lyapunov function in the Dini sense ensures stabilizability of (1) in the sample-and-hold sense.

Having described Lyapunov functions guaranteeing asymptotic stability for a given switching law σ, we now turn to the Lyapunov function characterization of stabilizability. This means that $\mu = \sigma(x)$ is not given a priori but we have the freedom to choose among all $\mu \in \{1, \dots, M\}$ when verifying the decrease condition for the Lyapunov function.

Definition 12. Consider a function $V : G \to \mathbb{R}$ and a positive definite function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$.

(a) If V satisfies conditions (i) and (ii) of Definition 8 and in addition the *decrease condition*

$$\min_{\mu \in \{1,\dots,M\}} \langle \nabla V(x), f_\mu(x) \rangle \le -\gamma(\|x\|) \quad \forall x \in G$$

holds, then V is called a *smooth control Lyapunov function* (CLF) for f_μ, $\mu = 1, \dots, M$.

(b) If V satisfies conditions (i) and (ii) of Definition 11 and in addition the *decrease condition*

$$\min_{\mu \in \{1,\dots,M\}} DV(x; f_\mu(x)) \le -\gamma(\|x\|) \quad \forall x \in G$$

holds, then V is called a *nonsmooth control Lyapunov function (CLF) in the Dini sense* for f_μ, $\mu = 1, \dots, M$.

For the Dini CLF, the following statement follows from (Sontag and Sussmann, 1995, Fact 2.8), (Sontag and Sussmann, 1996, Fact 6.1) and (Clarke et al., 1997, Theorem 1):

Proposition 13. System (1) is stabilizable in the sample-and-hold sense if and only if a nonsmooth CLF in the Dini sense exists.

We note that if we define $\sigma(x)$ to be one of the minimizing parameters μ in the respective decrease conditions, then for this switching law σ the CLF V will satisfy Definition 8 or Definition 11, respectively. For this reason, Proposition 9 yields that the existence of a smooth CLF implies stabilizability in the Filippov sense.

Based on these characterizations, we now consider the following four problems. For the solvability of the first and third problem, the existence of a smooth Lyapunov function is necessary, for the second and fourth problem, the existence of a smooth CLF is not necessary.

Problem 1: Given a switching law $\sigma : G \to \{1, \dots, M\}$, find a PWA Lyapunov function guaranteeing stability of (1) in the Filippov sense.

Problem 2: Given a switching law $\sigma : G \to \{1, \dots, M\}$, find a PWA CLF guaranteeing stabilizability of (1) in the sample-and-hold sense.

Problem 3: Given vector fields f_μ, $\mu \in \{1, \dots, M\}$, find a PWA CLF guaranteeing stabilizability of (1) in the Filippov sense and the stabilizing switching law $\sigma : G \to \{1, \dots, M\}$.

Problem 4: Given vector fields f_μ, $\mu \in \{1,\dots,M\}$, find a PWA CLF guaranteeing stabilizability of (1) in the sample-and-hold sense and the stabilizing switching law $\sigma : G \to \{1,\dots,M\}$.

3. PIECEWISE AFFINE APPROXIMATION

The computational domain G is subdivided into smaller regions via the *triangulation* $\mathcal{T} = \{T_\nu \mid \nu = 1,\dots,N\}$, where T_ν is a simplex and thus a convex hull of $n+1$ vertices p_i^ν, $i = 0,\dots,n$, such that $\bigcup_{\nu=1,\dots,N} T_\nu = G$. We collect all different vertices in the *set of vertices* $\mathcal{V}$ and renumber them as p_j, $j = 1,\dots,K$. The *barycenter* for a simplex T_ν is defined as $b(T_\nu) := \frac{1}{n+1}\sum_{i=0}^n p_i^\nu$. As in Baier et al. (2012, 2018) we assume that the intersection of two different simplices is either empty or a common k-face of both simplices.

For the class of considered switching laws we demand that the restriction $\sigma|_{\operatorname{int} T_\nu}$ is constant and equal to $\sigma(b(T_\nu))$ on the interior of each simplex $T_\nu \in \mathcal{T}$. At the boundary of the simplices, i.e., on the lower-dimensional faces, σ assumes either $\sigma|_{\operatorname{int} T_\nu}$ or one of the constant values $\sigma|_{\operatorname{int} T_{\bar\nu}}$ of adjacent simplices $T_{\bar\nu}$. Thus, the switching strategy can only be discontinuous at the boundary of the simplices. The computational challenge then lies in checking the decrease condition at the boundary of the simplices, where the switching law may be discontinuous.

To this end, two strategies are possible: the first checks if the value of $\sigma|_{\operatorname{int} T_\nu}$ also yields a decrease in all neighboring simplices $T_{\bar\nu}$ of T_ν. As shown in Baier and Hafstein (2014), this *neighboring decrease condition* yields a PWA CLF in the sense of generalized gradients. This guarantees asymptotic stabilizability in the Filippov sense, because of its equivalence to a smooth CLF following from Rifford (2001) or by direct arguments as in Baier and Hafstein (2014). However, the equivalence to a smooth CLF implies that the neighboring decrease condition is too strong a property when looking for a CLF in the Dini sense. For the second strategy, an observation from Baier et al. (2018) turns out to be useful: when the function V restricted to two adjacent simplices $T_\nu \cup T_{\bar\nu}$ is concave, then the decrease condition at the face $T_\nu \cap T_{\bar\nu}$ is satisfied for each of the switching values $\sigma|_{\operatorname{int} T_\nu}$ and $\sigma|_{\operatorname{int} T_{\bar\nu}}$ even if the neighboring decrease condition is violated. Only if V is not concave on $T_\nu \cup T_{\bar\nu}$ must this condition be enforced.

These considerations will be used in the algorithms in the next section. In the remainder of this section, we show how to verify the decrease conditions for PWA functions.

For a point $x \in G$, the *set of active indices of the triangulation* is defined as

$$I_{\mathcal{T}}(x) := \{\nu \in \{1,\dots,N\} \mid x \in T_\nu\}.$$

The following lemma assures the existence and uniqueness of the PWA approximant.

Lemma 14. The *PWA interpolation* $V_{\mathcal{T}}$ of a function $V : G \to \mathbb{R}$ on the triangulation $\mathcal{T}$ defined by

$$V_{\mathcal{T}}(p_j) = V(p_j) \quad (j = 1,\dots,K)$$

for all vertices $p_j \in \mathcal{V}$ is well-defined. The gradient ∇V_ν of $V_{\mathcal{T}}$ on the interior $\operatorname{int} T_\nu$ of some simplex T_ν is given by the solution of the linear equation

$$A_{\mathcal{T}} \nabla V_\nu = b_{\mathcal{T}}, \tag{11}$$

where $V_i^\nu = V(p_i^\nu)$, $i = 0,\dots,n$, and

$$A_{\mathcal{T}} = \Big(p_1^\nu - p_0^\nu, p_1^\nu - p_0^\nu, \dots, p_n^\nu - p_0^\nu\Big),$$

$$b_{\mathcal{T}} = \Big(V_1^\nu - V_0^\nu, V_2^\nu - V_0^\nu, \dots, V_n^\nu - V_0^\nu\Big)^\top.$$

Proof. We can fix the values of $V_{\mathcal{T}}$ on each vertex p_i^ν, $i = 1,\dots,n+1$, of the simplex T_ν. The well-definedness is due to the linear approximation and the assumption on the intersection of two neighboring simplices being a common k-face. The linear system is uniquely solvable, since the $n+1$ vertices p_k^ν are affinely independent and thus the matrix is invertible. □

Proposition 15. Consider the PWA interpolation $V_{\mathcal{T}}$ in Lemma 14 on a given triangulation $\mathcal{T}$. Then,

(i) $DV_{\mathcal{T}}(x; f_\mu(x)) = \langle \nabla V_\nu, f_\mu(x)\rangle$, where $x \in \operatorname{int} T_\nu$, $\nu \in I_{\mathcal{T}}(x)$, $\mu \in I_{\mathcal{G}}(x)$,
(ii) $DV_{\mathcal{T}}(p_k^\nu; f_\mu(p_k^\nu)) = \langle \nabla V_\rho, f_\mu(p_k^\nu)\rangle$, where $\nu, \rho = \rho(k,\nu,\mu) \in \{1,\dots,N\}$, $\mu \in I_{\mathcal{G}}(p_k^\nu)$ and there exists $\bar t > 0$ such that $p_k^\nu + t f_\mu(p_k^\nu) \in T_\rho$ for all $t \in [0, \bar t\,]$.

Proof. (i) follows analogously to (ii) with $\rho = \nu$.
(ii) $V_{\mathcal{T}}$ is Lipschitz so that for $x = p_k^\nu$

$$\begin{aligned} DV_{\mathcal{T}}(x; f_\mu(x)) &= \liminf_{t\downarrow 0} \frac{V_{\mathcal{T}}(x + t f_\mu(x)) - V_{\mathcal{T}}(x)}{t} \\ &= \liminf_{t\downarrow 0} \frac{\langle \nabla V_\rho, t f_\mu(x)\rangle}{t} = \langle \nabla V_\rho, f_\mu(x)\rangle. \end{aligned}$$

If the assumptions are fulfilled for another $\bar\rho$, then there exists $\widetilde{t} > 0$ such that $x, x + t f_\mu(x) \in T_\rho \cap T_{\bar\rho}$ lie in a common k-face of both triangles for $t \in [0, \widetilde{t}\,]$ and thus V has the same values on both points due to the well-definedness in Lemma 14. Hence, $\langle \nabla V_\rho, f_\mu(x)\rangle = \langle \nabla V_{\bar\rho}, f_\mu(x)\rangle$. □

The following explains how to discretize the decrease condition guaranteeing stability.

Proposition 16. Let $V : G \to \mathbb{R}$ be continuous, PWA on the triangulation $\mathcal{T} = \{T_\nu \mid \nu = 1,\dots,N\}$. Fix $q_\nu \in T_\nu$. Let us consider system (5) with a Lipschitz continuous function $f_\mu : G_\mu \to \mathbb{R}^n$ with modulus $L_\mu \geq 0$ for some $\mu \in I_{\mathcal{G}}(q_\nu)$ and $T_\nu \subset G_\mu$. If there exists $d > 0$ with

$$\langle \nabla V_\nu, f_\mu(q_\nu)\rangle + L_\mu \operatorname{diam}(T_\nu)\|\nabla V_\nu\| \;\leq -d, \tag{12}$$

then there exists $\delta > 0$ such that for all $x \in \mathcal{B}_\delta(q_\nu) \cap T_\nu$

$$\langle \nabla V_\nu, f_\mu(x)\rangle \leq -d.$$

Proof. Let $x \in T_\nu$, then $\mu \in I_{\mathcal{G}}(x)$, since $q_\nu \in T_\nu \subset G_\mu$ by the compatibility assumption for $\mathcal{T}$. Then,

$$\begin{aligned} &\langle \nabla V_\nu, f_\mu(x)\rangle = \langle \nabla V_\nu, f_\mu(q_\nu)\rangle + \langle \nabla V_\nu, f_\mu(x) - f_\mu(q_\nu)\rangle \\ &\leq \langle \nabla V_\nu, f_\mu(q_\nu)\rangle + \|\nabla V_\nu\| \cdot \|\, f_\mu(x) - f_\mu(q_\nu)\| \\ &\leq \langle \nabla V_\nu, f_\mu(q_\nu)\rangle + \|\nabla V_\nu\| \cdot L_\mu \|\, x - q_\nu\| \\ &\leq \langle \nabla V_\nu, f_\mu(q_\nu)\rangle + \|\nabla V_\nu\| \cdot L_\mu \operatorname{diam}(T_\mu) \leq -d. \end{aligned}$$ □

As discussed above, for checking the decrease condition at the boundaries of the simplices, concavity of V restricted to $T_\nu \cup T_{\bar\nu}$ is an important property. The next lemma shows how to check this property.

Lemma 17. (Baier et al., 2018, Lemma 1) Assume the conditions of Proposition 16 for the PWA function V. Consider two simplices $T_\nu, T_{\bar\nu} \in \mathcal{T}$ and the restriction

$V = V\,|_{T_\nu \cap T_{\bar\nu}}$.
Then V is concave if and only if

$$\langle \nabla V_\nu - \nabla V_{\bar\nu}, p_0^\nu\rangle + (b_\nu - b_{\bar\nu}) \le 0, \quad (13)$$
$$\langle \nabla V_{\bar\nu} - \nabla V_\nu, p_0^{\bar\nu}\rangle + (b_{\bar\nu} - b_\nu) \le 0 \quad (14)$$

holds, where $p_0^\nu \notin T_{\bar\nu}$, $p_0^{\bar\nu} \notin T_\nu$.

In (Rifford, 2001, Theorem 2) it was shown that asymptotic controllability for control systems with Lipschitz $F(x) = f(x, U)$ is equivalent to the existence of a semiconcave CLF. Note that a concave function is also semiconcave, see (Cannarsa and Sinestrari, 2004, Proposition 1.1.3).

4. NUMERICAL COMPUTATION

Since the decrease condition (12) may not be feasible around the equilibrium, we exclude in computations a small neighborhood $\mathcal{B}_\varepsilon(0)$ of the equilibrum (see Baier et al. (2012)). The following algorithm requires decrease in all neighboring simplices of a vertex, which according to the discussion at the beginning of Section 3 ensures stability in the Filippov sense and the existence of a smooth CLF.

Algorithm 1. (smooth CLF exists; given switching law) Consider Problem 1 with the system (5) stated on subregions from (4) and a given triangulation $\mathcal{T} = \{T_\nu \mid \nu = 1, \dots, N\}$ of $G \setminus \mathcal{B}_\varepsilon(0)$.

(i) For each vertex $p_j \in \mathcal{V}$, $j = 1, \dots, K$, we define a variable $V_j = V(p_j)$ and require $V_j \ge \|p_j\|$.
(ii) For each simplex T_ν, $\nu = 1, \dots, N$, we compute ∇V_ν by the linear equation (11) and set $V(x) = \langle \nabla V_\nu, x - p_0^\nu\rangle + V(p_0^\nu)$ on T_ν as PWA function. We demand $-C_\nu \le \nabla V_{\nu,r} \le C_\nu$ with variables C_ν for the coordinates $r = 1, \dots, n$ of the gradient.
(iii) For each vertex $p_j \in \mathcal{V}$, $j = 1, \dots, K$, with $p_j \in T_\nu$, we demand
$$\max_{\nu \in I_{\mathcal{T}}(p_j)} \max_{\mu \in I_{\mathcal{G}}(p_j)} \langle \nabla V_\nu, f_\mu(p_j)\rangle + L h_\nu C_\nu \le -\|p_j\|_2.$$

All the algorithms presented in this section use an objective function that minimizes the violation of the constraints. In (i) the inequality conditions are linear ones, since the vertices p_j are not optimization variables, but given a priori by the triangulation. The inequalities in (ii)–(iii) are also linear, since the gradient ∇V_ν depends linearly on the values $V(p_i^\nu)$, $i = 0, \dots, n$, by Lemma 14 and the function evaluations $f_\mu(p_j)$ are just values which are computed before the optimization run. The doubled maximum on the left-hand side of (iii) can be linearly modeled by writing the inequalities for each active indices μ and ν. Thus, Algorithm 1 states a linear optimization problem.

If a smooth CLF does not exist, as in Problem 2, we have to modify the decrease condition by using (10) together with Proposition 15 and using the second strategy described at the beginning of Section 3. The resulting problem is much more complicated than for a strong Lyapunov function as in Baier et al. (2012), see Baier et al. (2018) for more information of the formulation and the implementation.

Algorithm 2. (no smooth CLF exists; given switching law) Consider Problem 2 with the system (5) stated on subregions from (4) and, together with (i)–(ii) from Algorithm 1, we require that

(iv) For each simplex T_ν with all neighboring simplices $T_{\bar\nu}$, we demand the existence of $\mu \in I_{\mathcal{G}}(q)$ with
$$\langle \nabla V_\nu, f_\mu(q)\rangle + L h_\nu C_\nu \le -\|q\|_2 \quad (15)$$
for $q = q(\nu, \bar\nu) = b(T_\nu \cap T_{\bar\nu})$.
If f_μ points inside $\operatorname{int} T_{\bar\nu}$, i.e., $\bar\nu = \rho(k, \nu, \mu)$ from Proposition 15, we additionally demand either
$$\langle \nabla V_{\bar\nu}, f_\mu(q)\rangle + L h_\nu C_\nu \le -\|q\|_2$$
or the *local concavity conditions* (13)–(14) hold for all indices $\bar\nu$ with neighboring simplices $T_{\bar\nu}$.
Additionally, a *condition preventing local minima* of $V_{\mathcal{T}}$ is demanded by assuring for each $k \in \{0, \dots, n\}$ the existence of some indices $\bar\nu$ and $\bar k \in \{0, \dots, n\}$ with
$$V_{\mathcal{T}}(p_k^\nu) < V_{\mathcal{T}}(p_{\bar k}^{\bar\nu}).$$

In (iv) the existence of $\mu \in I_{\mathcal{G}}(q)$ in (15) can be modeled by the introduction of binary variables $z_{\nu,\mu} \in \{0, 1\}$ and by replacing the scalar product with $z_{\nu,\mu}\langle \nabla V_\nu, f_\mu(q)\rangle$ together with $\sum_{\mu \in I_{\mathcal{G}}(q)} z_{\nu,\mu} \ge 1$. The nonlinearity due to $z_{\nu,\mu}$ and the gradient can be resolved by the big-M method as
$$\langle \nabla V_\nu, f_\mu(q)\rangle + z_{\nu,\mu} M + L h_\nu C_\nu \le -\|q\|_2 + M,$$
where $M > 0$ is a big constant such that the inequality trivally holds if $z_{\nu,\mu} = 0$ (see Baier and Hafstein (2014) for more comments). This results in a mixed integer linear problem which is much harder to solve.

Algorithm 2 implements the second strategy discussed at the beginning of Section 3: for each two adjacent simplices, either we find a common decrease direction or $V_{\mathcal{T}}$ is concave on these simplices.

Now we turn to the algorithms for Problems 3 and 4 in which no switching strategy is given a priori.

Algorithm 3. (smooth CLF exists; unknown switching) Consider Problem 3 with the differential inclusion
$$\dot x \in F(x) := \operatorname{co}\{f_\mu(x) \mid \mu = 1, \dots, M\} \quad (16)$$
and require (i)–(ii) as in Algorithm 1 together with

(iii') For each vertex $p_j \in \mathcal{V}$, $j = 1, \dots, K$, with $p_j \in T_\nu$, we demand
$$\max_{\nu \in I_{\mathcal{T}}(p_j)} \min_{\mu = 1, \dots, M} \langle \nabla V_\nu, f_\mu(p_j)\rangle + L h_\nu C_\nu \le -\|p_j\|_2.$$

If the algorithm can compute a solution, there exists a smooth control Lyapunov function, cf. the discussion at the beginning of Section 3.

Algorithm 4. (no smooth CLF exists; unknown switching) Consider Problem 4 with the differential inclusion (16) and require (i)–(ii) as in Algorithm 1 together with

(iv') We use (iv) from Algorithm 2 with $I_{\mathcal{G}}(p_j) = \{1, \dots, M\}$ (in a formal way we consider $G_\mu = G$ for all indices μ).

We end this section with numerical examples. For their solution, we used Gurobi from Gurobi Optimization, LLC (2017) together with CVX, a package for specifying and solving convex programs Grant and Boyd (2014), Grant and Boyd (2008). Due to space limitations, we only provide examples for Algorithms 3 and 4.

Example 18. (Baier and Hafstein, 2014, Sec. IV)
We consider Problem 3 with

$$f_1(x) = \begin{pmatrix} x_2 \\ 4|x_2| - x_1 \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} x_2 \\ -4|x_2| - x_1 \end{pmatrix}.$$

on $G = \mathcal{B}_r(0) \subset \mathbb{R}^2$ and $r = 1.7$, $M = 2$. It was shown in (Clarke, 2011, Subsec. 8.1, Example) that this example admits a smooth CLF, hence we can apply Algorithm 3 which results in the PWA CLF depicted in Fig. 1 (left). The right subplot shows the switching strategy (triangles in red belong to $\mu(x) = 1$, triangles in blue belong to $\mu(x) = 2$).

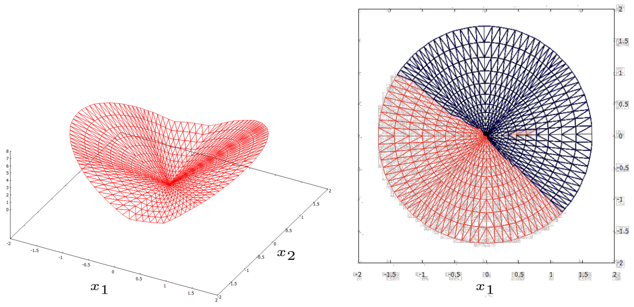

Fig. 1. Calculated PWA CLF for Example 18 (left); grid and switching strategy (right)

Example 19. (Artstein's circles).
We consider Problem 4 with $G = [-1, 1]^2$, $M = 2$ and the asymptotically controllable switched system with

$$f_1(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} -x_1^2 + x_2^2 \\ -2x_1x_2 \end{pmatrix}.$$

Fig. 2 shows the resulting PWA CLF computed by Algorithm 4 and the switching strategy on the grid. One clearly sees that the CLF is locally concave along the switching surface $x_1 = 0$.

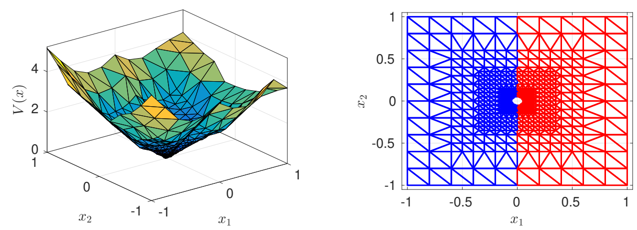

Fig. 2. Calculated PWA CLF for Example 19 (left); grid and switching strategy (right)

5. CONCLUSION

Our results in particular show that not only in Algorithm 3 and 4, in which the switching law is unknown, but also in Algorithm 2 nonlinearities appear that have to be resolved using a big-M linearization resulting in a difficult to solve mixed integer problem. In the case of unknown switching laws this is unavoidable since the problem of finding V and σ simultaneously is not convex. It is an open question whether the nonconvexity resulting from the concavity check for the Dini-sense Lyapunov function in Algorithm 2 could be avoided. This may be possible for more refined choices of σ and will be investigated in future research.

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