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OTT VILSON

Transformation properties and invariants in scalar-tensor theories of gravity

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Contents

List of publications

The thesis is based on the following five publications.

I L. Järv, P. Kuusk, M. Saal and O. Vilson “Invariant quantities in the scalar-tensor theories of gravitation” *Phys. Rev.* D **91**, 024041 (2015)
See Chapter 6, and arXiv:1411.1947 [INSPIRE] [ETIS]

II O. Vilson ”Some remarks concerning invariant quantities in scalar-tensor gravity” *Adv. Appl. Clifford Algebras* **27**, 321-32 (2017)
See Chapter 7, and arXiv:1509.02481 [INSPIRE] [ETIS]

III P. Kuusk, L. Järv and O. Vilson “Invariant quantities in the multiscalar-tensor theories of gravitation” *Int. J. Mod. Phys.* A **31**, 1641003 (2016)
See Chapter 8, and arXiv:1509.02903 [INSPIRE] [ETIS]

IV L. Järv, P. Kuusk, M. Saal and O. Vilson “Parametrizations in scalar-tensor theories of gravity and the limit of general relativity” *J. Phys.: Conf. Ser.* **532**, 012011 (2014)
See Chapter 9, and arXiv:1501.07781 [INSPIRE] [ETIS]

V L. Järv, P. Kuusk, M. Saal and O. Vilson “Transformation properties and general relativity regime in scalar–tensor theories”*Class. Quantum Grav.* **32**, 235013 (2015)
See Chapter 10, and arXiv:1504.02686 [INSPIRE] [ETIS]

Author’s contribution

I, Ott Vilson, have calculated and checked each and every equation in the papers. I wrote most of the manuscript for References I and III, and practically the whole manuscript for References II and V. In addition I participated actively in discussions, and took the time for proofreading the journal versions. I can not,

In accordance with a tradition in theoretical physics, the authors are ordered alphabetically, except when a paper is published in conference proceedings. In that case the person who gave the talk is also the first author of the paper.

however, claim sole right for any scientific result because it was indeed groupwork, and the synergy was especially strong during the time when we were writing the papers I and IV. The rest are aftershocks.

I have presented the results of the IV and V paper by giving a talk at the conferences Moduli Operads Dynamics II (2014, Tallinn) [ETIS], 10th International Conference on Clifford Algebras and their Applications in Mathematical Physics (2014, Tartu) [ETIS], and Geometric Foundations of Gravity in Tartu (2017, Tartu) [ETIS]. The content of the papers I and II was the subject of the talks I gave at the Tartu-Tuorla annual meeting 2015 (2015, Kubija) [ETIS], and at the seminar of the High Energy and Computational Physics workgroup in National Institute Of Chemical Physics And Biophysics (2016, Tallinn). The poster presented at the Funktsionaalsete materjalide ja tehnoloogiate doktorikooli teaduskonverents 2017 (The science conference of the graduate school of functional materials and technologies (unofficial loose translation)) (2017, Tartu) [ETIS] was also based on papers I and II.

Further publications to which I, Ott Vilson contributed

VI L. Järv, P. Kuusk, M. Saal and O. Vilson "The formalism of invariants in scalar-tensor and multiscalar-tensor theories of gravitation" proceedings of the Fourteenth Marcel Grossmann Meeting 1190–95 (2017)
arXiv:1512.09166 [INSPIRE] [ETIS]

VII P. Kuusk, M. Rünkla, M. Saal and O. Vilson "Invariant slow-roll parameters in scalar-tensor theories" *Class. Quantum Grav.* **33**, 195008 (2016)
arXiv:1605.07033 [INSPIRE] [ETIS] Read also CQG+ Insight.

VIII M. Hohmann, L. Järv, P. Kuusk, E. Randla and O. Vilson "Post-Newtonian parameter γ for multiscalar-tensor gravity with a general potential" *Phys. Rev.* D **94**, 124015 (2016)
arXiv:1607.02356 [INSPIRE] [ETIS]

IX L. Järv, M. Rünkla, M. Saal and O. Vilson "Nonmetricity formulation of general relativity and its scalar-tensor extension" *Phys. Rev.* D **97**, 124025 (2018)
arXiv:1802.00492 [INSPIRE] [ETIS]

X M. Rünkla and O. Vilson "Family of scalar-nonmetricity theories of gravity" *Phys. Rev.* D **98**, 084034 (2018)
arXiv:1805.12197 [INSPIRE] [ETIS]

Chapter 1

Introduction

Contents

1.1 Introduction

In the Einstein's general relativity the gravitational field is mediated by the metric tensor $g_{\mu\nu}$. Scalar-tensor theories are a class of extensions to general relativity where a scalar field Φ is added as an extra mediator of the gravitational field. Such theories in a sense date back to just a few years after the final formulation of general relativity, as already the Kaluza-Klein theories [1, 2] contain a scalar field. Scalar-tensor theories per se were studied by Jordan [3] and Fierz [4]. Their work was taken over by Brans and Dicke [5], and additionally generalized by Bergmann [6] and Wagoner [7], who promoted the constant coupling parameter, $\omega = \mathrm{const}$, of the original Brans-Dicke formulation into a dynamical function $\omega = \omega(\Phi)$ of the scalar field Φ. The latter is the starting point for the thesis, but not in its original formulation, but in the formulation by Flanagan [8] who completed the action functional with respect to the conformal transformation, introduced by Weyl [9], and scalar field redefinition. By the term 'completed' I mean that Flanagan considered an action functional where under the above mentioned transformations the form invariance of the action functional was explicit, because all four possible functional multipliers were kept arbitrary. Flanagan did not generalize the theory, as is also argued in the thesis, but his formulation is nevertheless useful and, prematurely, in my option the only straightforward way for studying the transformation properties.

One is justified to ask why should we consider the conformal transformation in the first place. The answer is not so clear, as for the generic case it is not a symmetry transformation of such theories. Hence in general, also no Noether current follows[1]. As I understand it is just a change of variables which in principle is one of the most used techniques for solving differential equations. In Dicke's interpretation the conformal transformation constitutes for changing one's units of measurement, which in that case might be spacetime point dependent. I am not a fan of such interpretation, because first such an approach would require a full understanding about measurements in such theories, and second the whole algebra of units breaks down. If the units are spacetime point dependent, then we cannot naïvely "pull them out" of the (differential) expressions. Perhaps the clearest reason for introducing a conformal transformation is the historical one. Roughly speaking, Jordan considered a five-dimensional flat space projected onto a four dimensional curved space. He obtained that such a projection naturally introduces a scalar field on the four dimensional space, however, the scalar field also entered the matter action functional thus breaking the usual local energy-momentum conservation. The scalar field was included in a particular manner, and its explicit presence in the matter action functional could be removed via a conformal transformation, as suggested by Pauli at the time (the description of the historical approach is loosely

[1] I thank Hardi Veermäe for making that clear at the right moment.

based on [10] and private communication with my supervisor D. Sc. Piret Kuusk).

When studying extensions of general relativity, one must, however, take into account that general relativity is in a very good agreement with the solar system experiments. Thus also the extensions must be close to general relativity in the weak field and/or late time regime. The first calculations in the parametrized post-Newtonian formulation were done by Nordtvedt [11] and later neatly generalized also to multiscalar case by Damour and Esposito-Farèse [12]. Their conclusion was that the coupling function $\omega(\Phi)$ must blow up, in order to obtain a general-relativity-like behavior of the scalar-tensor theory. The condition became even more significant when it turned out to yield general-relativity-like behavior also in the late time (Friedmann-type) cosmology [13]. In particular it was shown that there exists a fixed point in the $(\Phi, \dot{\Phi})$ space ("dot" denoting time derivative) that corresponds to general relativity and under certain circumstances the fixed point is an attractor, i.e., trajectories with different initial conditions converge towards it. Here, as different authors use different conformal frames, I have implicitly assumed that the results are invariant under the conformal transformation and scalar field reparametrization. Especially the latter needs a closer look, as diverging ω implies also the scalar field redefinition to be singular. (Perhaps the singular behavior is not so inadmissible if one takes into account that Liouville's theorem does not allow fixed points for Hamiltonian systems, and thus something must blow up [14][2].)

While deriving the described results, the authors did not consider the scalar field potential. It is therefore interesting to note that the post-Newtonian parameters approach their general relativity values also for sufficiently steep potential, leading to a massive scalar field [15] (see also Ref. VIII in List of publications). Similar conditions also render the fixed point, which corresponds to general relativity, to be an attractor, but only if matter is absent or is highly relativistic (the trace of the energy-momentum tensor vanishes). Let me use the term 'pseudo fixed point' for referring to the latter. In my opinion such results suggest a very nice speculation. (I heard at least a version of the thought from Dr. Margus Saal.) Let us consider an early universe, matter has not yet formed, and there is a possibility for a pseudo fixed point which is an attractor because the scalar field has high mass, presumably. Trajectories converge to that fixed point and already the mathematics used to study the nature of the fixed points via linearizing the equation suggest that in the vicinity of the point the time derivative of the scalar field is rather small. Potential of the scalar field dominates and we have slow-roll inflation. The weak field conditions are nevertheless rather close to general relativity because the scalar field is very massive and, thus, with short range. Let us further consider some unspecified mechanism which causes the scalar field to decay into ordinary matter. At first the matter is highly relativistic and the inflation continues. But once there exists some nonrelativistic matter, the pseudo fixed point

[2]I thank Mihkel Rünkla for pointing out that paper.

condition is not valid anymore, and in the phase space the trajectories slowly depart from what used to be a fixed point. Assuming that there is also a scalar field value where ω diverges, the trajectories now converge towards that point (if it is an attractor). The universe evolves from one inflationary period to another one, both corresponding to a fixed point in the phase space, and this is possible, because the nature of the first fixed point changes due to physical processes where matter is created. The speculation was, unfortunately for me, proven to be unphysical in a workgroup seminar by invoking the argument that due to the inflationary expansion of the universe the matter, the matter created simultaneously everywhere in space, would be ripped apart, thus keeping the density effectively zero.

The thesis, however, is about a particular mathematical problem concerning the transformation properties of expressions under the mentioned conformal transformation of the metric tensor, and under the scalar field redefinition, which becomes especially interesting if the latter is singular in the "vicinity" of general relativity. The mathematical problem could in principle be treated in its own right without any physical implications. Thus the introduction here is rather generic or even vague. Readers who are interested in further details concerning the scalar-tensor theories and the motivation for studying these should consult, first, the introductory sections of the attached papers, and second, more sophisticated and structured textbooks [10, 16].

1.2 Aim of the thesis and of the overview article

As briefly mentioned in the Introduction, the results ought to be covariant under the conformal transformation and under the scalar field reparametrization. To be more precise, while the question might seem trivial in some cases, it is important to check the correspondence also for singular transformations (as at least one particular case is related to the physically interesting general relativity regime), and perhaps most importantly, it is necessary to understand what must be done to impose the correspondence. A singular transformation in the case of the cosmological fixed point is studied in the attached paper V. The treatment there is generic and rather complete, thus I will not be studying it much in the following overview article. In the hope to ease the study we, I and Senior Researchers developed a formalism of quantities that are invariant under the conformal transformation and transform as scalar functions under the scalar field redefinition – the invariants. The thesis is mostly about these invariants and the overview article concentrates on an aspect which in my opinion is not clearly stated in the attached papers. Namely, it turns out that due to ambiguity in imposing the transformation properties, the formalism of invariants is not very useful when one is performing calculations in a particular fixed parametrization. An expression, in a fixed parametrization, by itself can always be considered to be such an invariant.

1.2.1 Statements

1. An action in a fixed parametrization is equivalent to the generic action rewritten in terms of the corresponding invariant pair. The actions differ only by the interpretation we assign to the quantities contained therein, and thus there is no physical or mathematical method for discriminating between the two.

2. The translation rules allow us to take an expression from a fixed parametrization and to rewrite it as an expression in the generic parametrization. These rules are just the equivalence, mentioned in the previous point, made explicit.

3. The translation rules, when applied on an expression without its context, are ambiguous because of the transformation properties. We cannot impose all possible transformation properties at once.

4. The transformation properties can be recovered by taking into account how the expression was derived in the first place, but such an approach is rather cumbersome.

5. A fixed parametrization is not a good setup for studying transformation properties. The Flanagan-like generic parametrization is much more convenient.

6. The previous statements have in principle nothing to do with the invariants. The latter, however, turned out to be useful for understanding the subtleties.

1.3 Structure of the overview article

The overview article contains of the current Introduction, which also includes the mathematical introduction. The content there is in principle just common knowledge, but further info may be found from [17, 18]. Chapter Introduction is followed by three chapters introducing the class of scalar-tensor theories of gravity in the generic parametrization in Chapter 2, in the Einstein frame canonical parametrization $\mathfrak{E}$ in Chapter 3, and in the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ in Chapter 4. In these three chapters altogether the material is presented trice, in hope that such exaggerated manner helps to make the statements more transparent. Especially the latter two are identical in their structure. The previous is followed by the Chapter 5 where I write down how the invariants, the formalism can be used in practice. In my opinion the list is complete, but I am nevertheless glad if somebody finds another uses. The overview article ends with the Chapter Summary. Each of the chapters 2, 3, 4 and 5 is preceded by a local Table of Contents.

1.4 Mathematical introduction

1.4.1 Christoffel symbols

In general relativity we define the curve of extremal length to be also the straightest one. In other words, varying the length of a curve between spacetime points $x_0 \equiv \{x_0^\mu\}_{\mu=0}^3$ and $x_1 \equiv \{x_1^\mu\}_{\mu=0}^3$, i.e.,

$$s = \int_{x_0}^{x_1} \mathrm{d}s = \int_{x_0}^{x_1} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda\,, \tag{1.1}$$

with respect to coordinates x^μ yields

$$\delta s = \int_{x_0}^{x_1} g_{\mu\omega}\left(\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\right)\delta x^\omega \mathrm{d}\lambda\,, \tag{1.2}$$

where I took into account that the boundaries are fixed, i.e., $\delta x_0 = 0 = \delta x_1$, and there is the possibility to choose $\mathrm{d}\lambda = \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$. The length of the curve is, thus, extremal if the coordinates x^μ solve the differential equations

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0\,, \tag{1.3}$$

where

$$\Gamma^\mu{}_{\sigma\rho} \equiv \frac{1}{2}g^{\mu\tau}\left(\partial_\sigma g_{\tau\rho} + \partial_\rho g_{\tau\sigma} - \partial_\tau g_{\sigma\rho}\right) \tag{1.4}$$

are the Christoffel symbols which foremost just make the equation covariant under a change of coordinates. Equation (1.3) is known as the geodesic equation. Here I used the spacetime metric tensor (in components) $g_{\mu\nu}$ with mostly plus signature, and the notation

$$\partial_\tau g_{\sigma\rho} \equiv \frac{\partial g_{\sigma\rho}}{\partial x^\tau}\,. \tag{1.5}$$

The choice of connection determines the straightest curves, and for general relativity the connection coefficients are the Christoffel symbols (1.4). Defining a vector field

$$u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{1.6}$$

allows us to write Eq. (1.3) as

$$\frac{\mathrm{d}x^\omega}{\mathrm{d}\lambda}\left(\partial_\omega u^\mu + \Gamma^\mu{}_{\omega\sigma}u^\sigma\right) = 0\,. \tag{1.7}$$

The expression in the parenthesis is the covariant derivative (generalization of the directional derivative), and the condition

$$\nabla_\omega u^\mu \equiv \partial_\omega u^\mu + \Gamma^\mu{}_{\omega\sigma}u^\sigma = 0 \tag{1.8}$$

states that the vector field u^μ is parallel transported along the coordinate line x^ω, i.e., it does not change along that line. Let us generalize covariant derivative to generic tensor fields as

$$\begin{aligned}\nabla_\nu T^{\lambda_1\ldots\lambda_n}_{\omega_1\ldots\omega_s} \equiv \partial_\nu T^{\lambda_1\ldots\lambda_n}_{\omega_1\ldots\omega_s} + \Gamma^{\lambda_1}{}_{\nu\alpha} T^{\alpha\ldots\lambda_n}_{\omega_1\ldots\omega_s} + \ldots + \Gamma^{\lambda_n}{}_{\nu\alpha} T^{\lambda_1\ldots\alpha}_{\omega_1\ldots\omega_s} \\ + \Pi^\beta{}_{\nu\omega_1} T^{\lambda_1\ldots\lambda_n}_{\beta\ldots\omega_s} + \ldots + \Pi^\beta{}_{\nu\omega_s} T^{\lambda_1\ldots\lambda_n}_{\omega_1\ldots\beta} \,. \end{aligned} \tag{1.9}$$

From the condition

$$\nabla_\mu \delta^\lambda_\omega \stackrel{!}{=} 0 \quad \Rightarrow \quad \Pi^\lambda{}_{\mu\nu} \equiv -\Gamma^\lambda{}_{\mu\nu} \,. \tag{1.10}$$

One can easily check that the Christoffel symbols (1.4) yield a torsionless, i.e., symmetric

$$\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} \,, \tag{1.11}$$

as well as metric compatible connection, i.e.,

$$\nabla_\mu g_{\sigma\rho} \equiv \partial_\mu g_{\sigma\rho} - \Gamma^\lambda{}_{\mu\sigma} g_{\lambda\rho} - \Gamma^\lambda{}_{\mu\rho} g_{\sigma\lambda} = 0 \,. \tag{1.12}$$

1.4.2 Riemann tensor, Ricci tensor and Ricci scalar

Covariant derivative is a generalization of the directional derivative. Hence, it shows how does a vector change along some curve. Let us study how does a vector field A^λ change when it is first infinitesimally shifted in the x^μ direction and then in the x^ν direction. This constitutes applying the covariant derivatives as

$$\begin{aligned}\nabla_\nu \nabla_\mu A^\lambda = \partial_\nu \partial_\mu A^\lambda - \Gamma^\omega{}_{\nu\mu} \partial_\omega A^\lambda + \Gamma^\lambda{}_{\nu\omega} \partial_\mu A^\omega + \left(\partial_\nu \Gamma^\lambda{}_{\mu\omega}\right) A^\omega \\ + \Gamma^\lambda{}_{\mu\omega} \partial_\nu A^\omega + \Gamma^\tau{}_{\nu\mu} \Gamma^\lambda{}_{\tau\omega} A^\omega + \Gamma^\lambda{}_{\nu\tau} \Gamma^\tau{}_{\mu\omega} A^\omega \,. \end{aligned} \tag{1.13}$$

Let us consider the other way, namely first the infinitesimal shift along x^ν and then along x^μ, yielding

$$\begin{aligned}\nabla_\mu \nabla_\nu A^\lambda = \partial_\mu \partial_\nu A^\lambda - \Gamma^\omega{}_{\mu\nu} \partial_\omega A^\lambda + \Gamma^\lambda{}_{\mu\omega} \partial_\nu A^\omega + \left(\partial_\mu \Gamma^\lambda{}_{\nu\omega}\right) A^\omega \\ + \Gamma^\lambda{}_{\nu\omega} \partial_\mu A^\omega + \Gamma^\tau{}_{\mu\nu} \Gamma^\lambda{}_{\tau\omega} A^\omega + \Gamma^\lambda{}_{\mu\tau} \Gamma^\tau{}_{\nu\omega} A^\omega \,. \end{aligned} \tag{1.14}$$

Comparing the difference yields

$$\nabla_\mu \nabla_\nu A^\lambda - \nabla_\nu \nabla_\mu A^\lambda = R^\lambda{}_{\omega\mu\nu} A^\omega \,, \tag{1.15}$$

where

$$R^\lambda{}_{\omega\mu\nu} = \partial_\mu \Gamma^\lambda{}_{\nu\omega} - \partial_\nu \Gamma^\lambda{}_{\mu\omega} + \Gamma^\tau{}_{\nu\omega} \Gamma^\lambda{}_{\mu\tau} - \Gamma^\tau{}_{\mu\omega} \Gamma^\lambda{}_{\nu\tau} \,, \tag{1.16}$$

is the Riemann curvature tensor. By contracting the latter we obtain the Ricci tensor

$$R_{\omega\nu} = \delta^\mu_\lambda R^\lambda{}_{\omega\mu\nu} \,, \tag{1.17a}$$

and the Ricci scalar

$$R = g^{\nu\omega} R_{\omega\nu} \,. \tag{1.17b}$$

The Christoffel symbols (1.4), Riemann tensor (1.16) and Ricci tensor and scalar (1.17) are all functionals of the metric tensor $g_{\mu\nu}$, i.e.,

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu}\left[g_{\sigma\rho}\right] \,, \qquad R^{\lambda}{}_{\omega\mu\nu} = R^{\lambda}{}_{\omega\mu\nu}\left[g_{\sigma\rho}\right] \,, \tag{1.18a}$$

$$R_{\omega\nu} = R_{\omega\nu}\left[g_{\sigma\rho}\right] \,, \qquad R = R\left[g_{\sigma\rho}\right] \,. \tag{1.18b}$$

1.4.3 Einstein-Hilbert action and Einstein equations

The field equations for general relativity can be derived from the Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_{\mathrm{N}}} \int_{M_4} \mathrm{d}^4x \sqrt{-g}\left(R\left[g_{\mu\nu}\right] - 16\pi G_{\mathrm{N}}\Lambda\right) + S_{\mathrm{m}}\left[g_{\mu\nu}\,, \chi\right] \,. \tag{1.19}$$

1. The action is an integral over the 4-dimensional manifold M_4.
2. The infinitesimal invariant integration measure $\mathrm{d}^4x\sqrt{-g}$ is given by the wedge product $\mathrm{d}^4x \equiv \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, multiplied by $\sqrt{-g}$ where g is the determinant of the metric tensor (for that procedure the components are written as a 4×4 matrix).
3. G_{N} is the Newton gravitational constant.
4. I am using the units where the speed of light $c \equiv 1$.
5. $R\left[g_{\mu\nu}\right]$ is the Ricci scalar (1.17b).
6. Λ is the cosmological constant.
7. The matter fields, described by the action S_{m}, are collectively denoted as χ.

Varying the action (1.19) with respect to the metric $g^{\mu\nu}$ yields

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G_{\mathrm{N}}} \int_{M_4} \mathrm{d}^4x \sqrt{-g} E^{(g)}_{\mu\nu} \delta g^{\mu\nu} \,. \tag{1.20}$$

Here I omitted the boundary terms as well as the equations of motion for matter fields. The Einstein field equations, i.e., second order differential equations for determining the components $g_{\mu\nu}$ are, thus,

$$E^{(g)}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 8\pi G_{\mathrm{N}}\, g_{\mu\nu}\Lambda - 8\pi G_{\mathrm{N}} T_{\mu\nu} = 0 \,, \tag{1.21}$$

where

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}\left[g_{\sigma\rho}, \chi\right]}{\delta g^{\mu\nu}} \tag{1.22}$$

is the energy-momentum tensor.

Chapter 2

Generic scalar-tensor theory

Contents

2.1 Generic theory

2.1.1 Notation

In the overview article I am going to use, and a bit modify the notation introduced in Ref. II. Mostly I shall introduce each quantity five times.

1. The quantities of the generic parametrization are denoted as

$$g_{\mu\nu}\,, \quad \Phi\,, \quad \mathcal{A}(\Phi)\,, \quad \mathcal{B}(\Phi)\,, \quad \mathcal{V}(\Phi)\,, \quad \alpha(\Phi)\,. \tag{2.1a}$$

2. The quantities of the Einstein frame canonical parametrization $\mathfrak{E}$ are denoted as

$$g^{\mathfrak{E}}_{\mu\nu}\,, \quad \Phi_{\mathfrak{E}}\,, \quad \mathcal{A}_{\mathfrak{E}}(\Phi_{\mathfrak{E}})\,, \quad \mathcal{B}_{\mathfrak{E}}(\Phi_{\mathfrak{E}})\,, \quad \mathcal{V}_{\mathfrak{E}}(\Phi_{\mathfrak{E}})\,, \quad \alpha_{\mathfrak{E}}(\Phi_{\mathfrak{E}})\,. \tag{2.1b}$$

3. The quantities of the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ are denoted as

$$g^{\mathfrak{J}}_{\mu\nu}\,, \quad \Phi_{\mathfrak{J}}\,, \quad \mathcal{A}_{\mathfrak{J}}(\Phi_{\mathfrak{J}})\,, \quad \mathcal{B}_{\mathfrak{J}}(\Phi_{\mathfrak{J}})\,, \quad \mathcal{V}_{\mathfrak{J}}(\Phi_{\mathfrak{J}})\,, \quad \alpha_{\mathfrak{J}}(\Phi_{\mathfrak{J}})\,. \tag{2.1c}$$

4. The quantities of the invariant Einstein frame canonical parametrization are denoted as

$$\hat{g}^{(\mathfrak{E})}_{\mu\nu}\,, \quad \mathcal{I}^{(\mathfrak{E})}_{\Phi}(\Phi)\,, \quad \mathcal{I}^{(\mathfrak{E})}_{\mathcal{A}}(\Phi)\,, \quad \mathcal{I}^{(\mathfrak{E})}_{\mathcal{B}}(\Phi)\,, \quad \mathcal{I}^{(\mathfrak{E})}_{\mathcal{V}}(\Phi)\,, \quad \mathcal{I}^{(\mathfrak{E})}_{\alpha}(\Phi)\,. \tag{2.1d}$$

5. The quantities of the invariant Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization are denoted as

$$\hat{g}^{(\mathfrak{J})}_{\mu\nu}\,, \quad \mathcal{I}^{(\mathfrak{J})}_{\Phi}(\Phi)\,, \quad \mathcal{I}^{(\mathfrak{J})}_{\mathcal{A}}(\Phi)\,, \quad \mathcal{I}^{(\mathfrak{J})}_{\mathcal{B}}(\Phi)\,, \quad \mathcal{I}^{(\mathfrak{J})}_{\mathcal{V}}(\Phi)\,, \quad \mathcal{I}^{(\mathfrak{J})}_{\alpha}(\Phi)\,. \tag{2.1e}$$

In addition we encounter some fixed parametrization $\mathfrak{P}$. See also Section 4 in Ref. II, page 99 in the current thesis. The main approach in the thesis is to show that essentially a fixed parametrization is equivalent to the corresponding invariant parametrization, up to the interpretation, and thus most of the results obtained in terms of invariants are just the already known expressions rewritten in terms of nice calligraphic fonts. However, in order to identify pairs of quantities, one must first distinguish these.

2.1.2 Action functional

Let me introduce a class of scalar-tensor theories, by postulating an action functional for the dynamical fields the metric tensor $g_{\mu\nu}$, the scalar field Φ, and the matter fields, denoted by χ, as [8]

$$S = S\left[g_{\mu\nu},\, \Phi,\, \chi\right] \tag{2.2a}$$
$$= \frac{1}{2\kappa^2}\int_{M_4} \mathrm{d}^4x\sqrt{-g}\big\{\mathcal{A}(\Phi)R[g_{\mu\nu}] - \mathcal{B}(\Phi)g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - 2\ell^{-2}\mathcal{V}(\Phi)\big\} + S_{\mathrm{m}}\left[\mathrm{e}^{2\alpha(\Phi)}g_{\mu\nu}, \chi\right]. \tag{2.2b}$$

Prematurely, let me stress that the action (2.2) with its four unspecified functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ and $\alpha(\Phi)$ is chosen, because under the conformal transformation and scalar field redefinition each expression in such generic formulation has a specific and unambiguous transformation rule. This is not the case for theories formulated in a fixed parametrization, although, as theories, such fixed parametrization formulations are equivalent to the generic formulation of the action (2.2).

1. The constants κ^2 and ℓ have the dimensions of the gravitational constant and length, respectively.

2. The action contains four dimensionless functions of the also dimensionless scalar field (by convention).

   (a) The Ricci scalar R, a functional of the metric $g_{\mu\nu}$, is multiplied by the nonminimal coupling function $\mathcal{A}(\Phi)$, introducing, roughly speaking, a "gravitational constant" $\propto \frac{\kappa^2}{\mathcal{A}(\Phi)}$ which through $\Phi(x^\mu)$ inherits the dependence on the spacetime point, labelled by x^μ.

   (b) The kinetic term $g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi \equiv g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$ is multiplied by the noncanonical kinetic coupling function $\mathcal{B}(\Phi)$.

   (c) The scalar field self-interaction potential $\mathcal{V}(\Phi)$ may contain also the cosmological constant Λ.

   (d) The matter fields couple to conformally rescaled metric $\mathrm{e}^{2\alpha(\Phi)}g_{\mu\nu}$, due to which the coupling function $\alpha(\Phi)$ enters also the continuity equation for the matter fields.

3. The matter action functional S_{m} describes matter fields, collectively denoted as χ, that only couple to the scalar field Φ via the above-mentioned conformal coupling.

2.1.3 Field equations

Varying the action (2.2) with respect to $g^{\mu\nu}$ and Φ reads (see also Section 2.1.2 in Ref. V, starting from page 146 in the current thesis)[1]

$$\delta S = \frac{1}{2\kappa^2}\int_{M_4} \mathrm{d}^4x\sqrt{-g}\left\{E^{(g)}_{\mu\nu}\delta g^{\mu\nu} + E^{(\Phi,R)}\delta\Phi\right\} . \tag{2.3}$$

Here I omitted the boundary terms as well as the equations of motion for matter fields, which at least formally are included in Ref. V.

The field equations are, thus,

$$\begin{aligned} E^{(g)}_{\mu\nu} \equiv\ & \mathcal{A}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \left(\frac{1}{2}\mathcal{B} + \mathcal{A}''\right) g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\Phi\nabla_\sigma\Phi \\ & - \left(\mathcal{B} + \mathcal{A}''\right)\nabla_\mu\Phi\nabla_\nu\Phi + \mathcal{A}'\left(g_{\mu\nu}\Box\Phi - \nabla_\mu\nabla_\nu\Phi\right) \\ & + \ell^{-2}g_{\mu\nu}\mathcal{V} - \kappa^2 T_{\mu\nu} = 0 \,, \end{aligned} \tag{2.4a}$$

$$E^{(\Phi,R)} \equiv \mathcal{A}'R + \mathcal{B}'g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + 2\mathcal{B}\Box\Phi - 2\ell^{-2}\mathcal{V}' + 2\kappa^2\alpha' T = 0 \,. \tag{2.4b}$$

Here

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}\left[\mathrm{e}^{2\alpha(\Phi)}g_{\sigma\rho},\chi\right]}{\delta g^{\mu\nu}} \qquad \text{and} \qquad T \equiv g^{\nu\mu}T_{\mu\nu} \,, \tag{2.5}$$

and

$$\begin{aligned} g^{\nu\mu}E^{(g)}_{\mu\nu} = & -\mathcal{A}R + \mathcal{B}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + 3\mathcal{A}''g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi \\ & + 3\mathcal{A}'\Box\Phi + \ell^{-2}4\mathcal{V} - \kappa^2 T \,. \end{aligned} \tag{2.6}$$

Combining (2.4a) and (2.4b) yields

$$E^{(\Phi)} \equiv E^{(\Phi,R)} + \frac{\mathcal{A}'}{\mathcal{A}}g^{\nu\mu}E^{(g)}_{\mu\nu} \tag{2.7a}$$

$$\begin{aligned} = & \frac{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}{\mathcal{A}}\Box\Phi + \frac{\left(2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2\right)'}{2\mathcal{A}}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi \\ & - \frac{2\left(\mathcal{A}\mathcal{V}' - 2\mathcal{V}\mathcal{A}'\right)}{\ell^2\mathcal{A}} + \frac{\kappa^2\left(2\mathcal{A}\alpha' - \mathcal{A}'\right)}{\mathcal{A}}T = 0 \,. \end{aligned} \tag{2.7b}$$

2.2 Generic transformations

2.2.1 Transformation of the variables

Let us consider a reparametrization of the scalar field as

$$\Phi \equiv \bar{f}(\bar{\Phi}) \,, \tag{2.8a}$$

[1] Note that with respect to Ref. V, I shifted the notation a bit. Equation (2.4b) was there denoted as $E^{(\Phi)}$, i.e., as Eq. (2.7) in the current overview article (cf. Eqs. (8), (9) and (13) in Ref. V, page 147 of the current thesis).

and a conformal transformation of the metric tensor (also known as a local Weyl rescaling [9])

$$g_{\mu\nu} \equiv \mathrm{e}^{2\bar{\gamma}(\bar{\Phi})}\bar{g}_{\mu\nu}\,. \tag{2.8b}$$

Along with (2.8b), we have

$$g^{\lambda\mu} = \mathrm{e}^{-2\bar{\gamma}(\bar{\Phi})}\bar{g}^{\lambda\mu}\,, \qquad\qquad \sqrt{-g} = \mathrm{e}^{4\bar{\gamma}(\bar{\Phi})}\sqrt{-\bar{g}}\,, \tag{2.9}$$

and the Christoffel symbols (1.4) transform as

$$\Gamma^{\lambda}{}_{\mu\nu}\left[g_{\sigma\rho}\right] = \bar{\Gamma}^{\lambda}{}_{\mu\nu}\left[\bar{g}_{\sigma\rho}\right] + \bar{\gamma}'\left(\delta^{\lambda}_{\mu}\partial_{\nu}\bar{\Phi} + \delta^{\lambda}_{\nu}\partial_{\mu}\bar{\Phi} - \bar{g}_{\mu\nu}\bar{g}^{\lambda\omega}\partial_{\omega}\bar{\Phi}\right)\,. \tag{2.10}$$

See, e.g., Eq. (7.106) in [18] while taking into account that due to (7.99) also in [18] the "barred" and "unbarred" quantities in the current thesis are interchanged with respect to [18]. The Ricci tensor and scalar (1.17) transform as

$$\begin{aligned}R_{\mu\nu}\left[g_{\sigma\rho}\right] = \bar{R}_{\mu\nu}\left[\bar{g}_{\sigma\rho}\right] &- 2\left(\bar{\gamma}'\right)^2\left(\bar{g}_{\mu\nu}\bar{g}^{\sigma\rho}\partial_{\sigma}\bar{\Phi}\partial_{\rho}\bar{\Phi} - \partial_{\mu}\bar{\Phi}\partial_{\nu}\bar{\Phi}\right) - \bar{\gamma}'2\bar{\nabla}_{\mu}\partial_{\nu}\bar{\Phi}\\ &- \bar{\gamma}'\bar{g}_{\mu\nu}\bar{g}^{\sigma\rho}\bar{\nabla}_{\sigma}\partial_{\rho}\bar{\Phi} - \bar{\gamma}''\left(\bar{g}_{\mu\nu}\bar{g}^{\sigma\rho}\partial_{\sigma}\bar{\Phi}\partial_{\rho}\bar{\Phi} + 2\partial_{\mu}\bar{\Phi}\partial_{\nu}\bar{\Phi}\right)\,,\end{aligned} \tag{2.11a}$$

$$\begin{aligned}R\left[g_{\sigma\rho}\right] = \mathrm{e}^{-2\bar{\gamma}(\bar{\Phi})}\Big\{\bar{R}\left[\bar{g}_{\sigma\rho}\right] &- 6\left(\bar{\gamma}'\right)^2\bar{g}^{\mu\nu}\partial_{\mu}\bar{\Phi}\partial_{\nu}\bar{\Phi} - 6\bar{\gamma}''\bar{g}^{\mu\nu}\partial_{\mu}\bar{\Phi}\partial_{\nu}\bar{\Phi}\\ &- 6\bar{\gamma}'\bar{g}^{\mu\nu}\bar{\nabla}_{\mu}\partial_{\nu}\bar{\Phi}\Big\}\,.\end{aligned} \tag{2.11b}$$

If along with the reparametrization of the scalar field (2.8a) and conformal transformation of the metric tensor (2.8b) one imposes the four arbitrary functions $\{\mathcal{A}(\Phi), \mathcal{B}(\Phi), \mathcal{V}(\Phi), \alpha(\Phi)\}$ to transform as [8]

$$\mathcal{A}\left(\bar{f}(\bar{\Phi})\right) = \mathrm{e}^{-2\bar{\gamma}(\bar{\Phi})}\bar{\mathcal{A}}(\bar{\Phi})\,, \tag{2.12a}$$

$$\mathcal{B}\left(\bar{f}(\bar{\Phi})\right) = \mathrm{e}^{-2\bar{\gamma}(\bar{\Phi})}\left(\bar{f}'\right)^{-2}\left(\bar{\mathcal{B}}(\bar{\Phi}) - 6\left(\bar{\gamma}'\right)^2\bar{\mathcal{A}}\left(\bar{\Phi}\right) + 6\bar{\gamma}'\bar{\mathcal{A}}'\right)\,, \tag{2.12b}$$

$$\mathcal{V}\left(\bar{f}(\bar{\Phi})\right) = \mathrm{e}^{-4\bar{\gamma}(\bar{\Phi})}\,\bar{\mathcal{V}}(\bar{\Phi})\,, \tag{2.12c}$$

$$\alpha\left(\bar{f}(\bar{\Phi})\right) = \bar{\alpha}(\bar{\Phi}) - \bar{\gamma}(\bar{\Phi})\,, \tag{2.12d}$$

then under (2.8) the action (2.2), S, preserves its form up to a boundary term, which we have always neglected in the published papers. However, the authors of Ref. [19][2] claim that the boundary term is neatly cancelled by the transformation of the Gibbons-Hawking-York boundary term, thus rendering the action to be completely form-invariant.

Let me stress that the transformations (2.12) are highly specific to the action (2.2) and I do not know of any fundamental meaning, either mathematical or physical, of such transformations.

[2]This paper was brought to my attention by Mihkel Rünkla, and I thank him for that.

The choice of barred and unbarred variables can be interchanged, in order to rewrite (2.8) and (2.12) as

$$\bar{\Phi} = f(\Phi)\,, \quad \bar{g}_{\mu\nu} = \mathrm{e}^{2\gamma(\Phi)} g_{\mu\nu}\,, \quad \bar{\mathcal{A}}(f(\Phi)) = \mathrm{e}^{-2\gamma(\Phi)}\mathcal{A}(\Phi)\,, \tag{2.13a}$$

$$\bar{\mathcal{V}}(f(\Phi)) = \mathrm{e}^{-4\gamma(\Phi)}\mathcal{V}(\Phi)\,, \qquad \bar{\alpha}(f(\Phi)) = \alpha(\Phi) - \gamma(\Phi)\,, \tag{2.13b}$$

$$\bar{\mathcal{B}}(f(\Phi)) = \mathrm{e}^{-2\gamma(\Phi)}\left(f'\right)^{-2}\left(\mathcal{B}(\Phi) - 6\left(\gamma'\right)^2\mathcal{A}(\Phi) + 6\gamma'\mathcal{A}'\right)\,. \tag{2.13c}$$

It is of utmost importance to understand that (2.8) does nothing more than just redistributes the already existing information between the fields $g_{\mu\nu}$ and Φ. One does not generate nor annihilate information. The transformations (2.12) are obtained by regrouping, i.e., from (2.11b) the Ricci scalar $R\left[g_{\mu\nu}\right]$ is substituted by $R = R\left[\bar{R},\, \bar{\Phi}\right]$, etc. The multiplier of $\sqrt{-\bar{g}}\bar{R}\left[\bar{g}_{\sigma\rho}\right]$ is defined to be the function $\bar{\mathcal{A}}(\bar{\Phi})$, etc.

2.2.2 Transformation of the field equations (2.4a), (2.4b) and (2.7)

Under a reparametrization (2.8a) of the scalar field Φ and conformal transformation (2.8b) of the metric tensor $g_{\mu\nu}$, the Eq. (2.4a) transforms as

$$E^{(g)}_{\mu\nu} = \mathrm{e}^{-2\bar{\gamma}}\bar{E}^{(\bar{g})}_{\mu\nu}\,, \tag{2.14}$$

while Eq. (2.4b) transforms as

$$E^{(\Phi,R)} = \left(\bar{f}'\right)^{-1}\mathrm{e}^{-4\bar{\gamma}}\left\{\bar{E}^{(\bar{\Phi},\bar{R})} + 2\bar{\gamma}'\bar{g}^{\mu\nu}\bar{E}^{(\bar{g})}_{\mu\nu}\right\}\,. \tag{2.15}$$

Combining the latter two yields

$$E^{(\Phi)} = \mathrm{e}^{-4\bar{\gamma}}\left(\bar{f}'\right)^{-1}\bar{E}^{(\bar{\Phi})} \tag{2.16}$$

to be the transformation of the Eq. (2.7). The transformation prescriptions (2.14), (2.15) and (2.16) are obtained by plugging the relations (2.8) and (2.12) into (2.4a), (2.4b) and (2.7), respectively.

Such transformation properties follow immediately from the Jacobian for (2.8) as a transformation of the variables. In particular

$$\begin{aligned}
\begin{pmatrix} \dfrac{\delta}{\delta\Phi} \\ \dfrac{\delta}{\delta g^{\sigma\rho}} \end{pmatrix} &= \begin{pmatrix} \dfrac{\delta\bar{\Phi}}{\delta\Phi} & \dfrac{\delta\bar{g}^{\mu\nu}}{\delta\Phi} \\ \dfrac{\delta\bar{\Phi}}{\delta g^{\sigma\rho}} & \dfrac{\delta\bar{g}^{\mu\nu}}{\delta g^{\sigma\rho}} \end{pmatrix} \begin{pmatrix} \dfrac{\delta}{\delta\bar{\Phi}} \\ \dfrac{\delta}{\delta\bar{g}^{\mu\nu}} \end{pmatrix} \\
&= \begin{pmatrix} \left(\bar{f}'\right)^{-1} & 2\bar{\gamma}'\left(\bar{f}'\right)^{-1}\bar{g}^{\mu\nu} \\ 0 & \mathrm{e}^{2\bar{\gamma}}\delta^{\mu}_{\sigma}\delta^{\nu}_{\rho} \end{pmatrix} \begin{pmatrix} \dfrac{\delta}{\delta\bar{\Phi}} \\ \dfrac{\delta}{\delta\bar{g}^{\mu\nu}} \end{pmatrix}\,.
\end{aligned} \tag{2.17}$$

See also Section 2.2.2 in Ref. V, starting from page 150 of the current thesis.

2.2.3 Covariance of the equations and solutions

The form-invariance of the action (2.2) under the transformations (2.8) encourages us to consider such transformations. However, let me point out, that any action can always be completed with respect to such transformations, as it is merely a choice of variables. By 'completed' I mean that we can add extra terms in order to impose the form-invariance, and consider these terms to be multiplied by functions that "happen" to be zero in the original formulation. Transformation prescriptions (2.14), (2.15) and (2.16) show that for regular transformations the pair $(g_{\mu\nu}, \Phi)$ is a solution to $E^{(g)}_{\mu\nu}$ and $E^{(\Phi)}$ if and only if the pair $(\bar{g}_{\mu\nu}, \bar{\Phi})$ solves $\bar{E}^{(\bar{g})}_{\mu\nu}$ and $\bar{E}^{(\bar{\Phi})}$. Singular cases must be considered separately and for an example see Section 4.2 in Ref. V, page 166 in the current thesis. Therefore, I conclude (as many others have concluded before), that considering the transformations (2.8) in the context of scalar-tensor theories is as useful as considering any other change of variables in any other theory. After all, it is a rather common technique for solving differential equations.

I must stress, however, that one must impose consistency, search for it. In the thesis I only consider the theory on classical level and then the problems do not appear. I am not an expert on the quantum level, but I would like to go through a simple example to illustrate the point. Let us consider a free 2-dimensional point particle with unit mass. The Hamiltonian reads

$$H(x, p_x, y, p_y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} \quad \Leftrightarrow \quad H(r, p_r, \varphi, p_\varphi) = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2}\,. \tag{2.18}$$

If we now naïvely promote the variables to operators (up to constant multiplier) as

$$\hat{H} \sim \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \not\Leftrightarrow \quad \hat{H} \sim \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \tag{2.19}$$

then the correspondence is lost but not because quantum mechanics is coordinate dependent, but because we did not search for consistency.

2.3 Invariants

The included papers I, II, and III are based on the observation that the quantities

$$\bar{\mathcal{I}}_1(\bar{\Phi}) \equiv \frac{\mathrm{e}^{2\bar{\alpha}(\bar{\Phi})}}{\bar{\mathcal{A}}(\bar{\Phi})} = \frac{\mathrm{e}^{2\alpha(\bar{f}(\bar{\Phi}))}}{\mathcal{A}(\bar{f}(\bar{\Phi}))} \equiv \mathcal{I}_1(\Phi)\ , \tag{2.20a}$$

$$\bar{\mathcal{I}}_2(\bar{\Phi}) \equiv \frac{\bar{\mathcal{V}}(\bar{\Phi})}{\left(\bar{\mathcal{A}}(\bar{\Phi})\right)^2} = \frac{\mathcal{V}(\bar{f}(\bar{\Phi}))}{\left(\mathcal{A}(\bar{f}(\bar{\Phi}))\right)^2} \equiv \mathcal{I}_2(\Phi)\,, \tag{2.20b}$$

$$\bar{\mathcal{I}}_3(\bar{\Phi}) \equiv \pm\int\sqrt{\bar{\mathcal{F}}(\bar{\Phi})}\mathrm{d}\bar{\Phi} = \pm\int\sqrt{\mathcal{F}(\bar{f}(\bar{\Phi}))}\mathrm{d}\Phi = \mathcal{I}_3(\Phi) \tag{2.20c}$$

are invariant with respect to the conformal transformation (2.8b), and transform as scalar functions under the reparametrization (2.8a) of the scalar field. Here

$$\mathcal{F} \equiv \frac{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}{4\mathcal{A}^2}\,, \qquad \mathcal{F} = \left(\bar{f}'\right)^{-2} \bar{\mathcal{F}}\,. \tag{2.21}$$

In what follows, I will write the expressions (2.20) always without bar, as the dependence on Φ or $\bar{\Phi}$ should be clear from the context. The numerical value of (2.20) with respect to a space-time point is invariant as well, and thus, e.g., $\partial_\mu \mathcal{I}_1$ is also and invariant. In addition to the scalar invariants (2.20) we may introduce geometrical invariants

$$\hat{g}^{(\mathfrak{E})}_{\mu\nu} \equiv \mathcal{A}(\Phi) g_{\mu\nu}\,, \qquad \hat{g}^{(\mathfrak{J})}_{\mu\nu} \equiv \mathrm{e}^{2\alpha(\Phi)} g_{\mu\nu}\,, \tag{2.22}$$

which, while being 2nd order tensors with respect to the change of tangent space basis, are due to (2.12) also invariant with respect to the conformal transformation (2.8b) and transform as scalar functions with respect to the reparametrization (2.8a) of the scalar field Φ. Mostly, in what follows, I shall drop the arguments of the invariants.

One can form infinitely many other invariants by constructing a function of invariants as

$$\mathcal{I}_k \equiv h(\{\mathcal{I}_i\}_{i\in\mathscr{I}})\,, \tag{2.23a}$$

where $\mathscr{I}$ is a set of some indices. Second option is to consider a quotient of derivatives as

$$\mathcal{I}_m \equiv \frac{\mathcal{I}_k'}{\mathcal{I}_l'} = \frac{\mathrm{d}\mathcal{I}_k}{\mathrm{d}\mathcal{I}_l}\,, \tag{2.23b}$$

where the second equality follows from the fact that with respect to Φ, the scalar invariants (such as (2.20)), are functions of one variable, i.e., derivative is given in terms of total differentials. Third technique is just the inverse of the second one as

$$\mathcal{I}_k \equiv \int \mathcal{I}_m \mathcal{I}_l' \mathrm{d}\Phi = \int \mathcal{I}_m \mathrm{d}\mathcal{I}_l\,, \tag{2.23c}$$

in the sense of an indefinite integral. See also the corresponding sections in the attached papers.

2.4 Parametrizations

Definition 2.4.1: Parametrization $\mathfrak{P} \in \{\mathfrak{E}, \mathfrak{J}, \ldots\}$

Let us consider the generic action functional (2.2), and the scalar invariants (2.20), together with further scalar invariants composed via (2.23).

The term 'fixed parametrization $\mathfrak{P}$' refers to a setup where the functional form of two and only two functions out of the four arbitrary functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ and $\alpha(\Phi)$ is specified in such a manner that also a scalar invariant $\mathcal{I}^{(\mathfrak{P})}(\Phi)$ has gained a nonconstant fixed functional form.

I have included the requirement of a fixed scalar invariant into the definition of a parametrization because the existence of such an invariant underlies the construction of the so called invariant pair, which furnishes the equivalence between the fixed parametrization and the generic parametrization. The scheme can be found from Section 4 in Ref. II, starting from page 99 in the current thesis. The invariant pair, in particular, is introduced in Theorem 4.2, page 101 of the current thesis.

2.4.1 Concerning notation

In a particular parametrization I shall denote the metric tensor and the scalar field as

$$g_{\mu\nu}|_{\mathfrak{P}} = g^{\mathfrak{P}}_{\mu\nu}\,, \qquad \Phi|_{\mathfrak{P}} = \Phi_{\mathfrak{P}}\,. \tag{2.24}$$

In a sense it is just renaming but it turns out that by fixing a parametrization to be $\mathfrak{P}$ we also introduce yet another metric tensor with components equal to $\hat{g}^{(\mathfrak{P})}_{\mu\nu}\Big|_{\mathfrak{P}} = g^{\mathfrak{P}}_{\mu\nu}$, as well as a scalar invariant $\mathcal{I}^{(\mathfrak{P})}_{\Phi}\Big|_{\mathfrak{P}} = \Phi_{\mathfrak{P}}$. See Theorem 4.2 in Ref. II, page (101) in the current thesis and in particular the invariant pair. This ambiguity is the reason for the equivalence of the generic parametrization and a fixed parametrization.

The four arbitrary functions (with two of them fixed) are denoted as

$$\mathcal{A}(\Phi)|_{\mathfrak{P}} = \mathcal{A}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})\,, \qquad \mathcal{B}(\Phi)|_{\mathfrak{P}} = \mathcal{B}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})\,, \tag{2.25a}$$

$$\mathcal{V}(\Phi)|_{\mathfrak{P}} = \mathcal{V}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})\,, \qquad \alpha(\Phi)|_{\mathfrak{P}} = \alpha_{\mathfrak{P}}(\Phi_{\mathfrak{P}})\,. \tag{2.25b}$$

2.4.2 Six possibilities for choosing a parametrization

The restriction of having a fixed scalar invariant $\mathcal{I}^{(\mathfrak{P})}(\Phi)$, however, just excludes two minor possibilities. There are 6 possibilities for fixing 2 functions out of 4.

1. If one chooses the parametrization to be $\mathfrak{P}$ by fixing the functional form of $\mathcal{A}(\Phi)|_{\mathfrak{P}} = \mathcal{A}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ and $\mathcal{B}(\Phi)|_{\mathfrak{P}} = \mathcal{B}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ then also the invariant

$$\mathcal{I}^{(\mathfrak{P})}(\Phi_{\mathfrak{P}}) = \mathcal{I}_3(\Phi)|_{\mathfrak{P}} = \pm \int \sqrt{\frac{2\mathcal{A}_{\mathfrak{P}}\mathcal{B}_{\mathfrak{P}} + 3(\mathcal{A}'_{\mathfrak{P}})^2}{4\mathcal{A}_{\mathfrak{P}}^2}}\, \mathrm{d}\Phi_{\mathfrak{P}} \tag{2.26a}$$

gains a fixed functional form with respect to the scalar field $\Phi_{\mathfrak{P}}$.

2. If one chooses the parametrization to be $\mathfrak{P}$ by fixing the functional form of $\mathcal{A}(\Phi)|_{\mathfrak{P}} = \mathcal{A}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ and $\mathcal{V}(\Phi)|_{\mathfrak{P}} = \mathcal{V}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ then also the invariant

$$\mathcal{I}^{(\mathfrak{P})}(\Phi_{\mathfrak{P}}) = \mathcal{I}_2(\Phi)|_{\mathfrak{P}} = \frac{\mathcal{V}_{\mathfrak{P}}}{\mathcal{A}_{\mathfrak{P}}^2} \tag{2.26b}$$

gains a fixed functional form with respect to the scalar field $\Phi_{\mathfrak{P}}$.

3. If one chooses the parametrization to be $\mathfrak{P}$ by fixing the functional form of $\mathcal{A}(\Phi)|_{\mathfrak{P}} = \mathcal{A}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ and $\alpha(\Phi)|_{\mathfrak{P}} = \alpha_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ then also the invariant

$$\mathcal{I}^{(\mathfrak{P})}(\Phi_{\mathfrak{P}}) = \mathcal{I}_1(\Phi)|_{\mathfrak{P}} = \frac{\mathrm{e}^{2\alpha_{\mathfrak{P}}}}{\mathcal{A}_{\mathfrak{P}}} \tag{2.26c}$$

gains a fixed functional form with respect to the scalar field $\Phi_{\mathfrak{P}}$.

4. If one chooses the parametrization to be $\mathfrak{P}$ by fixing the functional form of $\mathcal{V}(\Phi)|_{\mathfrak{P}} = \mathcal{V}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ and $\alpha(\Phi)|_{\mathfrak{P}} = \alpha_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ then also the invariant

$$\mathcal{I}^{(\mathfrak{P})}(\Phi_{\mathfrak{P}}) = \mathcal{I}_4(\Phi)|_{\mathfrak{P}} = \mathrm{e}^{-4\alpha_{\mathfrak{P}}}\mathcal{V}_{\mathfrak{P}} \tag{2.26d}$$

gains a fixed functional form with respect to the scalar field $\Phi_{\mathfrak{P}}$.

5. If one chooses the parametrization to be $\mathfrak{P}$ by fixing the functional form of $\mathcal{B}(\Phi)|_{\mathfrak{P}} = \mathcal{B}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ and $\alpha(\Phi)|_{\mathfrak{P}} = \alpha_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ then there are two possibilities.

   (a) If $\alpha_{\mathfrak{P}} = \mathrm{const}$, then

   $$\mathcal{I}^{(\mathfrak{P})}(\Phi_{\mathfrak{P}}) = \pm \int \sqrt{\mathcal{G}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})}\, \mathrm{d}\Phi_{\mathfrak{P}}\,, \tag{2.26e}$$

   where

   $$\begin{aligned} \mathcal{G}(\Phi) &\equiv \frac{2}{\mathcal{I}_1(\Phi)}\left(1 - 3\mathcal{I}_5\right)\left(\mathcal{I}'_3\right)^2 \\ &= \mathrm{e}^{-2\alpha}\mathcal{B} + 6\frac{(\alpha')^2}{\mathcal{I}_1} - 6\frac{\alpha'\mathcal{I}'_1}{\mathcal{I}_1^2}\,. \end{aligned} \tag{2.26e$'$}$$

   See also Eq. (14a) in Ref. III, page 117 of the current thesis as well as the invariant differential operator $\mathcal{D}_2$, in Table I of the attached paper I, page 79 in the thesis, and also the nearby Eq. (32).

(b) If $\mathcal{B}_{\mathfrak{P}} = 0$, then

$$\begin{aligned}\mathcal{I}^{(\mathfrak{P})}(\Phi_{\mathfrak{P}}) &= \int \left(\pm \mathcal{I}_3'(\Phi_{\mathfrak{P}}) + \sqrt{\frac{3}{4}} \left(\ln \mathcal{I}_1(\Phi_{\mathfrak{P}})\right)' \right) \mathrm{d}\Phi_{\mathfrak{P}} \\ &= \pm \mathcal{I}_3(\Phi_{\mathfrak{P}}) + \sqrt{\frac{3}{4}} \ln \mathcal{I}_1(\Phi_{\mathfrak{P}}) + \mathrm{const} \qquad (2.26\mathrm{f}) \\ &= \sqrt{3}\alpha_{\mathfrak{P}}(\Phi_{\mathfrak{P}}) + \mathrm{const}\end{aligned}$$

is the invariant gaining fixed functional form.

6. If the parametrization $\mathfrak{P}$ is obtained by fixing $\mathcal{B}(\Phi)|_{\mathfrak{P}} = \mathcal{B}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ and $\mathcal{V}(\Phi)|_{\mathfrak{P}} = \mathcal{V}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ then analogously to the previous case, there are two possibilities.

(a) If $\mathcal{V}_{\mathfrak{P}}(\Phi_{\mathfrak{P}}) = \mathrm{const}$, then

$$\mathcal{I}^{(\mathfrak{P})}(\Phi_{\mathfrak{P}}) = \int \sqrt{\mathcal{H}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})}\, \mathrm{d}\Phi_{\mathfrak{P}} \,, \qquad (2.26\mathrm{g})$$

where

$$\begin{aligned}\mathcal{H}(\Phi) &= \frac{2}{\sqrt{|\mathcal{I}_2|}} \left(\left(\mathcal{I}_3'\right)^2 - \frac{3}{4}\left(\left(\ln \sqrt{|\mathcal{I}_2|} \right)' \right)^2 \right) \\ &= \frac{\mathcal{B}}{|\mathcal{V}|} + \frac{3}{8\sqrt{|\mathcal{I}_2|}} \left(\frac{\mathcal{V}'}{\mathcal{V}} \right)^{-2} - \frac{3}{4}\frac{\mathcal{V}'}{\mathcal{V}} \frac{\mathcal{I}_2'}{\sqrt{|\mathcal{I}_2^3|}} \,. \qquad (2.26\mathrm{g}')\end{aligned}$$

(b) $\mathcal{B}_{\mathfrak{P}} = 0$, then

$$\begin{aligned}\mathcal{I}^{(\mathfrak{P})}(\Phi_{\mathfrak{P}}) &= \int \left(\pm \mathcal{I}_3'(\Phi_{\mathfrak{P}}) + \sqrt{\frac{3}{4}} \left(\ln \sqrt{|\mathcal{I}_2(\Phi_{\mathfrak{P}})|} \right)' \right) \mathrm{d}\Phi_{\mathfrak{P}} \\ &= \pm \mathcal{I}_3(\Phi_{\mathfrak{P}}) + \sqrt{\frac{3}{4}} \ln \sqrt{|\mathcal{I}_2(\Phi_{\mathfrak{P}})|} + \mathrm{const} \qquad (2.26\mathrm{f}) \\ &= \frac{\sqrt{3}}{4} \ln |\mathcal{V}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})| + \mathrm{const}\end{aligned}$$

is the invariant gaining a fixed functional form.

The cases 5 and 6 are distinct due to the transformation property (2.12b) which contains the third function $\mathcal{A}$, therefore spoiling the possibility for specifying two functions via two transformations. The order of the transformations (2.8) can be interchanged, and let us consider the conformal transformation (2.8b) to be the first one. The exceptions arise because after the conformal transformation the further transformation of one of the functions is neutralized. Namely, if α (analogously $\mathcal{V}$) is fixed to be a constant, then the scalar field transformation (2.8a) does not transform it further and the freedom can be used to fix $\mathcal{B}$. If $\mathcal{B}$ is fixed to be zero, then analogously the scalar field transformation (2.8a) does not alter $\mathcal{B}$, therefore allowing to fix α (analogously $\mathcal{V}$).

Chapter 3

Einstein frame canonical parametrization $\mathfrak{E}$

Contents

3.1 Definition and notation

Definition 3.1.1: Einstein frame canonical parametrization $\mathfrak{E}$

Let us specify the arbitrary functions in the generic action functional (2.2) to be

$$\mathcal{A}(\Phi)|_{\mathfrak{E}} \equiv \mathcal{A}_{\mathfrak{E}}(\Phi_{\mathfrak{E}}) \stackrel{!}{=} 1\,, \qquad \mathcal{B}(\Phi)|_{\mathfrak{E}} \equiv \mathcal{B}_{\mathfrak{E}}(\Phi_{\mathfrak{E}}) \stackrel{!}{=} 2\,, \tag{3.1a}$$

$$\mathcal{V}(\Phi)|_{\mathfrak{E}} \equiv \mathcal{V}_{\mathfrak{E}}(\Phi_{\mathfrak{E}})\,, \qquad \alpha(\Phi)|_{\mathfrak{E}} \equiv \alpha_{\mathfrak{E}}(\Phi_{\mathfrak{E}})\,. \tag{3.1b}$$

Such a setup is referred to as the Einstein frame canonical parametrization (EF can), denoted by $\mathfrak{E}$.

See also the subsection II.B in Ref. I (page 77 in the thesis), subsection 3.3 in Ref. IV (page 130 in the thesis), etc., for references and further information.

I shall denote the metric tensor, and the scalar field in the Einstein frame canonical parametrization $\mathfrak{E}$ as

$$g_{\mu\nu}|_{\mathfrak{E}} = g^{\mathfrak{E}}_{\mu\nu}\,, \qquad g^{\mu\nu}|_{\mathfrak{E}} = g^{\mu\nu}_{\mathfrak{E}}\,, \qquad \Phi|_{\mathfrak{E}} = \Phi_{\mathfrak{E}} \equiv \varphi\,. \tag{3.2}$$

The Christoffel symbols (1.4) corresponding to the metric in (3.2) are calculated as

$$\Gamma_{\mathfrak{E}}{}^{\lambda}{}_{\mu\nu} \equiv \Gamma^{\lambda}{}_{\mu\nu}\left[g^{\mathfrak{E}}_{\sigma\rho}\right] = \frac{1}{2} g^{\lambda\omega}_{\mathfrak{E}} \left(\partial_\mu g^{\mathfrak{E}}_{\omega\nu} + \partial_\nu g^{\mathfrak{E}}_{\omega\mu} - \partial_\omega g^{\mathfrak{E}}_{\mu\nu}\right)\,, \tag{3.3}$$

which allows to define the covariant derivative $\nabla^{\mathfrak{E}}$ as the one, where the particular Christoffel symbols (3.3) are used, and via (1.16), (1.17) lead us to the corresponding Riemann tensor, Ricci tensor and Ricci scalar

$$\begin{aligned} R^{\mathfrak{E}\sigma}{}_{\rho\mu\nu} \equiv R^{\sigma}{}_{\rho\mu\nu}\left[g^{\mathfrak{E}}_{\mu\nu}\right] &= \partial_\mu \Gamma_{\mathfrak{E}}{}^{\sigma}{}_{\nu\rho} - \partial_\nu \Gamma_{\mathfrak{E}}{}^{\sigma}{}_{\mu\rho} \\ &\quad + \Gamma_{\mathfrak{E}}{}^{\lambda}{}_{\nu\rho}\Gamma_{\mathfrak{E}}{}^{\sigma}{}_{\mu\lambda} - \Gamma_{\mathfrak{E}}{}^{\lambda}{}_{\mu\rho}\Gamma_{\mathfrak{E}}{}^{\sigma}{}_{\nu\lambda}\,, \end{aligned} \tag{3.4a}$$

$$R_{\mathfrak{E}\,\rho\nu} \equiv R_{\rho\nu}\left[g^{\mathfrak{E}}_{\mu\nu}\right] = \delta^{\mu}_{\sigma} R^{\mathfrak{E}\sigma}{}_{\rho\mu\nu}\,, \qquad R_{\mathfrak{E}} \equiv R\left[g^{\mathfrak{E}}_{\mu\nu}\right] = g^{\nu\rho}_{\mathfrak{E}} R_{\mathfrak{E}\,\rho\nu}\left[g^{\mathfrak{E}}_{\mu\nu}\right]\,. \tag{3.4b}$$

3.2 Action functional and field equations

The action functional (2.2) in the Einstein frame canonical parametrization $\mathfrak{E}$, Definition 3.1.1, thus reads

$$S_{\mathfrak{E}} \equiv S_{\mathfrak{E}}\left[g^{\mathfrak{E}}_{\mu\nu},\, \Phi_{\mathfrak{E}},\, \chi\right] \tag{3.5a}$$

$$\begin{aligned} &= \frac{1}{2\kappa^2}\int_{M_4} \mathrm{d}^4x \sqrt{-g^{\mathfrak{E}}}\left\{R_{\mathfrak{E}} - 2 g^{\mu\nu}_{\mathfrak{E}} \nabla^{\mathfrak{E}}_{\mu}\Phi_{\mathfrak{E}} \nabla^{\mathfrak{E}}_{\nu}\Phi_{\mathfrak{E}} - 2\ell^{-2}\mathcal{V}_{\mathfrak{E}}(\Phi_{\mathfrak{E}})\right\} \\ &\quad + S_{\mathrm{m}}\left[\mathrm{e}^{2\alpha_{\mathfrak{E}}(\Phi_{\mathfrak{E}})} g^{\mathfrak{E}}_{\mu\nu}, \chi\right]\,. \end{aligned} \tag{3.5b}$$

The field equations (2.4a), (2.4b) and (2.7) in the parametrization $\mathfrak{E}$ reduce to

$$
\begin{aligned}
E^{(g^{\mathfrak{E}})}_{\mu\nu} &\equiv \left(R_{\mathfrak{E}\,\mu\nu} - \frac{1}{2} g^{\mathfrak{E}}_{\mu\nu} R_{\mathfrak{E}}\right) + g^{\mathfrak{E}}_{\mu\nu} g^{\rho\sigma}_{\mathfrak{E}} \nabla^{\mathfrak{E}}_{\rho}\Phi_{\mathfrak{E}} \nabla^{\mathfrak{E}}_{\sigma}\Phi_{\mathfrak{E}} \\
&\qquad - 2\nabla^{\mathfrak{E}}_{\mu}\Phi_{\mathfrak{E}} \nabla^{\mathfrak{E}}_{\nu}\Phi_{\mathfrak{E}} + \ell^{-2} g^{\mathfrak{E}}_{\mu\nu} \mathcal{V}_{\mathfrak{E}} - \kappa^2 T^{\mathfrak{E}}_{\mu\nu} = 0\,, && (3.6a)\\
E^{(\Phi_{\mathfrak{E}}, R_{\mathfrak{E}})} &\equiv 4\Box^{\mathfrak{E}}\Phi_{\mathfrak{E}} - 2\ell^{-2}\mathcal{V}'_{\mathfrak{E}} + 2\kappa^2 \alpha'_{\mathfrak{E}} T^{\mathfrak{E}} = 0 && (3.6b)\\
&= E^{(\Phi_{\mathfrak{E}})}\,, && (3.6b')
\end{aligned}
$$

where

$$
T^{\mathfrak{E}}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g^{\mathfrak{E}}}} \frac{\delta S_{\mathrm{m}}\left[\mathrm{e}^{2\alpha_{\mathfrak{E}}(\Phi_{\mathfrak{E}})} g^{\mathfrak{E}}_{\sigma\rho}, \chi\right]}{\delta g^{\mu\nu}_{\mathfrak{E}}}\,, \qquad T^{\mathfrak{E}} \equiv g^{\nu\mu}_{\mathfrak{E}} T^{\mathfrak{E}}_{\mu\nu}\,, \tag{3.7}
$$

and prime as, e.g., in $\mathcal{V}'_{\mathfrak{E}}$ means derivative with respect to the Einstein frame scalar field $\Phi_{\mathfrak{E}}$, i.e.,

$$
\mathcal{V}'_{\mathfrak{E}} \equiv \frac{\mathrm{d}\mathcal{V}_{\mathfrak{E}}(\Phi_{\mathfrak{E}})}{\mathrm{d}\Phi_{\mathfrak{E}}}\,, \qquad \alpha'_{\mathfrak{E}} \equiv \frac{\mathrm{d}\alpha_{\mathfrak{E}}(\Phi_{\mathfrak{E}})}{\mathrm{d}\Phi_{\mathfrak{E}}}\,. \tag{3.8}
$$

3.3 Invariant Einstein frame canonical parametrization

3.3.1 The invariant pair

The invariant pair from Theorem 4.2 in Ref. II in the Einstein frame canonical parametrization $\mathfrak{E}$, i.e., example (4.12) in Ref. II, is

$$
\hat{g}^{(\mathfrak{E})}_{\mu\nu} \equiv \mathcal{A}(\Phi) g_{\mu\nu}\,, \qquad \mathcal{I}^{(\mathfrak{E})}_{\Phi} \equiv \pm\mathcal{I}_3 = \int \sqrt{\frac{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}{4\mathcal{A}^2}}\,\mathrm{d}\Phi\,. \tag{3.9}
$$

The integration constant in the indefinite integral is taken to be zero. Note that the definition is on the generic level, and therefore we do not write the quantities in the particular parametrization $\mathfrak{E}$. However, plugging the Definition 3.1.1 into Eq. (3.9) verifies

$$
\hat{g}^{(\mathfrak{E})}_{\mu\nu}\Big|_{\mathfrak{E}} = g^{\mathfrak{E}}_{\mu\nu}\,, \qquad \mathcal{I}^{(\mathfrak{E})}_{\Phi}\Big|_{\mathfrak{E}} = \Phi_{\mathfrak{E}}\,. \tag{3.10}
$$

The metric $\hat{g}^{(\mathfrak{E})}_{\mu\nu}$ is also know as the invariant Einstein frame metric (see also definition (18) in Ref. I on page 78 in the current thesis).

3.3.2 Four functions as invariants

Let us consider the invariant pair (3.9) to be a scalar field reparametrization and a conformal transformation as in (2.13), i.e., (2.8) backwards. The transformations are given as

$$
\mathrm{e}^{2\gamma(\Phi)} = \mathcal{A}(\Phi)\,, \qquad \bar{\Phi} = \pm\mathcal{I}_3(\Phi)\,, \tag{3.11}
$$

and since $\mathcal{A}$ itself transforms under the conformal transformation, the obtained (formally barred) quantities are invariants. The four functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ and $\alpha(\Phi)$, thus, transform into the four invariants

$$\mathcal{I}_{\mathcal{A}}^{(\mathfrak{E})} = \frac{\mathcal{A}(\Phi)}{\mathcal{A}(\Phi)} = 1\,, \qquad \mathcal{I}_{\mathcal{B}}^{(\mathfrak{E})} = 2\,, \tag{3.12a}$$

$$\mathcal{I}_{\mathcal{V}}^{(\mathfrak{E})} = \mathcal{I}_2\,, \qquad \mathcal{I}_{\alpha}^{(\mathfrak{E})} = \frac{1}{2}\ln\mathcal{I}_1\,. \tag{3.12b}$$

Also here, analogously to the case (3.10) of the invariant pair, as $\mathcal{A}_{\mathfrak{E}} \equiv 1$, it is natural, that

$$\mathcal{I}_{\mathcal{A}}^{(\mathfrak{E})}\Big|_{\mathfrak{E}} = \mathcal{A}_{\mathfrak{E}} = 1\,, \qquad \mathcal{I}_{\mathcal{B}}^{(\mathfrak{E})}\Big|_{\mathfrak{E}} = \mathcal{B}_{\mathfrak{E}} = 2\,, \tag{3.13a}$$

$$\mathcal{I}_{\mathcal{V}}^{(\mathfrak{E})}\Big|_{\mathfrak{E}} = \mathcal{V}_{\mathfrak{E}}\,, \qquad \mathcal{I}_{\alpha}^{(\mathfrak{E})}\Big|_{\mathfrak{E}} = \alpha_{\mathfrak{E}}\,. \tag{3.13b}$$

The invariants $\mathcal{I}_{\mathcal{A}}^{(\mathfrak{E})}$, $\mathcal{I}_{\mathcal{B}}^{(\mathfrak{E})}$, $\mathcal{I}_{\mathcal{V}}^{(\mathfrak{E})}$ and $\mathcal{I}_{\alpha}^{(\mathfrak{E})}$ are functions of $\mathcal{I}_{\Phi}^{(\mathfrak{E})} = \pm\mathcal{I}_3$. However, we do not need the explicit dependence, as the derivatives may be calculated as

$$\pm\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{E})}}{\mathrm{d}\mathcal{I}_3} = \pm\frac{\mathrm{d}\Phi}{\mathrm{d}\mathcal{I}_3}\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{E})}}{\mathrm{d}\Phi} = \pm\frac{\left(\mathcal{I}^{(\mathfrak{E})}\right)'}{\mathcal{I}_3'}\,. \tag{3.14}$$

See also the invariant differential operator $\mathcal{D}_3$ in the Table 1 of Ref. I, page 79 in the current thesis, as well as the nearby Eq. (33).

3.3.3 Invariant geometry of the Einstein frame

The metric $\hat{g}_{\mu\nu}^{(\mathfrak{E})}$ is an invariant, and hence, the corresponding Ricci tensor and scalar are as well. According to Eqs. (2.11)

$$\begin{aligned}\hat{R}_{\mu\nu}^{(\mathfrak{E})}\left[\hat{g}_{\mu\nu}^{(\mathfrak{E})}\right] = {} & R_{\mu\nu}\left[g_{\mu\nu}\right] + \frac{3}{2}\left(\frac{\mathcal{A}'}{\mathcal{A}}\right)^2\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\frac{\mathcal{A}''}{\mathcal{A}}g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\Phi\partial_\rho\Phi \\ & - \frac{\mathcal{A}''}{\mathcal{A}}\partial_\mu\Phi\partial_\nu\Phi - \frac{\mathcal{A}'}{\mathcal{A}}\left(\nabla_\mu\partial_\nu\Phi + \frac{1}{2}g_{\mu\nu}g^{\sigma\rho}\nabla_\sigma\partial_\rho\Phi\right)\,,\end{aligned} \tag{3.15a}$$

$$\begin{aligned}\hat{R}^{(\mathfrak{E})}\left[\hat{g}_{\mu\nu}^{(\mathfrak{E})}\right] = {} & \frac{1}{\mathcal{A}}R\left[g_{\mu\nu}\right] + \frac{3}{2}\frac{\left(\mathcal{A}'\right)^2}{\mathcal{A}^3}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \\ & - 3\frac{\mathcal{A}'}{\mathcal{A}^2}g^{\mu\nu}\nabla_\mu\partial_\nu\Phi - 3\frac{\mathcal{A}''}{\mathcal{A}^2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\,.\end{aligned} \tag{3.15b}$$

Note, first, that with respect to (2.11), the transformation is backwards, and second, that the expressions on the right hand side are indeed in the generic parametrization, and, third,

$$\hat{R}_{\mu\nu}^{(\mathfrak{E})}\left[g_{\mu\nu}^{(\mathfrak{E})}\right]\Big|_{\mathfrak{E}} = R_{\mathfrak{E}\,\mu\nu}\left[g_{\mu\nu}^{\mathfrak{E}}\right]\,, \qquad \hat{R}^{(\mathfrak{E})}\left[g_{\mu\nu}^{(\mathfrak{E})}\right]\Big|_{\mathfrak{E}} = R_{\mathfrak{E}}\left[g_{\mu\nu}^{\mathfrak{E}}\right]\,. \tag{3.16}$$

See also the Section IV.B in Ref. I, in particular the Eqs. (52) and (53), from page the 81 in the thesis.

3.3.4 Invariant Einstein frame action

By considering the invariant pair (3.9) as a particular scalar field redefinition and conformal transformation (2.8), and taking into account the results (3.12), we rewrite the generic action (2.2) as (see also Eq. (5.2) in Ref. II, page 103 of the thesis)

$$S = S\left[\hat{g}^{(\mathfrak{E})}_{\mu\nu}, \mathcal{I}^{(\mathfrak{E})}_{\Phi}, \chi\right] \tag{3.17a}$$

$$= \frac{1}{2\kappa^2}\int_{M_4} \mathrm{d}^4x\sqrt{-\hat{g}^{(\mathfrak{E})}}\left\{\hat{R}^{(\mathfrak{E})}\left[\hat{g}^{(\mathfrak{E})}_{\mu\nu}\right] - 2\hat{g}^{\mu\nu}_{(\mathfrak{E})}\hat{\nabla}^{(\mathfrak{E})}_{\mu}\mathcal{I}^{(\mathfrak{E})}_{\Phi}\hat{\nabla}^{(\mathfrak{E})}_{\nu}\mathcal{I}^{(\mathfrak{E})}_{\Phi} - 2\ell^{-2}\mathcal{I}^{(\mathfrak{E})}_{\mathcal{V}}\right\}$$
$$+ S_{\mathrm{m}}\left[\mathrm{e}^{2\mathcal{I}^{(\mathfrak{E})}_{\alpha}}\hat{g}^{(\mathfrak{E})}_{\mu\nu}, \chi\right] . \tag{3.17b}$$

The obtained action is just the action (2.2) in terms of different variables and, thus, as generic. On the other hand, however, comparing the Einstein frame invariant action (3.17) with the Einstein frame (noninvariant) action (3.5) reveals, that these two differ only by the meaning we assign to the quantities contained therein. The action is postulated, and therefore I conclude, that there is no way to distinguish the Einstein frame (noninvariant) action (3.5) from (3.17) (a priori). In other words, specifying the four functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ and $\alpha(\Phi)$ as in the Definition 3.1.1 is equivalent to rewriting the action via the invariant pair (3.9), and hence, the Einstein frame canonical parametrization $\mathfrak{E}$ is equivalent to the generic parametrization.

3.4 Translation rules from the Einstein frame canonical parametrization $\mathfrak{E}$ to the generic parametrization

The translation rules for the Einstein frame canonical parametrization $\mathfrak{E}$ are a set of essentially algebraic substitutions which allow us to rewrite an arbitrary expression in the Einstein frame as an expression in the generic parametrization. Therefore, these are just the transformation rules from the Einstein frame canonical parametrization $\mathfrak{E}$ to the generic parametrization and from there, of course, further to any other parametrization.

3.4.1 Rules for the invariant quantities

Under the assumption that the quantity under consideration is an invariant, the rules have been presented implicitly already in Ref. I. The explicit version, but on an abstract level was introduced in the last part of Section 5 in Ref. II, in particular on page 105 of the current thesis. The rules in the Einstein frame but for multiple scalar fields were introduced by Eq. (17) in Ref. III. More precisely one must revert the mappings in Eq. (17) on page 118 of the current thesis.

Therefore, the translation rules for the Einstein frame canonical parametrization $\mathfrak{E}$ are the algebraic substitutions

$$g^{\mathfrak{E}}_{\mu\nu} \mapsto \hat{g}^{(\mathfrak{E})}_{\mu\nu} \overset{(3.9)}{\equiv} \mathcal{A}(\Phi) g_{\mu\nu} \,, \qquad \Phi_{\mathfrak{E}} \mapsto \mathcal{I}^{(\mathfrak{E})}_{\Phi} \overset{(3.9)}{\equiv} \pm \mathcal{I}_3(\Phi) \,, \tag{3.18a}$$

$$\sqrt{-g^{\mathfrak{E}}} \mapsto \sqrt{-g^{(\mathfrak{E})}} = \mathcal{A}^2 \sqrt{-g} \,, \qquad \mathcal{V}_{\mathfrak{E}}(\Phi_{\mathfrak{E}}) \mapsto \mathcal{I}^{(\mathfrak{E})}_{\mathcal{V}}(\mathcal{I}^{(\mathfrak{E})}_{\Phi}) = \mathcal{I}_2(\Phi) \,, \tag{3.18b}$$

$$R_{\mathfrak{E}\,\mu\nu}\left[g^{\mathfrak{E}}_{\mu\nu}\right] \mapsto \hat{R}^{(\mathfrak{E})}_{\mu\nu}\left[\hat{g}^{(\mathfrak{E})}_{\mu\nu}\right] \,, \qquad \alpha_{\mathfrak{E}}(\Phi_{\mathfrak{E}}) \mapsto \mathcal{I}^{(\mathfrak{E})}_{\alpha}(\mathcal{I}^{(\mathfrak{E})}_{\Phi}) = \frac{1}{2} \ln \mathcal{I}_1(\Phi) \,, \tag{3.18c}$$

$$\nabla^{\mathfrak{E}}_{\mu} \mapsto \hat{\nabla}^{(\mathfrak{E})}_{\mu} \,, \qquad \frac{\mathrm{d}}{\mathrm{d}\Phi_{\mathfrak{E}}} \mapsto \frac{\mathrm{d}}{\mathrm{d}\mathcal{I}^{(\mathfrak{E})}_{\Phi}} = \pm \frac{(\ \ )'}{\mathcal{I}_3'} \,. \tag{3.18d}$$

Note that the right hand sides are essentially in the generic parametrization. Such mappings, thus, take us from the Einstein frame canonical parametrization $\mathfrak{E}$ to the generic parametrization. On the other hand, fixing the parametrization on the right hand side to be the Einstein frame canonical parametrization $\mathfrak{E}$ forces the mapping to reduce to identity.

3.4.2 Rules for noninvariant quantities

In addition to transforming invariant quantities one can also impose some particular transformation properties. In the Einstein frame canonical parametrization we rely on the fact that $\mathcal{A}_{\mathfrak{E}} = 1$. Hence, whenever we want to impose that something transforms as $\mathcal{A}$, we just use the translation rules (3.18) to obtain the invariant expression, and the multiply the latter by $\mathcal{A}$. Analogously

$$\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{E})}_{\Phi}}{\mathrm{d}\Phi} = \left(\bar{f}'\right)^{-1} \frac{\mathrm{d}\mathcal{I}^{(\mathfrak{E})}_{\Phi}}{\mathrm{d}\bar{\Phi}} \,, \quad \text{but} \quad \left.\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{E})}_{\Phi}}{\mathrm{d}\Phi}\right|_{\mathfrak{E}} = \frac{\mathrm{d}\Phi_{\mathfrak{E}}}{\mathrm{d}\Phi_{\mathfrak{E}}} = 1 \,. \tag{3.19}$$

Let us go through a number of examples.

1. Suppose we want a quantity that in the Einstein frame canonical parametrization $\mathfrak{E}$ has the same functional form as $\mathcal{A}_{\mathfrak{E}}$, and transforms as $\mathcal{A}$. The invariant corresponding to $\mathcal{A}_{\mathfrak{E}}$ is just the number 1. Multiplying the latter by $\mathcal{A}$ yields $\mathcal{A}$ which has the numerical value 1 in the Einstein frame canonical parametrization and under (2.8) transforms as $\mathcal{A}$. This example is rather trivial, so let us continue with more elaborate ones.

2. Let us consider the invariant which represents the function $\alpha_{\mathfrak{E}}$, i.e., $\mathcal{I}_{\alpha}^{(\mathfrak{E})}$. The transformation properties of α, given by Eq. (2.12d), are imposed by

$$\frac{1}{2}\ln\mathcal{A}=\frac{1}{2}\ln\bar{\mathcal{A}}-\bar{\gamma} \qquad \Rightarrow \qquad \mathcal{I}_{\alpha}^{(\mathfrak{E})}+\frac{1}{2}\ln\mathcal{A}=\alpha\,. \tag{3.20}$$

3. Analogously

$$\mathcal{A}^2\mathcal{I}_{\mathcal{V}}^{(\mathfrak{E})}=\mathcal{V}\,, \tag{3.21}$$

which reproduces the prescription (2.12c).

4. As the last case, let us consider the invariant which represents $\mathcal{B}_{\mathfrak{E}}$, i.e., the number 2. In order to impose the transformation properties (2.12b), let us take into account, that $\mathcal{A}_{\mathfrak{E}}=1$, and thus $\mathcal{A}_{\mathfrak{E}}'=0$. On the other hand

$$\mathcal{A}'=\left(\bar{f}'\right)^{-1}\mathrm{e}^{-2\bar{\gamma}}\left(\bar{\mathcal{A}}'-2\bar{\gamma}'\bar{\mathcal{A}}\right)\,, \tag{3.22a}$$

$$\left(\mathcal{A}'\right)^2=\left(\bar{f}'\right)^{-2}\mathrm{e}^{-4\bar{\gamma}}\left[\left(\bar{\mathcal{A}}'\right)^2-4\bar{\gamma}'\bar{\mathcal{A}}'\bar{\mathcal{A}}+4\left(\bar{\gamma}'\right)^2\bar{\mathcal{A}}^2\right]\,. \tag{3.22b}$$

A straightforward calculation shows the following

$$\begin{aligned}2-\frac{3}{2}\left(\frac{\mathcal{A}'}{\mathcal{A}}\right)^2\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}\right)^2 &= 2-\frac{3}{2}\left(\frac{\bar{\mathcal{A}}'}{\bar{\mathcal{A}}}\right)^2\left(\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}\right)^2\\ &\quad+\bar{\mathcal{A}}^{-1}\left(6\bar{\gamma}'\bar{\mathcal{A}}'-6\left(\bar{\gamma}'\right)^2\bar{\mathcal{A}}\right)\left(\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}\right)^2\,,\end{aligned} \tag{3.23}$$

while still

$$2-\frac{3}{2}\left(\frac{\mathcal{A}'}{\mathcal{A}}\right)^2\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}\right)^2\Bigg|_{\mathfrak{E}}=2\,. \tag{3.24}$$

Let us complete the transformation properties by multiplying the previous by

$$\begin{aligned}&\mathcal{A}\left(\frac{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}{\mathrm{d}\Phi}\right)^2\left[2-\frac{3}{2}\left(\frac{\mathcal{A}'}{\mathcal{A}}\right)^2\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}\right)^2\right]=\\ &=\mathrm{e}^{-2\bar{\gamma}}\left(\bar{f}'\right)^{-2}\left\{\bar{\mathcal{A}}\left(\frac{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}{\mathrm{d}\bar{\Phi}}\right)^2\left[2-\frac{3}{2}\left(\frac{\bar{\mathcal{A}}'}{\bar{\mathcal{A}}}\right)^2\left(\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}\right)^2\right]\right.\\ &\qquad\left.-6\left(\bar{\gamma}'\right)^2\bar{\mathcal{A}}+6\bar{\gamma}'\bar{\mathcal{A}}'\right\}\,,\end{aligned} \tag{3.25}$$

i.e., we reproduce (2.12b). In the generic parametrization we have

$$\mathcal{A}\left(\frac{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}{\mathrm{d}\Phi}\right)^{2}\left[2-\frac{3}{2}\left(\frac{\mathcal{A}'}{\mathcal{A}}\right)^{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\mathcal{I}_{\Phi}^{(\mathfrak{E})}}\right)^{2}\right]=\mathcal{B}\,. \tag{3.26}$$

The result, of course, is not surprising, as I have just inverted (3.12) under (3.11). However, I want to stress, that one does not need to substitute $2 \mapsto \mathcal{B}$, which would be ambiguous. We look for an invariant quantity that in the Einstein frame canonical parametrization is equal to $\mathcal{B}_{\mathfrak{E}}$, and after imposing the transformation properties, $\mathcal{B}$ emerges.

Note that the four possibilities are complete, as the basic rules (2.12) are reproduced, and everything else just follows.

3.4.3 Using the translation rules on the field equations (3.6)

Let us use the translation rules in order to rewrite the field equations (3.6) of the Einstein frame canonical parametrization $\mathfrak{E}$ in the generic parametrization. Using (3.18) for substitutions in (3.6a) (recall Eqs. (3.15) and (2.20) as well) yields

$$\hat{E}_{\mu\nu}^{(\hat{g}^{(\mathfrak{E})})}=\frac{1}{\mathcal{A}}E_{\mu\nu}^{(g)}\,. \tag{3.27}$$

Note that the Eq. (3.27) is invariant, and this is exactly the equation we obtain when varying the action (3.17) with respect to the invariant metric $\hat{g}_{\mu\nu}^{(\mathfrak{E})}$ (from the invariant pair (3.9)). Imposing the transformation properties (2.14), as discussed in the previous section, constitutes to multiplying by $\mathcal{A}$ (in the Einstein frame canonical parametrization $\mathfrak{E}$), thus, leading exactly to the generic field equation (2.4a).

Analogously, using (3.18) on (3.6b), leads us to

$$\hat{E}^{(\mathcal{I}_{\Phi}^{(\mathfrak{E})},\hat{R}^{(\mathfrak{E})})}=\frac{1}{\mathcal{A}^{2}\left(\mathcal{I}_{\Phi}^{(\mathfrak{E})}\right)'}E^{(\Phi)}\,, \tag{3.28}$$

i.e., to Eq. (2.7) instead of (2.4b). The reason is of course clear. In the Einstein frame canonical parametrization $\mathfrak{E}$ the Eqs. (2.4b) and (2.7) coincide, as indicated in Eqs. (3.6b) and (3.6b′). As before (3.28) is the one we obtain when varying the action (3.17) with respect to the scalar field $\mathcal{I}_{\Phi}^{(\mathfrak{E})}$ from the invariant pair (3.9). The equation is invariant, and imposing the transformation rule (2.16) constitutes multiplying by $\mathcal{A}^2\left(\mathcal{I}_{\Phi}^{(\mathfrak{E})}\right)'$.

We can easily reconstruct also the equation (2.4b) by imposing the suitable transformation rule (2.15), and hence just inverting (2.7). Note here the caveat. Let us consider an expression in a particular parametrization, e.g., the Einstein frame canonical parametrization $\mathfrak{E}$. In order to obtain the expression in the generic parametrization, we must know its transformation properties, we must know its origin, how it was derived. This is also the reason why in my opinion we cannot discriminate between the noninvariant Einstein frame action (3.5) and the invariant Einstein frame action (3.17). The action is postulated and thus it is not derived within the theory.

In the Sections IV.B and IV.C, as well as VI of the Ref. I (starting from the page 81 of the current thesis) in principle we have used the translation rules backwards and, based on later results. In these sections, first, we have done the calculations in the Einstein frame canonical parametrization $\mathfrak{E}$, and, second, these calculation independently do not prove the invariance (covariance) of the obtained quantities. However, the covariance of these results was expected from comparison with the Jordan frame results [23, 24], and later proven explicitly in Ref. V. In a sense, we just stumbled upon nearly invariant expressions, as many interesting results have rather simple transformation properties. To conclude, when one intends to study the transformation properties, the use of a particular parametrizations is not the best way.

Chapter 4

Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$

Contents

4.1 Definition and notation

Completely analogously to the Einstein frame case, let me proceed with the Jordan frame.

Definition 4.1.1: Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$

Let us specify the arbitrary functions in the generic action functional (2.2) to be

$$\mathcal{A}(\Phi)|_{\mathfrak{J}} \equiv \mathcal{A}_{\mathfrak{J}}(\Phi_{\mathfrak{J}}) \stackrel{!}{=} \Phi_{\mathfrak{J}}\,, \qquad \mathcal{B}(\Phi)|_{\mathfrak{J}} \equiv \mathcal{B}_{\mathfrak{J}}(\Phi_{\mathfrak{J}}) = \frac{\omega(\Phi_{\mathfrak{J}})}{\Phi_{\mathfrak{J}}}\,, \tag{4.1a}$$

$$\mathcal{V}(\Phi)|_{\mathfrak{J}} \equiv \mathcal{V}_{\mathfrak{J}}(\Phi_{\mathfrak{J}})\,, \qquad \alpha|_{\mathfrak{J}} \equiv \alpha_{\mathfrak{J}}(\Phi_{\mathfrak{J}}) \stackrel{!}{=} 0\,. \tag{4.1b}$$

Such a setup is referred to as the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization (JF BDBW), denoted by $\mathfrak{J}$.

See also the subsection II.B in Ref. I (page 77 in the thesis), subsection 3.1 in Ref. IV (page 129 in the thesis), etc., for references and further information.

I shall denote the metric tensor, and the scalar field in the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ as

$$g_{\mu\nu}|_{\mathfrak{J}} = g^{\mathfrak{J}}_{\mu\nu}\,, \qquad g^{\mu\nu}|_{\mathfrak{J}} = g^{\mu\nu}_{\mathfrak{J}}\,, \qquad \Phi|_{\mathfrak{J}} = \Phi_{\mathfrak{J}} \equiv \Psi\,. \tag{4.2}$$

The Christoffel symbols (1.4) corresponding to the metric in (4.2) are calculated as

$$\Gamma_{\mathfrak{J}}{}^{\lambda}{}_{\mu\nu} \equiv \Gamma^{\lambda}{}_{\mu\nu}\left[g^{\mathfrak{J}}_{\sigma\rho}\right] = \frac{1}{2} g^{\lambda\omega}_{\mathfrak{J}} \left(\partial_\mu g^{\mathfrak{J}}_{\omega\nu} + \partial_\nu g^{\mathfrak{J}}_{\omega\mu} - \partial_\omega g^{\mathfrak{J}}_{\mu\nu}\right)\,, \tag{4.3}$$

which allows to define the covariant derivative $\nabla^{\mathfrak{J}}$ as the one, where the particular Christoffel symbols (4.3) are used, and via (1.16), (1.17) lead us to the corresponding Riemann tensor, Ricci tensor and Ricci scalar

$$\begin{aligned} R^{\mathfrak{J}\sigma}{}_{\rho\mu\nu} \equiv R^{\sigma}{}_{\rho\mu\nu}\left[g^{\mathfrak{J}}_{\mu\nu}\right] &= \partial_\mu \Gamma_{\mathfrak{J}}{}^{\sigma}{}_{\nu\rho} - \partial_\nu \Gamma_{\mathfrak{J}}{}^{\sigma}{}_{\mu\rho} \\ &\quad + \Gamma_{\mathfrak{J}}{}^{\lambda}{}_{\nu\rho}\Gamma_{\mathfrak{J}}{}^{\sigma}{}_{\mu\lambda} - \Gamma_{\mathfrak{J}}{}^{\lambda}{}_{\mu\rho}\Gamma_{\mathfrak{J}}{}^{\sigma}{}_{\nu\lambda}\,, \end{aligned} \tag{4.4a}$$

$$R_{\mathfrak{J}\,\rho\nu} \equiv R_{\rho\nu}\left[g^{\mathfrak{J}}_{\mu\nu}\right] = \delta^{\mu}_{\sigma} R^{\mathfrak{J}\sigma}{}_{\rho\mu\nu}\,, \qquad R_{\mathfrak{J}} \equiv R\left[g^{\mathfrak{J}}_{\mu\nu}\right] = g^{\nu\rho}_{\mathfrak{J}} R_{\mathfrak{J}\,\rho\nu}\left[g^{\mathfrak{J}}_{\mu\nu}\right]\,. \tag{4.4b}$$

4.2 Action functional and field equations

The action functional (2.2) in the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ reads

$$S_{\mathfrak{J}} \equiv S_{\mathfrak{J}}\left[g^{\mathfrak{J}}_{\mu\nu},\, \Phi_{\mathfrak{J}},\, \chi\right] \tag{4.5a}$$

$$
= \frac{1}{2\kappa^2}\int_{M_4} \mathrm{d}^4x\sqrt{-g^{\mathfrak{J}}}\Big\{\Phi_{\mathfrak{J}}R_{\mathfrak{J}} - \frac{\omega(\Phi_{\mathfrak{J}})}{\Phi_{\mathfrak{J}}}g_{\mathfrak{J}}^{\mu\nu}\nabla_\mu^{\mathfrak{J}}\Phi_{\mathfrak{J}}\nabla_\nu^{\mathfrak{J}}\Phi_{\mathfrak{J}} - 2\ell^{-2}\mathcal{V}_{\mathfrak{J}}(\Phi_{\mathfrak{J}})\Big\}
+ S_{\mathrm{m}}\left[g^{\mathfrak{J}}_{\mu\nu},\chi\right] , \tag{4.5b}
$$

The field equations (2.4a), (2.4b) and (2.7) in the parametrization $\mathfrak{J}$ reduce to

$$
\begin{aligned}
E^{(g^{\mathfrak{J}})}_{\mu\nu} \equiv{}& \Phi_{\mathfrak{J}}\left(R_{\mathfrak{J}\,\mu\nu} - \frac{1}{2}g^{\mathfrak{J}}_{\mu\nu}R_{\mathfrak{J}}\right) + \frac{1}{2}\frac{\omega}{\Phi_{\mathfrak{J}}}g^{\mathfrak{J}}_{\mu\nu}g_{\mathfrak{J}}^{\sigma\rho}\nabla^{\mathfrak{J}}_\sigma\Phi_{\mathfrak{J}}\nabla^{\mathfrak{J}}_\rho\Phi_{\mathfrak{J}}\\
&- \frac{\omega}{\Phi_{\mathfrak{J}}}\nabla^{\mathfrak{J}}_\mu\Phi_{\mathfrak{J}}\nabla^{\mathfrak{J}}_\nu\Phi_{\mathfrak{J}} + g^{\mathfrak{J}}_{\mu\nu}\Box^{\mathfrak{J}}\Phi_{\mathfrak{J}} - \nabla^{\mathfrak{J}}_\mu\nabla^{\mathfrak{J}}_\nu\Phi_{\mathfrak{J}}\\
&+ \ell^{-2}g^{\mathfrak{J}}_{\mu\nu}\mathcal{V}_{\mathfrak{J}} - \kappa^2T^{\mathfrak{J}}_{\mu\nu} = 0\,,
\end{aligned} \tag{4.6a}
$$

$$
\begin{aligned}
E^{(\Phi_{\mathfrak{J}},R_{\mathfrak{J}})} \equiv{}& R_{\mathfrak{J}} + \frac{\omega'}{\Phi_{\mathfrak{J}}}g_{\mathfrak{J}}^{\mu\nu}\nabla^{\mathfrak{J}}_\mu\Phi_{\mathfrak{J}}\nabla^{\mathfrak{J}}_\nu\Phi_{\mathfrak{J}} - \frac{\omega}{\Phi_{\mathfrak{J}}^2}g_{\mathfrak{J}}^{\mu\nu}\nabla^{\mathfrak{J}}_\mu\Phi_{\mathfrak{J}}\nabla^{\mathfrak{J}}_\nu\Phi_{\mathfrak{J}}\\
&+ \frac{2\omega}{\Phi_{\mathfrak{J}}}\Box^{\mathfrak{J}}\Phi_{\mathfrak{J}} - 2\ell^{-2}\mathcal{V}'_{\mathfrak{J}} = 0\,,
\end{aligned} \tag{4.6b}
$$

$$
\begin{aligned}
E^{(\Phi_{\mathfrak{J}})} ={}& \frac{2\omega+3}{\Phi_{\mathfrak{J}}}\Box^{\mathfrak{J}}\Phi_{\mathfrak{J}} + \frac{\omega'}{\Phi_{\mathfrak{J}}}g_{\mathfrak{J}}^{\mu\nu}\nabla^{\mathfrak{J}}_\mu\Phi_{\mathfrak{J}}\nabla^{\mathfrak{J}}_\nu\Phi_{\mathfrak{J}}\\
&- 2\ell^{-2}\frac{\Phi_{\mathfrak{J}}\mathcal{V}'_{\mathfrak{J}} - 2\mathcal{V}_{\mathfrak{J}}}{\Phi_{\mathfrak{J}}} - \frac{\kappa^2}{\Phi_{\mathfrak{J}}}T^{\mathfrak{J}} = 0\,,
\end{aligned} \tag{4.6c}
$$

where

$$
T^{\mathfrak{J}}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g^{\mathfrak{J}}}}\frac{\delta S_{\mathrm{m}}\left[g^{\mathfrak{J}}_{\sigma\rho},\chi\right]}{\delta g_{\mathfrak{J}}^{\mu\nu}}\,, \qquad T^{\mathfrak{J}} \equiv g_{\mathfrak{J}}^{\nu\mu}T^{\mathfrak{J}}_{\mu\nu}\,, \tag{4.7}
$$

and prime as, e.g., in $\mathcal{V}'_{\mathfrak{J}}$ means derivative with respect to the Jordan frame scalar field $\Phi_{\mathfrak{J}}$, i.e.,

$$
\mathcal{V}'_{\mathfrak{J}} \equiv \frac{\mathrm{d}\mathcal{V}_{\mathfrak{J}}(\Phi_{\mathfrak{J}})}{\mathrm{d}\Phi_{\mathfrak{J}}}\,, \qquad \omega' \equiv \frac{\mathrm{d}\omega(\Phi_{\mathfrak{J}})}{\mathrm{d}\Phi_{\mathfrak{J}}}\,. \tag{4.8}
$$

4.3 Invariant Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization

4.3.1 The invariant pair

The invariant pair from Theorem 4.2 in Ref. II in the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$, i.e., example (4.11) in Ref. II, is

$$
\hat{g}^{(\mathfrak{J})}_{\mu\nu} \equiv \mathrm{e}^{2\alpha(\Phi)}g_{\mu\nu}\,, \qquad \mathcal{I}^{(\mathfrak{J})}_{\Phi} \equiv \mathcal{I}_1^{-1} = \mathrm{e}^{-2\alpha(\Phi)}\mathcal{A}(\Phi)\,. \tag{4.9}
$$

Plugging the Definition 4.1.1 into Eq. (4.9) verifies

$$
\hat{g}^{(\mathfrak{J})}_{\mu\nu}\Big|_{\mathfrak{J}} = g^{\mathfrak{J}}_{\mu\nu}\,, \qquad \mathcal{I}^{(\mathfrak{J})}_{\Phi}\Big|_{\mathfrak{J}} = \Phi_{\mathfrak{J}}\,. \tag{4.10}
$$

The metric $\hat{g}^{(\mathfrak{J})}_{\mu\nu}$ is also know as the invariant Jordan frame metric.

4.3.2 Four functions as invariants

Let us consider the invariant pair (4.9) to be a scalar field reparametrization and a conformal transformation as in (2.13), i.e., (2.8) backwards. The transformations are given as

$$\mathrm{e}^{2\gamma(\Phi)} = \mathrm{e}^{2\alpha(\Phi)} \,, \qquad \bar{\Phi} = \mathcal{I}_1^{-1}(\Phi) \,, \tag{4.11}$$

The four functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ and $\alpha(\Phi)$ transform into the four invariants

$$\mathcal{I}_{\mathcal{A}}^{(\mathfrak{J})} = \mathrm{e}^{-2\alpha(\Phi)}\mathcal{A}(\Phi) = \mathcal{I}_1^{-1} \,, \qquad \mathcal{I}_{\mathcal{B}}^{(\mathfrak{J})} = \frac{\mathcal{I}_1}{2}\left(\mathcal{I}_5^{-1} - 3\right) \,, \tag{4.12a}$$

$$\mathcal{I}_{\mathcal{V}}^{(\mathfrak{J})} = \mathcal{I}_4 \equiv \mathrm{e}^{-4\alpha(\Phi)}\mathcal{V}(\Phi) \,, \qquad \mathcal{I}_{\alpha}^{(\mathfrak{J})} = 0 \,. \tag{4.12b}$$

Here we introduced the invariants

$$\mathcal{I}_4(\Phi) \equiv \frac{\mathcal{I}_2}{\mathcal{I}_1^2} = \mathrm{e}^{-4\alpha(\Phi)}\mathcal{V}(\Phi) \,, \tag{4.13a}$$

$$\mathcal{I}_5(\Phi) \equiv \left(\frac{1}{2}\frac{\mathrm{d}\ln\mathcal{I}_1}{\mathrm{d}\mathcal{I}_3}\right)^2 = \frac{(2\mathcal{A}\alpha' - \mathcal{A}')^2}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2} \,, \tag{4.13b}$$

which make use of the rules (2.23a) and (2.23b). See also definitions (19) and (25) in Ref. I as well as Table I in there, page 79 in the current thesis.

Analogously to the case (4.10) of the invariant pair (recall also (3.13)), as $\alpha_{\mathfrak{J}} \equiv 0$, it is natural, that

$$\mathcal{I}_{\mathcal{A}}^{(\mathfrak{J})}\Big|_{\mathfrak{J}} = \mathcal{A}_{\mathfrak{J}} = \Phi_{\mathfrak{J}} \,, \qquad \mathcal{I}_{\mathcal{B}}^{(\mathfrak{J})}\Big|_{\mathfrak{E}} = \mathcal{B}_{\mathfrak{J}} = \frac{\omega(\Phi_{\mathfrak{J}})}{\Phi_{\mathfrak{J}}} \,, \tag{4.14a}$$

$$\mathcal{I}_{\mathcal{V}}^{(\mathfrak{J})}\Big|_{\mathfrak{J}} = \mathcal{V}_{\mathfrak{J}} \,, \qquad \mathcal{I}_{\alpha}^{(\mathfrak{J})}\Big|_{\mathfrak{E}} = \alpha_{\mathfrak{J}} = 0 \,. \tag{4.14b}$$

The invariants $\mathcal{I}_{\mathcal{A}}^{(\mathfrak{J})}$, $\mathcal{I}_{\mathcal{B}}^{(\mathfrak{J})}$, $\mathcal{I}_{\mathcal{V}}^{(\mathfrak{J})}$ and $\mathcal{I}_{\alpha}^{(\mathfrak{J})}$ are functions of $\mathcal{I}_{\Phi}^{(\mathfrak{J})} = \mathcal{I}_1^{-1}$. However, as in the case of (3.14) we do not need the explicit dependence, since the derivatives may be calculated as

$$\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{J})}}{\mathrm{d}\frac{1}{\mathcal{I}_1}} = -\mathcal{I}_1^2\frac{\left(\mathcal{I}^{(\mathfrak{J})}\right)'}{\left(\mathcal{I}_1\right)'} \,. \tag{4.15}$$

See also the invariant differential operator $\mathcal{D}_1$ in the Table 1 of Ref. I, page 79 in the current thesis, as well as the nearby Eq. (31).

4.3.3 Invariant geometry of the Jordan frame

The metric $\hat{g}_{\mu\nu}^{(\mathfrak{J})}$ is an invariant, and hence, the corresponding Ricci tensor and scalar are as well. According to Eqs. (2.11)

$$\begin{aligned}\hat{R}_{\mu\nu}^{(\mathfrak{J})}\left[\hat{g}_{\sigma\rho}^{(\mathfrak{J})}\right] = R_{\mu\nu}\left[g_{\sigma\rho}\right] - \left(2\left(\alpha'\right)^2 + \alpha''\right)g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\Phi\partial_\rho\Phi - \alpha' g_{\mu\nu}g^{\sigma\rho}\nabla_\sigma\partial_\rho\Phi \\ - \alpha' 2\nabla_\mu\partial_\nu\Phi + 2\left(\left(\alpha'\right)^2 - \alpha''\right)\partial_\mu\Phi\partial_\nu\Phi \,,\end{aligned} \tag{4.16a}$$

$$\hat{R}^{(\mathfrak{J})}\left[\hat{g}^{(\mathfrak{J})}_{\sigma\rho}\right] = \mathrm{e}^{-2\alpha}\Big\{R\left[g_{\sigma\rho}\right] - 6\left(\alpha'\right)^2 g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 6\alpha'' g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 6\alpha' g^{\mu\nu}\nabla_\mu\partial_\nu\Phi\Big\}\,. \tag{4.16b}$$

The expressions on the right hand sides are in the generic parametrization, and

$$\hat{R}^{(\mathfrak{J})}_{\mu\nu}\left[g^{(\mathfrak{J})}_{\mu\nu}\right]\Big|_{\mathfrak{J}} = R_{\mathfrak{J}\,\mu\nu}\left[g^{\mathfrak{J}}_{\mu\nu}\right]\,, \qquad \hat{R}^{(\mathfrak{J})}\left[g^{(\mathfrak{J})}_{\mu\nu}\right]\Big|_{\mathfrak{J}} = R_{\mathfrak{J}}\left[g^{\mathfrak{J}}_{\mu\nu}\right]\,. \tag{4.17}$$

See also Eq. (4.13) in Ref. II on page 102 in the current thesis.

4.3.4 Invariant Jordan frame action

By considering the invariant pair (4.9) as a particular scalar field redefinition and conformal transformation (2.8), and taking into account the results (4.12), we rewrite the generic action (2.2) as (see also Eq. (5.1) in Ref. II, page 103 of the thesis)

$$\begin{aligned} S &= S\left[\hat{g}^{(\mathfrak{J})}_{\mu\nu}, \mathcal{I}^{(\mathfrak{J})}_{\Phi}, \chi\right] &(4.18a)\\ &= \frac{1}{2\kappa^2}\int_{M_4}\mathrm{d}^4x\sqrt{-\hat{g}^{(\mathfrak{J})}}\Big\{\mathcal{I}^{(\mathfrak{J})}_{\Phi}\hat{R}^{(\mathfrak{J})}\left[\hat{g}^{(\mathfrak{J})}_{\mu\nu}\right] - \mathcal{I}^{(\mathfrak{J})}_{\mathcal{B}}\hat{g}^{\mu\nu}_{(\mathfrak{J})}\hat{\nabla}^{(\mathfrak{J})}_{\mu}\mathcal{I}^{(\mathfrak{J})}_{\Phi}\hat{\nabla}^{(\mathfrak{J})}_{\nu}\mathcal{I}^{(\mathfrak{J})}_{\Phi}\\ &\qquad - 2\ell^{-2}\mathcal{I}^{(\mathfrak{J})}_{\mathcal{V}}\Big\} + S_{\mathrm{m}}\left[\hat{g}^{(\mathfrak{J})}_{\mu\nu}, \chi\right]\,. &(4.18b) \end{aligned}$$

As in the case of the Einstein frame canonical parametrization $\mathfrak{E}$ invariant action (3.17), also the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ invariant action (4.18) is just the action (2.2) in terms of different variables. On the other hand, however, comparing the Jordan frame invariant action (4.18) with the Jordan frame (noninvariant) action (4.5) reveals, that these two differ only by the meaning we assign to the quantities contained therein. Specifying the four functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ and $\alpha(\Phi)$ as in the Definition 4.1.1 is, thus, equivalent to rewriting the action via the invariant pair (4.9), and hence, the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ is equivalent to the generic parametrization.

4.4 Translation rules from the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ to the generic parametrization

Identically to the Einstein frame canonical parametrization $\mathfrak{E}$ (see Section 3.4), the translation rules for the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ are a set of essentially algebraic substitutions which allow us to rewrite an arbitrary expression from the Jordan frame BDBW parametrization as an expression in the generic parametrization. Therefore, these are just the transformation rules from the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ to the generic parametrization and from there to any other parametrization.

4.4.1 Rules for the invariant quantities

Under the assumption that the quantity under consideration is an invariant, the rules have been presented implicitly already in Ref. I. The rules in the Jordan frame but for multiple scalar fields were introduced by Eq. (25) in Ref. III. More precisely one must revert the mappings in Eq. (25) on page 120 of the current thesis.

Therefore, recalling also the translation rules (3.18) for the Einstein frame canonical parametrization $\mathfrak{E}$, the translation rules for the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ are the algebraic substitutions

$$g^{\mathfrak{J}}_{\mu\nu} \mapsto \hat{g}^{(\mathfrak{J})}_{\mu\nu} \overset{(4.9)}{\equiv} \mathrm{e}^{2\alpha(\Phi)} g_{\mu\nu}\,, \qquad \Phi_{\mathfrak{J}} \mapsto \mathcal{I}^{(\mathfrak{J})}_{\Phi} \overset{(4.9)}{\equiv} \mathcal{I}_1^{-1}\,, \tag{4.19a}$$

$$\sqrt{-g^{\mathfrak{J}}} \mapsto \sqrt{-g^{(\mathfrak{J})}} = \mathrm{e}^{4\alpha(\Phi)}\sqrt{-g}\,, \quad \mathcal{V}_{\mathfrak{J}}(\Phi_{\mathfrak{E}}) \mapsto \mathcal{I}^{(\mathfrak{J})}_{\mathcal{V}}(\mathcal{I}^{(\mathfrak{J})}_{\Phi}) = \mathcal{I}_4(\Phi)\,, \tag{4.19b}$$

$$R^{\mathfrak{J}}_{\mu\nu}\left[g^{\mathfrak{J}}_{\mu\nu}\right] \mapsto \hat{R}^{(\mathfrak{J})}_{\mu\nu}\left[\hat{g}^{(\mathfrak{J})}_{\mu\nu}\right]\,, \qquad \omega(\Phi_{\mathfrak{J}}) \mapsto \mathcal{I}^{(\mathfrak{J})}_{\Phi}\mathcal{I}^{(\mathfrak{J})}_{\mathcal{B}} = \frac{1}{2}\left(\mathcal{I}_5^{-1} - 3\right)\,, \tag{4.19c}$$

$$\nabla^{\mathfrak{J}}_{\mu} \mapsto \hat{\nabla}^{(\mathfrak{J})}_{\mu}\,, \qquad \frac{\mathrm{d}}{\mathrm{d}\Phi_{\mathfrak{J}}} \mapsto \frac{\mathrm{d}}{\mathrm{d}\mathcal{I}^{(\mathfrak{J})}_{\Phi}} = -\mathcal{I}_1^2\frac{(\ )'}{\mathcal{I}_1'} \tag{4.19d}$$

Note that the right hand sides are essentially in the generic parametrization. Such mappings, thus, take us from the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ to the generic parametrization. On the other hand, fixing the parametrization on the right hand side to be the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ forces the mapping to reduce to identity.

4.4.2 Rules for noninvariant quantities

In the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization we rely on the fact that $\mathrm{e}^{2\alpha_{\mathfrak{J}}} = 1$. Hence, whenever we want to impose that something transforms as $\mathrm{e}^{2\alpha_{\mathfrak{J}}}$ (i.e, as $\mathcal{A}$), we just use the translation rules (4.19) to obtain the invariant expression, and the multiply the latter by $\mathrm{e}^{2\alpha}$. Analogously

$$\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{J})}_{\Phi}}{\mathrm{d}\Phi} = \left(\bar{f}'\right)^{-1}\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{J})}_{\Phi}}{\mathrm{d}\bar{\Phi}}\,, \quad \text{but} \quad \left.\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{J})}_{\Phi}}{\mathrm{d}\Phi}\right|_{\mathfrak{J}} = \frac{\mathrm{d}\Phi_{\mathfrak{J}}}{\mathrm{d}\Phi_{\mathfrak{J}}} = 1\,. \tag{4.20}$$

Let us proceed with examples.

1. Suppose we want a quantity that in the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ has the same functional form as $\mathcal{A}_{\mathfrak{J}}$, and transforms as $\mathcal{A}$. The invariant corresponding to $\mathcal{A}_{\mathfrak{J}}$ is $\mathcal{I}_1^{-1}$. Multiplying the latter by $\mathrm{e}^{2\alpha}$ yields $\mathcal{A}$.

2. Let us consider the invariant which represents the function $\alpha_{\mathfrak{J}}$, i.e., $\mathcal{I}_\alpha^{(\mathfrak{J})} = 0$. Hence just adding α, as a noninvariant function imposing the transformation properties, to zero, which is the invariant representing $\alpha_{\mathfrak{J}}$, yields α.

3. Analogously
$$\mathrm{e}^{4\alpha}\mathcal{I}_{\mathcal{V}}^{(\mathfrak{J})} = \mathcal{V}\,, \tag{4.21}$$
which reproduces the prescription (2.12c).

4. As the last case, let us consider the invariant in (4.12) which represents $\mathcal{B}_{\mathfrak{J}}$. In order to impose the transformation properties (2.12b), let us take into account, that $\alpha_{\mathfrak{J}} = 0$, and thus $\alpha_{\mathfrak{J}}' = 0$. On the other hand
$$\alpha' = \left(\bar{f}'\right)^{-1}\left(\bar{\alpha}' - \bar{\gamma}'\right)\,, \tag{4.22a}$$
$$\alpha'\mathcal{A}' - \mathcal{A}\left(\alpha'\right)^2 = \left(\bar{f}'\right)^{-2}\mathrm{e}^{-2\bar{\gamma}}\Big[\alpha'\mathcal{A}' - \mathcal{A}\left(\alpha'\right)^2 + \bar{\mathcal{A}}\left(\bar{\gamma}'\right)^2 - \bar{\gamma}'\bar{\mathcal{A}}'\Big]\,. \tag{4.22b}$$
A straightforward calculation shows the following
$$\begin{aligned}\frac{\mathcal{I}_1}{2}\left(\mathcal{I}_5^{-1} - 3\right) + 6\mathrm{e}^{-2\alpha}\left(\mathcal{A}\left(\alpha'\right)^2 - \alpha'\mathcal{A}'\right)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\mathcal{I}_\Phi^{(\mathfrak{J})}}\right)^2 &= \\ = \frac{\mathcal{I}_1}{2}\left(\mathcal{I}_5^{-1} - 3\right) + 6\mathrm{e}^{-2\bar{\alpha}}\left(\bar{\mathcal{A}}\left(\bar{\alpha}'\right)^2 - \bar{\alpha}'\bar{\mathcal{A}}'\right)\left(\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}\mathcal{I}_\Phi^{(\mathfrak{J})}}\right)^2 & \\ + \mathrm{e}^{-2\alpha}\left(\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}\mathcal{I}_\Phi^{(\mathfrak{J})}}\right)^2\left(-6\left(\bar{\gamma}'\right)^2 + 6\bar{\gamma}'\bar{\mathcal{A}}'\right) &\end{aligned} \tag{4.23}$$
while still
$$\left.\frac{\mathcal{I}_1}{2}\left(\mathcal{I}_5^{-1} - 3\right) + 6\mathrm{e}^{-2\alpha}\left(\mathcal{A}\left(\alpha'\right)^2 - \alpha'\mathcal{A}'\right)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\mathcal{I}_\Phi^{(\mathfrak{J})}}\right)^2\right|_{\mathfrak{J}} = \mathcal{B}_{\mathfrak{J}} = \frac{\omega(\Phi_{\mathfrak{J}})}{\Phi_{\mathfrak{J}}}\,. \tag{4.24}$$

Let us complete the transformation properties to reach

$$\begin{aligned} & \mathrm{e}^{-2\alpha}\left(\frac{\mathrm{d}\mathcal{I}_\Phi^{(\mathfrak{J})}}{\mathrm{d}\Phi}\right)^2\left[\frac{\mathcal{I}_1}{2}\left(\mathcal{I}_5^{-1}-3\right)\right. \\ & \qquad\qquad\qquad \left. +\, 6\mathrm{e}^{2\alpha}\left(\mathcal{A}\left(\alpha'\right)^2-\alpha'\mathcal{A}'\right)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\mathcal{I}_\Phi^{(\mathfrak{J})}}\right)^2\right]= \\ & = \mathcal{B}\,. \end{aligned} \tag{4.25}$$

Once more, as in the case of the Einstein frame canonical parametrization, here I have just inverted (4.12) under (4.11).

Note that the four possibilities are complete, as the basic rules (2.12) are reproduced, and everything else just follows.

4.4.3 Using the translation rules on the field equations (4.6)

Let us use the translation rules in order to rewrite the field equations (4.6) of the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ in the generic parametrization. Using (4.19) for substitutions in (4.6a) (recall Eqs. (4.16) and (2.20) as well) yields

$$\hat{E}_{\mu\nu}^{(\hat{g}^{(\mathfrak{J})})} = \mathrm{e}^{-2\alpha} E_{\mu\nu}^{(g)}\,. \tag{4.26}$$

Note that the Eq. (4.26) is invariant, and this is exactly the equation we obtain when varying the action (4.18) with respect to the invariant metric $\hat{g}_{\mu\nu}^{(\mathfrak{J})}$ (from the invariant pair (4.9)). Imposing the transformation properties (2.14), as discussed in the previous section, constitutes to multiplying by $\mathrm{e}^{2\alpha}$ (in the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$), thus, leading exactly to the generic field equation (2.4a).

Analogously, using (4.19) on (4.6b), leads us to

$$\hat{E}^{(\mathcal{I}_\Phi^{(\mathfrak{J})},\hat{R}^{(\mathfrak{J})})} = \frac{1}{\mathcal{A}'-2\mathcal{A}\alpha'}\mathrm{e}^{-2\alpha}\left(E^{(\Phi,R)}+2\alpha' g^{\nu\mu}E_{\mu\nu}^{(g)}\right)\,, \tag{4.27}$$

where one must also use (2.6). The equation (4.27) is invariant, and this is exactly the equation we obtain when varying the (invariant) action (4.18) with respect to the scalar field $\mathcal{I}_\Phi^{(\mathfrak{J})}$ from the invariant pair (4.9). Imposing the transformation rule (2.15) constitutes, first, multiplying by

$$\mathrm{e}^{4\alpha}\left(\frac{1}{\mathcal{I}_1}\right)' = \mathrm{e}^{2\alpha}\left(\mathcal{A}'-2\mathcal{A}\alpha'\right)\,, \tag{4.28}$$

and, second, subtracting $-2\alpha' g^{\nu\mu}E_{\mu\nu}^{(g)}$ which eventually yields (2.4b).

The substitutions (4.19) applied to (4.6c) yields

$$E^{(\mathcal{I}_\Phi^{\mathfrak{J}})} = \frac{1}{\mathcal{A}' - 2\mathcal{A}\alpha'} \mathrm{e}^{-2\alpha} E^{(\Phi)} \,. \tag{4.29}$$

Analogously to the previous, imposing the transformation rule (2.16) constitutes to multiplying by $\mathrm{e}^{2\alpha}\,(\mathcal{A}' - 2\mathcal{A}\alpha')$.

Chapter 5

Usefulness of the invariants

Contents

5.1 Relations between fixed parametrizations

Based on the Table I in Ref. IV (page 131 in the current thesis) the Einstein frame canonical parametrization $\mathfrak{E}$, Definition 3.1.1, and the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$, Definition 4.1.1, are related as

$$g^{\mathfrak{E}}_{\mu\nu} = \Phi_{\mathfrak{J}} g^{\mathfrak{J}}_{\mu\nu}\,, \qquad g^{\mathfrak{J}}_{\mu\nu} = \mathrm{e}^{2\alpha_{\mathfrak{E}}} g^{\mathfrak{E}}_{\mu\nu}\,, \tag{5.1a}$$

$$\left(\frac{\mathrm{d}\Phi_{\mathfrak{E}}}{\mathrm{d}\Phi_{\mathfrak{J}}}\right)^2 = \frac{2\omega(\Phi_{\mathfrak{J}})+3}{4\Phi_{\mathfrak{J}}^2}\,, \qquad \left(\frac{\mathrm{d}\Phi_{\mathfrak{J}}}{\mathrm{d}\Phi_{\mathfrak{E}}}\right)^2 = 4\mathrm{e}^{-4\alpha_{\mathfrak{E}}}\left(\alpha'_{\mathfrak{E}}\right)^2\,, \tag{5.1b}$$

$$\mathrm{e}^{2\alpha_{\mathfrak{E}}} = \Phi_{\mathfrak{J}}^{-1}\,, \qquad \frac{1}{2\omega(\Phi_{\mathfrak{J}})+3} = \left(\alpha'_{\mathfrak{E}}\right)^2\,. \tag{5.1c}$$

Let us now consider the relations between the invariant formulations of these parametrizations, i.e., the relation between the invariant pairs (3.9) and (4.9), and the invariant $\mathcal{I}_5$, defined by (4.13b)

$$\hat{g}^{(\mathfrak{E})}_{\mu\nu} = \mathcal{I}^{(\mathfrak{J})}_{\Phi}\hat{g}^{(\mathfrak{J})}_{\mu\nu}\,, \qquad \hat{g}^{(\mathfrak{J})}_{\mu\nu} = \mathrm{e}^{2\mathcal{I}^{(\mathfrak{E})}_{\alpha}}\hat{g}^{(\mathfrak{E})}_{\mu\nu}\,, \tag{5.2a}$$

$$\left(\frac{\mathrm{d}\mathcal{I}^{(\mathfrak{E})}_{\Phi}}{\mathrm{d}\mathcal{I}^{(\mathfrak{J})}_{\Phi}}\right)^2 = \mathcal{I}_1^2\left(4\mathcal{I}_5\right)^{-1}\,, \qquad \mathcal{I}_5(\Phi) = \frac{\left(2\mathcal{A}\alpha' - \mathcal{A}'\right)^2}{2\mathcal{A}\mathcal{B}+3\left(\mathcal{A}'\right)^2}\,. \tag{5.2b}$$

Let me point out that fixing the parametrization leads to

$$\mathcal{I}^{(\mathfrak{J})}_{\Phi}\Big|_{\mathfrak{E}} = \mathrm{e}^{-2\mathcal{I}^{(\mathfrak{E})}_{\alpha}}\Big|_{\mathfrak{E}} = \mathrm{e}^{-2\alpha_{\mathfrak{E}}}\,, \qquad \mathcal{I}^{(\mathfrak{J})}_{\Phi}\Big|_{\mathfrak{J}} = \mathrm{e}^{-2\mathcal{I}^{(\mathfrak{E})}_{\alpha}}\Big|_{\mathfrak{J}} = \Phi_{\mathfrak{J}}\,, \tag{5.3}$$

and thus both $\Phi_{\mathfrak{J}}$ and $\mathrm{e}^{-2\alpha_{\mathfrak{E}}}$ in (5.1a) represent the same invariant $\mathcal{I}_1^{-1}$. In addition

$$\mathcal{I}_5\big|_{\mathfrak{E}} = \left(\alpha'_{\mathfrak{E}}\right)^2\,, \qquad \mathcal{I}_5\big|_{\mathfrak{J}} = \frac{1}{2\omega(\Phi_{\mathfrak{J}})+3}\,. \tag{5.4}$$

which shows that both expressions in (5.1b) as well as in (5.1c) also just represent the same invariants in different parametrizations.

Hence, it turns out that the transformation relations in the Table I in Ref. IV (on page 131 of the thesis) relate invariants (which was not clear at the time, because the concept of such invariants was not developed), as only for invariants the numerical equivalence holds. As argued in Sections 3.4.2 and 4.4.2, imposing different transformation properties yields to different expressions, and therefore, in order to relate expressions in different parametrizations, we would need to know the transformation properties beforehand. For example, from the perspective of the generic parametrization, the scalar field $\Phi_{\mathfrak{J}}$ in the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$ has two meanings, it is the function $\mathcal{A}$ and the scalar field Φ. Those two have different transformation properties. One can not impose all possible transformation properties (of the same expression in a particular fixed parametrization $\mathfrak{P}$) at once, and thus it is clear that such a table can only be meaningful for somewhat canonical choice of transformation properties.

5.1.1 Example of the ambiguity

Let me go through an example to illustrate the ambiguity encountered, when one wants to know the generic form of an expression, that is written down in a fixed parametrization, without knowing the transformation rule beforehand. The same holds for quantities "translated" from one fixed parametrization to another fixed parametrization. Let us consider the expression

$$2\omega(\Phi_{\mathfrak{J}})+3 \tag{5.5}$$

in the Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization $\mathfrak{J}$, Definition 4.1.1. Let me write down three expressions in the generic parametrization, which all reduce to (5.5) if evaluated in the Jordan frame Brans-Dike-Bergmann-Wagoner parametrization $\mathfrak{J}$.

$$\mathcal{I}_5^{-1} \overset{(4.13b)}{=} \frac{2\mathcal{A}\mathcal{B}+3\left(\mathcal{A}'\right)^2}{\left(2\mathcal{A}\alpha'-\mathcal{A}'\right)^2}\,, \qquad \left.\mathcal{I}_5^{-1}\right|_{\mathfrak{J}} = 2\omega(\Phi_{\mathfrak{J}})+3\,, \tag{5.6a}$$

$$4\mathcal{A}^2\mathcal{F} \overset{(2.21)}{=} 2\mathcal{A}\mathcal{B}+3\left(\mathcal{A}'\right)^2\,, \qquad \left.\left(4\mathcal{A}^2\mathcal{F}\right)\right|_{\mathfrak{J}} = 2\omega(\Phi_{\mathfrak{J}})+3\,, \tag{5.6b}$$

$$2\mathcal{A}\mathcal{B}+3\,, \qquad \left.\left(2\mathcal{A}\mathcal{B}+3\right)\right|_{\mathfrak{J}} = 2\omega(\Phi_{\mathfrak{J}})+3\,, \tag{5.6c}$$

which of course originates from $\mathcal{A}'_{\mathfrak{J}}=1$, $\left(\mathcal{A}'_{\mathfrak{J}}\right)^2=1$, etc. The expressions in (5.6) differ by transformation properties as from (2.12)

$$\mathcal{I}_5^{-1}(\Phi) \overset{(2.8a)}{=} \mathcal{I}_5^{-1}(\bar{f}(\bar{\Phi})) = \bar{\mathcal{I}}_5^{-1}(\bar{\Phi})\,, \tag{5.7a}$$

$$4\mathcal{A}^2\mathcal{F} \overset{(2.21)}{=} \mathrm{e}^{-4\bar{\gamma}}\left(\bar{f}'\right)^{-2}4\bar{\mathcal{A}}^2\bar{\mathcal{F}}\,, \tag{5.7b}$$

$$\begin{aligned}2\mathcal{A}\mathcal{B}+3 \overset{(2.12b)}{=} {} & \mathrm{e}^{-4\bar{\gamma}}\left(\bar{f}'\right)^{-2}2\bar{\mathcal{A}}\bar{\mathcal{B}}+3 \\ & + e^{-4\bar{\gamma}(\bar{\Phi})}\left(\bar{f}'\right)^{-2}\left(12\bar{\gamma}'\bar{\mathcal{A}}'\bar{\mathcal{A}}-12\left(\bar{\gamma}'\right)^2\bar{\mathcal{A}}^2\right)\,.\end{aligned} \tag{5.7c}$$

The list is not complete, as we can form infinitely many of such expressions, all differing by the transformation properties.

In order to demonstrate that this is a generic problem and not related to the particular Jordan frame, let us also consider an expression

$$\left(\alpha'_{\mathfrak{E}}\right)^2 \tag{5.8}$$

in the Einstein frame canonical parametrization $\mathfrak{E}$, Definition 3.1.1. Analogusly to (5.6) the expressions

$$\mathcal{I}_5 \overset{(4.13b)}{=} \frac{\left(2\mathcal{A}\alpha'-\mathcal{A}'\right)^2}{\left(2\mathcal{A}\mathcal{B}+3\left(\mathcal{A}'\right)^2\right)}\,, \qquad \left.\mathcal{I}_5\right|_{\mathfrak{E}} = \left(\alpha'_{\mathfrak{E}}\right)^2\,, \tag{5.9a}$$

$$\frac{1}{4}\left(2\mathcal{A}\alpha'-\mathcal{A}'\right)^2\,, \qquad \left.\frac{1}{4}\left(2\mathcal{A}\alpha'-\mathcal{A}'\right)^2\right|_{\mathfrak{E}} = \left(\alpha'_{\mathfrak{E}}\right)^2\,, \tag{5.9b}$$

$$\left(\alpha'\right)^2\,, \qquad \left.\left(\alpha'\right)^2\right|_{\mathfrak{E}} = \left(\alpha'_{\mathfrak{E}}\right)^2\,. \tag{5.9c}$$

Also, analogously to (5.7),

$$\mathcal{I}_5(\Phi) = \mathcal{I}_5(\bar{f}(\bar{\Phi})) = \bar{\mathcal{I}}_5(\bar{\Phi})\,, \tag{5.10a}$$

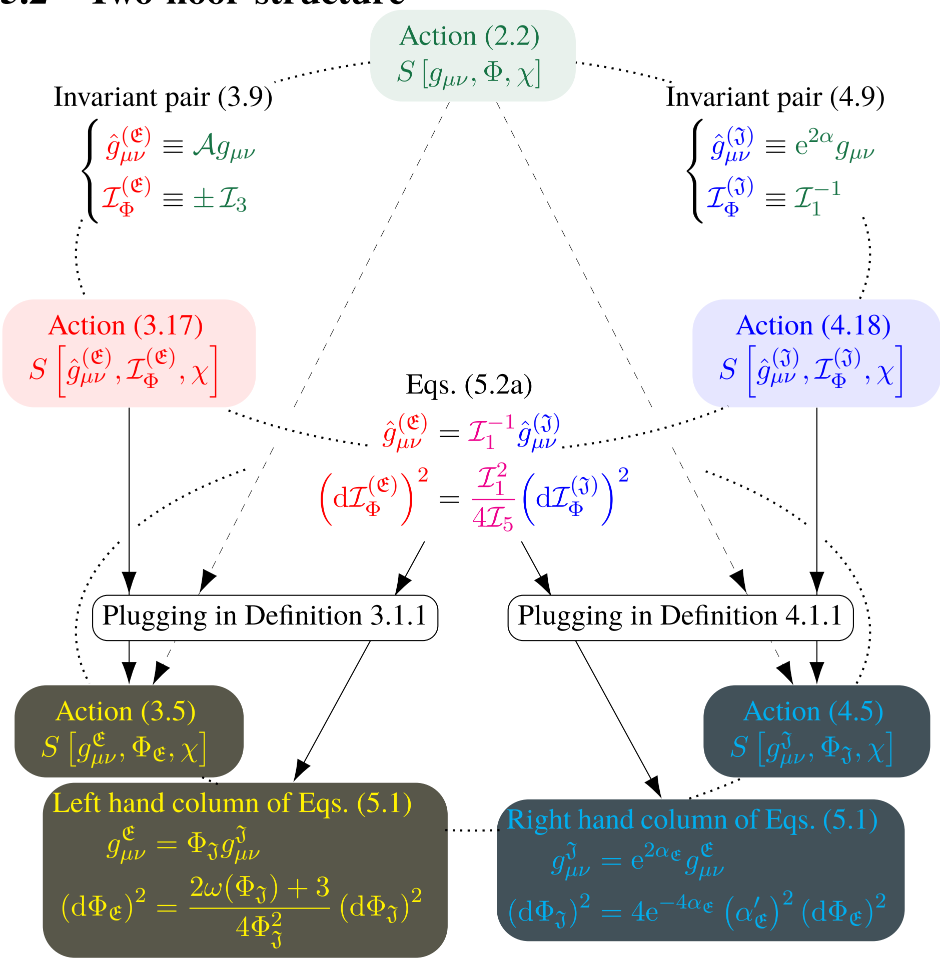
$$\frac{1}{4}\left(2\mathcal{A}\alpha' - \mathcal{A}'\right)^2 = \mathrm{e}^{-4\bar{\gamma}}\left(\bar{f}'\right)^{-2}\frac{1}{4}\left(2\bar{\mathcal{A}}\bar{\alpha}' - \bar{\mathcal{A}}'\right)^2\,, \tag{5.10b}$$

$$\left(\alpha'\right)^2 = \left(\bar{f}'\right)^{-2}\left(\bar{\alpha}' - \bar{\gamma}'\right)^2\,. \tag{5.10c}$$

Also, as before, there are infinitely many expressions of the generic parametrization that reduce to $(\alpha'_{\mathfrak{E}})^2$ when evaluated in the Einstein frame canonical parametrization $\mathfrak{E}$.

Let me point out, however, that mostly such problems might be unnoticed, because in practical calculations the expressions usually have rather simple transformation properties.

5.2 Two-floor-structure



We observe a two-floor-structure. On the upper floor there is the generic parametrization , and two invariant formulations (and , respectively) that are just rewritings of the generic parametrization and, thus, as generic. On the lower floor there are two fixed parametrizations (and) which correspond to the particular invariant rewritings above ($\leftrightarrow$ and $\leftrightarrow$, respectively). (In principle there are infinitely many inhabitants on both floors.) An invariant and the corresponding noninvariant formulation only differ in the interpretation we assign to the quantities contained therein. It might seem that by fixing the parametrization one restricts the theory (goes to the lower floor), but I conclude that a fixed parametrization is equivalent to the generic parametrization, because both are equivalent to the invariant formulation since there is no criteria for a priori discriminating between the invariant and noninvariant formulation. For each transformation between different invariant formulations there is a corresponding transformation between the noninvariant formulations, once more the only difference being the interpretation which is assigned by us. The same from a slightly different angle, invariant formulations inherit their mathematical properties from the corresponding initially fixed parametrizations, which are used for finding the invariant pairs, and hand these properties back to the noninvariant ordinary formulations once the parametrization is fixed.

This is the understanding that underlies the translation rules (3.18) and (4.19). From the viewpoint of such two-floor-structure, we lift the expressions to the upper floor (and also impose transformation properties if necessary). One the other hand, due to the same equivalence, the use of the invariant formulations as in the invariant pairs (3.9) and (4.9), as well as in the Sections IV.B, IV.C and VI of Ref. I is equivalent to the noninvariant formulations, e.g., actions (3.5) and (4.5), and therefore, by making use of the invariant formulation, we can not solve problems, that could not be solved within the noninvariant formulation.

5.3 The use of invariants in fixed parametrizations

5.3.1 Using invariants for transforming from a fixed parametrization $\mathfrak{P}$ to some other parametrization

As shown in Sections 3.4 and 4.4 for a fixed parametrization $\mathfrak{P}$ one can construct the translation rules, which allow us to rewrite every expression from the particular parametrization $\mathfrak{P}$ as an expression of the generic parametrization, given by action (2.2). Note, however, that in order to obtain the correct expression in the generic parametrization, we must know its transformation properties beforehand. Therefore, fixed parametrizations are not useful for studying transformation properties, because in a fixed parametrization $\mathfrak{P}$ each quantity can be considered to be an invariant. One might obtain some information about the transformation properties by considering two fixed parametrizations, and comparing the invariant results of the translation rules. For example, the Eqs. (3.6a) and (4.6a) are rewritten as (3.27) and (4.26), respectively. Comparing the latter two yields

$$\hat{E}^{(\hat{g}^{(\mathfrak{E})})}_{\mu\nu} = \mathcal{I}_1 \hat{E}^{(\hat{g}^{(\mathfrak{J})})}_{\mu\nu} \,, \tag{5.11}$$

where $\mathcal{I}_1^{-1}$ is exactly the conformal factor in (5.2a), and hence we recover the transformation rule (2.14). There is, however, a simple counterexample. In the Einstein frame $\mathcal{A}_{\mathfrak{E}} = 1$, and thus $\mathcal{A}''_{\mathfrak{E}} = 0$. In the Jordan frame $\mathcal{A}_{\mathfrak{J}} = \Phi_{\mathfrak{J}}$, and $\mathcal{A}''_{\mathfrak{J}} = 0$. The corresponding invariants are hence zero for both cases. However,

$$\begin{aligned} \mathcal{A}'' = \; & \left(\bar{f}'\right)^{-2} \mathrm{e}^{-2\bar{\gamma}} \left(\bar{\mathcal{A}}'' - 4\bar{\gamma}'\bar{\mathcal{A}}' + 4\left(\bar{\gamma}'\right)^2 \bar{\mathcal{A}} - \bar{\gamma}''\bar{\mathcal{A}}\right) \\ & - \left(\bar{f}'\right)^3 \bar{f}'' \mathrm{e}^{-2\bar{\gamma}} \left(\bar{\mathcal{A}}' - 2\bar{\gamma}'\bar{\mathcal{A}}\right) . \end{aligned} \tag{5.12}$$

Let me point out that the same problem haunts whenever whatever transformations are considered between fixed parametrizations. In order to find the correct expression, we must know the transformation properties.

5.3.2 Using invariants for transforming from some other parametrization to the fixed parametrization $\mathfrak{P}$

Once the invariant version of a parametrization $\mathfrak{P}$ is constructed, it is fairly easy to write down transformation rules from an arbitrary other fixed parametrization to the parametrization $\mathfrak{P}$. For an example, let us consider yet another parametrization

Definition 5.3.1: Jordan frame Boisseau-Esposito-Farèse-Polarski-Starobinsky parametrization $\mathfrak{B}$

Let us specify the arbitrary functions in the generic action functional (2.2) to be

$$\mathcal{A}(\Phi)|_{\mathfrak{B}} \equiv \mathcal{A}_{\mathfrak{B}}(\Phi_{\mathfrak{B}}) = F(\Phi_{\mathfrak{B}}) \,, \qquad \mathcal{B}(\Phi)|_{\mathfrak{B}} \equiv \mathcal{B}_{\mathfrak{B}}(\Phi_{\mathfrak{B}}) \stackrel{!}{=} 1 \,, \tag{5.13a}$$

$$\mathcal{V}(\Phi)|_{\mathfrak{B}} \equiv \mathcal{V}_{\mathfrak{B}}(\Phi_{\mathfrak{B}}) \,, \qquad \alpha(\Phi)|_{\mathfrak{B}} \equiv \alpha_{\mathfrak{B}}(\Phi_{\mathfrak{B}}) \stackrel{!}{=} 0 \,. \tag{5.13b}$$

Such a setup is referred to as the Jordan frame Boisseau-Esposito-Farèse-Polarski-Starobinsky parametrization, denoted by $\mathfrak{B}$.

See also the subsection 3.2 in Ref. IV (page 130 in the thesis), etc., for references and further information.

Without knowing anything but the Definition 5.3.1, i.e., without constructing the translation rules for this parametrization, let us consider the right hand side of the translation rules (3.18) for the Einstein frame canonical parametrization $\mathfrak{E}$.

In particular, evaluating the invariants in the second column in the Jordan frame Boisseau-Esposito-Farèse-Polarski-Starobinsky parametrization $\mathfrak{B}$ yields, first

$$\left(\frac{\mathrm{d}\mathcal{I}_\Phi^{(\mathfrak{E})}}{\mathrm{d}\Phi_\mathfrak{B}}\right)^2\Bigg|_\mathfrak{B} = \left(\frac{\mathrm{d}\Phi_\mathfrak{E}}{\mathrm{d}\Phi_\mathfrak{B}}\right)^2 = \left(\mathcal{I}_3'\right)^2\Big|_\mathfrak{B} = \frac{2\mathcal{AB}+3\left(\mathcal{A}'\right)}{4\mathcal{A}^2}\bigg|_\mathfrak{B} = \frac{2F+3\left(F'\right)^2}{4F^2}\,, \tag{5.14}$$

second

$$\mathcal{V}_\mathfrak{E} = \mathcal{I}_2|_\mathfrak{B} = \frac{\mathcal{V}_\mathfrak{B}}{F^2}\,, \tag{5.15}$$

and third

$$\alpha_\mathfrak{E} = \frac{1}{2}\ln\mathcal{I}_1\bigg|_\mathfrak{B} = -\frac{1}{2}\ln F\,. \tag{5.16}$$

which reproduces the second row of the third column from Table I in Ref. IV on page 131 in the thesis. Note that as before, these transformations are constructed for invariants.

5.4 The use of invariants in the generic parametrization

In the generic parametrization all the transformation properties are explicit and thus one can just directly study each expression and the result does not depend on how the particular expression was derived. I have been studying invariants in the generic parametrization in the context of the general relativity regime which itself was thoroughly studied in the papers [20, 21, 22, 23, 24, 25] (not authored by me) and my contribution to the subject (contained in Refs. IV–V) was to study the transformation properties. Unlike the formalism of invariants which developed over time as the understanding grew, the transformation properties of the general relativity limit are rather compactly and thoroughly studied in Ref. V which includes the important results from IV as well. I would like to highlight a few of the calculations where invariants are of importance. The reader interested in the general relativity regime should proceed to Ref. V.

5.4.1 Confirming the invariance of the general relativity regime point

The invariants were implicitly used in Ref. IV in order to argue for the invariance of the general relativity limit, where the noninvariant conditions (in principle Eqs. (4.2a), (4.2b) and (4.4) in Ref. IV, page 130–132 in the thesis, recall also Eqs. (2.20) and (2.21) in the current overview article) for determining a scalar field value Φ_0, i.e,

$$\frac{\mathcal{A}\mathcal{V}'-2\mathcal{A}'\mathcal{V}}{2\mathcal{AB}+3\left(\mathcal{A}'\right)^2}\bigg|_{\Phi_0} = \frac{\mathcal{A}}{4}\frac{\mathcal{I}_2'}{\mathcal{F}}\bigg|_{\Phi_0} = \frac{\mathcal{A}}{4}\frac{\mathcal{I}_2'}{\left(\mathcal{I}_3'\right)^2}\bigg|_{\Phi_0} \stackrel{!}{=} 0\,, \tag{5.17a}$$

$$\left.\frac{\mathcal{A}' - 2\mathcal{A}\alpha'}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}\right|_{\Phi_0} = -\left.\frac{1}{4\mathcal{A}}\frac{(\ln\mathcal{I}_1)'}{\mathcal{F}}\right|_{\Phi_0} = -\left.\frac{1}{4\mathcal{A}}\frac{(\ln\mathcal{I}_1)'}{(\mathcal{I}_3')^2}\right|_{\Phi_0} \stackrel{!}{=} 0\,, \tag{5.17b}$$

were replaced by invariant condition Eq. (4.7) in Ref. IV, page 132 in the thesis

$$\left.\frac{(\mathcal{A}\mathcal{V}' - 2\mathcal{A}'\mathcal{V})^2}{\mathcal{A}^4\left(2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2\right)}\right|_{\Phi_0} = \left.\left(\frac{1}{2}\frac{\mathcal{I}_2'}{\mathcal{I}_3'}\right)^2\right|_{\Phi_0} = \left.\left(\frac{1}{2}\frac{\mathrm{d}\mathcal{I}_2}{\mathrm{d}\mathcal{I}_3}\right)^2\right|_{\Phi_0} \stackrel{!}{=} 0\,, \tag{5.18a}$$

$$\left.\frac{(\mathcal{A}' - 2\mathcal{A}\alpha')^2}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}\right|_{\Phi_0} = \left.\left(\frac{1}{2}\frac{(\ln\mathcal{I}_1)'}{\mathcal{I}_3'}\right)^2\right|_{\Phi_0} = \left.\left(\frac{1}{2}\frac{\mathrm{d}\ln\mathcal{I}_1}{\mathrm{d}\mathcal{I}_3}\right)^2\right|_{\Phi_0} = \mathcal{I}_5|_{\Phi_0} \stackrel{!}{=} 0\,, \tag{5.18b}$$

since under certain rather reasonable assumptions the roots, i.e, the zeros are the same for both the invariant and noninvariant expressions.

5.4.2 Using invariants for gaining insight

Let us consider the equation for the scalar field and in particular the linearized version of it. The eigenvalues characterizing the latter are expected to be almost invariants (gaining just a multiplier under the transformation). From Eq. (89) in Ref. I, page 85 in the thesis, we had the insight what the invariant form of the eigenvalue should be like. (Note that in my opinion the calculation in Ref. I was just done in the Einstein frame canonical parametrization, although at the time it seemed to have a wider meaning, and in order to use that result for writing down the expression in the generic parametrization, one should know the transformation rule beforehand. Luckily for us this turned out to be rather simple. Recall that in a fixed parametrization each expression by itself might be considered to be invariant.) From the direct calculation in the generic parametrization (see Section 4 in Ref. V from page 163 in the thesis), linearizing the equation around Φ_0 for which (5.17) holds, we obtain the eigenvalue (91) in Section 4.1.4 of Ref. V, namely

$$\lambda^{\varepsilon}_{\pm} = \frac{1}{2}\left(-C^{\varepsilon}_1 \pm \sqrt{(C^{\varepsilon}_1)^2 + 2(2-M)C_2}\right)\,. \tag{5.19}$$

Let me denote the deviation as $x = \Phi - \Phi_0$. The constants in (5.19) are

$$C^{\varepsilon}_1 \equiv \left.\frac{\varepsilon}{\ell}\sqrt{3\mathcal{I}_2\mathcal{A}}\right|_{x=0}\,, \qquad C_2 \equiv -\left.\left(\frac{\mathcal{A}}{2\ell^2}\frac{\mathcal{I}_2'}{\mathcal{F}}\right)'\right|_{x=0}\,, \tag{5.20a}$$

$$M \equiv \lim_{x\to 0}\left\{\left(\frac{1}{\mathcal{F}}\right)'\mathcal{F}\cdot x\right\}\,, \tag{5.20b}$$

where $\epsilon = \pm 1$. (Currently I am interested in the transformation properties, and therefore the meaning of the quantities contained in (5.19) is not important. An interested reader should consult Section 4 in Ref. V.) The transformation rule (Eq. (116) in Ref. V, page 170 in the thesis)

$$\lambda^{\varepsilon}_{\pm} = \mathrm{e}^{-\bar{\gamma}}\Big|_0 \bar{\lambda}^{\varepsilon}_{\pm} \tag{5.21}$$

could be checked by considering the transformation properties for each expression in (5.20) separately, but one can also take into account the insight from Eq. (89) in Ref. I, page 85 in the thesis, and take into account that due to (5.17) the L'Hospital's rule yields

$$C_2 = -\left.\frac{\mathcal{A}}{2\ell^2}\right|_{x=0} \lim_{x\to 0}\left[\frac{\mathcal{I}_2'}{\mathcal{F}}\Big/ x\right] , \tag{5.22}$$

in order to write (see Eq. (115) in Ref. V, page 170 in the thesis)

$$(2-M)C_2 = -\,\ell^{-2}\mathcal{A}\left.\frac{\mathrm{d}^2\mathcal{I}_2}{\mathrm{d}\mathcal{I}_3^2}\right|_{x=0} . \tag{5.23}$$

Hence, Eq. (5.19) reads

$$\lambda^{\varepsilon}_{\pm} = \sqrt{\mathcal{A}}\frac{1}{2\ell}\left.\left(-\varepsilon\sqrt{3\mathcal{I}_2} \pm \sqrt{3\mathcal{I}_2 - 2\frac{\mathrm{d}^2\mathcal{I}_2}{\mathrm{d}\mathcal{I}_3^2}}\right)\right|_{x=0} , \tag{5.24}$$

which immediately verifies the rule (5.21).

5.5 Critique and loose ends

The formalism of invariants in scalar-tensor theories of gravity (Refs. I, II and III) is up to now not yet finished. First, the three invariants (2.20) have been used in order to generate all other invariants we have encountered via the three rules (2.23), but nevertheless there is no mathematical proof that these three form a complete set in that sense. Dr. Laur Järv used an elaborate computer program in 2014–2015, and his result was that at least up to the specified order of complexity there are no invariants that could not be constructed from (2.20) by making use of (2.23). D. Sc. Piret Kuusk has performed some calculations based on the monographs [26, 27] to investigate infinitesimal operators for finite transformations, but the calculations have not been finalized.

Therefore, in my opinion, while there is a prescription for deriving translation rules as (3.18) and (4.19), there is no proof for the uniqueness of such rules. The whole prescription makes use of invariants which reduce to certain functions once the parametrization has been fixed, e.g.,

$$\mathcal{I}^{(\mathfrak{E})}_{\Phi}\Big|_{\mathfrak{E}} = \Phi_{\mathfrak{E}} , \qquad \mathcal{I}^{(\mathfrak{E})}_{\mathcal{V}}\Big|_{\mathfrak{E}} = \mathcal{V}_{\mathfrak{E}} , \qquad \mathcal{I}^{(\mathfrak{E})}_{\alpha}\Big|_{\mathfrak{E}} = \alpha_{\mathfrak{E}} . \tag{5.25}$$

However, one should also prove

$$\left(\mathcal{I}_1^{(\mathfrak{E})} - \mathcal{I}_2^{(\mathfrak{E})}\right)\Big|_{\mathfrak{E}} = 0 \quad \overset{?}{\Rightarrow} \quad \mathcal{I}_1^{(\mathfrak{E})} = \mathcal{I}_2^{(\mathfrak{E})}\,, \tag{5.26}$$

in order to complete the formalism. Let me stress that I do not claim that the basic invariants (2.20) do not form a complete set. I actually believe that they do, but in my opinion this has not been mathematically proven.

Yet another goal would be to put forward a scheme that would easily allow us to identify the actions which are obtained from the generic action (2.2) via some (2.8). For fixed parametrizations the prescription introduced in Ref. II is perhaps cumbersome but nevertheless straightforward. However, once we extend the treatment, there is no routine. Let me proceed with an example provided to me by Mihkel Rünkla. Let us consider a fixed parametrization $\mathfrak{P}_1$ where the functions $\mathcal{A}_{\mathfrak{P}_1}$ and $\alpha_{\mathfrak{P}_1}$ have gained a fixed functional form. Hence also $\mathcal{I}_1$ has gained a fixed functional form. Let us consider the case where the latter is dynamical, as then we have a fixed parametrization. Using an additional conformal transformation to fix the potential (in the new frame $\mathfrak{P}_2$ (abuse of notation!)) to be constant, i.e., $\mathcal{V}_{\mathfrak{P}_2} = \mathrm{const}$ takes us to a setup where one of the functions, $\mathcal{V}_{\mathfrak{P}_2}$, is determined, and two other functions, $\mathcal{A}_{\mathfrak{P}_2}$ and $\mathrm{e}^{2\alpha_{\mathfrak{P}_2}}$, contain the same undetermined multiplier via $\mathcal{V}_{\mathfrak{P}_1}$. The invariant $\mathcal{I}_1$ still has a fixed functional form and the parametrization $\mathfrak{P}_2$ can be derived from the generic parametrization by rewriting the action (2.2) in terms of different variables. However, currently the formalism of invariants does not provide proper tools for analyzing or recognizing such situations.

Summary

In the attached papers I, II and III underlying the thesis we have developed the formalism of invariants, i.e., we have constructed quantities that remain invariant under a conformal transformation (2.8b) and transform as scalar functions under a scalar field redefinition (2.8a). The research was originated by the idea, hypothesis (from Laur Järv, if I remember correctly) that physically measurable quantities should be such invariants. However, most of the research in literature was and is done in a fixed parametrization (see Definition 2.4.1). Thus, when it turned out that in a fixed parametrization each expression can be considered to be such an invariant, the significance of such quantities in the context of fixed parametrizations was diminished. The correspondence for rewriting an expression from a fixed parametrization as an invariant in the generic parametrization, presented in Table I of Ref. I, was elaborated as abstract translation rules in Ref. II (see also Ref. III for examples in the multiscalar case). The nuance that perhaps has not been stressed enough in the attached papers is that in a fixed parametrization the transformation rules are implicit, i.e., for an expression without context these are lost, but one can recover these if the derivation of the result is taken into account. (See, for example, Eq. (2.17)) An expression by itself, however, contains ambiguity, because imposing the transformation properties in a fixed parametrization constitutes to multiplying by and adding functions that have numerical value 1 and 0, respectively, if evaluated in that particular parametrization. The aim of the overview article is to stress this subtlety.

The invariants can be used to construct transformation properties from a fixed parametrization to some other parametrization, and also from some other parametrization to that fixed parametrization, but as before, if we do not know the transformation property of the original expression, then the resulting expression is ambiguous. Note that the same applies if transformation rules between fixed parametrizations are applied without invariants (as in my opinion is mostly done in the literature). Hence, for example, Table I in Ref. IV contains invariants, since we can impose numerical equality only for these. Due to these ambiguities, while it is good to have the formalism, because it makes easier to group things appropriately, in my opinion there is no need to study the formalism of invariants any further. If one is interested in studying the transformation properties, then the generic parametrization is the way to go. The attached paper V is a good example. In the generic parametrization one does not need to know invariants, because one can calculate the transformation properties explicitly.

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Kokkuvõte (in Estonian)

Väitekirja alusobjektiks on Flanagani mõjufunktsionaal, mis kirjeldab skalaartensortüüpi gravitatsiooniteooriaid. See mõjufunktsionaal on samaväärne varem näiteks Bergmanni ja Wagoneri poolt kasutatud mõjufunktsionaalidega. Viimastes postuleeritakse teooriate klass, mis sisaldab kaht vaba funktsiooni. Täiendusena, kuid mitte üldistusena, on Flanagan mõjufunktsionaali postuleerinud nelja vaba funktsiooniga. Selle tulemusel muutub mõjufunktsionaal ilmutatult kujuinvariantseks meetrilise tensori konformse teisenduse ning skalaarvälja ümberdefineerimise suhtes. Neli eelmainitud vaba funktsiooni omavad selget ja unikaalset teisenemiseeskirja. Varasemate, kahe vaba funktsiooniga teooriate puhul see nii polnud ning avaldise teisenemiseeskiri oli kodeeritud avaldise tuletuskäiku. Seega avaldistel endil, väljaspool konteksti, puudus varasemate formuleeringute puhul ühene teisenemiseeskiri. Väitekirja laiem eesmärk on sellisest üldisest formuleeringust lähtudes uurida objekte, nii skalaarseid kui ka tensoriaalseid, mis meetrika konformse teisenduse suhtes on invariantsed ning skalaarvälja ümberdefineerimisel teisenevad kui skalaarsed funktsioonid. Kitsam eesmärk on juhtida tähelepanu justnimelt asjaolule, et fikseeritud parametriseering, milles põhiosa valdkonna teadustööst on formuleeritud, ei sobi teisenemiseeskirjade uurimiseks. (Täpsemalt öeldes, teisenemiseeskirjade uurimine sellistes formuleeringutes pole võimatu, vaid lihtsalt ebamugav. Kirjanduses küll uuritakse usutavasti ainult invariante.)

Aluspublikatsioonidest esimeses, viites I tutvustatakse avaldisi, mis meetrilise tensori konformsel teisendusel ei muutu ja skalaarvälja ümberdefineerimisl teisenevad skalaarsete funktsioonidena. Selles artiklis tutvustatakse kolme skalaarset baasinvarianti, kolme reeglit edasiste avaldiste formuleerimiseks ning tulemused on põhiosas kokku võetud sealses Tabelis 1. Hilisemate teadmiste valguses olgu öeldud, et mainitud tabel sisaldab ainult invariante erinevates parametriseeringutes. Mitteinvariantsete suurustega seal tegeletud pole. Küll aga võib etteruttavalt öelda, et viites I toodud Tabel 1 täielikult hõlmab ka viites IV esitatud Tabelit 1. Viimane on aga väidetavalt teisenemiseeskirjade tabel ja siit juba järeldubki, et kirjanduses (enamasti) kasutatavad teisenemiseeskirjad pole mitte teisenemiseeskirjad, vaid invariantide avaldised erinevates fikseeritud parametriseeringutes ja põhjus on sellise üldisema formuleeringu seisukohalt ilmne. Kontekstist välja rebitud avaldis fikseeritud parametriseeringus ei oma üheselt määratletavat teisenemiseeskirja.

Aluspublikatsioonidest teises, viites II näidatakse abstraktsel tasemel fikseeritud parametriseeringu ja üldise parametriseeringu ekvivalentsust ning põhjendatakse sellega „tõlkereegleid“. Viimased on algebraliste asendamiste kogu, millega on võimalik fikseeritud parametriseeringu avaldis (teadaoleva teisenemiseeskirjaga) ümber kirjutada üldise parametriseeringu avaldisena. Sellise protseduuri võimalikkus ja ka see protseduur ise järelduvad justnimelt kahe formuleeringu ekvivalentsusest.

Aluspublikatsioonidest kolmandas, viites III on mõnevõrra eraldiseisvana esitatud invariantide formuleering multiskalaar-tensor tüüpi gravitatsiooniteooriates. Skalaarväljade ruum on nüüd mitmemõõtmeline ning üks invariantidest omandab selge geomeetrilise interpretatsiooni, olles seotud invariantse infinitesimaalse ruumalaelemendiga selles skalaarväljade ruumis.

Aluspublikatsioonidest neljandas, viites IV, kõige varasemas, on toodud esimesed invariandid (neid nii nimetamata) ning juba mainitud teisenemiseeskirjad kolme fikseeritud parametriseeringu vahel ehk Tabel 1, mis hilisemate teadmiste valguses on hoopis invariantide avaldised erinevates parametriseeringutes.

Aluspublikatsioonidest viiendas, viites V on uuritud teisenemiseeskirju üldises formuleeringus. Kõige uudsema tulemusena on seal näidatud, et ka singulaarne skalaarvälja ümberdefineerimine üldrelatiivsusteooria piiril ei riku vastavust parametriseeringute vahel. Minu hinnangul igasugune teisenemiseeskirjade uurimine peaks toimuma just selle artikli vaimus.

Ülevaateartiklis ehk aluspublikatsioone täiendavas teaduslikus materjalis, mis põhimõtteliselt sisaldub vähemalt vihjamisi ka artiklites, keskendutakse justnimelt mitteinvariantsete objektide käsitlemisele. Selleks esitatakse sama teooria kolm korda, sealjuures kahel juhul toimub veel materjali dubleerimine. Seega põhimõtteliselt viis formuleeringut. Juhtmõtteks on seal loosung, et objektide samastamiseks peab neid kõigepealt eristama. Põhiliseks tulemuseks on arusaamine, et mitteinvariantsete objektide korral „tõlkereeglid“ pole ühesed, sest kontekstist välja võetud avaldise korral ei saa me samaaegselt postuleerida kõikvõimalikke teisenemiseeskirju. Seega pole fikseeritud parametriseeringu avaldised („tõlkereeglite“ kasutusala) konteksti teadmata sobivad teisenemiseeskirjade uurimiseks. Viimaste teadasaamiseks on mõistlik teooria formuleerida Flanagani-tüüpi mõjufunktsionaali kasutades, sest siis jäävad teisenemiseeskirjad ilmutatuks ning avaldise teisenemine on üheselt määratud ja ei sõltu kontekstist. Kuna viimasel juhul pole tarvidust invariante teada ning seoses „tõlkereeglite“ mitteühesusega pole minu arvates vajadust täiendavalt invariantide formalismi uurida.

Attached publications

Chapter 6

Invariant quantities in the scalar-tensor theories of gravitation

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Contents

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Invariant quantities in the scalar-tensor theories of gravitation

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We consider the general scalar-tensor gravity without derivative couplings. By rescaling of the metric and reparametrization of the scalar field, the theory can be presented in different conformal frames and parametrizations. In this work we argue that while due to the freedom to transform the metric and the scalar field, the scalar field itself does not carry a physical meaning (in a generic parametrization), there are functions of the scalar field and its derivatives which remain invariant under the transformations. We put forward a scheme to construct these invariants, discuss how to formulate the theory in terms of the invariants, and show how the observables like parametrized post-Newtonian parameters and characteristics of the cosmological solutions can be neatly expressed in terms of the invariants. In particular, we describe the scalar field solutions in Friedmann-Lemaître-Robertson-Walker cosmology in Einstein and Jordan frames and explain their correspondence despite the approximate equations turning out to be linear and nonlinear in different frames.

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I. INTRODUCTION

Scalar-tensor gravity (STG) [1–4] introduces a scalar field that is nonminimally coupled to curvature and, thus, can be interpreted as an additional mediator, besides the usual metric tensor, of gravitational interaction. Such theories provide a simple but versatile extension to general relativity. They arise naturally in constructions involving higher dimensions and are featured in attempts to construct scale-invariant fundamental physics [5]. The theory can be generalized further by allowing various derivative couplings and higher-order derivative terms in the action [6]. It has received a lot of attention in phenomenological model building: inflation and dark energy [7] and more recently Higgs inflation [8].

Since the early paper by Dicke [9] it has been well known that by rescaling of the metric and reparametrization of the scalar field, the theory can be presented in different conformal frames and parametrizations [4]. Despite an extensive use of this property as a convenient calculational tool, there lingers a conceptual issue of what is the precise relation of different frames and parametrizations to the observable world and to each other.

In the former, it is a question of whether physical measurements choose one frame which defines the units used in physical observations, i.e. which metric defines the measured lengths (for early references see Refs. [3,10]; for more recent papers, see Refs. [11]). From an alternative point of view, allowing the units to rescale inversely with the metric neutralizes the effect of conformal transformation [9,12], and the question of physical frame becomes superfluous. This can be interpreted by generalizing the underlying geometry from the Riemann into the Weyl integrable [13].

The latter aspect means a mathematical problem, addressing whether the different formulations are mathematically equivalent. Here the common wisdom about the subject says that different frames are equivalent on the level of classical action (although one must be careful in the limit where the transformation becomes singular [14]). However, things get more complicated and warrant careful consideration and debate on the level of e.g. cosmological perturbations [15] and quantum corrections [16,17].

One may view different conformal frames and parametrizations of the theory as arising from a change of coordinates in some abstract generalized field space. Then the discrepancies can be attributed to the circumstance that the theory has not been formulated in a covariant way with respect to that abstract space [17]. Therefore, some authors have strived to formulate the theory in terms of invariant variables. The idea has been to focus upon the conformal transformation and express all observables in terms of frame-invariant combinations of the theory parameters and variables, as well as the units [18,19].

In the present paper we complement this line of thought by introducing invariant quantities of the scalar field. The scalar field is amenable to reparametrization; therefore, in a generic parametrization it cannot carry a physical meaning (cannot be measured directly). However, it is possible to combine the functions of the scalar field and their derivatives into quantities which remain invariant under the conformal transformations and field redefinitions and, therefore, should have a more direct relevance to observable physics. Indeed, using these quantities we show how the parametrized post-Newtonian (PPN) parameters and the

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qualitative features of the scalar field cosmological solutions like convergence properties and periods of oscillation are independent of the frame and parametrization. These invariant quantities also enable us to write the equations of motion and the action in a manifestly invariant form, and ease the conversion of calculational results from one frame and parametrization into another. A few preliminary efforts in this approach were presented earlier in a conference note [20].

The outline of the paper is as follows. First, we recall the general action for scalar-tensor gravity and the rules of transformation under conformal rescaling and field reparametrization. In the next section we introduce three basic invariant quantities of the scalar field and outline how to construct many other invariants from them. In Sec. IV we invoke an invariant metric that helps to write the field equations and the action in terms of the invariants. As an application in Sec. V we convert the PPN parameters into an invariant form and check that they reproduce the results for particular parametrizations known in the literature. In Sec. VI we focus upon the flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe without matter and study the scalar field solutions near the fixed points. The conditions for the fixed points as well as the eigenvalues determining the approximate solutions turn out to be invariant. Yet, for a specific situation it is interesting to see how a linear result in the Einstein frame can actually correspond to a nonlinear result in the Jordan frame. Finally, in Sec. VII we conclude with a brief summary and outlook.

II. GENERAL ACTION FUNCTIONAL AND DIFFERENT PARAMETRIZATIONS

A. General action functional

Let us consider the general action functional for a scalar-tensor theory of gravity written down by Flanagan [4],

$$S = \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-g}\{\mathcal{A}(\Phi)R - \mathcal{B}(\Phi)g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - 2\ell^{-2}\mathcal{V}(\Phi)\} + S_m[e^{2\alpha(\Phi)}g_{\mu\nu},\chi]. \tag{1}$$

It contains four arbitrary functions of the dimensionless scalar field Φ: curvature coupling function $\mathcal{A}(\Phi)$, generic kinetic coupling of the scalar field $\mathcal{B}(\Phi)$, self-interaction potential of the scalar field $\mathcal{V}(\Phi)$ and conformal coupling $e^{2\alpha(\Phi)}$ between the metric $g_{\mu\nu}$ and matter fields χ. Functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ and $\alpha(\Phi)$ are dimensionless, and fixing them all gives us some concrete theory. In the rest of the text, we drop the arguments of functions unless confusion might arise.

If we impose a physical condition that gravitational interaction is always finite and attractive, the curvature coupling function must satisfy $0 < \mathcal{A}(\Phi) < \infty$. We also assume from physical considerations that self-interaction potential is non-negative, $0 \leq \mathcal{V}(\Phi) < \infty$. We will use the units where $c = 1$, but we do not fix the values of the nonvariable part of the effective gravitational "constant" κ^2 and a positive constant parameter ℓ with the dimension of length, e.g. the Planck length. Note that from a convention $[\kappa^2] = 1$ it follows that $[S] = [\hbar] = L^2$ and from a convention $[S] = [\hbar] = 1$ it follows that $[\kappa^2] = L^2$.

It is well known that two out of the four arbitrary functions $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$ can be fixed by transformations that contain two functional degrees of freedom,

$$g_{\mu\nu} = e^{2\bar{\gamma}(\bar{\Phi})}\bar{g}_{\mu\nu}, \tag{2}$$

$$\Phi = \bar{f}(\bar{\Phi}). \tag{3}$$

We shall refer to first of them as the change of the frame and the second one the reparametrization of the scalar field. The change of the frame is in fact a conformal rescaling of the metric. We assume that the function $\bar{\gamma}(\bar{\Phi})$ and its first and second derivative, $d\bar{\gamma}/d\bar{\Phi}$ and $d^2\bar{\gamma}/d\bar{\Phi}^2$ respectively, do not diverge at any permitted $\bar{\Phi}$, because otherwise we would introduce geometrical singularities via conformal transformation. (Note that this excludes the interesting possibility of "conformal continuation" [21].) We also assume the function $\bar{f}(\bar{\Phi})$ to be at least directionally continuous, but retain a possibility that Jacobian $\bar{f}' \equiv d\Phi/d\bar{\Phi}$ of this coordinate transformation in one-dimensional field space may be singular at some isolated value of the scalar field $\bar{\Phi}$.

Under the transformation (2), (3) the action functional (1) preserves its structure up to the boundary term (total divergence)

$$\begin{aligned}\bar{S} = \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-\bar{g}}\{\bar{\mathcal{A}}(\bar{\Phi})\bar{R} - \bar{\mathcal{B}}(\bar{\Phi})\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\Phi}\bar{\nabla}_\nu\bar{\Phi} \\ - 2\ell^{-2}\bar{\mathcal{V}}(\bar{\Phi})\} + \bar{S}_m[e^{2\bar{\alpha}(\bar{\Phi})}\bar{g}_{\mu\nu},\chi] \\ - \frac{1}{2\kappa^2}\int_{V_4} d^4x\partial_\mu(6\bar{\gamma}'\sqrt{-\bar{g}}\bar{\mathcal{A}}\bar{g}^{\mu\nu}\partial_\nu\bar{\Phi}),\end{aligned} \tag{4}$$

with transformed functions [4]

$$\begin{aligned}\bar{\mathcal{A}}(\bar{\Phi}) &= e^{2\bar{\gamma}(\bar{\Phi})}\mathcal{A}(\bar{f}(\bar{\Phi})), \\ \bar{\mathcal{B}}(\bar{\Phi}) &= e^{2\bar{\gamma}(\bar{\Phi})}((\bar{f}')^2\mathcal{B}(\bar{f}(\bar{\Phi})) - 6(\bar{\gamma}')^2\mathcal{A}(\bar{f}(\bar{\Phi})) - 6\bar{\gamma}'\bar{f}'\mathcal{A}'), \\ \bar{\mathcal{V}}(\bar{\Phi}) &= e^{4\bar{\gamma}(\bar{\Phi})}\mathcal{V}(\bar{f}(\bar{\Phi})), \\ \bar{\alpha}(\bar{\Phi}) &= \alpha(\bar{f}(\bar{\Phi})) + \bar{\gamma}(\bar{\Phi}).\end{aligned} \tag{5}$$

Here we have adopted a convention that prime at a quantity with a bar denotes derivative with respect to $\bar{\Phi}$, e.g. $\bar{f}' \equiv \frac{d\bar{f}(\bar{\Phi})}{d\bar{\Phi}}$, and prime at a quantity without a bar denotes derivative with respect to Φ, e.g. $\mathcal{A}' \equiv \frac{d\mathcal{A}(\Phi)}{d\Phi}$. If we denote the backward transformations as

$$\bar{g}_{\mu\nu} = e^{2\gamma(\Phi)} g_{\mu\nu}, \tag{6}$$

$$\bar{\Phi} = f(\Phi), \tag{7}$$

then $\gamma(\bar{f}(\bar{\Phi})) = -\bar{\gamma}(\bar{\Phi})$.

Under the assumptions on $\bar{\gamma}$ and its derivatives mentioned above, the transformation rules (5) imply the following.

(i) The conditions on curvature coupling function, $0<\mathcal{A}<\infty$, and self-interaction potential, $0\leq\mathcal{V}<\infty$, are preserved, i.e. $0<\bar{\mathcal{A}}<\infty$ and $0\leq\bar{\mathcal{V}}<\infty$.

(ii) If in some frame $\alpha = 0$, then in any other frame $|\bar{\alpha}| < \infty$.

(iii) If we want to avoid ghosts, i.e. if there is a frame where the tensorial and scalar part of the gravitational interaction are separated with $\mathcal{A} = 1$ and $\mathcal{B} > 0$, then in any related frame and parametrization it follows that $2\bar{\mathcal{A}}\bar{\mathcal{B}}+3(\bar{\mathcal{A}}')^2$ is non-negative. In this text we assume this quantity to be also nonvanishing. In other words we assume a strict inequality

$$\bar{\mathcal{F}} \equiv \frac{2\bar{\mathcal{A}}\bar{\mathcal{B}}+3(\bar{\mathcal{A}}')^2}{4\bar{\mathcal{A}}^2} > 0. \tag{8}$$

However, we do not impose a condition that the quantity $\bar{\mathcal{F}}$ is bounded from above.

B. Different parametrizations

In the literature mostly such action functionals are considered where two out of the four arbitrary functions $\{\mathcal{A},\mathcal{B},\mathcal{V},\alpha\}$ are fixed. If the latter can be derived from the action functional (1) by using the transformations (2) and (3) then the corresponding theory retains its generality up to some details. We use the term 'fixed parametrization' to refer to the case when two arbitrary functions out of four are fixed by the transformations. Fixing the remaining two functions gives a specific theory in this parametrization. The most common parametrizations are the following:

(i) The Jordan frame action in the Brans-Dicke-Bergmann-Wagoner parametrization (JF BDBW) [2] for the scalar field Ψ fixes $\mathcal{A}=\Psi$, $\alpha=0$, while keeping $\mathcal{B}=\omega(\Psi)/\Psi$, $\mathcal{V}=\mathcal{V}(\Psi)$.

(ii) The Jordan frame action in the parametrization used by e.g. Boisseau, Esposito-Farèse, Polarski and Starobinsky (JF BEPS) [22] for the scalar field ϕ is obtained by taking $\mathcal{B}=1$, $\alpha=0$, while having $\mathcal{A}=F(\phi)$, $\mathcal{V}=\mathcal{V}(\phi)$.

(iii) The Einstein frame action in canonical parametrization (EF canonical) [2,9] for the scalar field φ fixes $\mathcal{A}=1$, $\mathcal{B}=2$, while keeping $\alpha=\alpha(\varphi)$ and $\mathcal{V}=\mathcal{V}(\varphi)$. This is the parametrization that was meant when no ghost condition (8) was discussed.

In the Jordan frame the metric tensor that is used to construct geometrical objects is the same that enters the matter part of the action functional. Therefore, freely falling particles follow the geodesics of the corresponding geometry. In the Einstein frame, scalar and tensor degrees of freedom are separated and a well-posed initial value formulation is guaranteed by general theorems [23].

III. INVARIANTS

A. Constructing invariants

A closer look at the transformations (5) allows us to write out four quantities that under the rescaling (2) and reparametrization (3) gain a multiplier but otherwise preserve their structure. Namely,

$$\bar{\mathcal{A}} = e^{2\bar{\gamma}}\mathcal{A}, \tag{9}$$

$$e^{2\bar{\alpha}} = e^{2\bar{\gamma}}e^{2\alpha}, \tag{10}$$

$$\bar{\mathcal{V}} = e^{4\bar{\gamma}}\mathcal{V}, \tag{11}$$

$$\bar{\mathcal{F}} \equiv \frac{2\bar{\mathcal{A}}\bar{\mathcal{B}}+3(\bar{\mathcal{A}}')^2}{4\bar{\mathcal{A}}^2} = (\bar{f}')^2\frac{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2}{4\mathcal{A}^2} \equiv (\bar{f}')^2\mathcal{F}. \tag{12}$$

From these we can construct three independent quantities that are invariant under a local rescaling of the metric tensor and transform as scalar functions under the scalar field redefinition:

$$\mathcal{I}_1(\Phi) \equiv \frac{e^{2\alpha(\Phi)}}{\mathcal{A}(\Phi)}, \tag{13}$$

$$\mathcal{I}_2(\Phi) \equiv \frac{\mathcal{V}(\Phi)}{(\mathcal{A}(\Phi))^2}, \tag{14}$$

$$\mathcal{I}_3(\Phi) \equiv \pm\int\sqrt{\mathcal{F}(\Phi)}d\Phi. \tag{15}$$

Note that at any spacetime point $x\in V_4$, the scalar field values are related to each other via Eq. (3) and, therefore, we are actually dealing with spacetime point invariants,

$$\bar{\mathcal{I}}_i(\bar{\Phi}(x)) = \mathcal{I}_i(\bar{f}(\bar{\Phi}(x))) = \mathcal{I}_i(\Phi(x)). \tag{16}$$

This means that their numerical value at some fixed spacetime point is preserved under the transformation (3) while their functional form with respect to the scalar field as an argument changes under that transformation. We shall refer to these quantities as invariants. As the conformal transformation or the scalar field redefinition are, in principle, unrelated to a coordinate transformation, it follows that spacetime derivatives of invariants,

$$\partial_\mu(\bar{\mathcal{I}}_i(\bar{\Phi}(x))) = \partial_\mu(\mathcal{I}_i(\bar{f}(\bar{\Phi}(x)))), \tag{17}$$

are also invariants in this sense.

The quantity $\mathcal{I}_1$ can be used to define the notion of nonminimal coupling in an invariant way. If $\mathcal{I}_1$ is a constant, then the scalar field is minimally coupled. For example, quintessence in general relativity has $\mathcal{A}=1$, $\alpha=0$, thus $\mathcal{I}_1\equiv 1$ which holds in any frame and parametrization, i.e. in "veiled" [24] or "Weyled" [25] general relativity. We say that the scalar field is nonminimally coupled if $\mathcal{I}_1'\not\equiv 0$. Later at Eq. (50) it becomes clear that a nonminimally coupled scalar field is sourced by the matter energy-momentum in any frame. If $\mathcal{I}_2\equiv 0$ then the scalar field has a vanishing potential, a property that is not affected by a conformal transformation or reparametrization. Invariant $\mathcal{I}_3$ is given as an indefinite integral and, therefore, it is constant only if the integrand is identically zero. From Eq. (12) we see that this could only happen if the theory has minimal coupling to curvature and no kinetic term for the scalar field. So, in a generic scalar-tensor theory, the invariants (13)–(15) are dynamical functions of Φ, independent of each other.

The assumptions on $\mathcal{A}$ and α, listed in subsection II A bring along a constraint, $0<\mathcal{I}_1<\infty$. Another useful assumption for the ensuing presentation is to demand that $\mathcal{I}_1'$ and $\mathcal{I}_1''$ do not diverge. In a similar vein also $0\leq\mathcal{I}_2<\infty$, and it makes sense to assume further that $|\mathcal{I}_2'|<\infty$. Constraints on the derivatives are not invariant themselves and, therefore, we are actually restricting the possible forms of these functions.

We can also introduce an additional invariant object,

$$\hat{g}_{\mu\nu}\equiv\mathcal{A}(\Phi)g_{\mu\nu}, \tag{18}$$

which can be used to express geometrical quantities via invariants. In principle, $\hat{g}_{\mu\nu}$ can be considered to be a metric tensor. Note that the choice $\mathcal{A}(\Phi)g_{\mu\nu}$ is not unique. Namely, we could have multiplied the metric tensor with any other function of the scalar field which has a suitable transformation property, e.g. $e^{2\alpha}g_{\mu\nu}$ [4]. The assumption that the first and second derivative of $\mathcal{A}$ do not diverge guarantees that we do not introduce geometrical singularities by defining the invariant metric (18).

The fact that any function of the invariants is also an invariant can be used to construct further invariants. For example, we may define

$$\mathcal{I}_4\equiv\frac{\mathcal{I}_2}{\mathcal{I}_1^2}=\frac{\mathcal{V}}{e^{4\alpha}}. \tag{19}$$

The transformation of a derivative of these quantities with respect to the scalar field is given by a chain rule,

$$\bar{\mathcal{I}}_i'\equiv\frac{d\bar{\mathcal{I}}_i(\bar{\Phi})}{d\bar{\Phi}}=\frac{d\mathcal{I}_i(\Phi)}{d\Phi}\frac{d\Phi}{d\bar{\Phi}}\equiv\mathcal{I}_i'\bar{f}'. \tag{20}$$

This result is consistent with the transformation properties of a differential of the scalar field,

$$d\bar{\Phi}=\frac{d\bar{\Phi}}{d\Phi}d\Phi\equiv(\bar{f}')^{-1}d\Phi, \tag{21}$$

in a sense that the integration should cancel the differentiation. Indeed,

$$\bar{\mathcal{I}}_i(\bar{\Phi})=\int\bar{\mathcal{I}}_i'd\bar{\Phi}=\int\mathcal{I}_i'\bar{f}'(\bar{f}')^{-1}d\Phi=\int\mathcal{I}_i'd\Phi=\mathcal{I}_i(\Phi). \tag{22}$$

Note that we have already used that logic to construct the invariant $\mathcal{I}_3$. From Eq. (20) we conclude that a quotient of the derivatives of invariants is also an invariant,

$$\mathcal{I}_k=\frac{\mathcal{I}_i'}{\mathcal{I}_j'}, \tag{23}$$

while obviously

$$\mathcal{I}_i=\int\mathcal{I}_k\mathcal{I}_j'd\Phi. \tag{24}$$

The expressions (15) and (24) are given in the sense of an antiderivative, meaning that they also contain an integration constant. Therefore, only their change with respect to some variable should carry physical information.

By using the rule (23) and the possibility to form arbitrary functions, let us define

$$\mathcal{I}_5\equiv\left(\frac{\mathcal{I}_1'}{2\mathcal{I}_1\mathcal{I}_3'}\right)^2=\frac{(2\alpha'\mathcal{A}-\mathcal{A}')^2}{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2}. \tag{25}$$

This invariant helps to distinguish between different theories described by the action functional (1), for instance, for the minimally coupled scalar field $\mathcal{I}_5\equiv 0$. For the O'Hanlon-type action functional $(\mathcal{B}=0,\alpha=0)$ [26], which corresponds to the $f(R)$ gravity [7], an easy calculation shows that $\mathcal{I}_5\equiv\frac{1}{3}$. The JF BEPS parametrization is applicable in the range $0\leq\mathcal{I}_5<\frac{1}{3}$, while JF BDBW and EF canonical parametrizations cover $0\leq\mathcal{I}_5<\infty$. It has been noted before that in order to match the BDBW parameter range of $-\frac{3}{2}<\omega<0$, the BEPS parametrization should have the sign of the kinetic term flipped [22]. Violation of the "no ghosts" assumption (8), corresponding to BDBW $\omega<-\frac{3}{2}$, renders $\mathcal{I}_3$ imaginary and $\mathcal{I}_5$ negative.

In the calculations we sometimes encounter another invariant,

$$\mathcal{I}_6\equiv\left(\frac{\mathcal{I}_2'}{2\mathcal{I}_3'}\right)^2=\frac{(\mathcal{V}'\mathcal{A}-2\mathcal{V}\mathcal{A}')^2}{\mathcal{A}^4(2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2)}. \tag{26}$$

The invariants are conveniently summarized in Table I.

TABLE I. Invariants in different parametrizations.

Invariant	General parametrization	JF BDBW	JF BEPS	EF canonical
$\mathcal{I}_1$	$\frac{e^{2\alpha(\Phi)}}{\mathcal{A}(\Phi)}$	$\frac{1}{\Psi}$	$\frac{1}{F(\phi)}$	$e^{2\alpha(\varphi)}$
$\mathcal{I}_2$	$\frac{\mathcal{V}(\Phi)}{\mathcal{A}(\Phi)^2}$	$\frac{\mathcal{V}(\Psi)}{\Psi^2}$	$\frac{\mathcal{V}(\phi)}{F(\phi)^2}$	$\mathcal{V}(\varphi)$
$\mathcal{I}_3$	$\pm\int\sqrt{\frac{2\mathcal{A}(\Phi)\mathcal{B}(\Phi)+3(\mathcal{A}'(\Phi))^2}{4\mathcal{A}(\Phi)^2}}d\Phi$	$\pm\int\sqrt{\frac{2\omega(\Psi)+3}{4\Psi^2}}d\Psi$	$\pm\int\sqrt{\frac{2F(\phi)+3(F'(\phi))^2}{4F(\phi)^2}}d\phi$	$\pm\varphi+\mathrm{const}$
$\mathcal{I}_4\equiv\frac{\mathcal{I}_2}{\mathcal{I}_1^2}$	$\frac{\mathcal{V}(\Phi)}{e^{4\alpha(\Phi)}}$	$\mathcal{V}(\Psi)$	$\mathcal{V}(\phi)$	$\frac{\mathcal{V}(\varphi)}{e^{4\alpha(\varphi)}}$
$\mathcal{I}_5\equiv(\frac{\mathcal{I}_1'}{2\mathcal{I}_1\mathcal{I}_3'})^2$	$\frac{(2\alpha'(\Phi)\mathcal{A}(\Phi)-\mathcal{A}'(\Phi))^2}{2\mathcal{A}(\Phi)\mathcal{B}(\Phi)+3(\mathcal{A}'(\Phi))^2}$	$\frac{1}{2\omega(\Psi)+3}$	$\frac{(F'(\phi))^2}{2F(\phi)+3(F'(\phi))^2}$	$(\alpha'(\varphi))^2$
$\mathcal{I}_6\equiv(\frac{\mathcal{I}_2'}{2\mathcal{I}_3'})^2$	$\frac{(\mathcal{V}'(\Phi)\mathcal{A}(\Phi)-2\mathcal{V}(\Phi)\mathcal{A}'(\Phi))^2}{\mathcal{A}(\Phi)^4(2\mathcal{A}(\Phi)\mathcal{B}(\Phi)+3(\mathcal{A}'(\Phi))^2)}$	$\frac{(\mathcal{V}'(\Psi)\Psi-2\mathcal{V}(\Psi))^2}{\Psi^4(2\omega(\Psi)+3)}$	$\frac{(\mathcal{V}'(\phi)F(\phi)-2\mathcal{V}(\phi)F'(\phi))^2}{F(\phi)^4(2F(\phi)+3(F'(\phi))^2)}$	$\frac{(\mathcal{V}'(\varphi))^2}{4}$
$\mathcal{D}_1$	$\frac{e^{2\alpha(\Phi)}}{\mathcal{A}'(\Phi)-2\mathcal{A}(\Phi)\alpha'(\Phi)}\frac{d}{d\Phi}$	$\frac{d}{d\Psi}$	$\frac{1}{F'(\phi)}\frac{d}{d\phi}$	$-\frac{e^{2\alpha(\varphi)}}{2\alpha'(\varphi)}\frac{d}{d\varphi}$
$\mathcal{D}_2$	$\frac{\pm e^{\alpha(\Phi)}}{\sqrt{\mathcal{B}(\Phi)-6(\alpha'(\Phi))^2\mathcal{A}(\Phi)+6\alpha'(\Phi)\mathcal{A}'(\Phi)}}\frac{d}{d\Phi}$	$\pm\sqrt{\frac{\Psi}{\omega(\Psi)}}\frac{d}{d\Psi}$	$\pm\frac{d}{d\phi}$	$\frac{\pm 2^{-\frac{1}{2}}e^{\alpha(\varphi)}}{\sqrt{1-3(\alpha'(\varphi))^2}}\frac{d}{d\varphi}$
$\mathcal{D}_3$	$\pm\frac{2\mathcal{A}(\Phi)}{\sqrt{2\mathcal{A}(\Phi)\mathcal{B}(\Phi)+3(\mathcal{A}'(\Phi))^2}}\frac{d}{d\Phi}$	$\pm\frac{2\Psi}{\sqrt{2\omega(\Psi)+3}}\frac{d}{d\Psi}$	$\pm\frac{2F(\phi)}{\sqrt{2F(\phi)+3(F'(\phi))^2}}\frac{d}{d\phi}$	$\pm\frac{d}{d\varphi}$
$e^{2\alpha}g_{\mu\nu}$	$e^{2\alpha(\Phi)}g_{\mu\nu}$	$g_{\mu\nu}$	$g_{\mu\nu}$	$e^{2\alpha(\varphi)}g_{\mu\nu}$
$\mathcal{A}g_{\mu\nu}$	$\mathcal{A}(\Phi)g_{\mu\nu}$	$\Psi g_{\mu\nu}$	$F(\phi)g_{\mu\nu}$	$g_{\mu\nu}$

B. Invariant differential operators

Knowledge about the transformation properties of the differential (21) allows us to write out invariant differential operators for taking derivatives with respect to the scalar field. These will be in the following form:

$$\frac{1}{\bar{\mathcal{I}}_j'}\frac{d}{d\bar{\Phi}}=\frac{1}{\mathcal{I}_j'}\frac{d}{d\Phi}. \tag{27}$$

If we apply that operator to an invariant, then the result is also an invariant. For example,

$$\frac{1}{\mathcal{I}_j'}\frac{d}{d\Phi}\mathcal{I}_i=\frac{\mathcal{I}_i'}{\mathcal{I}_j'}, \tag{28}$$

and we have once again obtained Eq. (23). Note that as these invariants have the same argument, this result could also be written as a derivative of one invariant with respect to another,

$$\frac{\mathcal{I}_i'}{\mathcal{I}_j'}\equiv\frac{d\Phi}{d\mathcal{I}_j}\frac{d\mathcal{I}_i}{d\Phi}=\frac{d\mathcal{I}_i}{d\mathcal{I}_j}. \tag{29}$$

Previous knowledge becomes handy when we want to "translate" the results from a distinct parametrization into a general one. This procedure is based on fact that in common parametrizations any quantity or operator can be replaced by an invariant which in this parametrization functionally coincides with that quantity or operator. Namely, if for a fixed parametrization there is an invariant which is a fixed function, then we can construct an invariant differentiation operator (27) which in this parametrization functionally coincides with the derivative with respect to scalar field. For example, let us take a look at the JF BDBW parametrization. We have $\mathcal{I}_1=\frac{1}{\Psi}$, where $\Psi=\frac{1}{\mathcal{I}_1}$. Therefore,

$$\frac{d}{d\Psi}=\frac{d\Phi}{d\Psi}\frac{d}{d\Phi}=\frac{1}{\frac{d\Psi}{d\Phi}}\frac{d}{d\Phi}=\frac{1}{\frac{d}{d\Phi}\left(\frac{1}{\mathcal{I}_1}\right)}\frac{d}{d\Phi}. \tag{30}$$

Although that last equality holds only in this parametrization, it allows us to define the invariant differentiation operator,

$$\mathcal{D}_1\equiv\frac{1}{\frac{d}{d\Phi}\left(\frac{1}{\mathcal{I}_1}\right)}\frac{d}{d\Phi}=-\frac{\mathcal{I}_1^2}{\mathcal{I}_1'}\frac{d}{d\Phi}=\frac{e^{2\alpha}}{\mathcal{A}'-2\mathcal{A}\alpha'}\frac{d}{d\Phi}, \tag{31}$$

which in JF BDBW coincides with $\frac{d}{d\Psi}$. Analogically, in JF BEPS, $\frac{d}{d\phi}$ coincides with

$$\begin{aligned}\mathcal{D}_2&\equiv\frac{\sqrt{\mathcal{I}_1}}{\mathcal{I}_3'\sqrt{2(1-3\mathcal{I}_5)}}\frac{d}{d\Phi}\\&=\pm\frac{e^{\alpha}}{\sqrt{\mathcal{B}-6(\alpha')^2\mathcal{A}+6\alpha'\mathcal{A}'}}\frac{d}{d\Phi},\end{aligned} \tag{32}$$

and in EF canonical parametrization, $\frac{d}{d\varphi}$ coincides with

$$\mathcal{D}_3\equiv\frac{1}{\mathcal{I}_3'}\frac{d}{d\Phi}=\pm\frac{2\mathcal{A}}{\sqrt{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2}}\frac{d}{d\Phi}. \tag{33}$$

These results are also gathered in Table I. Equation (32) tells us once again that JF BEPS is narrower than the other two. The term under the square root must be non-negative and, therefore, $\mathcal{I}_5<\frac{1}{3}$.

C. Invariants in different parametrizations

The invariants and their functional forms in three common parametrizations are presented in Table I which can be used to obtain transformation rules between different parametrizations and the most general one. For example if one wants to find a relation between the JF BDBW scalar field Ψ and the EF canonical scalar field φ in terms of the JF BDBW variables, then one has to search for an invariant counterpart of the derivative with respect to the EF canonical scalar field $\frac{d}{d\varphi}$. From that row in Table I, one can write out

$$\pm\frac{2\Psi}{\sqrt{2\omega(\Psi)+3}}\frac{d}{d\Psi}=\pm\frac{d}{d\varphi}, \tag{34}$$

hence,

$$\left(\frac{d\Psi}{d\varphi}\right)^2=\frac{4\Psi^2}{2\omega(\Psi)+3}, \tag{35}$$

which can be integrated to obtain $\varphi(\Psi)$. If we want the same in terms of the EF variables, we should look for an invariant counterpart of the derivative with respect to the JF BDBW scalar field,

$$\frac{d}{d\Psi}=-\frac{e^{2\alpha(\varphi)}}{2\alpha'(\varphi)}\frac{d}{d\varphi}, \tag{36}$$

where

$$\left(\frac{d\Psi}{d\varphi}\right)^2=e^{-4\alpha(\varphi)}4(\alpha'(\varphi))^2, \tag{37}$$

which integrates to $\Psi(\varphi)$. The relation between the right-hand sides of Eqs. (35) and (37) can be also acquired by combining the rows for $\mathcal{I}_1$ and $\mathcal{I}_5$.

Table I can also be used for transforming invariant quantities from a distinct parametrization to the general one. For example, one may want to quickly find out how the expression $\frac{(\mathcal{V}'\Psi-2\mathcal{V})^2}{\Psi^4(2\omega+3)}$ written in JF BDBW reads in the general parametrization. Here one should take $\mathcal{I}_4$ which in JF BDBW has the same functional form as potential $\mathcal{V}$, and then apply invariant differentiation $\mathcal{D}_1$ on $\mathcal{I}_4$ to get the invariant counterpart for $\mathcal{V}'$. Further, the invariant $\mathcal{I}_1$ is in this parametrization identical to $\frac{1}{\Psi}$, while $\frac{1}{2\omega+3}$ should be replaced by $\mathcal{I}_5$. So combining these pieces together gives the whole expression,

$$(\mathcal{I}_1^{-1}\mathcal{D}_1\mathcal{I}_4-2\mathcal{I}_4)^2\mathcal{I}_1^4\mathcal{I}_5=\mathcal{I}_6, \tag{38}$$

where some manipulation and definitions of the invariants have been used on the lhs. However, if the quantity we want to transform is not invariant, some caution is needed since undetermined multiplicative factors of the transformation functions $\bar{f}$ and $\bar{\gamma}$ can be left out in the procedure.

D. Scalar field Φ as function of $\mathcal{I}_3$

In each parametrization we can, in principle, express Φ as a function of any invariant $\mathcal{I}_i$. Considering the scalar field equation of motion (59) later in the paper, it is useful to express Φ as a function of $\mathcal{I}_3$. In EF canonical parametrization $\mathcal{I}_3\sim\varphi$, but for some other parametrizations (e.g. JF BDBW and JF BEPS) $\mathcal{I}_3$ is given in the form of an indefinite integral (15) and finding an inverse can be complicated. However, under certain conditions we can always approximate $\Phi=\Phi(\mathcal{I}_3)$ as a Taylor expansion around some value Φ_0.

We start by noticing that $\mathcal{I}_3$ as an indefinite integral contains an integration constant which, in principle, can be chosen so that $\mathcal{I}_3|_{\Phi_0}=0$. Recall that

$$\frac{d}{d\mathcal{I}_3}=\frac{1}{\mathcal{I}_3'}\frac{d}{d\Phi}\equiv\pm\frac{1}{\sqrt{\mathcal{F}}}\frac{d}{d\Phi}. \tag{39}$$

Therefore, the Taylor expansion reads as follows:

$$\begin{aligned}\Phi(\mathcal{I}_3)-\Phi_0&=\frac{d\Phi}{d\mathcal{I}_3}\bigg|_{\Phi_0}\cdot\mathcal{I}_3+\frac{d^2\Phi}{d\mathcal{I}_3^2}\bigg|_{\Phi_0}\cdot\frac{\mathcal{I}_3^2}{2!}+\ldots\\&=\frac{1}{\mathcal{I}_3'}\bigg|_{\Phi_0}\cdot\mathcal{I}_3+\left[\frac{1}{\mathcal{I}_3'}\frac{d}{d\Phi}\left(\frac{1}{\mathcal{I}_3'}\right)\right]_{\Phi_0}\cdot\frac{\mathcal{I}_3^2}{2!}+\ldots\\&=\pm\frac{1}{\sqrt{\mathcal{F}}}\bigg|_{\Phi_0}\cdot\mathcal{I}_3+\left[\frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'\right]_{\Phi_0}\cdot\frac{\mathcal{I}_3^2}{2!}+\ldots,\end{aligned} \tag{40}$$

where we have used

$$\frac{1}{\sqrt{\mathcal{F}}}\frac{d}{d\Phi}\left(\frac{1}{\sqrt{\mathcal{F}}}\right)\equiv\frac{1}{\sqrt{\mathcal{F}}}\left(\frac{1}{\sqrt{\mathcal{F}}}\right)'=-\frac{1}{2}\frac{\mathcal{F}'}{\mathcal{F}^2}=\frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'. \tag{41}$$

One can show that the coefficients in the Taylor series (40) do not diverge, and at least some of them are nonvanishing if

$$0\leq\frac{1}{\mathcal{F}}<\infty, \tag{42}$$

$$-\infty<\left(\frac{1}{\mathcal{F}}\right)^{\overbrace{\prime\ldots\prime}^{n\text{-times}}}\equiv\frac{d^n}{d\Phi^n}\left(\frac{1}{\mathcal{F}}\right)<\infty, \tag{43}$$

$$\text{if}\quad\frac{1}{\mathcal{F}}\bigg|_{\Phi_0}=0,\quad\text{then}\quad\left(\frac{1}{\mathcal{F}}\right)'\bigg|_{\Phi_0}\neq0. \tag{44}$$

The same assumptions arose in the context of Friedmann cosmology [27]. They restrict the possible forms of

$\mathcal{F} \equiv (\mathcal{I}_3')^2$ and the scalar field dynamics. These assumptions complement the restrictions on $\mathcal{I}_1$, $\mathcal{I}_2$ and their derivatives discussed earlier. A few comments follow.

First, the assumption (44) imposes that $\frac{1}{\mathcal{F}}|_{\Phi_0} = 0$ is not an extremum. Therefore, if the scalar field Φ evolved through value Φ_0, then $\mathcal{F}$ would be negative, thereby violating the condition (42), i.e. (8). A consistent theory would avoid this happening. Indeed, if the linear term in the Taylor expansion (40) vanishes due to $\frac{1}{\mathcal{F}} = 0$, then the assumption (44) guarantees that the coefficient of the quadratic term is definitely nonvanishing,

$$\Phi(\mathcal{I}_3) - \Phi_0 \approx \frac{1}{4}\left(\frac{1}{\mathcal{F}}\right)'\bigg|_{\Phi_0} \mathcal{I}_3^2. \tag{45}$$

Hence, the possible scalar field Φ values are never smaller (higher) than Φ_0 if $(\frac{1}{\mathcal{F}})'|_{\Phi_0}$ is positive (negative), which means that the scalar field Φ can approach Φ_0 from above (form below).

Second, since here $\mathcal{I}_3$ is an invariant infinitesimal quantity, we can use it as a scale to compare the order of magnitude of the perturbation $\Phi(\mathcal{I}_3) - \Phi_0$ in different parametrizations. In the parametrization where $\mathcal{F}|_{\Phi_0}$ is regular, the Taylor expansion (40) starts with a linear term and the perturbation $\Phi(\mathcal{I}_3) - \Phi_0$ is the same order small as $\mathcal{I}_3$. While expanding at $\bar{\Phi}_0 = f(\Phi_0)$ in another parametrization, if $\bar{\mathcal{F}}|_{\bar{\Phi}_0}$ diverges, the corresponding perturbation $\bar{\Phi}(\mathcal{I}_3) - \bar{\Phi}_0$ is quadratically small compared to $\mathcal{I}_3$, as in Eq. (45).

Third, if the leading coefficient in the Taylor series (40) vanishes, then $\nabla_\mu \Phi|_{\Phi_0} = 0$ because $\mathcal{I}_3|_{\Phi_0} = 0$,

$$\begin{aligned}\nabla_\mu \Phi|_{\Phi_0} &\approx \nabla_\mu \left[\frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'\bigg|_{\Phi_0} \cdot \frac{\mathcal{I}_3^2}{2!}\right]_{\Phi_0} \\ &= \left[\frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'\bigg|_{\Phi_0} \cdot \mathcal{I}_3\right]_{\mathcal{I}_3|_{\Phi_0}} \nabla_\mu \mathcal{I}_3|_{\Phi_0} = 0,\end{aligned} \tag{46}$$

even if $\nabla_\mu \mathcal{I}_3|_{\Phi_0} \neq 0$.

Finally, we may remark that since in the Einstein frame canonical parametrization $\mathcal{I}_3 = \pm\varphi + \text{const}$, all discussion in this subsection is equivalent to the Taylor expansion of the general scalar field Φ as a function of the EF canonical scalar field φ.

IV. EQUATIONS OF MOTION

A. Equations of motion in the general parametrization

Varying the action (1) with respect to the metric tensor gives a tensor equation,

$$\begin{aligned}&\mathcal{A}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) + \left(\frac{1}{2}\mathcal{B} + \mathcal{A}''\right) g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \Phi \nabla_\sigma \Phi \\ &\quad - (\mathcal{B} + \mathcal{A}'') \nabla_\mu \Phi \nabla_\nu \Phi + \mathcal{A}'(g_{\mu\nu} \Box \Phi - \nabla_\mu \nabla_\nu \Phi) \\ &\quad + \frac{1}{\ell^2} g_{\mu\nu} \mathcal{V} - \kappa^2 T_{\mu\nu} = 0,\end{aligned} \tag{47}$$

where the matter energy-momentum tensor is

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{48}$$

Analogously, varying the action (1) with respect to the scalar field gives us an equation of motion for the scalar field,

$$R\mathcal{A}' + \mathcal{B}' g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi + 2\mathcal{B} \Box \Phi - 2\ell^{-2} \mathcal{V}' + 2\kappa^2 \alpha' T = 0, \tag{49}$$

where $T \equiv g^{\mu\nu} T_{\mu\nu}$. Using the trace of the tensor equation (47) to eliminate R from the scalar field equation (49) yields

$$\begin{aligned}&\frac{2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2}{\mathcal{A}} \Box \Phi + \frac{(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2)'}{2\mathcal{A}} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi \\ &\quad - \frac{2(\mathcal{A}\mathcal{V}' - 2\mathcal{A}'\mathcal{V})}{\ell^2 \mathcal{A}} + \frac{\kappa^2 (2\mathcal{A}\alpha' - \mathcal{A}')}{\mathcal{A}} T = 0.\end{aligned} \tag{50}$$

As alluded to before in subsection III A, one way to define the meaning of nonminimal coupling is that the scalar field in Eq. (50) is sourced by the contracted matter energy-momentum tensor T. Inspection of the last term on the lhs confirms the claim that nonminimal coupling is realized when $\mathcal{I}_1' \not\equiv 0$. The continuity equation

$$\nabla^\mu T_{\mu\nu} = \alpha' T \nabla_\nu \Phi \tag{51}$$

tells us that the usual matter energy-momentum is covariantly conserved in those parametrizations where $\alpha(\Phi) = \text{const}$.

B. Equations of motion in terms of the invariants

We have noted that the invariant object $\hat{g}_{\mu\nu} \equiv \mathcal{A} g_{\mu\nu}$, introduced in Eq. (18), can be taken as a metric tensor and, therefore, it is possible to calculate Christoffel symbols with respect to it,

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{\mathcal{A}'}{2\mathcal{A}} (\delta^\lambda_\mu \partial_\nu \Phi + \delta^\lambda_\nu \partial_\mu \Phi - g_{\mu\nu} g^{\lambda\sigma} \partial_\sigma \Phi). \tag{52}$$

Mathematically this result is the well-known transformation rule for Christoffel symbols under the conformal transformation [3,23] or the definition corresponding to Weyl-integrable geometry [13,25]. But here the point is simply that $\hat{\Gamma}^\lambda_{\mu\nu}$ remains invariant under the transformations

(2) and (3). Now we can use (52) to define the covariant derivative with respect to $\hat{g}_{\mu\nu}$, e.g. $\hat{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \hat{\Gamma}^\nu_{\mu\lambda} V^\lambda$, etc. Similarly, the objects $\hat{\Gamma}^\lambda_{\mu\nu}$ can be employed to build the Riemann-Christoffel tensor $\hat{R}^\lambda{}_{\mu\rho\nu}$, which in this case is manifestly invariant under conformal transformation and scalar field reparametrization. Therefore, we can also construct the Einstein tensor $\hat{G}_{\mu\nu} \equiv \hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R}$, which can be expressed in terms of $g_{\mu\nu}$ and $\mathcal{A}(\Phi)$ as

$$\begin{aligned}\hat{G}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{\mathcal{A}''}{\mathcal{A}}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\Phi\nabla_\sigma\Phi - \frac{\mathcal{A}''}{\mathcal{A}}\nabla_\mu\Phi\nabla_\nu\Phi \\ &\quad + \frac{\mathcal{A}'}{\mathcal{A}}g_{\mu\nu}\Box\Phi - \frac{\mathcal{A}'}{\mathcal{A}}\nabla_\mu\nabla_\nu\Phi - \frac{3(\mathcal{A}')^2}{4\mathcal{A}^2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\Phi\nabla_\sigma\Phi \\ &\quad + \frac{3(\mathcal{A}')^2}{2\mathcal{A}^2}\nabla_\mu\Phi\nabla_\nu\Phi. \end{aligned} \tag{53}$$

In the same spirit we can define an energy-momentum tensor that is invariant under conformal transformation and scalar field reparametrization,

$$\begin{aligned}\hat{T}_{\mu\nu} &\equiv -\frac{2}{\sqrt{-\hat{g}}}\frac{\delta S_m}{\delta\hat{g}^{\mu\nu}} = -\frac{2}{\mathcal{A}^2\sqrt{-g}}\frac{\delta g^{\mu\nu}}{\delta\hat{g}^{\mu\nu}}\frac{\delta S_m}{\delta g^{\mu\nu}} \\ &= \frac{1}{\mathcal{A}}\left\{-\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\right\} \equiv \frac{1}{\mathcal{A}}T_{\mu\nu}. \end{aligned} \tag{54}$$

Comparing the result (53) with Eq. (47), while taking into account the definitions (12), (14), (15) and (54), allows us to rewrite Eq. (47) as follows,

$$\begin{aligned}\mathcal{A}\{\hat{G}_{\mu\nu} &+ \hat{g}_{\mu\nu}\hat{g}^{\rho\sigma}\hat{\nabla}_\rho\mathcal{I}_3\hat{\nabla}_\sigma\mathcal{I}_3 - 2\hat{\nabla}_\mu\mathcal{I}_3\hat{\nabla}_\nu\mathcal{I}_3 \\ &+ \ell^{-2}\hat{g}_{\mu\nu}\mathcal{I}_2 - \kappa^2\hat{T}_{\mu\nu}\} = 0, \end{aligned} \tag{55}$$

and the scalar field equation (50) as

$$4\mathcal{A}^2\left\{\mathcal{F}\hat{\Box}\Phi + \frac{1}{2}\mathcal{F}'\hat{g}^{\mu\nu}\hat{\nabla}_\mu\Phi\hat{\nabla}_\nu\Phi - \frac{1}{2\ell^2}\mathcal{I}_2' + \frac{\kappa^2}{4}\frac{\mathcal{I}_1'}{\mathcal{I}_1}\hat{T}\right\} = 0. \tag{56}$$

Here $\Box$ operator with respect to $\hat{g}_{\mu\nu}$ is defined by

$$\hat{\Box}\Phi \equiv \frac{1}{\sqrt{-\hat{g}}}\partial_\mu(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu\Phi) = \frac{\mathcal{A}'}{\mathcal{A}^2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + \frac{1}{\mathcal{A}}\Box\Phi. \tag{57}$$

Due to the identity

$$\mathcal{I}_3'\hat{\Box}\mathcal{I}_3 = \mathcal{F}\hat{\Box}\Phi + \frac{1}{2}\mathcal{F}'\hat{g}^{\mu\nu}\hat{\nabla}_\mu\Phi\hat{\nabla}_\nu\Phi, \tag{58}$$

we may write the scalar field equation (56) as

$$4\mathcal{A}^2\mathcal{I}_3'\left\{\hat{\Box}\mathcal{I}_3 - \frac{1}{2\ell^2}\frac{\mathcal{I}_2'}{\mathcal{I}_3'} + \frac{\kappa^2}{4\mathcal{I}_1}\frac{\mathcal{I}_1'}{\mathcal{I}_3'}\hat{T}\right\} = 0. \tag{59}$$

Since by the assumption neither $\mathcal{A}$ nor $\mathcal{I}_3' = \pm\sqrt{\mathcal{F}}$ can vanish, we can divide the last equation with the term in front of the braces and obtain an equation where each term is an invariant,

$$\hat{\Box}\mathcal{I}_3 - \frac{1}{2\ell^2}\frac{d\mathcal{I}_2}{d\mathcal{I}_3} + \frac{\kappa^2}{4}\frac{d\ln\mathcal{I}_1}{d\mathcal{I}_3}\hat{T} = 0. \tag{60}$$

The logic of differentiation used here was introduced before Eq. (29).

C. Action in terms of the invariants

The definition of the conformally invariant metric tensor $\hat{g}_{\mu\nu} \equiv \mathcal{A}g_{\mu\nu}$ was based on the knowledge about the transformation properties of $\mathcal{A}$ given by (5), which were read off from the transformed action functional (4). Therefore, it is natural that we can also rewrite the action functional in terms of invariants up to a boundary term, namely,

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-\hat{g}}\{\hat{R} - 2\hat{g}^{\mu\nu}\hat{\nabla}_\mu\mathcal{I}_3\hat{\nabla}_\nu\mathcal{I}_3 - 2\ell^{-2}\mathcal{I}_2\} \\ &\quad + S_m[\mathcal{I}_1\hat{g}_{\mu\nu},\chi] + \frac{3}{2\kappa^2}\int_{V_4} d^4x\partial_\mu\left(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu\ln\mathcal{A}\right). \end{aligned} \tag{61}$$

Varying (61) with respect to $\hat{g}^{\mu\nu}$ and $\mathcal{I}_3$ gives us invariant expressions that coincide with the terms in braces in, respectively, Eqs. (55) and (59).

As already mentioned, the choice $\hat{g}_{\mu\nu} \equiv \mathcal{A}g_{\mu\nu}$ is not unique; it just seems to give the equations in the simplest form. Note that these expressions remind us of the Einstein frame equations because in the Einstein frame the invariant metric $\hat{g}_{\mu\nu}|_{EF}$ coincides with the Einstein frame metric $g_{\mu\nu}$, while the invariant $\mathcal{I}_2|_{EF}$ coincides with the Einstein frame potential $\mathcal{V}$. If we had chosen $\hat{g}_{\mu\nu} = e^{2\alpha}g_{\mu\nu}$, then the equations would have been more similar to the Jordan frame equations.

V. PPN PARAMETERS

A. PPN parameters in the JF BDBW parametrization

The aim of this section is to use Table I for writing the effective gravitational constant $G_{\rm eff}$ and the parametrized post-Newtonian parameters γ and β in terms of the invariants and thereby obtain a form which easily allows us to get the PPN parameters in any other parametrization. We start from the JF BDBW parametrization where the most general calculation was recently accomplished [28], expanding earlier Refs. [29]. Table I contains all possible objects occurring in that parametrization. We proceed under the premise that the PPN parameters are invariants and must be determined uniquely. It does not matter whether we use the transformations (5) to obtain the results in the general parametrization or substitute in the respective invariants from Table I in order to get an invariant which in a parametrization at hand functionally coincides with PPN parameters.

So, from Ref. [28] we take the following results calculated in the JF BDBW parametrization. The PPN ansatz assumes that in the absence of any perturbation we have flat Minkowski geometry as a background, which leads to the conditions $\mathcal{V}=0$ and $\mathcal{V}'=0$. Taking these conditions into account in the calculation gives a result which is expressed in terms of the scalar field effective mass,

$$m_\Psi \equiv \frac{1}{\ell}\sqrt{\frac{2\Psi}{2\omega(\Psi)+3}\frac{d^2\mathcal{V}}{d\Psi^2}}. \tag{62}$$

The effective gravitational constant that in an experimental setup multiplies the nonvarying constant $\frac{\kappa^2}{8\pi}$, and the PPN parameters are given by

$$G_{\mathrm{eff}} = \frac{1}{\Psi}\left(1+\frac{e^{-m_\Psi r}}{2\omega+3}\right), \tag{63}$$

$$\gamma - 1 = -\frac{2e^{-m_\Psi r}}{G_{\mathrm{eff}}\Psi(2\omega+3)}, \tag{64}$$

$$\beta - 1 = \frac{\frac{d\omega}{d\Psi}e^{-2m_\Psi r}}{G_{\mathrm{eff}}^2\Psi(2\omega+3)^3} - \frac{m_\Psi r}{G_{\mathrm{eff}}^2\Psi^2(2\omega+3)}\beta(r), \tag{65}$$

where the extra radius dependent contribution in β,

$$\begin{aligned}\beta(r) = &\frac{1}{2}e^{-2m_\Psi r} + (m_\Psi r + e^{m_\Psi r})\mathrm{Ei}(-2m_\Psi r)\\ &- e^{-m_\Psi r}\ln(m_\Psi r) + \frac{3\Psi}{2(2\omega+3)}\left(\frac{\frac{d^3V}{d\Psi^3}}{3\frac{d^2V}{d\Psi^2}} - \frac{1}{\Psi} - \frac{\frac{d\omega}{d\Psi}}{2\omega+3}\right)\\ &\times (e^{m_\Psi r}\mathrm{Ei}(-3m_\Psi r) - e^{-m_\Psi r}\mathrm{Ei}(-m_\Psi r)),\end{aligned} \tag{66}$$

involves exponential integrals $\mathrm{Ei}(m_\Psi r)$. It is understood in these formulas that Ψ and the functions $\omega(\Psi)$, $V(\Psi)$, etc., are all evaluated at the spatially asymptotic constant background value of Ψ.

B. PPN parameters in terms of the invariants

Let us rewrite the previous result in terms of invariants by making use of Table I. The first constraint arising from Minkowskian boundary conditions, $\mathcal{V}=0$, translates into $\mathcal{I}_4 \equiv \frac{\mathcal{I}_2}{\mathcal{I}_1^2} = 0$, which implies $\mathcal{I}_2 = 0$. The second condition $\mathcal{V}'=0$ gives $\mathcal{D}_1\mathcal{I}_4|_{\mathcal{I}_2=0} \equiv \frac{\mathcal{I}_2'}{\mathcal{I}_1'} = 0$. Similarly, the scalar field effective mass reads

$$m_\Phi = \frac{1}{\ell}\sqrt{2\mathcal{I}_1^{-1}\mathcal{I}_5\mathcal{D}_1^2\mathcal{I}_4} = \frac{1}{\ell}\sqrt{\frac{\mathcal{I}_2''}{2\mathcal{I}_1(\mathcal{I}_3')^2}}. \tag{67}$$

Here in order to preserve a simpler form of the expression on the rhs, we have substituted the Minkowskian boundary conditions written in terms of the invariants. The quantity on the rhs is invariant only under these conditions. The effective gravitational constant and the PPN parameters γ and β turn out to be

$$G_{\mathrm{eff}} = \mathcal{I}_1(1+\mathcal{I}_5 e^{-m_\Phi r}), \tag{68}$$

$$\gamma - 1 = -\frac{2e^{-m_\Phi r}}{G_{\mathrm{eff}}}\mathcal{I}_1\mathcal{I}_5, \tag{69}$$

$$\begin{aligned}\beta - 1 &= \frac{e^{-2m_\Phi r}}{G_{\mathrm{eff}}^2\mathcal{I}_1^{-1}}\mathcal{I}_5^3\left[\mathcal{D}_1\left(\frac{1}{2}\left(\frac{1}{\mathcal{I}_5}-3\right)\right)\right] - \frac{m_\Phi r}{G_{\mathrm{eff}}^2\mathcal{I}_1^{-2}}\mathcal{I}_5\beta(r)\\ &= \frac{1}{2}\frac{\mathcal{I}_1^3\mathcal{I}_5}{G_{\mathrm{eff}}^2}\frac{\mathcal{I}_5'}{\mathcal{I}_1'}e^{-2m_\Phi r} - \frac{m_\Phi r}{G_{\mathrm{eff}}^2}\mathcal{I}_1^2\mathcal{I}_5\beta(r),\end{aligned} \tag{70}$$

where

$$\begin{aligned}\beta(r) = &\frac{1}{2}e^{-2m_\Phi r} + (m_\Phi r + e^{m_\Phi r})\mathrm{Ei}(-2m_\Phi r) - e^{-m_\Phi r}\ln(m_\Phi r)\\ &+\frac{3\mathcal{I}_5}{2\mathcal{I}_1}\left(\frac{1}{3}\frac{\mathcal{D}_1^3\mathcal{I}_4}{\mathcal{D}_1^2\mathcal{I}_4} - \mathcal{I}_1 - \mathcal{I}_5\mathcal{D}_1\left(\frac{1}{2}\left(\frac{1}{\mathcal{I}_5}-3\right)\right)\right)\\ &\times (e^{m_\Phi r}\mathrm{Ei}(-3m_\Phi r) - e^{-m_\Phi r}\mathrm{Ei}(-m_\Phi r)).\end{aligned} \tag{71}$$

Note that the quantity $m_\Phi r$ is invariant in our conventions.

C. PPN parameters in the general parametrization

In the general parametrization, expressing the invariants in terms of the functions $\{\mathcal{A},\mathcal{B},\mathcal{V},\alpha\}$, the result is [20]

$$m_\Phi = \frac{1}{\ell}\sqrt{e^{-2\alpha}\frac{2\mathcal{A}\mathcal{V}''}{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2}} \tag{72}$$

and

$$G_{\mathrm{eff}} = \frac{e^{2\alpha}}{\mathcal{A}}\left(1+\frac{(2\alpha'\mathcal{A}-\mathcal{A}')^2}{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2}e^{-m_\Phi r}\right), \tag{73}$$

$$\gamma - 1 = -\frac{2e^{-m_\Phi r}}{G_{\mathrm{eff}}}\frac{e^{2\alpha}}{\mathcal{A}}\frac{(2\alpha'\mathcal{A}-\mathcal{A}')^2}{(2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2)}, \tag{74}$$

$$\begin{aligned}\beta - 1 = &\frac{e^{4\alpha}}{2\mathcal{A}G_{\mathrm{eff}}^2}\frac{(2\alpha'\mathcal{A}-\mathcal{A}')}{(2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2)}\left(\frac{(2\alpha'\mathcal{A}-\mathcal{A}')^2}{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2}\right)'e^{-2m_\Phi r}\\ &-\frac{m_\Phi r}{G_{\mathrm{eff}}^2}\frac{e^{4\alpha}}{\mathcal{A}^2}\frac{(2\alpha'\mathcal{A}-\mathcal{A}')^2}{(2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2)}\beta(r).\end{aligned} \tag{75}$$

Now it is easy to check that there is a match with the Einstein frame calculation [30,31] and the corresponding expression in the JF BEPS parametrization [22]. The effective mass (72) differs from the one obtained in Ref. [31] by the factor $e^{-\alpha}$, but in the conventions of Ref. [31] this is precisely the factor that relates the masses in the Jordan and Einstein frames.

VI. COSMOLOGICAL SOLUTIONS

A. Equations for flat FLRW cosmology without matter

Let us start with the flat ($k = 0$) Friedmann-Lemaître-Robertson-Walker (FLRW) line element

$$\begin{aligned} ds^2 &\equiv g_{\mu\nu}dx^\mu dx^\nu \\ &= -dt^2 + (a(t))^2\{dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2\}. \end{aligned} \tag{76}$$

Now take the conformally invariant metric tensor $\hat{g}_{\mu\nu} \equiv \mathcal{A}g_{\mu\nu}$, Eq. (18), where $g_{\mu\nu}$ is in the FLRW form. In order to have $\hat{g}_{\mu\nu}$ also in that form, we should make a coordinate transformation and the scale factor redefinition

$$\frac{d}{d\hat{t}} \equiv \frac{1}{\sqrt{\mathcal{A}}}\frac{d}{dt}, \tag{77}$$

$$\hat{a}(\hat{t}) \equiv \sqrt{\mathcal{A}}a(t). \tag{78}$$

The Hubble parameter $\hat{H}$ calculated in terms of the invariant variables is related to the Hubble parameter H calculated in the frame defined by $g_{\mu\nu}$ as

$$\hat{H} \equiv \frac{1}{\sqrt{\mathcal{A}}}\left(H + \frac{1}{2}\frac{\mathcal{A}'}{\mathcal{A}}\dot{\Phi}\right). \tag{79}$$

Plugging the invariant form of the FLRW metric (76) into Eqs. (55) and (60) yields

$$\hat{H}^2 = \frac{1}{3}\left(\frac{d}{d\hat{t}}\mathcal{I}_3\right)^2 + \frac{1}{3\ell^2}\mathcal{I}_2, \tag{80}$$

$$2\frac{d}{d\hat{t}}\hat{H} + 3\hat{H}^2 = -\left(\frac{d}{d\hat{t}}\mathcal{I}_3\right)^2 + \frac{1}{\ell^2}\mathcal{I}_2, \tag{81}$$

$$\frac{d}{d\hat{t}}\left(\frac{d}{d\hat{t}}\mathcal{I}_3\right) = -3\hat{H}\frac{d}{d\hat{t}}\mathcal{I}_3 - \frac{1}{2\ell^2}\frac{\mathcal{I}_2'}{\mathcal{I}_3'}. \tag{82}$$

We have dropped the matter terms, i.e. $\hat{T}_{\mu\nu} \equiv 0$. By doing this we have truncated the theory by omitting α; thus, we are left with only three arbitrary functions $\{\mathcal{A}, \mathcal{B}, \mathcal{V}\}$.

B. Scalar field equation as a dynamical system

The first equation of the system (80)–(82) is a constraint; therefore, we may focus only upon Eq. (82) where the geometrical quantity $\hat{H}$ has been substituted from Eq. (80),

$$\frac{d}{d\hat{t}}\left(\frac{d}{d\hat{t}}\mathcal{I}_3\right) = -\varepsilon\sqrt{3\left(\frac{d}{d\hat{t}}\mathcal{I}_3\right)^2 + \frac{3}{\ell^2}\mathcal{I}_2}\,\frac{d}{d\hat{t}}\mathcal{I}_3 - \frac{1}{2\ell^2}\frac{d\mathcal{I}_2}{d\mathcal{I}_3}, \tag{83}$$

where $\varepsilon = +1$ ($\varepsilon = -1$) corresponds to an expanding (contracting) universe with respect to the metric $\hat{g}_{\mu\nu}$. In order to learn about the general features of the cosmological solutions it is instructive to write the scalar field equation as a dynamical system and ask whether there are any fixed points and what are their properties. For Φ_0 to give a fixed point we must insist that $\frac{d}{d\hat{t}}\mathcal{I}_3|_{\Phi_0} = 0$ and $\frac{d^2}{d\hat{t}^2}\mathcal{I}_3|_{\Phi_0} = 0$. From Eq. (83) we see that this occurs when

$$\frac{d\mathcal{I}_2}{d\mathcal{I}_3} \equiv \left.\frac{\mathcal{I}_2'}{\mathcal{I}_3'}\right|_{\Phi_0} = 0. \tag{84}$$

Hereby we may distinguish two types of the scalar field values Φ_0:

$$\Phi_\bullet:\ \mathcal{I}_2'|_{\Phi_\bullet} = 0, \qquad \left.\frac{1}{\mathcal{I}_3'}\right|_{\Phi_\bullet} \neq 0, \tag{85}$$

$$\Phi_\star:\ \left.\frac{1}{\mathcal{I}_3'}\right|_{\Phi_\star} = 0. \tag{86}$$

Note that the condition (84) for a fixed point is invariant, while the distinction (85), (86) is not. Therefore, if a fixed point occurs in some parametrization, then a corresponding fixed point will be present in any parametrization. However, whether the fixed point satisfies (85) or (86) might depend on the parametrization.

Linearizing Eq. (83) around a fixed point ($\mathcal{I}_3(\Phi_0) = 0$, $\frac{d}{d\hat{t}}\mathcal{I}_3 = 0$) gives

$$\frac{d}{d\hat{t}}\left(\frac{d}{d\hat{t}}\mathcal{I}_3\right) = -\varepsilon\left.\sqrt{\frac{3}{\ell^2}\mathcal{I}_2}\right|_{\Phi_0}\cdot\frac{d}{d\hat{t}}\mathcal{I}_3 - \frac{1}{2\ell^2}\left.\frac{d^2\mathcal{I}_2}{d\mathcal{I}_3^2}\right|_{\Phi_0}\cdot\mathcal{I}_3 \tag{87}$$

or, written as a dynamical system,

$$\begin{pmatrix} \frac{d}{d\hat{t}}\mathcal{I}_3 \\ \frac{d}{d\hat{t}}\Pi \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2\ell^2}\frac{d^2\mathcal{I}_2}{d\mathcal{I}_3^2} & -\varepsilon\sqrt{\frac{3}{\ell^2}\mathcal{I}_2} \end{bmatrix}_{\Phi_0}\begin{pmatrix} \mathcal{I}_3 \\ \Pi \end{pmatrix}, \tag{88}$$

where $\Pi \equiv \frac{d}{d\hat{t}}\mathcal{I}_3$.

C. Solution to the linearized equation

Solutions of the linearized equation (87) are determined by the eigenvalues of the matrix in Eq. (88). A straightforward calculation shows that the eigenvalues are

$$\lambda_{\pm}^{\varepsilon} = \frac{1}{2\ell}\left[-\varepsilon\sqrt{3\mathcal{I}_2} \pm \sqrt{3\mathcal{I}_2 - 2\frac{d^2\mathcal{I}_2}{d\mathcal{I}_3^2}}\right]_{\Phi_0}. \tag{89}$$

It is clear that these eigenvalues are invariant. As the properties of a fixed point, i.e. the characteristic features of the solutions near that point, are determined by the real and imaginary parts of the eigenvalues, we can infer that if a fixed point is an attractor in one parametrization, it will be an attractor in any parametrization, etc. The qualitative features of the solutions like convergence and periods of oscillation are independent of the parametrization. Writing the eigenvalues in terms of the arbitrary functions $\{\mathcal{A},\mathcal{B},\mathcal{V}\}$ gives

$$\lambda_{\pm}^{\varepsilon} = \frac{1}{\ell\sqrt{\mathcal{A}(\Phi_0)}}\left[-\varepsilon\sqrt{\frac{3\mathcal{V}}{4\mathcal{A}}} \pm \sqrt{\frac{3\mathcal{V}}{4\mathcal{A}} - 2\frac{(\mathcal{V}'\mathcal{A}-2\mathcal{V}\mathcal{A}')'}{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2} - \left(\frac{1}{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2}\right)'(\mathcal{V}'\mathcal{A}-2\mathcal{V}\mathcal{A}')}\right]_{\Phi_0}. \tag{90}$$

Here under the second square root we have realized that if $\frac{\mathcal{I}_2'}{\mathcal{I}_3'}=0$ then also $\frac{\mathcal{I}_2'}{(\mathcal{I}_3')^2}=0$ due to the assumption (42). From the eigenvalues (90) we see that if $\frac{1}{\mathcal{F}}\equiv\frac{4\mathcal{A}^2}{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2}=0$ and $(\frac{1}{\mathcal{F}})'\equiv(\frac{4\mathcal{A}^2}{2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2})'=0$ at the same value Φ_0, then one of the eigenvalues is zero, hence its real part is also zero and the fixed point is nonhyperbolic. Therefore, the assumptions (42)–(44) are necessary conditions for studying the properties of the fixed points by using linearization.

If the eigenvalues are different, then the general solution for Eq. (87) reads

$$\mathcal{I}_3(\hat{t}) = M_1 e^{\lambda_+^{\varepsilon}\hat{t}} + M_2 e^{\lambda_-^{\varepsilon}\hat{t}}, \tag{91}$$

where M_1 and M_2 are constants of integration. We can make use of the Taylor expansion (40) to write out the solution for scalar field Φ from (91). If the scalar field value at that fixed point is determined by the condition (85), then the leading term in the Taylor expansion is linear and gives

$$\Phi(\hat{t}) - \Phi_{\bullet} \approx \pm\frac{1}{\sqrt{\mathcal{F}}}\bigg|_{\Phi_{\bullet}}\mathcal{I}_3(\hat{t}). \tag{92}$$

On the other hand, if the scalar field value is determined by the condition (86), then the first coefficient of the Taylor expansion (40) vanishes and the leading term is of the second order (45),

$$\Phi(\hat{t}) - \Phi_{\star} \approx 0 + \frac{1}{4}\left(\frac{1}{\mathcal{F}}\right)'\bigg|_{\Phi_{\star}}\cdot\mathcal{I}_3^2(\hat{t}). \tag{93}$$

In the latter case the solution is

$$\Phi(\hat{t}) - \Phi_{\star} \approx \frac{1}{4}\left(\frac{1}{\mathcal{F}}\right)'\bigg|_{\Phi_{\star}}(M_1 e^{\lambda_+^{\varepsilon}\hat{t}} + M_2 e^{\lambda_-^{\varepsilon}\hat{t}})^2. \tag{94}$$

Here the underlying perturbed equation for Φ could not have been a linear one, and this is exactly in accord with the approach in Ref. [32]. See also the discussion around Eq. (45).

The redefinition of time $\hat{t}\to t$ should rigorously be given as an integral due to Eq. (77). Since $\mathcal{A}$ is assumed to be always positive, nondiverging and nonvanishing, we conclude that we can just substitute $\hat{t}=\sqrt{\mathcal{A}}t$ because this has no effect on the properties of the fixed point.

Analysis of the $\lambda_+^{\varepsilon}=\lambda_-^{\varepsilon}$ case can be handled in a similar manner.

D. Eigenvalues in different parametrizations

Writing the eigenvalues in the general parametrization (90) in terms of the Jordan frame BDBW parametrization gives

$$\lambda_{\pm}^{\varepsilon(BDBW)} = \frac{1}{\ell\sqrt{\Psi_0}}\left[-\varepsilon\sqrt{\frac{3\mathcal{V}}{4\Psi}} \pm \sqrt{\frac{3\mathcal{V}}{4\Psi} - 2\frac{(\mathcal{V}'\Psi-2\mathcal{V})'}{2\omega+3} - \left(\frac{1}{2\omega+3}\right)'(\mathcal{V}'\Psi-2\mathcal{V})}\right]_{\Psi_0}. \tag{95}$$

For the more usual fixed point at $\Phi_{\bullet}$, this result coincides with the eigenvalues found in Refs. [27,33], while for the nonlinear situation of $\Phi_{\star}$ this result matches the solutions obtained in Ref. [34]. The eigenvalues (90) expressed in the JF BEPS parametrization read

$$\lambda_{\pm}^{\varepsilon(\mathrm{BEPS})} = \frac{1}{\ell\sqrt{F(\phi_0)}}\left[-\varepsilon\sqrt{\frac{3\mathcal{V}}{4F}} \pm \sqrt{\frac{3\mathcal{V}}{4F} - 2\frac{\mathcal{V}''F^2-2\mathcal{V}((F')^2+FF'')}{F(2F+3(F')^2)}}\right]_{\phi_0}. \tag{96}$$

For instance these can be compared to the present accelerating epoch in the model with specific curvature coupling function but general potential in Ref. [35]. The fixed point stability condition is determined by the real part of the eigenvalues. Note that in the JF BEPS parametrization only the $\Phi_\bullet$ case (85) can be realized. The last remark holds true also for the Einstein frame canonical parametrization for which the eigenvalues,

$$\lambda_{\pm}^{\varepsilon(\mathrm{EF\,can})} = \frac{1}{2\ell}[-\varepsilon\sqrt{3\mathcal{V}} \pm \sqrt{3\mathcal{V} - 2\mathcal{V}''}]_{\varphi_0}, \tag{97}$$

obtained from (90) are in accord with the results for the general potential case analyzed in Ref. [36], as well as the solutions in Ref. [37].

VII. CONCLUSION

We have considered general scalar-tensor gravity without derivative couplings. Using the transformation properties of four arbitrary functions $\{\mathcal{A}(\Phi), \mathcal{B}(\Phi), \mathcal{V}(\Phi), \alpha(\Phi)\}$, we have constructed three functions $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ of the scalar field Φ that are invariant under a local rescaling of the metric tensor and the scalar field reparametrization. These three invariants can be used to define infinitely many analogical invariants via three procedures: (i) forming arbitrary functions of these, (ii) introducing quotients of derivatives $\mathcal{I}_m \equiv \frac{\mathcal{I}'_k}{\mathcal{I}'_l}$, and (iii) integrating in the sense of the indefinite integral $\mathcal{I}_r \equiv \int \mathcal{I}_n \mathcal{I}'_p d\Phi$. Using these invariants we have written down the rules that easily allow us to transform invariant quantities from three distinct parametrizations (JF BDBW, JF BEPS and EF canonical) into the general one. Useful formulas are gathered into Table I. By introducing an invariant object $\hat{g}_{\mu\nu} \equiv \mathcal{A} g_{\mu\nu}$, we can write the equations of motion and the action in terms of invariants.

We argue that physical observables appear as invariant quantities. This is illustrated by PPN parameters and the features of cosmological solutions near scalar field fixed points. We demonstrate that these invariant expressions accommodate the results obtained in earlier literature for distinct conformal frames and reparametrizations of the scalar field. In a particular case, this formalism provides a nice explanation of the correspondence of linear and nonlinear approximate solutions in the Einstein and Jordan frames.

In the future it would be interesting to see, whether the invariant variables proposed here would help to clarify the contested issues of the frame dependence of cosmological perturbations and quantum corrections in STG. As an extension one may consider whether an analogous reasoning can be carried out for more general scalar-tensor theories of gravity with derivative couplings and higher-order derivatives in action [6], where the role of conformal transformation seems to be taken over by disformal transformation [38].

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Chapter 7

Some remarks concerning invariant quantities in scalar-tensor gravity

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Some remarks concerning invariant quantities in scalar-tensor gravity

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Abstract

The aim of the current paper is to clarify some aspects of the formalism used for describing the scalar-tensor gravity characterized by four arbitrary local functionals of the scalar field. We recall the objects that are invariant with respect to a spacetime point under the local Weyl rescaling of the metric and under the scalar field redefinition. We phrase and prove a theorem that allows to link such an object to each quantity in a theory where two out of the four arbitrary local functionals of the scalar field are specified in a suitable manner. Based on these results we phrase and reason the existence of the so called translation rules.

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Keywords. Invariants, scalar-tensor theory of gravity

1 Introduction

The history of scalar-tensor theories of gravity (STG) is long, starting with the works of Jordan [6], later developed by Brans and Dicke [1], [2]. The original idea was purely theoretical since there were no observational contradictions to Einstein's general relativity (GR). In about a decade ago astronomers claimed that the Universe is expanding in an accelerating manner

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and explained that in the context of GR with a nonvanishing cosmological constant. This needs finetuning which we would like to avoid in a fundamental theory. Due to the latter studying the extensions of GR, STG being one of them, is still popular.

The aim of the current paper is to clarify some mathematical issues concerning the invariant quantities in general STG and the so called translation rules that were proposed in our recent paper [8]. A more detailed introduction and references to the literature on that subject can also be found there.

The outline of the paper is the following. In Section 2 we recall the general framework for STG mostly relying on the paper by Flanagan [4]. Section 3 summarizes the results of Ref. [8] that will be used in the current paper. In Section 4 we phrase and prove a lemma and a theorem claiming the existence of the so called invariant pair. In Section 5 we point out an important corollary of the latter. Based on these results we formulate and reason the existence of the so called translation rules proposed in Ref. [8].

2 Parametrizations in scalar-tensor theories of gravity

In a scalar-tensor theory of gravity the gravitational interaction is characterized by a metric tensor $g_{\mu\nu}(x^\mu)$ of a curved spacetime $x^\mu \in V_4$ and a scalar field $\Phi(x^\mu)$. In the current paper we consider a family of scalar-tensor theories of gravity by postulating a general action functional [4]

$$S = \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-g}\left\{\mathcal{A}(\Phi)R - \mathcal{B}(\Phi)g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - 2\ell^{-2}\mathcal{V}(\Phi)\right\} + S_m\left[e^{2\alpha(\Phi)}g_{\mu\nu},\chi\right] \tag{2.1}$$

which contains four arbitrary local functionals $\{\mathcal{A}(\Phi),\, \mathcal{B}(\Phi),\, \mathcal{V}(\Phi),\, \alpha(\Phi)\}$ of the dimensionless scalar field $\Phi(x^\mu)$. Out of the four the local functional $\mathcal{A}(\Phi)$ is multiplied by the Ricci scalar R and occasionally the term 'curvature coupling' is used to refer to $\mathcal{A}(\Phi)$. Analogically 'kinetic coupling' refers to $\mathcal{B}(\Phi)$, i.e. to the multiplier of the kinetic term for the scalar field $\Phi(x^\mu)$. The local functional $\mathcal{V}(\Phi)$ is known as the scalar field potential and from the particle physics viewpoint it contains the scalar field self-interactions. For a general case the matter action functional S_m depends on the metric tensor $g_{\mu\nu}$ via conformal coupling $e^{2\alpha(\Phi)}$, i.e. the spacetime indexes in the Lagrangian for the matter fields, collectively denoted as χ, are contracted by $e^{2\alpha(\Phi)}g_{\mu\nu}$ and its inverse. The term 'matter coupling' is frequently used to refer to $\alpha(\Phi)$. Due to suitably chosen dimensionful constants κ^2 and ℓ^{-2} the four arbitrary local functionals $\{\mathcal{A}(\Phi), \mathcal{B}(\Phi), \mathcal{V}(\Phi), \alpha(\Phi)\}$ are dimensionless and if the functional form w.r.t. $\Phi(x^\mu)$ of each of them is fixed then the theory is fixed. Let us point out that all local functionals of $\Phi(x^\mu)$ inherit a dependence on x^μ and hence are functions of a spacetime point as well.

Proposition 2.1. *If under the local Weyl rescaling of the metric tensor and under the scalar field redefinition*

$$g_{\mu\nu} = e^{2\bar{\gamma}(\bar{\Phi})}\bar{g}_{\mu\nu}\,, \tag{2.2}$$

$$\Phi = \bar{f}(\bar{\Phi}) \tag{2.3}$$

the four arbitrary local functionals are imposed to transform as

$$\mathcal{A}\left(\bar{f}(\bar{\Phi})\right) = e^{-2\bar{\gamma}(\bar{\Phi})}\bar{\mathcal{A}}(\bar{\Phi})\,, \tag{2.4a}$$

$$\mathcal{B}\left(\bar{f}(\bar{\Phi})\right) = e^{-2\bar{\gamma}(\bar{\Phi})}\left(\bar{f}'\right)^{-2}\left(\bar{\mathcal{B}}(\bar{\Phi}) - 6\left(\bar{\gamma}'\right)^2\bar{\mathcal{A}}\left(\bar{\Phi}\right) + 6\bar{\gamma}'\bar{\mathcal{A}}'\right)\,, \tag{2.4b}$$

$$\mathcal{V}\left(\bar{f}(\bar{\Phi})\right) = e^{-4\bar{\gamma}(\bar{\Phi})}\,\bar{\mathcal{V}}(\bar{\Phi})\,, \tag{2.4c}$$

$$\alpha\left(\bar{f}(\bar{\Phi})\right) = \bar{\alpha}(\bar{\Phi}) - \bar{\gamma}(\bar{\Phi}) \tag{2.4d}$$

then the action functional (2.1) *is invariant under the transformations* (2.2)-(2.3) *up to a boundary term [4].*

Here and in the following we shall drop the arguments of the functionals unless confusion might arise. Let us also adopt a notation where prime as a superscript of a “barred" local functional of the scalar field means variational derivative w.r.t. the “barred" scalar field $\bar{\Phi}(x^\mu)$ and prime as a superscript of such a quantity without “bar" means variational derivative w.r.t. the “unbarred" scalar field $\Phi(x^\mu)$, e.g. $\bar{f}' \equiv \dfrac{\delta\bar{f}(\bar{\Phi})}{\delta\bar{\Phi}}$ and $\mathcal{A}' \equiv \dfrac{\delta\mathcal{A}(\Phi)}{\delta\Phi}$ respectively. Note that due to the inherited dependence on a spacetime point one can differentiate functionals of Φ w.r.t. x^μ via ordinary partial derivatives.

The relations (2.4) are obtained by rewriting the action functional (2.1) using $\bar{g}_{\mu\nu}$ and $\bar{\Phi}$ as dynamical fields. In the current paper we assume the affine connection to be the Levi-Civita one. Due to the latter such a rewriting of the action functional (2.1) also introduces a boundary term but here and in the following we shall drop boundary terms. We also assume the premiss of Proposition 2.1 to hold and whenever Eqs. (2.2)-(2.3) are recalled also Eqs. (2.4) are taken into account.

Definition 2.2 (parametrization)**.** If the functional form w.r.t. Φ of exactly two out of the four arbitrary local functionals $\{\mathcal{A},\, \mathcal{B},\, \mathcal{V},\, \alpha\}$ is fixed then we say that the theory is given in a specific frame and parametrization.

The term 'reparametrization' refers to the scalar field redefinition (2.3) while the Weyl rescaling (2.2) is the change of the 'frame'. Roughly speaking both of these transformations can be used to fix the functional form of one arbitrary local functional out of the four. A closer look on the transformation properties (2.4) reveals that all four arbitrary local functionals transform under the Weyl rescaling (2.2) but it might be the case that not all of them transform under the scalar field redefinition (2.3) (e.g. $\mathcal{A} = 1$). Therefore it is convenient to think that first the frame is chosen, i.e. we specify the metric tensor, and then the parametrization is chosen. In that sense the latter involves the former and in the following an explicit reference to the chosen frame is suppressed.

Example. The Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization (JF BDBW) with the scalar field denoted as Ψ is given by [1], [3], [5]:

$$\mathcal{A} \equiv \Psi \quad , \quad \mathcal{B} \equiv \frac{\omega(\Psi)}{\Psi} \quad , \quad \mathcal{V} \equiv \mathcal{V}_{\mathfrak{J}}(\Psi) \quad , \quad \alpha \equiv 0\,. \tag{2.5}$$

The Einstein frame canonical parametrization (EF canonical) with the scalar field denoted as φ is given by [2], [3], [5]:

$$\mathcal{A} \equiv 1 \quad , \quad \mathcal{B} \equiv 2 \quad , \quad \mathcal{V} \equiv \mathcal{V}_{\mathfrak{E}}(\varphi) \quad , \quad \alpha \equiv \alpha_{\mathfrak{E}}(\varphi)\,. \tag{2.6}$$

A parametrization is in principle meaningful without considering the Weyl rescaling (2.2) and the scalar field redefinition (2.3) at all but nevertheless in a generic case these transformations can be used to transform an arbitrary set of functionals $\{\mathcal{A}(\Phi),\, \mathcal{B}(\Phi),\, \mathcal{V}(\Phi),\, \alpha(\Phi)\}$ into e.g. JF BDBW parametrization (2.5). Hence a chosen parametrization is not a unique description of a theory.

Example. In order to transform from JF BDBW parametrization (2.5) to EF canonical parametrization (2.6) we consider the relations

$$e^{2\bar{\gamma}(\varphi)} = e^{2\alpha_{\mathfrak{E}}(\varphi)}\,, \quad \left(\frac{\delta\Psi}{\delta\varphi}\right)^2 = 4e^{-4\alpha_{\mathfrak{E}}(\varphi)}\left(\frac{\delta\alpha_{\mathfrak{E}}(\varphi)}{\delta\varphi}\right)^2 \to \Psi = \Psi(\varphi) \tag{2.7}$$

in the case when EF canonical parametrization quantities are considered to be the "barred" ones. For the reverse transformation we choose

$$e^{2\bar{\gamma}(\Psi)} = \Psi\,, \quad \left(\frac{\delta\varphi}{\delta\Psi}\right)^2 = \frac{2\omega(\Psi)+3}{4\Psi^2} \to \varphi \equiv \varphi(\Psi) \tag{2.8}$$

if instead JF BDBW parametrization quantities are considered to be the "barred" ones [3].

3 Invariants

Let us recall three basic objects introduced in our recent paper [8]

$$\mathcal{I}_1(\Phi) \equiv \frac{e^{2\alpha(\Phi)}}{\mathcal{A}(\Phi)}\,, \qquad \mathcal{I}_2(\Phi) \equiv \frac{\mathcal{V}(\Phi)}{\mathcal{A}(\Phi)^2}\,, \tag{3.1}$$

$$\mathcal{I}_3(\Phi) \equiv \pm \int \sqrt{\frac{2\mathcal{A}(\Phi)\mathcal{B}(\Phi) + 3\left(\mathcal{A}'(\Phi)\right)^2}{4\mathcal{A}(\Phi)^2}}\delta\Phi\,. \tag{3.2}$$

In Eq. (3.2) the integrand is a local functional of Φ but, as there is no dependence on the derivatives of Φ, for such a case $\delta\Phi$ coincides with $d\Phi$ and the expression under consideration is in principle an ordinary indefinite integral.

Eqs. (3.1)-(3.2) define functions of a spacetime point through three compositional steps:

i) $\mathcal{I}_i \equiv \mathcal{I}_i(\{\mathcal{A},\, \mathcal{B},\, \mathcal{V},\, \alpha\})$, e.g. $\mathcal{I}_1 \equiv \mathcal{I}_1(\mathcal{A},\, \alpha) \equiv \frac{e^{2\alpha}}{\mathcal{A}}$.
The structure of $\mathcal{I}_i$ w.r.t. $\{\mathcal{A},\, \mathcal{B},\, \mathcal{V},\, \alpha\}$ is preserved under the Weyl rescaling of the metric tensor (2.2) and the scalar field redefinition (2.3).

ii) $\mathcal{I}_i \equiv \mathcal{I}_i(\Phi) \Leftarrow \mathcal{A} \equiv \mathcal{A}(\Phi)$ etc.
Under the Weyl rescaling $\mathcal{I}_i$ preserves its functional form w.r.t. the scalar field Φ, i.e. $\bar{\mathcal{I}}_i(\bar{\Phi}) \equiv \mathcal{I}_i(\Phi \equiv \bar{\Phi})$. If also the scalar field Φ is redefined then $\bar{\mathcal{I}}_i(\bar{\Phi}) \equiv \left(\mathcal{I}_i \circ \bar{f}\,\right)(\bar{\Phi})$.

iii) $\mathcal{I}_i \equiv \mathcal{I}_i(x^\mu) \Leftarrow \Phi \equiv \Phi(x^\mu)$.
$\mathcal{I}_i$ is an invariant w.r.t. a spacetime point $x^\mu \in V_4$ which follows from the fact that under the transformations (2.2)-(2.3) the numerical value of the four arbitrary local functionals at a spacetime point changes due to multiplicative and additive terms in Eqs. (2.4). For $\mathcal{I}_i$ the extra terms and factors cancel out and hence the numerical value of $\mathcal{I}_i$ at a spacetime point is preserved under the transformations (2.2)-(2.3). In the same spirit we conclude that $\partial_\mu \mathcal{I}_i$ is also an invariant w.r.t. x^μ.

Corollary 3.1. *One may define arbitrarily many quantities having the same transformation properties as $\mathcal{I}_1$ etc. via three procedures*

i) Introducing an arbitrary functional h

$$\mathcal{I}_i \equiv h\left(\{\mathcal{I}_j\}_{j\in\mathscr{J}}\right) \tag{3.3}$$

where $\mathscr{J}$ is some set of indices.

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ii) Introducing a quotient of derivatives

$$\mathcal{I}_j \equiv \frac{\mathcal{I}_i'}{\mathcal{I}_k'} \equiv \frac{\delta \mathcal{I}_i}{\delta \Phi} \Big/ \frac{\delta \mathcal{I}_k}{\delta \Phi} = \frac{\delta \mathcal{I}_i}{\delta \mathcal{I}_k} \,. \tag{3.4}$$

iii) Integrating over the scalar field Φ

$$\mathcal{I}_i \equiv \int \mathcal{I}_j \mathcal{I}_k' \delta \Phi \tag{3.5}$$

in the sense of an indefinite integral.

We shall refer to such quantities as invariants.

Example.

$$\mathcal{I}_4(\Phi) \equiv \frac{\mathcal{I}_2(\Phi)}{\mathcal{I}_1(\Phi)^2} \,, \qquad \mathcal{I}_5(\Phi) \equiv \left(\frac{\mathcal{I}_1'(\Phi)}{2\,\mathcal{I}_1(\Phi)\,\mathcal{I}_3'(\Phi)} \right)^2 \,. \tag{3.6}$$

Let us introduce an 'invariant metric' as

$$\hat{g}^{(\cdot)}_{\mu\nu} \equiv \mathcal{I}_i \mathcal{A} g_{\mu\nu} \,. \tag{3.7}$$

Here the precise definition depends on the choice of $\mathcal{I}_i$ and we shall distinguish between different invariant metrics by using some superscript $(\cdot)$. By Eq. (3.7) we have defined an object which under the Weyl rescaling of the metric tensor (2.2) and under the scalar field redefinition (2.3) transforms as $\mathcal{I}_1$ etc. due to suitable transformation properties of $\mathcal{A}$ given by (2.4a). Nevertheless it is a metric tensor, e.g. it can be used to raise and lower spacetime indices.

We define the Levi-Civita connection with respect to $\hat{g}^{(\cdot)}_{\mu\nu}$ as

$$\begin{aligned}\hat{\Gamma}^{\sigma}_{\mu\nu} \equiv \Gamma^{\sigma}_{\mu\nu} + \frac{\mathcal{A}'}{2\mathcal{A}} \left(\delta^{\sigma}_{\mu} \partial_{\nu} \Phi + \delta^{\sigma}_{\nu} \partial_{\mu} \Phi - g_{\mu\nu} g^{\sigma\rho} \partial_{\rho} \Phi \right) + \\ + \frac{1}{2\,\mathcal{I}_i} \left(\delta^{\sigma}_{\mu} \partial_{\nu} \mathcal{I}_i + \delta^{\sigma}_{\nu} \partial_{\mu} \mathcal{I}_i - g_{\mu\nu} g^{\sigma\rho} \partial_{\rho} \mathcal{I}_i \right)\end{aligned} \tag{3.8}$$

where $\Gamma^{\sigma}_{\mu\nu}$ are the Levi-Civita connection coefficients for the metric $g_{\mu\nu}$.

Remark 3.2. The definition (3.8) is in a sense identical to the well known transformation rule of the Levi-Civita connection coefficients under the Weyl rescaling of the metric tensor $g_{\mu\nu}$ [9] but here the key idea is that we introduce additional terms to cancel the effect of the Weyl rescaling on $\Gamma^{\sigma}_{\mu\nu}$.

The definitions (3.7) and (3.8) can be used to construct geometrical objects, such as $\hat{R}^{(\cdot)}$, that are invariant under the Weyl rescaling of the metric tensor (2.2).

4 Invariants and parametrizations

In what follows we shall work with three formulations of STG:

i) The generic case described by the action functional (2.1) where non of the four arbitrary local functionals $\{\mathcal{A},\, \mathcal{B},\, \mathcal{V},\, \alpha\}$ of Φ have gained a fixed functional form. We denote these variables as denoted in (2.1), i.e.

$$g_{\mu\nu}\,,\ \ \Phi\,,\ \text{etc.} \tag{4.1}$$

ii) An arbitrary parametrization $\mathfrak{P}$, see Definition 2.2, where we shall add a superscript $\mathfrak{P}$ to the metric tensor and a subscript $\mathfrak{P}$ to all other objects as

$$g^{\mathfrak{P}}_{\mu\nu}\,,\ \ \Phi_{\mathfrak{P}}\,,\ \ \mathcal{A}_{\mathfrak{P}} \equiv \mathcal{A}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})\,,\ \text{etc.} \tag{4.2}$$

iii) The invariant case determined by a parametrization $\mathfrak{P}$. There we use an invariant metric (3.7) and other invariants

$$\hat{g}^{(\mathfrak{P})}_{\mu\nu}\,,\ \ \mathcal{I}^{(\mathfrak{P})}(\Phi)\,,\ \text{etc.} \tag{4.3}$$

Here $\mathfrak{P}$ as a superscript in parentheses emphasizes that the quantity under consideration is determined by the parametrization $\mathfrak{P}$ but does not have to be evaluated in that parametrization. It could be calculated in any other parametrization or instead considered in the generic case. What it means to be determined by a parametrization $\mathfrak{P}$ will be clarified in the following pages.

There are six possibilities to fix two out of the four arbitrary functionals $\{\mathcal{A},\, \mathcal{B},\, \mathcal{V},\, \alpha\}$, i.e. to choose a parametrization. For four possibilities out of the six a quick glimpse on (2.4) reveals that also one invariant gains a fixed functional form. Namely

i) $\mathcal{A}$ and α are fixed: $\mathcal{I}_1(\Phi_{\mathfrak{P}}) \equiv \frac{e^{2\alpha_{\mathfrak{P}}}}{\mathcal{A}_{\mathfrak{P}}}$,

ii) $\mathcal{A}$ and $\mathcal{V}$ are fixed: $\mathcal{I}_2(\Phi_{\mathfrak{P}}) \equiv \frac{\mathcal{V}_{\mathfrak{P}}}{\mathcal{A}^2_{\mathfrak{P}}}$,

iii) $\mathcal{A}$ and $\mathcal{B}$ are fixed: $\mathcal{I}_3(\Phi_{\mathfrak{P}}) \equiv \pm \int \sqrt{\frac{2\mathcal{A}_{\mathfrak{P}}\mathcal{B}_{\mathfrak{P}} + 3\left(\mathcal{A}'_{\mathfrak{P}}\right)^2}{4\mathcal{A}^2_{\mathfrak{P}}}}\delta\Phi_{\mathfrak{P}}$,

iv) $\mathcal{V}$ and α are fixed: $\mathcal{I}_4(\Phi_{\mathfrak{P}}) \equiv \frac{\mathcal{V}_{\mathfrak{P}}}{e^{4\alpha_{\mathfrak{P}}}}$.

The case where $\mathcal{B}$ and α (analogically $\mathcal{B}$ and $\mathcal{V}$) have a fixed functional form is more complicated: the corresponding invariant (if it exists) depends on the exact functional form of $\mathcal{B}$ and α and is not the same for all possible choices. For an example see JF BEPS in [8].

Lemma 4.1. *Let us assume that in a parametrization $\mathfrak{P}$ an invariant $\mathcal{I}_{fix}(\Phi_{\mathfrak{P}})$ has gained a fixed functional form. If $\mathcal{I}_{fix}$ is a nonconstant local functional then there exists a functional $\mathcal{K}^{(\mathfrak{P})}(\Phi)$ which in the parametrization $\mathfrak{P}$ is equal to 1 and in the generic case transforms as $\mathcal{A}(\Phi)$, i.e. according to (2.4a).*

Note that by writing $\mathcal{K}^{(\mathfrak{P})}(\Phi)$ we abuse the notation (4.3) since it is not an invariant but we make an exception because it is determined by a parametrization $\mathfrak{P}$ and yet does not have to be evaluated in $\mathfrak{P}$.

Proof. Let us consider a parametrization $\mathfrak{P}$. If the premiss is fulfilled then $\mathcal{I}_{fix}(\Phi_{\mathfrak{P}}) = h(\Phi_{\mathfrak{P}})$ is a known nonconstant local functional. We invert the latter to obtain a possibly multivalued relation $\Phi_{\mathfrak{P}} = h^{-1}(\mathcal{I}_{fix})$. In the current paper we do not consider the consequences of multivaluedness. According to the Corollary 3.1 a functional of an invariant is also an invariant and therefore in the parametrization $\mathfrak{P}$ it is meaningful to write $\Phi_{\mathfrak{P}} = \mathcal{I}^{(\mathfrak{P})}$ where $\mathcal{I}^{(\mathfrak{P})} \equiv h^{-1}(\mathcal{I}_{fix})$. Note that $\mathcal{I}^{(\mathfrak{P})}$ is determined by the parametrization $\mathfrak{P}$ but otherwise is an ordinary invariant. In a sense $\Phi_{\mathfrak{P}} = \mathcal{I}^{(\mathfrak{P})}(\Phi)$ relates the scalar field $\Phi_{\mathfrak{P}}$ to a generic scalar field Φ.

According to the Definition 2.2 two out of the four arbitrary local functionals $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$ of Φ have gained a fixed functional form. Therefore one must be either $\mathcal{A}$, $\mathcal{V}$ or α.

First let us consider the case where the functional form of $\mathcal{A}_{\mathfrak{P}}(\Phi_{\mathfrak{P}}) \equiv \mathcal{A}(\Phi)|_{\mathfrak{P}}$ is fixed. We make use of the result $\Phi_{\mathfrak{P}} = \mathcal{I}^{(\mathfrak{P})}$ and replace the argument of $\mathcal{A}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$ as $\mathcal{A}_{\mathfrak{P}} \equiv \mathcal{A}_{\mathfrak{P}}(\mathcal{I}^{(\mathfrak{P})})$. The Corollary 3.1 states that the obtained quantity is an invariant. By making use of the notation introduced in (4.3) we write $\mathcal{A}^{(\mathfrak{P})}(\Phi) \equiv \mathcal{A}_{\mathfrak{P}}(\mathcal{I}^{(\mathfrak{P})}(\Phi))$ to denote an invariant with the property $\mathcal{A}^{(\mathfrak{P})}(\Phi)\big|_{\mathfrak{P}} = \mathcal{A}_{\mathfrak{P}}(\Phi_{\mathfrak{P}})$. Hence $\mathcal{A}^{(\mathfrak{P})}(\Phi)$ is an invariant which is determined by the parametrization $\mathfrak{P}$ but can be considered in whatever case. In the generic case the quotient

$$\mathcal{K}^{(\mathfrak{P})}(\Phi) \equiv \frac{\mathcal{A}(\Phi)}{\mathcal{A}^{(\mathfrak{P})}(\Phi)} \tag{4.4}$$

is a local functional of Φ that transforms as $\mathcal{A}(\Phi)$ and in the parametrization $\mathfrak{P}$ we obtain that $\mathcal{K}^{(\mathfrak{P})}\big|_{\mathfrak{P}} = 1$.

The proof in the case when the functional form of either $\mathcal{V}$ or α is fixed proceeds analogically. □

Note that formally each functional that transforms as $\mathcal{A}$, i.e. according to (2.4a), can be written as a product of $\mathcal{A}$ and some invariant, e.g. $e^{2\alpha} \equiv \mathcal{A}\mathcal{I}_1$.

Example. We consider a parametrization $\mathfrak{P}$ where $\mathcal{A}_{\mathfrak{P}} \equiv \Phi_{\mathfrak{P}}$ and $e^{2\alpha_{\mathfrak{P}}} \equiv 1 + \lambda\Phi_{\mathfrak{P}}$. Here λ is some constant parameter. The scalar field $\Phi_{\mathfrak{P}}$ can be expressed as a local functional of the fixed invariant $\mathcal{I}_1$ as follows

$$\Phi_{\mathfrak{P}} = \frac{1}{\mathcal{I}_1 - \lambda} \equiv \mathcal{I}^{(\mathfrak{P})} . \tag{4.5}$$

Hence $\mathcal{A}^{(\mathfrak{P})} \equiv \mathcal{I}^{(\mathfrak{P})}$ and the quotient

$$\mathcal{K}^{(\mathfrak{P})}(\Phi) \equiv \frac{\mathcal{A}(\Phi)}{\mathcal{A}^{(\mathfrak{P})}(\Phi)} \equiv \left(\mathcal{I}_1(\Phi) - \lambda\right) \mathcal{A}(\Phi) \tag{4.6}$$

has the demanded properties. A direct calculation shows that if we use an analogous procedure but consider $e^{2\alpha_{\mathfrak{P}}}$ instead of $\mathcal{A}_{\mathfrak{P}}$ then we get the same result.

The result for JF BDBW parametrization (2.5) is obtained by fixing $\lambda \equiv 0$. In that case the result (4.6) reduces to

$$\mathcal{K}^{(\mathfrak{J})} \equiv \mathcal{A}\mathcal{I}_1 \equiv e^{2\alpha} \tag{4.7}$$

which in JF BDBW parametrization is indeed equal to one and in the generic case transforms as $\mathcal{A}$. For EF canonical parametrization (2.6) $\mathcal{K}^{(\mathfrak{E})} \equiv \mathcal{A}$.

The relation $\Phi_{\mathfrak{P}} = \mathcal{I}^{(\mathfrak{P})}$ in the parametrization $\mathfrak{P}$, obtained in the proof of the Lemma 4.1, introduces an another object which in the parametrization $\mathfrak{P}$ is equal to one but has a specific transformation property. Namely in the parametrization $\mathfrak{P}$

$$1 = \frac{\delta\Phi_{\mathfrak{P}}}{\delta\Phi_{\mathfrak{P}}} = \frac{\delta\mathcal{I}^{(\mathfrak{P})}}{\delta\Phi_{\mathfrak{P}}} . \tag{4.8}$$

In the generic case $\bar{\mathcal{I}}^{(\mathfrak{P})\prime} = \bar{f}'\mathcal{I}^{(\mathfrak{P})\prime}$.

Theorem 4.2. *If, due to specifying the parametrization to be $\mathfrak{P}$, an invariant gains a fixed nonconstant functional form then there exists an 'invariant pair'*

$$\left(\hat{g}^{(\mathfrak{P})}_{\mu\nu}, \mathcal{I}^{(\mathfrak{P})}\right) \tag{4.9}$$

which in the parametrization $\mathfrak{P}$ functionally coincides with the pair $\left(g^{\mathfrak{P}}_{\mu\nu}, \Phi_{\mathfrak{P}}\right)$.

Proof. Let us consider a parametrization $\mathfrak{P}$. If the premiss holds then the Lemma 4.1 proposes the existence of the functional $\mathcal{K}^{(\mathfrak{P})}(\Phi)$ which has the properties: $\mathcal{K}^{(\mathfrak{P})} = e^{-2\bar{\gamma}}\bar{\mathcal{K}}^{(\mathfrak{P})}$ and $\mathcal{K}^{(\mathfrak{P})}(\Phi)\big|_{\mathfrak{P}} = 1$. Hence $\hat{g}^{(\mathfrak{P})}_{\mu\nu} \equiv \mathcal{K}^{(\mathfrak{P})} g_{\mu\nu}$ is an invariant metric (3.7) and in the parametrization $\mathfrak{P}$

$$\hat{g}^{(\mathfrak{P})}_{\mu\nu}\Big|_{\mathfrak{P}} = g^{\mathfrak{P}}_{\mu\nu} \,. \tag{4.10}$$

In the same spirit $\mathcal{I}^{(\mathfrak{P})}\big|_{\mathfrak{P}} = \Phi_{\mathfrak{P}}$ holds by the definition introduced in the proof of the Lemma 4.1. □

Example. In JF BDBW parametrization (2.5)

$$\left(\hat{g}^{(\mathfrak{J})}_{\mu\nu}, \frac{1}{\mathcal{I}_1}\right)\Bigg|_{\mathfrak{J}} \equiv \left(e^{2\alpha} g_{\mu\nu}, \frac{1}{\mathcal{I}_1}\right)\Bigg|_{\mathfrak{J}} = (g^{\mathfrak{J}}_{\mu\nu}, \Psi) \,. \tag{4.11}$$

In EF canonical parametrization (2.6)

$$\left(\hat{g}^{(\mathfrak{E})}_{\mu\nu}, \pm\mathcal{I}_3\right)\Big|_{\mathfrak{E}} \equiv (\mathcal{A} g_{\mu\nu}, \pm\mathcal{I}_3)|_{\mathfrak{E}} = (g^{\mathfrak{E}}_{\mu\nu}, \varphi) \,. \tag{4.12}$$

Let us take the metric tensor from the invariant pair (4.9), determined by some parametrization $\mathfrak{P}$, and calculate the invariant Ricci scalar $\hat{R}^{(\mathfrak{P})}$ for that metric tensor. In the parametrization $\mathfrak{P}$ the invariant Ricci scalar $\hat{R}^{(\mathfrak{P})}$ functionally coincides with the Ricci scalar $R_{\mathfrak{P}}$ that is calculated using the metric tensor $g^{\mathfrak{P}}_{\mu\nu}$.

Example. Let us consider JF BDBW parametrization (2.5) that determines the invariant pair (4.11). One can show that [9]

$$e^{2\alpha}\hat{R}^{(\mathfrak{J})} = R - 6g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\alpha - 6g^{\mu\nu}\nabla_\mu\nabla_\nu\alpha \tag{4.13}$$

where the r.h.s. is calculated for the generic case (4.1). Restricting Eq. (4.13) to the JF BDBW parametrization (2.5) ($\alpha \equiv 0$) gives us the equality

$$\hat{R}^{(\mathfrak{J})}\Big|_{\mathfrak{J}} = R_{\mathfrak{J}} \,. \tag{4.14}$$

The result (4.13) resembles the transformation of the Ricci scalar under the Weyl rescaling. Here, in the spirit of the Remark 3.2, we introduce additional terms to cancel the effect of the conformal transformation on the Ricci scalar.

5 The relation between the generic case and a chosen parametrization revisited. The translation rules.

Let us consider an invariant pair (4.9) determined by a parametrization $\mathfrak{P}$. If one rewrites the action functional (2.1) using the components of the invariant pair (4.9) as the dynamical variables then four invariants, which we shall denote as $\left\{\mathcal{I}_{\mathcal{A}}^{(\mathfrak{P})},\, \mathcal{I}_{\mathcal{B}}^{(\mathfrak{P})},\, \mathcal{I}_{\mathcal{V}}^{(\mathfrak{P})},\, \mathcal{I}_{\alpha}^{(\mathfrak{P})}\right\}$, appear into the positions of the four arbitrary local functionals $\{\mathcal{A},\, \mathcal{B},\, \mathcal{V},\, \alpha\}$.

Such a claim can be reasoned as follows. Using the invariant metric $\hat{g}_{\mu\nu}^{(\mathfrak{P})}$ to calculate geometrical quantities guarantees that the latter are invariant under the transformations (2.2)-(2.3). In the same spirit the kinetic term for $\mathcal{I}^{(\mathfrak{P})}$ is invariant as well. Therefore there is no mixing of the additive terms in the action functional $S\left[\hat{g}_{\mu\nu}^{(\mathfrak{P})},\, \mathcal{I}^{(\mathfrak{P})},\, \chi\right]$ under the transformations (2.2)-(2.3). We conclude that for such an action functional each additive term must be an invariant by itself because we have assumed the action functional (2.1) to be invariant. Each of the four arbitrary local functionals $\{\mathcal{A},\, \mathcal{B},\, \mathcal{V},\, \alpha\}$ multiplies an object which after rewriting is replaced by an invariant. Therefore during the rewriting process the four arbitrary local functionals must be replaced by invariants as well.

Example. First let us consider JF BDBW parametrization (2.5). Rewriting the action functional (2.1) in terms of the invariant pair (4.11) reads

$$S = \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-\hat{g}^{(\mathfrak{J})}}\left\{\frac{1}{\mathcal{I}_1}\hat{R}^{(\mathfrak{J})} - \mathcal{I}_1\frac{1}{2}\left(\frac{1}{\mathcal{I}_5}-3\right)\hat{g}^{(\mathfrak{J})\mu\nu}\hat{\nabla}_{\mu}^{(\mathfrak{J})}\frac{1}{\mathcal{I}_1}\hat{\nabla}_{\nu}^{(\mathfrak{J})}\frac{1}{\mathcal{I}_1}\right.$$
$$\left. - 2\ell^{-2}\mathcal{I}_4\right\} + S_m\left[\hat{g}_{\mu\nu}^{(\mathfrak{J})},\, \chi\right]. \tag{5.1}$$

Here we have made use of the definitions (3.1)-(3.2) and (3.6) and of the result (4.13). Hence $\mathcal{I}_{\mathcal{A}}^{(\mathfrak{J})} = \frac{1}{\mathcal{I}_1}$, $\mathcal{I}_{\mathcal{B}}^{(\mathfrak{J})} = \mathcal{I}_1\frac{1}{2}\left(\frac{1}{\mathcal{I}_5}-3\right)$, $\mathcal{I}_{\mathcal{V}}^{(\mathfrak{J})} = \mathcal{I}_4$ and $\mathcal{I}_{\alpha}^{(\mathfrak{J})} = 0$.

Second let us consider EF canonical parametrization (2.6) and the corresponding invariant pair (4.12). One can rewrite the action functional (2.1) as

$$S = \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-\hat{g}^{(\mathfrak{E})}}\left\{\hat{R}^{(\mathfrak{E})} - 2\hat{g}^{(\mathfrak{E})\mu\nu}\hat{\nabla}_{\mu}^{(\mathfrak{E})}\mathcal{I}_3\hat{\nabla}_{\nu}^{(\mathfrak{E})}\mathcal{I}_3 - 2\ell^{-2}\mathcal{I}_2\right\}$$
$$+ S_m\left[\mathcal{I}_1\hat{g}_{\mu\nu}^{(\mathfrak{E})},\, \chi\right]. \tag{5.2}$$

In this example $\mathcal{I}_{\mathcal{A}}^{(\mathfrak{E})} = 1$, $\mathcal{I}_{\mathcal{B}}^{(\mathfrak{E})} = 2$, $\mathcal{I}_{\mathcal{V}}^{(\mathfrak{E})} = \mathcal{I}_2$ and $\mathcal{I}_{\alpha}^{(\mathfrak{E})} = \frac{1}{2}\ln \mathcal{I}_1$.

Rewriting the action functional (2.1) in terms of an invariant pair (4.9) retains the generality of the theory up to some minor details that we shall not discuss in the current paper.

Corollary 5.1. *Let us rewrite the general action functional $S = S\left[g_{\mu\nu}, \Phi, \chi\right]$, defined by* (2.1)*, using the components of an invariant pair $\left(\hat{g}_{\mu\nu}^{(\mathfrak{P})}, \mathcal{I}^{(\mathfrak{P})}\right)$, determined by a parametrization $\mathfrak{P}$, as dynamical variables. We end up with an action functional $S = S\left[\hat{g}_{\mu\nu}^{(\mathfrak{P})}, \mathcal{I}^{(\mathfrak{P})}, \chi\right]$ involving a boundary term which we shall neglect. Let us focus upon the action functional in terms of the invariants. If we specify the theory by fixing the parametrization to be $\mathfrak{P}$ then each invariant quantity is mapped to the corresponding noninvariant quantity in the parametrization $\mathfrak{P}$ as follows*

$$
\begin{array}{rclcrcl}
\hat{g}_{\mu\nu}^{(\mathfrak{P})} & \mapsto & g_{\mu\nu}^{\mathfrak{P}} & , & \mathcal{I}_{\mathcal{A}}^{(\mathfrak{P})} & \mapsto & \mathcal{A}_{\mathfrak{P}}\,, \\
\sqrt{-\hat{g}^{(\mathfrak{P})}} & \mapsto & \sqrt{-g^{\mathfrak{P}}} & , & \mathcal{I}_{\mathcal{B}}^{(\mathfrak{P})} & \mapsto & \mathcal{B}_{\mathfrak{P}}\,, \\
\hat{R}^{(\mathfrak{P})} & \mapsto & R_{\mathfrak{P}} & , & \mathcal{I}_{\mathcal{V}}^{(\mathfrak{P})} & \mapsto & \mathcal{V}_{\mathfrak{P}}\,, \\
\hat{\nabla}_{\mu}^{(\mathfrak{P})} & \mapsto & \nabla_{\mu}^{\mathfrak{P}} & , & \mathcal{I}_{\alpha}^{(\mathfrak{P})} & \mapsto & \alpha_{\mathfrak{P}}\,, \\
\mathcal{I}^{(\mathfrak{P})} & \mapsto & \Phi_{\mathfrak{P}} & . & & &
\end{array}
\tag{5.3}
$$

Example. First let us consider JF BDBW parametrization (2.5). The action functional (2.1) rewritten in terms of the invariant pair (4.11) is given by (5.1). A straightforward calculation shows that fixing the parametrization to be JF BDBW parametrization implies

$$
\left.\frac{1}{\mathcal{I}_1}\right|_{\mathfrak{J}} = \Psi \equiv \mathcal{A}_{\mathfrak{J}}\,, \quad \left.\mathcal{I}_1 \frac{1}{2}\left(\frac{1}{\mathcal{I}_5} - 3\right)\right|_{\mathfrak{J}} = \frac{\omega(\Psi)}{\Psi} \equiv \mathcal{B}_{\mathfrak{J}}\,, \tag{5.4}
$$

$$
\left.\mathcal{I}_4\right|_{\mathfrak{J}} = \mathcal{V}_{\mathfrak{J}}(\Psi)\,, \quad \mathcal{I}_{\alpha}^{(\mathfrak{J})} = 0 = \alpha_{\mathfrak{J}}\,. \tag{5.5}
$$

Second let us consider EF canonical parametrization (2.6). The invariant pair (4.12) gives rise to the action functional (5.2). A direct calculation shows that

$$
1 \equiv \mathcal{A}_{\mathfrak{E}}\,, \quad 2 \equiv \mathcal{B}_{\mathfrak{E}}\,, \quad \left.\mathcal{I}_2\right|_{\mathfrak{E}} = \mathcal{V}_{\mathfrak{E}}(\varphi)\,, \quad \left.\frac{1}{2}\ln \mathcal{I}_1\right|_{\mathfrak{E}} = \alpha_{\mathfrak{E}}(\varphi)\,. \tag{5.6}
$$

Remark 5.2. Let us consider the case where we have two action functionals S_1 and S_2. The action $S_1 \equiv S_1\left[g_{\mu\nu}^{\mathfrak{P}}, \Phi_{\mathfrak{P}}, \chi\right]$ is obtained from (2.1) by fixing the parametrization to be $\mathfrak{P}$ and $S_2 \equiv S_2\left[\hat{g}_{\mu\nu}^{(\mathfrak{P})}, \mathcal{I}^{(\mathfrak{P})}, \chi\right]$ is obtained by rewriting the action functional (2.1) in terms of the invariant pair (4.9) that is determined by $\mathfrak{P}$. Suppose that we are given an action functional S_3 and we know that S_3 is either S_1 or S_2. Due to the one to one correspondence (5.3) we cannot determine whether S_3 is S_1 or S_2 without a priori knowing how the quantities contained in S_3 transform, i.e. whether the transformation of the quantities obey Eqs. (2.2)-(2.4d) or the rules described after Eq. (3.2). Therefore without a priori given transformation rules the action functionals S_1 and S_2 cannot be distinguished.

Let us point out that the redefinition of the scalar field can be seen as choosing a different invariant to be the dynamical variable.

Example. Lets us consider JF BDBW parametrization (2.5) scalar field Ψ as a local functional of the EF canonical parametrization (2.6) scalar field φ. By comparing the invariant pairs (4.11) and (4.12) we obtain that this corresponds to

$$\frac{1}{\mathcal{I}_1} \equiv \frac{1}{\mathcal{I}_1(\mathcal{I}_3)}\,. \tag{5.7}$$

Hence

$$\left(\frac{\delta\Psi}{\delta\varphi}\right)^2 = \left(\frac{\delta\frac{1}{\mathcal{I}_1}}{\delta\mathcal{I}_3}\right)^2 = \left(\frac{\mathcal{I}_1'}{\mathcal{I}_1^2\,\mathcal{I}_3'}\right)^2 = \frac{4\mathcal{I}_5}{\mathcal{I}_1^2} \tag{5.8}$$

where we made use of the definition (3.6). If the result is evaluated in EF canonical parametrization (2.6) then it agrees with Eq. (2.7). If Eq. (5.8) is evaluated in JF BDBW parametrization (2.5) then it agrees with (2.8).

The one to one correspondence (5.3) gives rise to the 'translation rules' that were first implicitly used in Ref. [7] and more thoroughly studied in Ref. [8]. The translation rules can be used to rewrite the results obtained in some parametrization $\mathfrak{P}$ as the results of the generic case described by the action functional (2.1). The key idea can be phrased as follows.

i) Calculate the invariant pair (4.9) determined by a parametrization $\mathfrak{P}$.

ii) Rewrite the action functional (2.1) in terms of the obtained invariant pair and determine the l.h.s. of the correspondence (5.3).

iii) Replace each quantity in the parametrization $\mathfrak{P}$ by the corresponding invariant, i.e. use the mapping (5.3) backwards.

iv) Evaluate the obtained invariant quantities in terms of the four arbitrary local functionals $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$ and use a generic metric tensor $g_{\mu\nu}$ and a generic scalar field Φ as dynamical variables.

Instead of following the second rule of the aforementioned prescription one can use the transformations (2.4) to obtain the invariants that correspond to the four local functionals $\{\mathcal{A}_{\mathfrak{P}}, \mathcal{B}_{\mathfrak{P}}, \mathcal{V}_{\mathfrak{P}}, \alpha_{\mathfrak{P}}\}$ in a parametrization $\mathfrak{P}$.

Namely, let us consider the quantities of the invariant case to be formally the "barred" ones. The definition of the invariant metric in the invariant pair (4.9), i.e. $\hat{g}^{(\mathfrak{P})}_{\mu\nu} \equiv \mathcal{K}^{(\mathfrak{P})} g_{\mu\nu}$ can be seen as a Weyl rescaling of the metric tensor (2.2) where $e^{2\bar{\gamma}(\mathcal{I}^{(\mathfrak{P})})} = \left(\mathcal{K}^{(\mathfrak{P})}(\Phi(\mathcal{I}^{(\mathfrak{P})}))\right)^{-1}$. The crucial point is that for generic case

$$\left(\bar{\mathcal{K}}^{(\mathfrak{P})}\right)^{-1} = e^{-2\bar{\gamma}} \left(\mathcal{K}^{(\mathfrak{P})}\right)^{-1} . \tag{5.9}$$

Therefore using $e^{2\bar{\gamma}} = \left(\mathcal{K}^{(\mathfrak{P})}\right)^{-1}$ for performing the transformations (2.4) actually, in the spirit of the Remark 3.2, introduces extra terms with suitable transformation properties to cancel the effect of the Weyl rescaling on the arbitrary local functionals $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$. Analogically $\bar{f}' = \left(\mathcal{I}^{(\mathfrak{P})\prime}\right)^{-1}$.

There are noninvariant objects that in a parametrization $\mathfrak{P}$ are equal to one, e.g. (4.4) and (4.8) and various combinations of these. Therefore the translation rules cannot directly determine the transformation properties and hence can work fluently only in the case of invariant quantities. There are indirect ways to obtain the transformation properties as well, e.g. comparing the results calculated from different parametrizations.

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Chapter 8

Invariant quantities in the multiscalar-tensor theories of gravitation

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Invariant quantities in the multiscalar-tensor theories of gravitation

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The aim of the current paper is to study the multiscalar-tensor theories of gravity without derivative couplings. We construct a few basic objects that are invariant under a Weyl rescaling of the metric and transform covariantly when the scalar fields are redefined. We introduce rules to construct further such objects and put forward a scheme that allows to express the results obtained either in the Einstein frame or in the Jordan frame as general ones. These so-called "translation" rules are used to show that the parametrized post-Newtonian approximation results obtained in the aforementioned two frames indeed are the same if expressed in a general frame.

Keywords: Multiscalar-tensor theories of gravity; invariants; conformal frames.

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1. Introduction

Multiscalar-tensor gravity (MSTG)[1,2] generalizes the well known Jordan-Brans-Dicke scalar-tensor gravity (STG) by including more scalar fields non-minimally coupled to curvature. In recent years these theories have mostly attracted attention by providing models for inflation,[3–10] dark energy,[11–15] and relativistic stars.[16] To make a reliable use of these models the details of mathematical correspondence and physical interpretation of different MSTG conformal frames need to be understood, e.g. in the context of cosmological perturbations,[8,9] gravitational particle production,[10] or one-loop divergences.[17]

As has been argued recently for a single field case the mathematical comparisons between the results obtained in different frames are greatly facilitated by quantities which remain invariant under the conformal Weyl rescaling of the metric and scalar field reparametrization.[18–20] In this brief note we generalize the formalism of invariants to the multiscalar case.

The paper is organized as follows. In Sec. 2 we postulate an action functional for multiscalar-tensor theories of gravity, invoke the Weyl rescaling of the metric and redefinition of the scalar fields in order to study the transformation properties of the unspecified functions contained in the action. The equations of motion and the Einstein and the Jordan frame are introduced in Sec. 3. Next, Sec. 4 is devoted to the functions of the scalar fields as well as to the metric tensors for the space of scalar fields that are invariant under the local Weyl rescaling of the (spacetime) metric. We note that a spacetime Weyl rescaling induces a disformal transformation in the space of scalar fields. Based on these results, in Sec. 5 we construct the so-called "translation" rules for both the Einstein frame and the Jordan frame. These are used in Sec. 6 in order to express the parametrized post-Newtonian approximation results[1,2,21] in a generic frame.

2. Action Functional and Transformations

Let us start by postulating an action functional (generalizing Refs. 18 and 22)

$$S = \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-g}\left\{\mathcal{A}(\Phi)\,R - \mathcal{B}_{AB}(\Phi)\,g^{\mu\nu}\nabla_\mu\Phi^A\nabla_\nu\Phi^B - 2\ell^{-2}\mathcal{V}(\Phi)\right\}$$
$$+ S_{\mathrm{m}}\left[e^{2\alpha(\Phi)}g_{\mu\nu},\chi\right], \tag{1}$$

describing a generic multiscalar-tensor theory of gravity without derivative couplings.[1–3] It contains three unspecified functions $\mathcal{A}(\Phi)$, $\mathcal{V}(\Phi)$, $\alpha(\Phi)$ and one invertible symmetric square matrix function $\mathcal{B}(\Phi)_{AB}$ of order n. Each of the three unspecified functions as well as the entries of the matrix $\mathcal{B}(\Phi)_{AB}$ in general depend on all n scalar fields denoted by the set $\Phi \equiv \left\{\Phi^A\right\}_{A=1}^{n}$ as an argument of these quantities. We consider the scalar fields Φ^A, the functions $\mathcal{A}(\Phi)$, $\mathcal{V}(\Phi)$, $\alpha(\Phi)$ and the entries of $\mathcal{B}_{AB}(\Phi)$ to be dimensionless. In order for the latter to be consistent with $c = 1$, while $\hbar$ and G_N are left unspecified, we have introduced constants κ^2 having the dimension of the Newtonian gravitational constant G_N and $\ell > 0$ having the dimension of length. The matter fields, collectively denoted by χ, are described by the action S_{m}.

The functions $\mathcal{A}(\Phi)$ and $\alpha(\Phi)$ characterize the scalar coupling to curvature and to matter, respectively. The interactions between the scalar fields are gathered into $\mathcal{V}(\Phi)$ which is often referred to as potential. The matrix $\mathcal{B}(\Phi)_{AB}$ gives the kinetic couplings of the scalar fields.

One might want to apply the local Weyl rescaling to the metric tensor $g_{\mu\nu}$ and reparametrize the scalar fields Φ^A as

$$g_{\mu\nu} = e^{2\bar{\gamma}(\bar{\Phi})}\bar{g}_{\mu\nu}\,, \tag{2a}$$

$$\Phi^A = \bar{f}^A(\bar{\Phi})\,. \tag{2b}$$

Often the term change of the frame is used to refer to the conformal transformation (2a) of the metric tensor, while (2b) is dubbed the change of the parametrization. If under the transformations (2) the arbitrary functions of the scalar fields, contained in the action (1), are imposed to transform as

$$\mathcal{A}(\bar{f}(\bar{\Phi})) = e^{-2\bar{\gamma}(\bar{\Phi})}\bar{\mathcal{A}}(\bar{\Phi})\,, \tag{3a}$$

$$\mathcal{V}(\bar{f}(\bar{\Phi})) = e^{-4\bar{\gamma}(\bar{\Phi})}\bar{\mathcal{V}}(\bar{\Phi})\,, \tag{3b}$$

$$\alpha(\bar{f}(\bar{\Phi})) = \bar{\alpha}(\bar{\Phi}) - \bar{\gamma}(\bar{\Phi})\,, \tag{3c}$$

$$\begin{aligned}\mathcal{B}_{AB}(\bar{f}(\bar{\Phi})) = e^{-2\bar{\gamma}(\bar{\Phi})}\left(\bar{f}^{C}{}_{,A}\right)^{-1}\left(\bar{f}^{D}{}_{,B}\right)^{-1}\Big\{&\bar{\mathcal{B}}_{CD}(\bar{\Phi}) - 6\bar{\gamma}_{,C}\bar{\gamma}_{,D}\bar{\mathcal{A}}(\bar{\Phi})\\ &+ 3\left(\bar{\gamma}_{,D}\bar{\mathcal{A}}_{,C} + \bar{\gamma}_{,C}\bar{\mathcal{A}}_{,D}\right)\Big\}\,,\end{aligned} \tag{3d}$$

where $\bar{f}(\bar{\Phi}) \equiv \left\{\bar{f}^{A}(\bar{\Phi})\right\}_{A=1}^{n}$, then the action functional (1) is invariant up to a boundary term which we shall neglect.

Here and in the following we shall make use of the convention where the "barred" ("unbarred") quantities are functions of the "barred" ("unbarred") scalar fields $\left\{\bar{\Phi}^{A}\right\}$ ($\left\{\Phi^{A}\right\}$). In addition, each index A written after a comma denotes a partial derivative with respect to (w.r.t.) a scalar field. If such a combination is a subscript of a "barred" ("unbarred") quantity then the partial derivative is taken w.r.t. the "barred" ("unbarred") scalar field, e.g.

$$\bar{\gamma}_{,A} \equiv \frac{\partial\bar{\gamma}(\bar{\Phi})}{\partial\bar{\Phi}^{A}}\,, \quad \mathcal{A}_{,A} \equiv \frac{\partial\mathcal{A}(\Phi)}{\partial\Phi^{A}}\,. \tag{4}$$

We have introduced all these conventions in order to be able to drop the arguments of the functions without generating ambiguities.

The transformation (2b) can be considered as a coordinate transformation in the n-dimensional space of scalar fields. Then

$$\bar{f}^{A}{}_{,C} \equiv \frac{\partial\bar{f}^{A}}{\partial\bar{\Phi}^{C}} \equiv \frac{\partial\Phi^{A}}{\partial\bar{\Phi}^{C}} \tag{5}$$

is a Jacobian matrix and $\left(\bar{f}^{C}{}_{,B}\right)^{-1} \equiv \partial\bar{\Phi}^{C}/\partial\Phi^{B}$ is its inverse.

3. Equations of Motion, Frames and Parametrizations

Varying the action functional (1) w.r.t. the metric $g^{\mu\nu}$ and w.r.t. the scalar fields Φ^{C} gives us the following equations of motion:

$$\begin{aligned}&\mathcal{A}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \mathcal{A}_{,A}\left(g_{\mu\nu}\Box\Phi^{A} - \nabla_{\mu}\nabla_{\nu}\Phi^{A}\right) - \left(\mathcal{B}_{AB} + \mathcal{A}_{,AB}\right)\nabla_{\mu}\Phi^{A}\nabla_{\nu}\Phi^{B}\\ &\quad + g_{\mu\nu}\left(\frac{1}{2}\mathcal{B}_{AB} + \mathcal{A}_{,AB}\right)g^{\sigma\rho}\nabla_{\sigma}\Phi^{A}\nabla_{\rho}\Phi^{B} + \ell^{-2}g_{\mu\nu}\mathcal{V} - \kappa^{2}T_{\mu\nu} = 0\,,\end{aligned} \tag{6a}$$

$$\begin{aligned}&\mathcal{A}_{,C}R + 2\mathcal{B}_{AC}\Box\Phi^{A} + \left(2\mathcal{B}_{BC,A} - \mathcal{B}_{AB,C}\right)g^{\mu\nu}\nabla_{\mu}\Phi^{A}\nabla_{\nu}\Phi^{B} - 2\ell^{-2}\mathcal{V}_{,C}\\ &\quad + 2\kappa^{2}\alpha_{,C}T = 0\,.\end{aligned} \tag{6b}$$

Often in the literature some of the unspecified functions contained in the action (1) are given a fixed functional form, e.g. in order to have a more straightforward physical interpretation. Let us recall the two setups that are often used.

- For the Einstein frame as used in Ref. 1 let us denote the metric tensor as $g^{\mathfrak{E}}_{\mu\nu}$ and specify the scalar functions as

$$\mathcal{A} \equiv 1 \equiv \mathcal{A}_{\mathfrak{E}}\,, \quad \mathcal{B}_{AB} \equiv 2\mathcal{B}^{\mathfrak{E}}_{AB}\,, \quad \mathcal{V} \equiv \mathcal{V}_{\mathfrak{E}}\,, \quad \alpha \equiv \alpha_{\mathfrak{E}}\,. \tag{7}$$

  A closer look to the equations of motion (6) reveals that if $\mathcal{A} \equiv 1$ then Eq. (6a) does not contain the second derivatives of the scalar fields Φ^A and hence purely describes the propagation of the metric tensor $g^{\mathfrak{E}}_{\mu\nu}$. Analogously Eq. (6b) does not contain the second derivatives of the metric tensor $g^{\mathfrak{E}}_{\mu\nu}$ and hence describes the propagation of the scalar fields Φ^A. One can further separate the scalar fields by multiplying Eq. (6b) with the inverse matrix $\mathcal{B}^{\mathfrak{E}\,BC}$ where $\mathcal{B}^{\mathfrak{E}\,BC}\mathcal{B}^{\mathfrak{E}}_{CA} \equiv \delta^B_A$. It is said that the equations are fully debraided.[23]
- For the Jordan frame in the Brans-Dicke-Bergmann-Wagoner (BDBW) parametrization as used in Refs. 11, 21 let us denote the metric tensor as $g^{\mathfrak{J}}_{\mu\nu}$ and distinguish one scalar field Ψ while the others are denoted as $\bar{\Phi}^a$, $a, b = 1 \ldots n-1$, where

$$\mathcal{A} \equiv \Psi \equiv \bar{\mathcal{A}}_{\mathfrak{J}}\,, \quad \mathcal{V} \equiv \bar{\mathcal{V}}_{\mathfrak{J}}\,, \quad \alpha \equiv 0 = \bar{\alpha}_{\mathfrak{J}}\,, \tag{8a}$$

$$\mathcal{B}_{ab} \equiv \bar{\mathcal{B}}^{\mathfrak{J}}_{ab}\,, \quad \mathcal{B}_{na} \equiv 0 = \bar{\mathcal{B}}^{\mathfrak{J}}_{na}\,, \quad \mathcal{B}_{nn} \equiv \bar{\mathcal{B}}_{nn} \equiv \frac{\omega(\bar{\Phi}^1, \ldots, \bar{\Phi}^{n-1}, \Psi)}{\Psi}\,. \tag{8b}$$

  The quantities in the Jordan frame are "barred" for the sake of notational consistency and the reason will be made clear in Subsec. 5.2. In this frame the action S_{m} for the matter fields functionally depends on the geometrical metric $g^{\mathfrak{J}}_{\mu\nu}$ and hence freely falling particles follow the geodesics of the metric $g^{\mathfrak{J}}_{\mu\nu}$.

4. Invariants and the Metric for the Space of Scalar Fields

Just as in the case of one scalar field,[18] a closer inspection of the transformation rules (3) allows us to write out two quantities that are invariant under a Weyl rescaling of the metric (2a) and transform as scalar functions under the scalar fields reparametrization (2b)

$$\mathcal{I}_1(\Phi) \equiv \frac{e^{2\alpha(\Phi)}}{\mathcal{A}(\Phi)}\,, \quad \mathcal{I}_2(\Phi) \equiv \frac{\mathcal{V}(\Phi)}{(\mathcal{A}(\Phi))^2}\,. \tag{9}$$

We shall call them invariants. Also an arbitrary function of these, e.g.

$$\mathcal{I}_4 \equiv \frac{\mathcal{I}_2}{\mathcal{I}_1^2} = e^{-4\alpha}\mathcal{V} \tag{10}$$

is an invariant.[18] Note that these quantities are also invariants of a spacetime point. In comparison with the one scalar field case,[18] the third invariant $\mathcal{I}_3$ and the other two rules for constructing further invariants do not generalize so straightforwardly to the case of n scalar fields. To address this issue, a few preluding remarks about the metric of the space of scalar fields are in order.

One could take $\mathcal{B}_{AB}$ to be the metric of the space of scalar fields and indeed if only the scalar fields reparametrizations (2b) are considered then $\mathcal{B}_{AB}$ transforms as a second order covariant tensor. However, if also the local Weyl rescaling of the (spacetime) metric tensor is utilized, then $\mathcal{B}_{AB}$ gains additive terms. Our aim is to construct quantities that are invariant under a Weyl rescaling (2a) and transform covariantly under scalar fields reparametrizations (2b). Thus, we introduce the metric of the space of scalar fields and its transformation rule as

$$\mathcal{F}_{AB} \equiv \frac{2\mathcal{A}\mathcal{B}_{AB} + 3\mathcal{A}_{,A}\mathcal{A}_{,B}}{4\mathcal{A}^2}\,, \quad \mathcal{F}_{AB} = \left(\bar{f}^C{}_{,A}\right)^{-1}\left(\bar{f}^D{}_{,B}\right)^{-1}\bar{\mathcal{F}}_{CD}\,. \tag{11}$$

This allows us to generalize the third invariant $\mathcal{I}_3$ as an indefinite integral

$$\mathcal{I}_3(\Phi) \equiv \int \sqrt{\det|\mathcal{F}_{AB}|}d\Phi^1 \wedge \ldots \wedge d\Phi^n\,. \tag{12}$$

We assume $\mathcal{F}_{AB}$ to be an invertible matrix and denote its inverse as $\mathcal{F}^{AB}$. Introducing a covariant derivative in the space of scalar fields via the metric $\mathcal{F}_{AB}$ guarantees that the obtained differential operator is invariant under the Weyl rescaling (2a) of the spacetime metric. Note that $\mathcal{F}^{AB}$ can be used to contract indexes as e.g.

$$\mathcal{I}_5 \equiv \frac{1}{4}\mathcal{F}^{AB}\left(\ln\mathcal{I}_1\right)_{,A}\left(\ln\mathcal{I}_1\right)_{,B} \tag{13}$$

and thereby allows us to introduce further invariants.

It is possible to define other objects that transform exactly as $\mathcal{F}_{AB}$. Namely, one could consider an invariant (Eq. (9) etc.) as a scalar function defined on the space of scalar fields and invoke a (special) disformal transformation[24] of $\mathcal{F}_{AB}$

$$\begin{aligned}\mathcal{G}_{AB} &\equiv \frac{2}{\mathcal{I}_1}\mathcal{F}_{AB} - \frac{3}{2\mathcal{I}_1}\left(\ln\mathcal{I}_1\right)_{,A}\left(\ln\mathcal{I}_1\right)_{,B} \\ &= e^{-2\alpha}\left(\mathcal{B}_{AB} - 6\mathcal{A}\alpha_{,A}\alpha_{,B} + 3\left(\alpha_{,A}\mathcal{A}_{,B} + \alpha_{,B}\mathcal{A}_{,A}\right)\right)\,, \end{aligned} \tag{14a}$$

$$\mathcal{G}^{BC} = \frac{\mathcal{I}_1}{2}\mathcal{F}^{BC} + \frac{\mathcal{I}_1}{2}\left(1 - 3\mathcal{I}_5\right)^{-1}\frac{3}{4}\mathcal{F}^{BE}\left(\ln\mathcal{I}_1\right)_{,E}\mathcal{F}^{CF}\left(\ln\mathcal{I}_1\right)_{,F}\,, \tag{14b}$$

where the inverse is calculated by making use of the knowledge about disformal transformations (cf. Appendix A in Ref. 25). Therefore the matrix $\mathcal{G}_{AB}$ also fulfils the requirements of the metric of the space of scalar fields, and can be invoked to construct invariants analogously to Eqs. (12) and (13).

Let us take a closer look to the relation between metrics (11), (14a) of the space of scalar fields and $\mathcal{B}_{AB}$. If one chooses to work within the Einstein frame defined by Eq. (7), then $\mathcal{F}_{AB}|_{\mathfrak{E}} = \mathcal{B}^{\mathfrak{E}}_{AB}$. If instead the Jordan frame, defined by Eq. (8), is considered then $\mathcal{G}_{AB}\big|_{\mathfrak{J}} = \mathcal{B}^{\mathfrak{J}}_{AB}$. In this sense, we see that a Weyl rescaling (conformal transformation) in the spacetime introduces a disformal transformation in the space of scalar fields (cf. also Ref. 26). However this relation is somewhat formal because $\mathcal{I}_1$ does not have a dynamics of its own.

5. Translation Rules

For one scalar field case a prescription was developed[18,20] how to easily "translate" the results obtained in a particular frame and parametrization to the general one. The idea is to write the action in terms of invariant quantities in the form resembling a particular frame and parametrization and read off the correspondences. In the current paper we generalize this approach to the multiscalar field case.

5.1. *Einstein frame*

Let us consider the Einstein frame setup (7) used by Damour and Esposito-Farèse.[1] We start by defining a spacetime metric

$$\hat{g}^{(\mathfrak{E})}_{\mu\nu} \equiv \mathcal{A} g_{\mu\nu} \,, \tag{15}$$

that in the Einstein frame ($\mathcal{A} = 1$) coincides with the metric $g^{\mathfrak{E}}_{\mu\nu}$. Note that due to suitable transformation properties of $\mathcal{A}$, given by Eq. (3a), the metric $\hat{g}^{(\mathfrak{E})}_{\mu\nu}$ does not transform under the local Weyl rescaling (2a). Because of that we shall use the term invariant metric to refer to $\hat{g}^{(\mathfrak{E})}_{\mu\nu}$ and other metric tensors having the same transformation properties. Expressing the action (1) in terms of the new dynamical metric $\hat{g}^{(\mathfrak{E})}_{\mu\nu}$, while neglecting the boundary term, we get

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int_{V_4} d^4x \sqrt{-\hat{g}^{(\mathfrak{E})}} \left\{ \hat{R}^{(\mathfrak{E})} - 2\mathcal{F}_{AB} \hat{g}^{(\mathfrak{E})\mu\nu} \hat{\nabla}^{(\mathfrak{E})}_{\mu} \Phi^A \hat{\nabla}^{(\mathfrak{E})}_{\nu} \Phi^B - 2\ell^{-2} \mathcal{I}_2 \right\} \\ + S_{\mathrm{m}} \left[\mathcal{I}_1 \hat{g}^{(\mathfrak{E})}_{\mu\nu}, \chi \right] . \end{aligned} \tag{16}$$

The obtained action has preserved all degrees of freedom and hence is as general as action (1). However, if one fixes the frame to be the Einstein frame (Eq. (7)) then we have the following mapping

$$\begin{array}{rcl|rcl} \hat{g}^{(\mathfrak{E})}_{\mu\nu} & \mapsto & g^{\mathfrak{E}}_{\mu\nu} & 1 & \mapsto & 1 \equiv \mathcal{A}_{\mathfrak{E}} \\ \sqrt{-\hat{g}^{(\mathfrak{E})}} & \mapsto & \sqrt{-\hat{g}^{\mathfrak{E}}} & \mathcal{F}_{AB} \equiv \frac{2\mathcal{A}\mathcal{B}_{AB} + 3(\mathcal{A})_{,A}(\mathcal{A})_{,B}}{4\mathcal{A}^2} & \mapsto & \mathcal{B}^{\mathfrak{E}}_{AB} \\ \hat{R}^{(\mathfrak{E})} & \mapsto & R_{\mathfrak{E}} & \mathcal{I}_2 \equiv \frac{\mathcal{V}}{\mathcal{A}^2} & \mapsto & \mathcal{V}_{\mathfrak{E}} \\ \hat{\nabla}^{(\mathfrak{E})}_{\mu} & \mapsto & \nabla^{\mathfrak{E}}_{\mu} & \frac{1}{2}\ln \mathcal{I}_1 \equiv \frac{1}{2} \ln\left(\frac{e^{2\alpha}}{\mathcal{A}}\right) & \mapsto & \alpha_{\mathfrak{E}} \end{array} \,, \tag{17}$$

where $(\mathfrak{E})$ as a superscript or a subscript denotes that the quantity under consideration is calculated via the invariant metric defined by Eq. (15) and $\mathfrak{E}$ (without parenthesis) denotes that the quantity is expressed in the Einstein frame.[20]

When one wants to express an invariant quantity calculated in the Einstein frame as a general result then one has to use the mapping (17) backwards and evaluate everything in terms of $g_{\mu\nu}$, $\mathcal{A}$, $\mathcal{V}$, α and $\mathcal{B}_{AB}$. Note that as no scalar fields redefinition is used for obtaining action (16) also the derivative $\frac{\partial}{\partial \Phi^A}$ is mapped to itself in both directions.

5.2. *Jordan frame*

Now let us consider the Jordan frame in the Brans-Dicke-Bergmann-Wagoner type parametrization (8).[11,21] Analogously to the previous case, we define an invariant metric

$$\hat{g}^{(\mathfrak{J})}_{\mu\nu} = e^{2\alpha} g_{\mu\nu} \,. \tag{18}$$

Expressing the action functional (1) in terms of $\hat{g}^{(\mathfrak{J})}_{\mu\nu}$ we get

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int_{V_4} d^4x \sqrt{-\hat{g}^{(\mathfrak{J})}} \left\{ \frac{1}{\mathcal{I}_1} \hat{R}^{(\mathfrak{J})} - \mathcal{G}_{AB}\, \hat{g}^{(\mathfrak{J})\mu\nu} \hat{\nabla}^{(\mathfrak{J})}_{\mu} \Phi^A \hat{\nabla}^{(\mathfrak{J})}_{\nu} \Phi^B - 2\ell^{-2} \mathcal{I}_4 \right\} \\ + S_{\mathrm{m}} \left[\hat{g}^{(\mathfrak{J})}_{\mu\nu}, \chi \right] . \end{aligned} \tag{19}$$

As before, we have neglected the boundary term. The action functional (19) could be used to read off the "translation rules" for Jordan frame in a generic parametrization. However, in the current paper we also consider the case where the parametrization is chosen to be the BDBW parametrization as given by Eq. (8).

In Refs. 11, 21 one scalar field has been made distinct by defining $\bar{\mathcal{A}}_{\mathfrak{J}} = \Psi = \frac{1}{\mathcal{I}_1}\big|_{\mathfrak{J}}$ which multiplies the Ricci scalar. Following that line of thought we redefine the scalar fields $\left\{\Phi^A\right\} \to \left\{\bar{\Phi}^1, \ldots, \bar{\Phi}^{n-1}, 1/\mathcal{I}_1\right\}$ in order to distinguish $1/\mathcal{I}_1$ as a scalar field that has vanishing kinetic coupling to the other scalar fields, thereby mimicking conditions (8b). Therefore for the latter the condition

$$\bar{\mathcal{G}}^{an} \equiv \frac{\partial \bar{\Phi}^a}{\partial \Phi^A} \mathcal{G}^{AB} \left(\frac{1}{\mathcal{I}_1} \right)_{,B} = 0 \,, \qquad a = 1 \ldots n-1 \tag{20}$$

must hold. Note that this is just a transformation of $\mathcal{G}^{AB}$ under a change of coordinates in the space of scalar fields. In the same spirit

$$\bar{\mathcal{G}}^{nn} = \mathcal{G}^{AB} \left(\frac{1}{\mathcal{I}_1} \right)_{,A} \left(\frac{1}{\mathcal{I}_1} \right)_{,B} = \frac{2\mathcal{I}_5}{\mathcal{I}_1 \left(1 - 3\mathcal{I}_5\right)} = \left(\bar{\mathcal{G}}_{nn}\right)^{-1} , \tag{21}$$

where the last equality follows from the condition (20). We also made use of Eqs. (14) to write the expression in terms of $\mathcal{F}^{AB}$ hidden in $\mathcal{I}_5$. Hence we see that the kinetic term for $1/\mathcal{I}_1$ is an invariant by itself,[20]

$$\bar{\mathcal{G}}_{nn} \hat{g}^{(\mathfrak{J})\mu\nu} \hat{\nabla}^{(\mathfrak{J})}_{\mu} \frac{1}{\mathcal{I}_1} \hat{\nabla}^{(\mathfrak{J})}_{\nu} \frac{1}{\mathcal{I}_1} = \frac{\mathcal{I}_1 \left(1 - 3\mathcal{I}_5\right)}{2\mathcal{I}_5} \hat{g}^{(\mathfrak{J})\mu\nu} \hat{\nabla}^{(\mathfrak{J})}_{\mu} \frac{1}{\mathcal{I}_1} \hat{\nabla}^{(\mathfrak{J})}_{\nu} \frac{1}{\mathcal{I}_1} \,. \tag{22}$$

In addition, due to the condition (20) and to the result (21) it holds that

$$\bar{\mathcal{G}}^{nn} \frac{\partial \Phi^A}{\partial \frac{1}{\mathcal{I}_1}} + \bar{\mathcal{G}}^{an} \frac{\partial \Phi^A}{\partial \bar{\Phi}^a} = \frac{2\mathcal{I}_5}{\mathcal{I}_1 \left(1 - 3\mathcal{I}_5\right)} \frac{\partial \Phi^A}{\partial \frac{1}{\mathcal{I}_1}} = \mathcal{G}^{AB} \left(\frac{1}{\mathcal{I}_1} \right)_{,B} . \tag{23}$$

Thereby we can introduce a differential operator

$$\frac{\partial}{\partial \frac{1}{\mathcal{I}_1}} = \frac{\partial \Phi^A}{\partial \frac{1}{\mathcal{I}_1}} \frac{\partial}{\partial \Phi^A} = \bar{\mathcal{G}}_{nn} \mathcal{G}^{AB} \left(\frac{1}{\mathcal{I}_1} \right)_{,B} \frac{\partial}{\partial \Phi^A} = -\frac{\mathcal{I}_1}{4\mathcal{I}_5} \mathcal{F}^{AB} \left(\ln \mathcal{I}_1\right)_{,B} \frac{\partial}{\partial \Phi^A} \,, \tag{24}$$

that gives an invariant if acted upon an invariant.

The "translation" rules can be read out from the mapping

$$\begin{array}{rcl|rcl}
\hat{g}^{(\mathfrak{J})}_{\mu\nu} & \mapsto & g^{\mathfrak{J}}_{\mu\nu} & \frac{1}{\mathcal{I}_1} & \mapsto & \Psi \equiv \bar{\mathcal{A}}_{\mathfrak{J}} \\
\sqrt{-\hat{g}^{(\mathfrak{J})}} & \mapsto & \sqrt{-g^{\mathfrak{J}}} & \frac{\mathcal{I}_1(1-3\mathcal{I}_5)}{2\mathcal{I}_5} & \mapsto & \frac{\omega}{\Psi} \equiv \bar{\mathcal{B}}^{\mathfrak{J}}_{nn} \\
\hat{R}^{(\mathfrak{J})} & \mapsto & R_{\mathfrak{J}} & \mathcal{I}_4 \equiv e^{-4\alpha}\mathcal{V} & \mapsto & \bar{\mathcal{V}}_{\mathfrak{J}} \\
\hat{\nabla}^{(\mathfrak{J})}_{\mu} & \mapsto & \nabla^{\mathfrak{J}}_{\mu} & 0 & \mapsto & 0 = \bar{\alpha}_{\mathfrak{J}} \\
\hat{R}^{(\mathfrak{J})}_{\mu\nu} & \mapsto & R_{\mathfrak{J}\,\mu\nu} & -\frac{\mathcal{I}_1}{4\mathcal{I}_5}(\ln \mathcal{I}_1)^{,A}\frac{\partial}{\partial\Phi^A} & \mapsto & \frac{\partial}{\partial\Psi}
\end{array} , \tag{25}$$

where, analogously to the previous case the superscript $(\mathfrak{J})$ denotes a quantity calculated via the invariant metric (18) and super or subscript $\mathfrak{J}$ indicates that the quantity under consideration is evaluated in the Jordan frame. The mapping for $\frac{\omega}{\Psi}$ follows from Eq. (21), while $\frac{\partial}{\partial\Psi}$ is Eq. (24) where the indexes are raised with $\mathcal{F}^{AB}$. Similarly to the Einstein frame, if one wants to "translate" invariant quantities then one has to invoke the mapping (25) backwards. Note that the rules given by Eq. (25) are not complete but they are sufficient for showing how the formalism works as is done in the next section.

6. Parametrized Post-Newtonian Approximation

Each theory must be confronted with experiments. For metric gravity theories a prescription named the parametrized post-Newtonian approximation (PPN) has been constructed in order to be able to test the viability of a theory via experiments carried out in the solar system.

In the current paper we do not calculate the PPN parameters but rather show that the results obtained in different frames generalize to the same invariant and hence are frame-independent. We start by writing out the results from Ref. 1 where the Einstein frame (without potential) was considered:

$$G_{eff} \equiv \frac{\kappa^2}{8\pi} e^{2\alpha_{\mathfrak{E}}} \Big(1 + \mathcal{B}^{\mathfrak{E}\,AB}(\alpha_{\mathfrak{E}})_{,A}(\alpha_{\mathfrak{E}})_{,B}\Big) , \tag{26a}$$

$$\gamma - 1 \equiv -2\left(\frac{\mathcal{B}^{\mathfrak{E}\,AB}(\alpha_{\mathfrak{E}})_{,A}(\alpha_{\mathfrak{E}})_{,B}}{1+\mathcal{B}^{\mathfrak{E}\,AB}(\alpha_{\mathfrak{E}})_{,A}(\alpha_{\mathfrak{E}})_{,B}}\right) , \tag{26b}$$

$$\beta - 1 \equiv \frac{\mathcal{B}^{\mathfrak{E}\,AC}(\alpha_{\mathfrak{E}})_{,C}\big((\alpha_{\mathfrak{E}})_{,AB} - \Gamma^F_{AB}(\alpha_{\mathfrak{E}})_{,F}\big)\mathcal{B}^{\mathfrak{E}\,BD}(\alpha_{\mathfrak{E}})_{,D}}{2\big(1+\mathcal{B}^{\mathfrak{E}\,AB}(\alpha_{\mathfrak{E}})_{,A}(\alpha_{\mathfrak{E}})_{,B}\big)^2} , \tag{26c}$$

where Γ^F_{AB} are the Christoffel symbols for $\mathcal{B}^{\mathfrak{E}}_{AB}$.

Second, we write out the results from Ref. 21 obtained in the Jordan frame (without potential):

$$G_{eff} \equiv \frac{\kappa^2}{8\pi}\frac{1}{\Psi}\left(1+\frac{1}{2\omega+3}\right), \quad \gamma - 1 \equiv -\frac{2}{(2\omega+3)\big(1+\frac{1}{2\omega+3}\big)} , \tag{27a}$$

$$\beta - 1 \equiv \frac{\Psi\frac{\partial\omega}{\partial\Psi}}{\big(1+\frac{1}{2\omega+3}\big)^2}\left(\frac{1}{2\omega+3}\right)^3 . \tag{27b}$$

Here the expression for β differs from the one presented in Ref. 21, because we inverted the normalization $\frac{\kappa^2}{8\pi}\frac{1}{\Psi}\left(1+\frac{1}{2\omega+3}\right)\equiv 1$ in order to get rid of κ^2. This result also matches the early computation[2] of γ in the Jordan frame with constant $\mathcal{B}^{\mathfrak{J}}_{AB}$, as well as the general result for a single scalar field with a potential.[27]

One can show that if for Eqs. (26) we use the mapping (17) backwards and for Eqs. (27) we use the mapping (25) backwards then both generalize to

$$G_{eff} \equiv \frac{\kappa^2}{8\pi}\mathcal{I}_1\left(1+\mathcal{I}_5\right), \quad \gamma - 1 \equiv -2\left(\frac{\mathcal{I}_5}{1+\mathcal{I}_5}\right), \tag{28a}$$

$$\beta - 1 \equiv \frac{(\ln\mathcal{I}_1)^{,A}(\ln\mathcal{I}_1)^{,B}\left((\ln\mathcal{I}_1)_{,AB} - \frac{1}{2}\mathcal{F}_{AB,C}(\ln\mathcal{I}_1)^{,C}\right)}{16(1+\mathcal{I}_5)^2}, \tag{28b}$$

where the indexes are raised with $\mathcal{F}^{AB}$. Hence, it is evident that physical observables are frame and parametrization independent since they transform as invariants.

7. Summary

We studied general multiscalar-tensor theories of gravity without derivative couplings. By introducing quantities that are invariant under a local Weyl rescaling of the spacetime metric and transform covariantly if the scalar fields are reparametrized, we generalized the formalism of the invariants that has been developed in the case of a single scalar field.[18,20] Just as in the latter, we were able to construct rather simple "translation" rules in the context of multiscalar-tensor theories of gravity as well. By invoking the prescription, one can neatly compare the results obtained in different frames and parametrizations by "translating" an expression under consideration to a generic frame. As an example we used the formalism to show that the results of the parametrized post-Newtonian approximation, calculated in the Einstein frame[1] and in the Jordan frame,[2,21] indeed are the same if expressed in a generic frame.

It would be interesting to see, how the formalism of invariant quantities generalizes and could help to explore the next generation of MSTG, namely the theories of multi-Galileons, multi-Horndeski and beyond,[28–30] where the conformal transformation of the metric seems to generalize to multi-disformal.[31]

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Chapter 9

Parametrizations in scalar-tensor theories of gravity and the limit of general relativity

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Parametrizations in scalar-tensor theories of gravity and the limit of general relativity

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Abstract. We consider a general scalar-tensor theory of gravity and review briefly different forms it can be presented (different conformal frames and scalar field parametrizations). We investigate the conditions under which its field equations and the parametrized post-Newtonian parameters coincide with those of general relativity. We demonstrate that these so-called limits of general relativity are independent of the parametrization of the scalar field, although the transformation between scalar fields may be singular at the corresponding value of the scalar field. In particular, the limit of general relativity can equivalently be determined and investigated in the commonly used Jordan and Einstein frames.

1. Introduction

The still unknown nature of the phenomena of dark matter and dark energy facilitates continued interest in the alternatives to Einstein's general relativity, for a comprehensive review see e.g [1]. A simple and straightforward extension of general relativity (GR) is provided by the Jordan-Brans-Dicke theory [2, 3] and its generalization scalar-tensor gravity (STG) [4, 5], where an additional scalar field participates in the gravitational interaction. As was proposed by Dicke, it is possible via a conformal rescaling of the metric (frame change) and reparametrization of the scalar field to transform the STG action into another representation, equivalent to the original one if the units of measurement are also appropriately rescaled [6]. However, the precise significance of this transformation and the physical and mathematical equivalence of different representations are still a topic for an ongoing debate, for a glimpse of the most recent papers, see e.g [7, 8, 9] and [Stabile A *et al* arXiv:1310.7097]. The present work contributes to the discussion by following our earlier study [10] in asking what happens when the transformation from one parametrization of the scalar field to another is singular.

In the following section we write down the most general STG action involving four free functions, the transformations that leave this action invariant, and the ensuing field equations. In Sec. 3 we recall some of the most used STG frames and parametrizations, and collect some useful relations between them. Then Sec. 4 argues that the limit that reduces the STG field equations into those of general relativity does not depend on the particular frame and parametrization, in particular, it is not affected by a possible singularity in reparametrizing the scalar field. Sec. 5 gives the parametrized post-Newtonian (PPN) parameters for STG in its most general form as well as for the most used special cases introduced before, in order to witness again that the conditions for these parameters to coincide with those of GR are independent of the parametrization. Therefore if in some parametrization there is a certain value of the scalar field that takes STG to its GR limit, there is necessarily a corresponding value (or values) of the reparametrized field, that does the same in another frame and parametrization. This conclusion is illustrated by an example in Sec. 6 and summarized in Sec. 7.

2. General action functional and field equations

The most general action functional for a scalar-tensor theory of gravity including scalar self-interaction only through scalar field but not its derivatives was written down by Flanagan [11],

$$S = \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-g}\left\{\mathcal{A}(\Phi)R - \mathcal{B}(\Phi)g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - 2\kappa^2 V(\Phi)\right\} + S_m\left(e^{2\alpha(\Phi)}g_{\mu\nu},\chi\right) \quad (2.1)$$

It contains four arbitrary functions of the dimensionless scalar field Φ: nonminimal coupling function $\mathcal{A}(\Phi)$, generic kinetic coupling of the scalar field $\mathcal{B}(\Phi)$, self-interaction potential of the scalar field $V(\Phi)$ and conformal coupling $e^{2\alpha(\Phi)}$ between the metric $g_{\mu\nu}$ and matter fields χ. Note that $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$ and $\alpha(\Phi)$ are dimensionless, but for the convenience of notation in cosmology the scalar potential is assumed to be of the dimension of energy density, $[V] = [\rho]$ in units $c = 1$. If we impose a physical condition that gravitational interaction is always finite and attractive, the nonminimal coupling function must satisfy $0 < \mathcal{A} < \infty$. We also assume from physical considerations that self-interaction potential is non-negative, $0 \leq V(\Phi) < \infty$.

As demonstrated by Flanagan [11], two of the four arbitrary functions can be fixed by transformations that contain two arbitrary functions $\bar\gamma(\bar\Phi)$, $\bar f(\bar\Phi)$ and leave the structure of action functional (2.1) invariant:

$$g_{\mu\nu} = e^{2\bar\gamma(\bar\Phi)}\bar g_{\mu\nu}\,, \quad \Phi = \bar f(\bar\Phi) \quad (2.2)$$

We will call the first transformation the change of the frame and the second one the reparametrization of the scalar field. The change of the frame is in fact a conformal rescaling of the metric and we assume that it is reasonable, i.e the function $\bar\gamma(\bar\Phi)$ and its derivative $d\bar\gamma/d\bar\Phi$ do not diverge at any $\bar\Phi$.

The transformed action functional (2.1) retains its form

$$\bar S = \frac{1}{2\kappa^2}\int_{V_4} d^4x\sqrt{-\bar g}\{\bar{\mathcal{A}}(\bar\Phi)\bar R - \bar{\mathcal{B}}(\bar\Phi)\bar g^{\mu\nu}\bar\nabla_\mu\bar\Phi\bar\nabla_\nu\bar\Phi - 2\kappa^2\bar V(\bar\Phi)\} + S_m\left(e^{2\bar\alpha(\bar\Phi)}\bar g_{\mu\nu},\chi\right) \quad (2.3)$$

with transformed functions [11]

$$\begin{aligned}&\bar{\mathcal{A}}(\bar\Phi) = e^{2\bar\gamma(\bar\Phi)}\mathcal{A}\left(\bar f(\bar\Phi)\right), \quad \bar V(\bar\Phi) = e^{4\bar\gamma(\bar\Phi)}\,V\left(\bar f(\bar\Phi)\right), \quad \bar\alpha(\bar\Phi) = \alpha\left(\bar f(\bar\Phi)\right) + \bar\gamma(\bar\Phi)\\ &\bar{\mathcal{B}}(\bar\Phi) = e^{2\bar\gamma(\bar\Phi)}\left(\mathcal{B}\left(\bar f(\bar\Phi)\right)\left(\bar f'\right)^2 - 6\left(\bar\gamma'\right)^2\mathcal{A}\left(\bar f(\bar\Phi)\right) - 6\bar\gamma'\bar f'\mathcal{A}'\right)\end{aligned} \quad (2.4)$$

where

$$\bar f' \equiv \frac{d\bar f(\bar\Phi)}{d\bar\Phi}, \quad \mathcal{A}' \equiv \frac{d\mathcal{A}(\Phi)}{d\Phi} \quad \text{etc} \quad (2.5)$$

Note that at the conformal transformation (2.2) the energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \quad (2.6)$$

transforms as $\bar T_{\mu\nu} = e^{2\bar\gamma}T_{\mu\nu}$ and its trace as $\bar T = e^{4\bar\gamma}T$.

From these transformation rules we can notice the following.

- The conditions on nonminimal coupling function $0 < \mathcal{A} < \infty$ and self-interaction potential $0 \leq V(\Phi) < \infty$ are preserved, i.e $0 < \bar{\mathcal{A}} < \infty$ and $0 \leq \bar V(\bar\Phi) < \infty$.

- If in some frame $\alpha = 0$, then in any other frame $|\bar{\alpha}| < \infty$.
- If we want to avoid ghosts, i.e if there is a frame where the tensorial and scalar part of the gravitational interaction are separated with $\mathcal{A} = 1$ and $\mathcal{B} > 0$, then in any related frame and parametrization it follows that

$$2\bar{\mathcal{A}}\bar{\mathcal{B}} + 3(\bar{\mathcal{A}}')^2 > 0 \tag{2.7}$$

We assume this relation to hold.

The field equations can be derived from the general action functional (2.1) by varying with respect to metric tensor $g^{\mu\nu}$ and scalar field Φ, respectively:

$$\mathcal{A}G_{\mu\nu} + \left[\frac{1}{2}\mathcal{B} + \mathcal{A}''\right] g_{\mu\nu}\nabla_\rho\Phi\nabla^\rho\Phi - \left[\mathcal{B} + \mathcal{A}''\right]\nabla_\mu\Phi\nabla_\nu\Phi + \mathcal{A}'\left[g_{\mu\nu}\Box\Phi - \nabla_\mu\nabla_\nu\Phi\right] + \kappa^2 g_{\mu\nu}V = \kappa^2 T_{\mu\nu} \tag{2.8}$$

$$\frac{1}{2}R\mathcal{A}' + \frac{1}{2}\mathcal{B}' g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + \mathcal{B}\Box\Phi - \kappa^2 V' = -\kappa^2\alpha' T \tag{2.9}$$

Upon substituting the scalar curvature R from the first equation into the second one and multiplying by $2\mathcal{A}$, the equation for the scalar field reads

$$\left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)\Box\Phi + \frac{\left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)'}{2} g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - 2\kappa^2\left(\mathcal{A}V' - 2\mathcal{A}'V\right) = \kappa^2\left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)T \tag{2.10}$$

A direct calculation demonstrates that upon the transformation (2.4) all terms in the equation of the metric tensor (2.8) acquire a common factor $e^{2\bar{\gamma}}$, which we have assumed to be regular. However, the transformed equation for the scalar field (2.10) gets a common factor $e^{6\bar{\gamma}}\bar{f}'$. If the transformation is regular, i.e $\bar{f}' \neq 0$, $\bar{f}' \neq \infty$, this equation should yield an equivalent account of the same physics in different parametrizations. What happens for the points where $\bar{f}'$ fails to be finite needs extra attention.

Finally, from the field equations (2.8) and (2.9) a continuity equation follows:

$$\nabla_\mu T^{\mu\nu} = \alpha' T\nabla^\nu\Phi \tag{2.11}$$

If $\alpha' = 0$ the right-hand side vanishes and the usual conservation of energy law holds; let us call the $\alpha = 0$ case the Jordan frame. Another well-known frame is the Einstein frame with $\mathcal{A} = 1$ and in general $\alpha' \neq 0$.

3. Some widely used action functionals

Sometimes in the literature one may encounter treatments which fix the frame (i.e fix $\alpha(\Phi)$, e.g $\alpha = 0$), but leave the parametrization of the scalar field unfixed, thus keeping three arbitrary functions in the STG action functional. But most often one meets a few distinct forms of the STG action functional obtained from the general action (2.1) by fixing two of the four arbitrary functions. These are the following.

1. The Jordan frame action in the Brans-Dicke-Bergmann-Wagoner parametrization (JF BDBW) [3, 4, 5] for the scalar field Ψ fixes $\mathcal{A} = \Psi$, $\alpha = 0$, while keeping $\mathcal{B} = \omega(\Psi)/\Psi$, $V = V(\Psi)$:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\Psi R - \frac{\omega(\Psi)}{\Psi}\nabla^\rho\Psi\nabla_\rho\Psi - 2\kappa^2 V(\Psi)\right] + S_m\left(g_{\mu\nu}, \chi\right) \tag{3.1}$$

The original Brans-Dicke gravity (JF BD) [3] with a potential is a special case where $\omega = const.$,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\Psi R - \frac{\omega}{\Psi}\nabla^\rho\Psi\nabla_\rho\Psi - 2\kappa^2 V(\Psi)\right] + S_m\left(g_{\mu\nu},\chi\right) \tag{3.2}$$

2. The Jordan frame action in the parametrization used by Boisseau, Esposito-Farèse, Polarski and Starobinsky (JF BEPS) [12, 13] for the scalar field as ϕ is obtained by taking $\mathcal{B} = 1$, $\alpha = 0$, while having $\mathcal{A} = F(\phi)$, $V = V(\phi)$:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[F(\phi)R - \nabla^\rho\phi\nabla_\rho\phi - 2\kappa^2 V(\phi)\right] + S_m\left(g_{\mu\nu},\chi\right) \tag{3.3}$$

In the so-called nonminimal coupling case (JF nm), the function F has a distinct form $F(\phi) = 1 - \xi\phi^2$, where ξ is a dimensionless parameter:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\left(1-\xi\phi^2\right)R - \nabla^\rho\phi\nabla_\rho\phi - 2\kappa^2 V(\phi)\right] + S_m\left(g_{\mu\nu},\chi\right) \tag{3.4}$$

3. The Einstein frame action in canonical parametrization (EF can) [6, 4, 5] for the scalar field denoted as φ, fixes $\mathcal{A} = 1$, $\mathcal{B} = 2$, while keeping $\alpha = \alpha(\varphi)$ and $V = V(\varphi)$:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - 2g^{\mu\nu}\,\nabla_\mu\varphi\nabla_\nu\varphi - 2\kappa^2\,V(\varphi)\right] + S_m\left(e^{2\alpha(\varphi)}g_{\mu\nu},\chi\right) \tag{3.5}$$

The well known Einstein gravity with minimally coupled scalar field (EF min) can be viewed as a special case here with $\alpha = 0$,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - 2g^{\mu\nu}\,\nabla_\mu\varphi\nabla_\nu\varphi - 2\kappa^2\,V(\varphi)\right] + S_m\left(g_{\mu\nu},\chi\right) \tag{3.6}$$

However, in the latter case the scalar field equation (2.10) does not contain matter energy-momentum T as a source and strictly speaking the scalar field is not mediating the gravitational interaction any more.

The transformations between these most common frames and parametrizations are presented in Table 1. Note that the mutual derivatives of the scalar field in different parametrizations included in the Table 1 are in fact just $\bar{f}'$ which should satisfy the conditions $\bar{f}' \neq 0$, $\bar{f}' \neq \infty$ for a transformation to be regular.

4. Field equations and the limit of general relativity

Let us investigate the conditions under which a STG coincides with GR. Since the latter one does not involve a dynamical scalar field, a natural assumption is $\Phi = \text{const.}$, $\nabla_\mu\Phi = 0$. However, this condition should be made consistent by requiring that the source term for the scalar field also vanishes, otherwise constant Φ can not be maintained. Rewriting the scalar field equation (2.10) as

$$\Box\Phi + \frac{1}{2}\frac{\left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)'}{2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi = \kappa^2\frac{\left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)T + 2\left(\mathcal{A}V' - 2\mathcal{A}'V\right)}{2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2} \tag{4.1}$$

it becomes clear that the STG equations can concur with those of GR at the values of Φ where the term on the RHS of (4.1) vanishes. Given that $\mathcal{A}$ is everywhere regular there are several possibilities.

The first and most obvious case is realized for $\Phi_\bullet$ which should simultaneously satisfy

$$\left(\mathcal{A}' - 2\alpha'\mathcal{A}\right)|_{\Phi_\bullet}T = 0 \tag{4.2a}$$

Table 1. Transformations between frames and parametrizations

	JF BDBW (Ψ)	JF BEPS (ϕ)	EF can (φ)
JF BDBW (Ψ)	Identity	$F(\phi) = \Psi$ $(\frac{d\phi}{d\Psi})^2 = \frac{\omega(\Psi)}{\Psi}$ $(\frac{dF}{d\phi})^2 = \frac{\Psi}{\omega(\Psi)}$	$\alpha(\varphi) = -\frac{1}{2}\ln\Psi$ $(\frac{d\varphi}{d\Psi})^2 = \frac{2\omega(\Psi)+3}{4\Psi^2}$ $(\frac{d\alpha}{d\varphi})^2 = \frac{1}{2\omega(\Psi)+3}$
JF BD (Ψ)	$\omega = \text{const.}$	$F(\phi) = \Psi$ $(\frac{d\phi}{d\Psi})^2 = \frac{\omega}{\Psi}$ $(\frac{dF}{d\phi})^2 = \frac{\Psi}{\omega}$	$\alpha(\varphi) = -\frac{1}{2}\ln\Psi$ $(\frac{d\varphi}{d\Psi})^2 = \frac{2\omega+3}{4\Psi^2}$ $(\frac{d\alpha}{d\varphi})^2 = \frac{1}{2\omega+3}$
JF BEPS (ϕ)	$\Psi = F(\phi)$ $\frac{d\Psi}{d\phi} = \frac{dF}{d\phi}$ $\omega(\Psi) = F(\phi)\frac{1}{(\frac{dF}{d\phi})^2}$	Identity	$\alpha(\varphi) = -\frac{1}{2}\ln F(\phi)$ $(\frac{d\varphi}{d\phi})^2 = \frac{3}{4}(\frac{d\ln F(\phi)}{d\phi})^2$ $+\frac{1}{2F(\phi)}$
JF nm (ϕ)	$\Psi = 1-\xi\phi^2$ $\frac{d\Psi}{d\phi} = -2\xi\phi$ $\omega(\Psi) = \frac{\Psi}{4\xi(1-\Psi)} = \frac{1-\xi\phi^2}{4\xi^2\phi^2}$	$F(\phi) = 1-\xi\phi^2$	$\alpha(\varphi) = \frac{1}{2}\ln\left(\frac{1}{1-\xi\phi^2}\right)$ $(\frac{d\varphi}{d\phi})^2 = \frac{1-\xi\phi^2+6\xi^2\phi^2}{2(1-\xi\phi^2)^2}$ $\phi^2 = \frac{1}{\xi}(1-e^{-2\alpha(\varphi)})$
EF can (φ)	$\Psi = e^{-2\alpha(\varphi)}$ $(\frac{d\Psi}{d\varphi})^2 = 4e^{-4\alpha(\varphi)}(\frac{d\alpha}{d\varphi})^2$ $\omega(\Psi) = \frac{1}{2}\left(\frac{1}{(\frac{d\alpha}{d\varphi})^2} - 3\right)$	$F(\phi) = e^{-2\alpha(\varphi)}$ $\left(\frac{d\phi}{d\varphi}\right)^2 = 2e^{-2\alpha(\varphi)}\times$ $\times\left(1-3\left(\frac{d\alpha}{d\varphi}\right)^2\right)$ $(\frac{dF}{d\phi})^2 = \frac{2e^{-2\alpha(\varphi)}\left(\frac{d\alpha}{d\varphi}\right)^2}{1-3\left(\frac{d\alpha}{d\varphi}\right)^2}$	Identity
EF min (φ)	$\Psi = 1$	$F(\phi) = 1$	$\alpha = 0$

$$\left(\mathcal{A}V' - 2\mathcal{A}'V\right)|_{\Phi_\bullet} = 0 \tag{4.2b}$$

while $\mathcal{B}|_{\Phi_\bullet}$ is finite and nonvanishing. In addition, for the full compliance with GR the factors in front of the kinetic terms in the Einstein equation (2.8) and scalar field equation (4.1) should remain regular, hence $\mathcal{A}''|_{\Phi_\bullet}$ and $\mathcal{B}'|_{\Phi_\bullet}$ should not diverge. If the latter is not the case, then the STG does not allow a solution which behaves exactly as GR, but it may still be possible to have solutions which dynamically approach GR as a limiting process, provided

$$\mathcal{A}''\nabla_\mu\Phi\nabla_\nu\Phi \to 0 \quad \text{or/and} \quad \frac{\mathcal{B}'}{\mathcal{B}}\nabla_\mu\Phi\nabla_\nu\Phi \to 0 \quad \text{as} \quad \Phi \to \Phi_\bullet \tag{4.3}$$

The result is the GR Einstein equation with $V|_{\Phi_\bullet}$ effectively playing the role of the cosmological constant. It is instructive to observe that for JF BEPS parametrization the condition (4.2a), (4.2b) translates into $F'|_{\phi_\bullet} = 0$, $V'|_{\phi_\bullet} = 0$, for the nonminimal coupling case into $\phi_\bullet = 0$, $V'|_{\phi_\bullet} = 0$, and for the EF canonical parametrization into $\alpha'|_{\varphi_\bullet} = 0$, $V'|_{\varphi_\bullet} = 0$. But in the JF BDBW parametrization the condition (4.2a) can not be realized for general matter ($T \neq 0$) at all since $\mathcal{A}' \equiv 1$.

The second possibility to make the RHS of (4.1) to vanish is by having a value $\Phi_\star$ for which

$$\frac{1}{\mathcal{B}}\Big|_{\Phi_\star} = 0 \tag{4.4}$$

while $\mathcal{A}'|_{\Phi_\star}$, $\alpha'|_{\Phi_\star}$, and $V'|_{\Phi_\star}$ do not diverge. An important difference with the previous case is that here we do not have a constant solution for the scalar field, but only a process of approaching to that value. For this process to correctly yield the GR, we must demand that $\mathcal{B}\nabla_\mu\Phi\nabla_\nu\Phi \to 0$ as $\Phi \to \Phi_\star$. In addition, if $\mathcal{B}'|_{\Phi_\star}$ or $\mathcal{A}''|_{\Phi_\star}$ happen to be singular as well, only the solutions with

$$\frac{\mathcal{B}'}{\mathcal{B}}\nabla_\mu\Phi\nabla_\nu\Phi \to 0, \quad \mathcal{A}''\nabla_\mu\Phi\nabla_\nu\Phi \to 0 \tag{4.5}$$

lead to GR as a limit. For a later remark we note that if the Einstein equation (2.8) and the scalar field equation (4.1) converge to the GR limit at the same "rate", i.e if

$$\mathcal{B}\nabla_\mu\Phi\nabla_\nu\Phi \propto \frac{\mathcal{B}'}{\mathcal{B}}\nabla_\mu\Phi\nabla_\nu\Phi \tag{4.6}$$

then $\frac{\mathcal{B}'}{\mathcal{B}^2}$ is finite. Among the particular forms of STG the condition (4.4) can be only realized in JF BDBW parametrization where it translates into $\frac{1}{\omega}|_{\Psi_\star} = 0$. In the JF BEPS and nonminimal, and EF canonical parametrizations the function $\mathcal{B}$ is fixed to a constant value which precludes (4.4).

At first there seems to be also a third option to make the RHS of (4.1) to vanish by letting $\frac{1}{\mathcal{A}'} = 0$. However by looking at the scalar field equation (2.9) this case turns out to be problematic. Namely, for GR it is well known that spacetime curvature and matter energy momentum are proportional to each other, $R \propto T$. But if $\mathcal{A}' \to \infty$ then finite T would correspond to vanishing R, unless α' also blows up. The latter would complicate the continuity equation (2.11). Hereby we restrict our attention to the cases where $\mathcal{A}'$ and α' are not singular at the same value of scalar field Φ, and therefore to achieve a GR-like behaviour we do not consider this possibility. Furthermore, for the sake of mathematical simplicity we also leave aside the rather fine-tuned theories where the conditions (4.2a), (4.2b) and (4.4) are realized together, or where both the numerator and denominator of the RHS term of (4.1) vanish simultaneously thus requiring a much more thorough analysis.

Now, given the rather different conditions (4.2a), (4.2b) and (4.4), one is entitled to ask whether there is any connection between them. It is interesting to note that there is. Under the transformations (2.4) the quantities

$$\frac{(\mathcal{A}' - 2\mathcal{A}\alpha')^2}{2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2} \quad \text{and} \quad \frac{(\mathcal{A}V' - 2\mathcal{A}'V)^2}{\mathcal{A}^4\,(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2)} \tag{4.7}$$

retain their form, i.e do not acquire extra terms or common factors. Therefore, under a generic transformation the condition that the RHS of (4.1) vanishes remains invariant, i.e if some Φ in a certain frame and parametrization satisfies it, then the corresponding $\bar{\Phi}$ in another frame and parametrization will also satisfy it. Although, it is completely feasible that $\Phi_\bullet$ obeying (4.2a), (4.2b) may get translated into $\bar{\Phi}_\star$ obeying (4.4).

5. Parametrized post-Newtonian approximation

For being viable, the scalar-tensor theory of gravity must pass the tests on local scales, e.g, give a good account of the motions in our solar system. A natural framework for such a check is provided by the parametrized post-Newtonian (PPN) formalism adapted to slow motions in a weak field. To compare GR and STG there are two nonvanishing PPN parameters γ and β. They both have value 1 for Einstein's general relativity which is also favored by current observations. For an STG, the PPN parameters can deviate from unity as they depend on the spatially asymptotic background value of the scalar field [14, 15]. When STG has a potential, the PPN parameters cease to be constants as they also acquire an extra dependence on the distance r from the source [16, 17, 18, 19]. It is useful to express the result in units where the Newtonian potential $U_N = \frac{\kappa^2 M}{8\pi r}$ is dimensionless, while the dimensionless constant $G_{\text{eff}}(\Phi, r)$ modifies multiplicatively Newton's gravitational constant $G_N = \frac{\kappa^2}{8\pi}$ and determines the Cavendish force.

It is possible to translate the general results [19] from JF BDBW parametrization into a generic representation of $\mathcal{A}$, $\mathcal{B}$, V, α by using the transformations (2.2) and (2.4) where $\bar{\mathcal{A}} = \Psi$, $\bar{\mathcal{B}} = \frac{\omega(\Psi)}{\Psi}$, and $\bar{\alpha} = 0$. It follows that $\bar{\gamma} = -\alpha$, $\bar{\gamma}' = -\alpha' \bar{f}'$, while the other necessary quantity $\bar{f}'$ can be expressed by taking the derivative of the first line of Eq. (2.4). Since the PPN ansatz assumes flat Minkowski spacetime in spatial infinity, the internal consistency requires $V = 0$, $V' = 0$ (all values of the functions taken at the asymptotic value of the scalar field). In terms of the constant related to the scalar field effective mass,

$$m_\Phi = \kappa \sqrt{\frac{2\mathcal{A}}{e^{2\alpha}\left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)} \frac{d^2V}{d\Phi^2}} \tag{5.1}$$

the results are

$$G_{\text{eff}} = \frac{e^{2\alpha}}{\mathcal{A}} \left(1 + \frac{\left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)^2 e^{-m_\Phi r}}{2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2} \right) \tag{5.2a}$$

$$\gamma - 1 = -\frac{2e^{2\alpha}\left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)^2 e^{-m_\Phi r}}{G_{\text{eff}} \mathcal{A}\left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)} \tag{5.2b}$$

$$\begin{aligned} \beta - 1 = {} & \frac{\left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)^2 \left[\left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)' \left(\mathcal{A}' - 2\mathcal{A}\alpha'\right) - 2\left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)\left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)' \right] e^{-2m_\Phi r}}{2e^{-4\alpha} G_{\text{eff}}^2 \mathcal{A} \left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)^3} \\ & - \frac{\left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)^2 m_\Phi r}{e^{-4\alpha} G_{\text{eff}}^2 \left(2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2\right)} \left[e^{-m_\Phi r} \ln m_\Phi r + \ldots \right] \end{aligned} \tag{5.2c}$$

Among the number of r-dependent terms in the square brackets on the last line only the contribution that is leading for large $m_\Phi r$ is given. For different frames and parametrizations the corresponding expressions can be found in Table 2. These can be deduced by specifying the functions in the formulas above, or using the transformations (2.4) and the information in Table 1.

The conceptual difference with the previous section is that now we have a static configuration and the functions of the scalar field are taken at their spatially asymptotic values. However, the analysis of the limit where the STG PPN parameters coincide with those of general relativity, viz. $G_{\text{eff}} = 1$, $\gamma = 1$, $\beta = 1$ proceeds quite analogously. The first option is provided by the condition (4.2a), where in addition $\mathcal{B}$ is finite and

$$\left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)^2 \left(\mathcal{A}' - 2\mathcal{A}\alpha'\right)' = 0 \tag{5.3}$$

Note that the twin condition (4.2b) is automatically satisfied due to the PPN ansatz. The second option would be given by (4.4), with α', α'' not infinite, and $\frac{\mathcal{B}'}{\mathcal{B}^3} = 0$. Comparing the latter with

Table 2. PPN parameters in different frames and parametrizations

JF BDBW	$G_{\text{eff}} = \frac{1}{\Psi}\left(1+\frac{e^{-m_\Psi r}}{2\omega+3}\right)$	$m_\Psi = \kappa\sqrt{\frac{2\Psi}{2\omega(\Psi)+3}\frac{d^2V}{d\Psi^2}}$
	$\gamma - 1 = -\frac{2e^{-m_\Psi r}}{G_{\text{eff}}\Psi(2\omega+3)}$	
	$\beta - 1 = \frac{\frac{d\omega}{d\Psi}e^{-2m_\Psi r}}{G_{\text{eff}}^2\Psi(2\omega+3)^3} - \frac{m_\Psi r}{G_{\text{eff}}^2\Psi^2(2\omega+3)}\left[e^{-m_\Psi r}\ln(m_\Psi r)+\ldots\right]$	
JF BEPS	$G_{\text{eff}} = \frac{1}{F}\left(1+\frac{\left(\frac{dF}{d\phi}\right)^2 e^{-m_\phi r}}{2F+3\left(\frac{dF}{d\phi}\right)^2}\right)$	$m_\phi = \kappa\sqrt{\frac{2F}{2F+3\left(\frac{dF}{d\phi}\right)^2}\frac{d^2V}{d\phi^2}}$
	$\gamma - 1 = -\frac{2\left(\frac{dF}{d\phi}\right)^2 e^{-m_\phi r}}{G_{\text{eff}}F\left(2F+3\left(\frac{dF}{d\phi}\right)^2\right)}$	
	$\beta - 1 = \frac{\left(\frac{dF}{d\phi}\right)^2\left(\left(\frac{dF}{d\phi}\right)^2-2F\frac{d^2F}{d\phi^2}\right)e^{-2m_\phi r}}{G_{\text{eff}}^2F\left(2F+3\left(\frac{dF}{d\phi}\right)^2\right)^3} - \frac{\left(\frac{dF}{d\phi}\right)^2 m_\phi r}{G_{\text{eff}}^2F^2\left(2F+3\left(\frac{dF}{d\phi}\right)^2\right)}\left[e^{-m_\phi r}\ln(m_\phi r)+\ldots\right]$	
JF nm	$G_{\text{eff}} = \frac{1}{1-\xi\phi^2}\left(1+\frac{2\xi^2\phi^2e^{-m_\phi r}}{1-\xi\phi^2+6\xi^2\phi^2}\right)$	$m_\phi = \kappa\sqrt{\frac{2(1-\xi\phi^2)}{1-\xi\phi^2+6\xi^2\phi^2}\frac{d^2V}{d\phi^2}}$
	$\gamma - 1 = -\frac{4\xi^2\phi^2e^{-m_\phi r}}{G_{\text{eff}}(1-\xi\phi^2)(1-\xi\phi^2+6\xi^2\phi^2)}$	
	$\beta - 1 = \frac{2\xi^3\phi^2e^{-2m_\phi r}}{(1-\xi\phi^2)(1-\xi\phi^2+6\xi^2\phi^2)^3} - \frac{2\xi^2\phi^2 m_\phi r}{(1-\xi\phi^2)(1-\xi\phi^2+6\xi^2\phi^2)}\left[e^{-m_\phi r}\ln(m_\phi r)+\ldots\right]$	
EF can	$G_{\text{eff}} = e^{2\alpha}\left(1+\left(\frac{d\alpha}{d\varphi}\right)^2e^{-m_\varphi r}\right)$	$m_\varphi = \kappa\sqrt{\frac{1}{2e^{2\alpha}}\frac{d^2V}{d\varphi^2}}$
	$\gamma - 1 = -\frac{2e^{2\alpha}\left(\frac{d\alpha}{d\varphi}\right)^2e^{-m_\varphi r}}{G_{\text{eff}}}$	
	$\beta - 1 = \frac{e^{4\alpha}\left(\frac{d\alpha}{d\varphi}\right)^2\frac{d^2\alpha}{d\varphi^2}e^{-2m_\varphi r}}{2G_{\text{eff}}^2} - \frac{e^{4\alpha}\left(\frac{d\alpha}{d\varphi}\right)^2 m_\varphi r}{G_{\text{eff}}^2}\left[e^{-m_\varphi r}\ln(m_\varphi r)+\ldots\right]$	
EF min	$G_{\text{eff}} = 1$	$m_\varphi = \kappa\sqrt{\frac{1}{2}\frac{d^2V}{d\varphi^2}}$
	$\gamma - 1 = 0$	
	$\beta - 1 = 0$	

the discussion in the previous section we may note that the condition on $\mathcal{B}'$ to achieve the GR limit is marginally less strict in PPN than the one obtained from the equations of motion, i.e $\frac{\mathcal{B}'}{\mathcal{B}^2}$ finite. (A similar observation in the case of cosmology was made in Ref. [20]). The third option is realized by giving the scalar field a very large effective mass, i.e

$$\frac{1}{m_\Phi}\Big|_{\Phi_\blacklozenge} = \left(\frac{2\kappa^2\mathcal{A}}{e^{2\alpha}\left(2\mathcal{A}\mathcal{B}+3(\mathcal{A}')^2\right)}\frac{d^2V}{d\Phi^2}\right)^{-\frac{1}{2}}\Big|_{\Phi_\blacklozenge} = 0 \tag{5.4}$$

However, in that case it is not so obvious what the corresponding condition arising from the general equations of motion would be.

6. Example

To have an illustration let us take a look at a specific simple example. Let the JF BDBW functions be given by

$$\mathcal{A} = \Psi, \quad \mathcal{B} = \frac{\omega(\Psi)}{\Psi} = \frac{3}{2(1-\Psi)}, \quad V = \frac{1}{\left(\frac{1}{2}-\Psi\right)^2}, \quad \alpha = 0 \tag{6.1}$$

The attactive gravitation condition ($\mathcal{A} > 0$) and no ghosts condition (2.7) delimit $0 < \Psi \leq 1$. Recalling the discussions in Secs. 4 and 5 we may find that the field equations and PPN parameters reduce to those of general relativity in several different occasions.

- The first is realized when (4.4) holds, i.e $\Psi_\star = 1$, while $\frac{\mathcal{B}'}{\mathcal{B}^2} = \frac{2}{3}$ is finite. Here the PPN parameters also reduce to their general relativity values, as expected, since $\frac{\mathcal{B}'}{\mathcal{B}^3} = 0$.
- The second possibility to reduce the field equations to GR only occurs when the trace of matter energy-momentum tensor $T = 0$ and the condition (4.2a) does not apply. Then (4.2b) is satisfied at $\Psi_\bullet = \frac{1}{4}$. The PPN parameters, however, do not coincide with those of GR now.
- Finally, it is possible to draw the PPN parameters into the GR values by satisfying (5.4) with an extremely massive scalar field, $\Psi_\blacklozenge = \frac{1}{2}$. But now the field equations do not agree with those of general relativity.

We can transform the theory from the BDBW parametrization with Ψ into the BEPS parametrization with ϕ by using Table 1. Integrating

$$\mp\frac{d\phi}{d\Psi} = \sqrt{\frac{\omega(\Psi)}{\Psi}} = \sqrt{\frac{3}{2(1-\Psi)}} \tag{6.2}$$

gives (neglecting the additive integration constant)

$$\pm\phi = \sqrt{6(1-\Psi)}, \quad \Psi = 1 - \frac{1}{6}\phi^2 \tag{6.3}$$

and we see it is actually the nonminimal coupling subclass of BEPS, where

$$\mathcal{A} = F(\phi) = 1 - \frac{1}{6}\phi^2, \quad \mathcal{B} = 1, \quad V = \frac{1}{\left(\frac{1}{2} - \frac{\phi^2}{6}\right)^2}, \quad \alpha = 0 \tag{6.4}$$

Note that Ψ is mapped doubly to ϕ, as $\Psi \in (0,1]$ translates into $\phi \in (-\sqrt{6}, 0]$ and $\phi \in [0, \sqrt{6})$. Again, there are several possibilities to achive the general relativity limit of the field equations and PPN parameters.

- First, the field equations reduce to the ones of GR when Eq. (4.2a), given by $\mathcal{A}' = 0$, and (4.2b), given by $\mathcal{A}V' - 2\mathcal{A}'V = 0$, are satisfied. The only common solution is $\phi_\bullet = 0$. A glance to Table 2 reveals that the PPN parameters also trivially fall into their GR limit. By a direct comparison via (6.3) it becomes obvious that this value of ϕ corresponds to the first case in the BDBW case.
- If matter $T = 0$ and the condition (4.2a) is not enforced, the condition (4.2b) alone has also the solution $\pm\phi_\bullet = \frac{3}{\sqrt{2}}$. This does not lead the PPN parameters to their GR values. A direct check by (6.3) tells that the corresponding case in the BDBW parametrization was the second one.

- Last, when the scalar field acquires an extremely large mass by (5.4) at $\pm\phi_\blacklozenge = \sqrt{3}$ the PPN parameters reduce to those of GR, but the field equations do not. It corresponds to the third case above.

We may transform the same theory from JF BDBW parametrization into EF canonical parametrization by integrating

$$\mp\frac{d\varphi}{d\Psi} = \sqrt{\frac{2\omega(\Psi)+3}{4\Psi^2}} = \sqrt{\frac{3}{4\Psi^2(1-\Psi)}} \tag{6.5}$$

which gives (neglecting the additive integration constant)

$$\pm\varphi = \sqrt{3}\,\mathrm{arctanh}\sqrt{1-\Psi}\,, \quad \Psi = 1 - \tanh^2\frac{\varphi}{\sqrt{3}} \tag{6.6}$$

The functions characterizing the frame and parametrization are

$$\mathcal{A} = 1, \quad \mathcal{B} = 2, \quad V = \frac{1}{\left(\frac{1}{2} - \tanh^2\frac{\varphi}{\sqrt{3}}\right)^2\left(1 - \tanh^2\frac{\varphi}{\sqrt{3}}\right)^2}\,, \quad \alpha = -\frac{1}{2}\ln\left(1 - \tanh^2\frac{\varphi}{\sqrt{3}}\right) \tag{6.7}$$

Alternatively, one may embark from the JF BEPS parametrization and integrate

$$\pm\frac{d\varphi}{d\phi} = \sqrt{\frac{3}{4}\left(\frac{d\ln F(\phi)}{d\phi}\right)^2 + \frac{1}{2F(\phi)}} = \frac{1}{\sqrt{2}\left(1-\frac{\phi^2}{6}\right)} \tag{6.8}$$

to obtain (again, neglecting the additive integration constant)

$$\pm\varphi = \sqrt{3}\,\mathrm{arctanh}\frac{\phi}{\sqrt{6}}\,, \quad \pm\phi = \sqrt{6}\,\tanh\frac{\varphi}{\sqrt{3}} \tag{6.9}$$

The mapping into EF canonical parametrization is again double for JF BDBW, as $\Psi \in (0,1]$ translates into $\phi \in (-\infty, 0]$ and $\phi \in [0,\infty)$, while JF BEPS $\phi \in (-\sqrt{6}, \sqrt{6})$ translates into $\phi \in (-\infty,\infty)$ and equivalently into $-\phi \in (-\infty,\infty)$ according to the sign in Eq. (6.8). Analogously with the other parametrizations we can discuss the general relativity limit of the field equations and PPN parameters in three cases.

- When the conditions (4.2a) and (4.2b) both hold, i.e $\alpha' = 0$ and $V' = 0$, the value of the scalar field is $\varphi_\bullet = 0$. It takes the PPN parameters to their GR limit and by direct check using (6.6) and (6.9) one can conclude it corresponds to the first cases discussed above.
- For absent or radiative matter with $T = 0$ the condition (4.2a) does not apply and (4.2b) alone is also solved by $\pm\varphi_\bullet = \sqrt{3}\,\mathrm{arctanh}\left(\sqrt{3}/2\right)$. The PPN parameters differ from those of GR and it is straightforward to check that this value of φ corresponds to the second cases above.
- The scalar field mass diverges at $\pm\varphi_\blacklozenge = \sqrt{3}\,\mathrm{arctanh}\left(1/\sqrt{2}\right)$, satisfying (5.4) and reducing the PPN parameters to their GR values. The field equations still differ from those of GR and we recognize correspondence to the third cases described above.

We see that the derivative of the transformation function $\bar{f}'$ relating different scalar field parametrizations gets singular for different values of the field. For a transformation from JF BDBW to BEPS (6.2) it is singular at $\Psi = 1$, $\phi = 0$, for a transformation from JF BDBW to EF canonical (6.5) it is singular at $\Psi = 1$, $\varphi = 0$ and $\Psi = 0$, $\varphi = \pm\infty$, while for a transformation from JF BEPS to EF canonical (6.8) it is singular at $\phi = \pm\sqrt{6}$, $\varphi = \pm\infty$. The value $\Psi = 0$, $\phi = \pm\sqrt{6}$, $\varphi = \pm\infty$ also makes the the conformal transformation singular. But strictly speaking, this value is actually outside the range of the assumed validity of the theory, since for JF BDBW and BEPS it violates the attractive gravity assumption, while for EF canonical the infinite value of the field is arguably unphysical since α becomes singular.

So, it is only the singularity of transformation and the GR limit occuring at JF BDBW $\Psi = 1$ that is possibly problematic. However, we saw that despite the transformation becoming singular the general relativity limit in terms of the field equations and PPN parameters, namely the first cases discussed above, does also occur in the corresponding JF BEPS value $\phi = 0$ and EF canonical parametrization value $\varphi = 0$. It is interesting that in the JF BDBW parametrization this GR limit is realized by satisfying the condition (4.4), while in the JF BEPS and EF canonical parametrizations it comes from the conditions (4.2a), (4.2b). This confirms the discussion in Secs. 4 and 5 that the existence of the GR limit is invariant of the parametrization.

7. Conclusion

We studied general scalar-tensor gravity involving four free functions in different conformal frames and scalar field parametrizations. We investigated its general relativity limits in the sense of field equations and the values of PPN parameters coinciding with those of general relativity. Despite the transformation of the scalar field from one representation to another may possess a singularity, it turned out that the existence of general relativity limits is independent of the parametrization.

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Chapter 10

Transformation properties and general relativity regime in scalar-tensor theories

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Transformation properties and general relativity regime in scalar–tensor theories

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Abstract

We consider first generation scalar–tensor theories of gravitation in a completely generic form, keeping the transformation functions of the local rescaling of the metric and the scalar field redefinition explicitly distinct from the coupling functions in the action. It is well known that in Jordan frame Brans–Dicke type parametrization, the diverging kinetic coupling function $\omega \to \infty$ can lead to the general relativity regime, however the transformation functions to other parametrizations then typically become singular, possibly spoiling the correspondence between different parametrizations. We give a detailed analysis of the transformation properties of field equations with arbitrary metrics and also in the Friedmann cosmology, and provide sufficient conditions under which the correspondence between different parametrizations is retained, even if the transformation is singular. It is interesting to witness the invariance of the notion of the general relativity regime and the correspondence of the perturbed cosmological equations as well as their solutions in different parametrizations, despite the fact that in some cases the perturbed equation turns out to be linear in one parametrization and nonlinear in another.

Keywords: scalar–tensor gravity, general relativity limit
Friedmann cosmology, transformation properties

1. Introduction

The history of the scalar–tensor theory of gravitation (STG) [1, 2] as an extension to Einstein's general relativity (GR) started in principle with the works by Kaluza and Klein. Complementary ideas were pursued by Jordan and Fierz [3], developed by Brans and Dicke [4] and further generalized by Bergmann and Wagoner [5, 6]. Nowadays, the aforementioned can be called first generation scalar–tensor theories. The Horndeski theory [7], which also

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allows derivative couplings and possesses equations of motion with up to second order derivatives of the metric and scalar field, may be considered to be the second generation. Healthy, ghost free theories going beyond Horndeski can be referred to as the third generation [8].

Soon after his joint work with Brans [4], Dicke published another paper [9] where he recalled the local Weyl rescaling of the metric tensor, interpreted it as a transformation of the units and claimed that physics must be invariant under this transformation [10, 11]. From this viewpoint STG is a natural extension of GR because rewriting the Einstein–Hilbert action in terms of a Weyl rescaled metric tensor introduces an action functional having a structure that resembles the one used for STG [12, 13]. Namely, on the level of the rewritten action functional the scalar field entering via the Weyl rescaling is coupled to curvature and to matter etc. Of course in that case, the functions describing the coupling of the scalar field to curvature etc are related to each other in a specific way which implies that the scalar field equation of motion is an identity $0 \equiv 0$, and the scalar field is hence not a physical degree of freedom. Nevertheless if one considers an analogous action functional but without the relations between the coupling functions, then the resulting theory is STG, congruent with Weyl integrable geometry [14].

Jordan [3] has already pointed out that for scalar–tensor theories with a constant kinetic coupling parameter ω the equations of motion reduce to those of GR if $\omega = \infty$. In the framework of the parametrized post-Newtonian approximation it has been shown that for the theory with a dynamical $\omega \equiv \omega(\Psi)$ [5, 6], conditions for the theory to comply with GR are again $\omega(\Psi) \to \infty$ as well as $\frac{\omega'(\Psi)}{\omega(\Psi)^3} \to 0$ [15, 16]. In the context of the Friedmann cosmology, Damour and Nordtvedt [17, 18] showed that for a wide family of theories the limit $\omega(\Psi) \to \infty$ is an attractor. To be more precise, there exists a mechanism ending scalar field evolution at a constant value thereby rendering the remaining dynamical degrees of freedom identical to those of GR. In the current paper we shall use the term 'GR regime' to refer to such a situation. Due to these results, a dynamical approach to the GR regime has been studied by a number of authors, e.g., [19–28].

Damour and Nordtvedt noted that the points in field space where $\omega(\Psi) = \infty$ enter the theory as mathematically singular boundary points [18]. They used the local Weyl rescaling of the metric tensor and redefined (reparametrized) the scalar field $\varphi = \varphi(\Psi)$ in order to rewrite the theory in the so-called Einstein frame where all functions are regular. However, the singularity in $\omega(\Psi) \to \infty$ (in the so-called Jordan frame) is then absorbed by the scalar field redefinition, hence rendering the transformation to be singular instead, i.e. $\frac{\mathrm{d}\varphi}{\mathrm{d}\Psi} \to \infty$. Therefore, it is not so obvious that these transformations can be trusted at all and one must take extra caution when applying the transformation in the vicinity of the GR regime [29].

Note that in the literature, when the equivalence of the parametrizations is discussed, the transformation functions are often assumed to be regular [10, 11], which in principle is easily achievable when a suitable choice of coupling functions in the Jordan frame is considered. However, in our recent paper [30] we showed that the scalar field Ψ in the Jordan frame is equivalent to the invariant notion of the nonminimal coupling while the Einstein frame scalar field φ is equivalent to the invariant notion of the scalar field space volume. Therefore $\frac{\mathrm{d}\Psi}{\mathrm{d}\varphi} = 0$ in the GR regime is not due to an unfortunate choice of coupling functions, but is a crucial part of the notion of the GR regime, stating via invariants that the nonminimal coupling vanishes. We hence conclude that the singular scalar field redefinition is physically meaningful and deserves a closer look.

In the current paper we intend to clarify the question of whether or not such a singular transformation is permitted by first studying the transformation properties of the action, the equations of motion and the Friedmann cosmology. Afterwards, we focus upon the transformations in the neighbourhood of the GR regime, which corresponds to a critical point of the scalar field equation of motion. We argue that the conditions for critical points in general, as well as in the Friedmann cosmology, are preserved under the scalar field redefinition, even if the latter is singular. Most importantly we show in detail that the perturbed equation, approximating the dynamics in the vicinity of the GR regime, transforms well despite the fact that in the case of the singular scalar field redefinition a nonlinear perturbed equation is transformed into a linear one. The transformation of the solutions also shows an interesting analogous correspondence. To give a completely generic treatment of the transformations between all possible parametrizations we adopt the notation introduced by Flanagan [10]. The paper is in accordance with the spirit of recent works [31, 32] etc, where the correspondence between the Jordan and Einstein frames is discussed in explicit detail.

The outline of the paper is as follows: in section 2 we write down the action functional, derive the equations of motion and plug in the Friedmann–Lemaître–Robertson–Walker (FLRW) line element in order to obtain the general Friedmann cosmology in the context of STG. In section 3 we introduce the notion of the general relativity regime by examining the necessary conditions for maintaining the constancy of the scalar field once it has been obtained. Section 4 completes the line of thought of [29, 33–36] by considering a dynamical approach to the general relativity regime in the context of the potential dominated epoch of the Friedmann cosmology. In the current paper, the latter serves as an example for showing the equivalence of different parametrizations on the level of the perturbed equations. It turns out to be nontrivial and we have included a lot of calculational details in order to keep the treatment as traceable as possible.

From the structural point of view the paper is divided into three sections, each of which is split into two halves. In the first halves of the sections a relatively complete theory in an arbitrary parametrization, starting with the action functional and ending with the solutions in the context of the Friedmann cosmology is given. The second halves follow the first halves by providing the corresponding transformation properties under the local Weyl rescaling of the metric tensor and under the scalar field redefinition. Therefore, subsections numbered as $i.1.j$ contain the theory and $i.2.k$ discuss the transformation properties of the quantities introduced in $i.1.j$.

2. General theory

In this section we write down an action functional and derive the equations of motion. The general Friedmann cosmology is also discussed.

2.1. Theory: part I

2.1.1. Action functional. Let us consider a family of theories of gravitation by postulating an action functional [10, 37]

$$S = \frac{1}{2\kappa^2}\int_{V_4} \mathrm{d}^4x\sqrt{-g}\left\{\mathcal{A}(\Phi)R - \mathcal{B}(\Phi)g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - 2\ell^{-2}\mathcal{V}(\Phi)\right\} + S_{\mathrm{m}}\left[\mathrm{e}^{2\alpha(\Phi)}g_{\mu\nu}, \chi^A\right]. \tag{1}$$

There are two unspecified constants: κ^2 yields the dimension for the gravitational 'constant' and $\ell > 0$ has the dimension of length. We make use of the convention $c \equiv 1$ and have chosen constants κ^2 and ℓ^{-2} suitably in order to consider the scalar field Φ and the four arbitrary functions $\{\mathcal{A}(\Phi), \mathcal{B}(\Phi), \mathcal{V}(\Phi), \alpha(\Phi)\}$ of it to be dimensionless, regardless of whether in addition either $\kappa^2 \equiv 1$ or $\hbar \equiv 1$ is imposed.

Note that in the general case, the action functional S_m for the matter fields χ^A, where different components are labelled by the superscript A, functionally depends on the metric tensor $\hat{g}_{\mu\nu} = \mathrm{e}^{2\alpha(\Phi)} g_{\mu\nu}$. Nevertheless, the coupling of the matter fields to the geometry described by $g_{\mu\nu}$ is universal and therefore one of the basic principles underlying general relativity is fulfilled.

In order to consider a concrete theory one must specify each of the four arbitrary functions $\{\mathcal{A}(\Phi), \mathcal{B}(\Phi), \mathcal{V}(\Phi), \alpha(\Phi)\}$. However, in the literature the action functionals which have mostly been considered are those in which the functional form of two out of the four arbitrary functions has been specified. This is because in such cases the calculations are easier, while the corresponding action functional has retained its generality up to some details [30, 38]. In the current paper we shall use 'parametrization' to refer to these setups and hereby recall the two most well-known ones:

- The Jordan frame action in the Brans–Dicke–Bergmann–Wagoner parametrization (JF BDBW) [4–6] for the scalar field denoted as Ψ is obtained as follows:

$$\mathcal{A} = \Psi, \; \mathcal{B} = \frac{\omega(\Psi)}{\Psi}, \; \mathcal{V} = \mathcal{V}_\mathrm{J}(\Psi), \; \alpha = 0. \tag{2}$$

- The Einstein frame action in canonical parametrization (EF can) [5, 6, 9] for the scalar field denoted as φ is obtained as follows:

$$\mathcal{A} = 1, \; \mathcal{B} = 2, \; \mathcal{V} = \mathcal{V}_\mathrm{E}(\varphi), \; \alpha = \alpha_\mathrm{E}(\varphi). \tag{3}$$

Here, and in the following, we shall drop the arguments of the arbitrary functions $\{\mathcal{A}(\Phi), \mathcal{B}(\Phi), \mathcal{V}(\Phi), \alpha(\Phi)\}$ unless confusion could arise. We also adopt a convention where prime means derivative w.r.t. the scalar field, e.g.,

$$\mathcal{A}' \equiv \frac{\mathrm{d}\mathcal{A}(\Phi)}{\mathrm{d}\Phi}, \; \mathcal{B}' \equiv \frac{\mathrm{d}\mathcal{B}(\Phi)}{\mathrm{d}\Phi}, \text{ etc.} \tag{4}$$

In the current paper we shall use the so-called mostly plus signature for the metric tensor $g_{\mu\nu}$ and always assume the affine connection to be the Levi-Civita one. The other unspecified conventions are as, e.g., in the textbook by Carroll [39].

2.1.2. Equations of motion. Varying the action (1) while considering $g^{\mu\nu}$, Φ and χ^A to be the dynamical fields reads

$$\begin{aligned}\delta S = &\frac{1}{2\kappa^2} \int_{V_4} \mathrm{d}^4x \sqrt{-g} \left\{ E^{(g)}_{\mu\nu} \delta g^{\mu\nu} + E^{(\Phi)} \delta\Phi + 2\kappa^2 \mathrm{e}^{4\alpha} E^{(\chi)}_A \delta\chi^A \right\} \\ &+ \frac{1}{2\kappa^2} \int_{V_4} \mathrm{d}^4x \partial_\sigma \left(\sqrt{-g} \left[\mathscr{B}^{\sigma}_{(g)} + \mathscr{B}^{\sigma}_{(\Phi)} + 2\kappa^2 \mathrm{e}^{4\alpha} \mathscr{B}^{\sigma}_{(\chi)} \right] \right)\end{aligned} \tag{5}$$

where

$$\sqrt{-g}\, \mathscr{B}^{\sigma}_{(g)} = \sqrt{-g} \left\{ \mathcal{A} g_{\mu\nu} g^{\sigma\lambda} \nabla_\lambda \delta g^{\mu\nu} - \mathcal{A} \nabla_\mu \delta g^{\sigma\mu} - g^{\sigma\lambda} \left(\nabla_\lambda \mathcal{A} \right) g_{\mu\nu}\, \delta g^{\mu\nu} + \left(\nabla_\mu \mathcal{A} \right) \delta g^{\mu\sigma} \right\}, \tag{6}$$

$$\sqrt{-g}\ \mathscr{B}^{\sigma}_{(\Phi)} = -\sqrt{-g}\ 2\mathcal{B}g^{\sigma\mu}\left(\nabla_{\mu}\Phi\right)\delta\Phi \tag{7}$$

and $\sqrt{-g}\ e^{4\alpha}\mathscr{B}^{\sigma}_{(\chi)}$ are eventually the boundary terms arising from varying w.r.t. the metric tensor $g^{\mu\nu}$, the scalar field Φ and the matter fields respectively. The boundary terms have been written out explicitly for the sake of completeness, although they do not make a contribution to the equations of motion. Therefore, by making use of the minimal action principle $\delta S = 0$ we obtain the equations of motion as follows:

$$E^{(g)}_{\mu\nu} \equiv \mathcal{A}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \left(\frac{1}{2}\mathcal{B} + \mathcal{A}''\right)g_{\mu\nu}g^{\rho\sigma}\nabla_{\rho}\Phi\nabla_{\sigma}\Phi - \left(\mathcal{B} + \mathcal{A}''\right)\nabla_{\mu}\Phi\nabla_{\nu}\Phi$$
$$+ \mathcal{A}'\left(g_{\mu\nu}\Box\Phi - \nabla_{\mu}\nabla_{\nu}\Phi\right) + \ell^{-2}g_{\mu\nu}\mathcal{V} - \kappa^{2}T_{\mu\nu} = 0, \tag{8}$$

$$E^{(\Phi)} \equiv R\mathcal{A}' + \mathcal{B}'g^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi + 2\mathcal{B}\Box\Phi - 2\ell^{-2}\mathcal{V}' + 2\kappa^{2}\alpha' T = 0, \tag{9}$$

$$E^{(\chi)}_{A} \equiv E^{(\chi)}_{A}\left[e^{2\alpha}g_{\mu\nu},\ \chi^{C}\right] = 0. \tag{10}$$

Here

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} \tag{11}$$

is the matter energy-momentum tensor, $T \equiv g^{\mu\nu}T_{\mu\nu}$ is its contraction and $\Box \equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$. In the current paper we are not directly interested in the equations of motion for the matter fields χ^{A}, i.e. we do not specify either (10) or the corresponding boundary terms. However, including them provides us with a complete picture—at least on the schematic level—and allows us to stress an important point. Namely, the matter fields χ^{A} 'feel' the geometry determined by $\hat{g}_{\mu\nu} \equiv e^{2\alpha}g_{\mu\nu}$. Therefore, freely falling material objects follow the corresponding geodesics. Hence, if one intends to measure the geometry determined by $g_{\mu\nu}$ using reference objects built out of the matter fields then, in the spirit of Dicke [9], correction factors must be applied.

In the literature, the contraction of (8), i.e.,

$$g^{\mu\nu}E^{(g)}_{\mu\nu} \equiv -\mathcal{A}R + \mathcal{B}g^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi + 3\mathcal{A}''g^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi + 3\mathcal{A}'\Box\Phi + 4\ell^{-2}\mathcal{V} - \kappa^{2}T = 0 \tag{12}$$

is usually used to eliminate the Ricci scalar R from (9) in order to obtain an equation of motion for the scalar field Φ that does not contain the second derivatives of the metric tensor $g_{\mu\nu}$ and therefore purely describes the propagation of the scalar field. The result reads

$$\frac{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^{2}}{\mathcal{A}}\Box\Phi + \frac{\left(2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^{2}\right)'}{2\mathcal{A}}g^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi$$
$$-\frac{2\left(\mathcal{A}\mathcal{V}' - 2\mathcal{V}\mathcal{A}'\right)}{\ell^{2}\mathcal{A}} + \frac{\kappa^{2}\left(2\mathcal{A}\alpha' - \mathcal{A}'\right)}{\mathcal{A}}T = 0. \tag{13}$$

This procedure is also known as 'debraiding'—see for example a recent paper by Bettoni *et al* [40] for comments and further references. Note that due to $\nabla_{\mu}\nabla_{\nu}\Phi$ in (8) it is not possible to make an analogous substitution in order to obtain an equation that would describe solely the evolution of the metric tensor. In some sense this is the underlying motivation for the Einstein frame canonical parametrization (3). Last but not least, combining (9) and the covariant divergence of the tensor equation (8) leads us to

$$E_{\nu}^{(c)} \equiv \nabla^{\mu}E_{\mu\nu}^{(g)} + \frac{1}{2}E^{(\Phi)}\nabla_{\nu}\Phi = -\kappa^2\nabla^{\mu}T_{\mu\nu} + \kappa^2\alpha' T\nabla_{\nu}\Phi = 0 \tag{14}$$

which is the well-known continuity equation.

2.1.3. Friedmann cosmology. Let us consider the FLRW line element in spherical coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (a(t))^2\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right) \tag{15}$$

defined in an arbitrary parametrization. Here t and $a(t)$ are the cosmological time and the scale factor connected to the chosen parametrization respectively. The constant k takes values -1, 0 and $+1$, determining the spatial geometry to be hyperbolic, flat or spherical respectively. The dependence on the two angles is gathered into $\mathrm{d}\Omega^2$. Due to the homogeneity and isotropy assumption underlying the Friedmann cosmology, the scalar field can only depend on the cosmological time $\Phi \equiv \Phi(t)$. The equations of motion (8), (13) and (14) in the case of the FLRW metric read

$$H^2 = -\frac{\mathcal{A}'}{\mathcal{A}}H\dot{\Phi} + \frac{\mathcal{B}}{6\mathcal{A}}\dot{\Phi}^2 + \frac{1}{3\ell^2\mathcal{A}}\mathcal{V} + \frac{\kappa^2}{3\mathcal{A}}\rho - \frac{k}{a^2}, \tag{16}$$

$$2\dot{H} + 3H^2 = -2\frac{\mathcal{A}'}{\mathcal{A}}H\dot{\Phi} - \left(\frac{\mathcal{B}}{2\mathcal{A}} + \frac{\mathcal{A}''}{\mathcal{A}}\right)\dot{\Phi}^2 - \frac{\mathcal{A}'}{\mathcal{A}}\ddot{\Phi} + \frac{1}{\ell^2\mathcal{A}}\mathcal{V} - \frac{\kappa^2}{\mathcal{A}}p - \frac{k}{a^2}, \tag{17}$$

$$\begin{aligned}\ddot{\Phi} = &- 3H\dot{\Phi} - \frac{1}{2}\frac{\left(2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2\right)'}{\left(2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2\right)}\dot{\Phi}^2 - 2\ell^{-2}\frac{\mathcal{A}\mathcal{V}' - 2\mathcal{V}\mathcal{A}'}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}\\ &- \kappa^2\frac{\left(2\mathcal{A}\alpha' - \mathcal{A}'\right)}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}(\rho - 3p),\end{aligned} \tag{18}$$

$$\dot{\rho} = -3H(\rho + p) + \alpha'(\rho - 3p)\dot{\Phi}, \tag{19}$$

where dot means derivative with respect to the cosmological time t and $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter. We have assumed the matter to be a perfect fluid with the energy density ρ and pressure p.

2.2. Transformations: part I

2.2.1. Transformation of the action functional. It is well known that the action functional (1) preserves its structure up to a boundary term under the transformations that contain two functional degrees of freedom

$$g_{\mu\nu} = \mathrm{e}^{2\bar{\gamma}(\bar{\Phi})}\bar{g}_{\mu\nu}, \tag{20}$$

$$\Phi = \bar{f}\left(\bar{\Phi}\right). \tag{21}$$

The first of them is known as the Weyl rescaling which is a distinct case of the conformal transformation of the metric tensor $g_{\mu\nu}$. We shall occasionally refer to it as the change of the 'frame'. The second one is the redefinition of the scalar field Φ, also known as 'reparametrization'. The transformed action functional reads

$$\bar{S} = \frac{1}{2\kappa^2}\int_{V_4} \mathrm{d}^4x\sqrt{-\bar{g}}\,\{\bar{\mathcal{A}}(\bar{\Phi})\bar{R} - \bar{\mathcal{B}}(\bar{\Phi})\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\Phi}\bar{\nabla}_\nu\bar{\Phi} - 2\ell^{-2}\bar{\mathcal{V}}(\bar{\Phi})\Big\} + \bar{S}_{\mathrm{m}}\Big[\mathrm{e}^{2\bar{\alpha}(\bar{\Phi})}\bar{g}_{\mu\nu},\,\chi^A\Big] + \frac{1}{2\kappa^2}\int_{V_4} \mathrm{d}^4x\partial_\sigma\Big(\sqrt{-\bar{g}}\,\bar{\mathscr{B}}^{\sigma}_{(\bar{S})}\Big) \tag{22}$$

where

$$\bar{\mathscr{B}}^{\sigma}_{(\bar{S})} = -6\bar{\gamma}'\bar{\mathcal{A}}\bar{g}^{\sigma\mu}\bar{\nabla}_\mu\bar{\Phi} \tag{23}$$

is a negligible boundary term. Here we have made use of the following notation [10]

$$\bar{\mathcal{A}}(\bar{\Phi}) = \mathrm{e}^{2\bar{\gamma}(\bar{\Phi})}\mathcal{A}\big(\bar{f}(\bar{\Phi})\big), \tag{24a}$$

$$\bar{\mathcal{B}}(\bar{\Phi}) = \mathrm{e}^{2\bar{\gamma}(\bar{\Phi})}\Big(\big(\bar{f}'\big)^2\mathcal{B}\big(\bar{f}(\bar{\Phi})\big) - 6\big(\bar{\gamma}'\big)^2\mathcal{A}\big(\bar{f}(\bar{\Phi})\big) - 6\bar{\gamma}'\bar{f}'\mathcal{A}'\Big), \tag{24b}$$

$$\bar{\mathcal{V}}(\bar{\Phi}) = \mathrm{e}^{4\bar{\gamma}(\bar{\Phi})}\,\mathcal{V}\big(\bar{f}(\bar{\Phi})\big), \tag{24c}$$

$$\bar{\alpha}(\bar{\Phi}) = \alpha\big(\bar{f}(\bar{\Phi})\big) + \bar{\gamma}(\bar{\Phi}), \tag{24d}$$

and refined convention (4) in order to distinguish between derivatives w.r.t. the 'barred' scalar field $\bar{\Phi}$ and the 'unbarred' scalar field Φ in the following manner:

$$\bar{f}' \equiv \frac{\mathrm{d}\bar{f}(\bar{\Phi})}{\mathrm{d}\bar{\Phi}}, \qquad \bar{\gamma}' \equiv \frac{\mathrm{d}\bar{\gamma}(\bar{\Phi})}{\mathrm{d}\bar{\Phi}}, \qquad \bar{\mathcal{A}}' \equiv \frac{\mathrm{d}\bar{\mathcal{A}}(\bar{\Phi})}{\mathrm{d}\bar{\Phi}}, \qquad \mathcal{A}' \equiv \frac{\mathrm{d}\mathcal{A}(\Phi)}{\mathrm{d}\Phi}, \qquad \text{etc.} \tag{25}$$

If we impose a condition in which the action functional (1) is invariant under the local Weyl rescaling of the metric tensor (20) and under the scalar field redefinition (21), then equations (24) are the transformation properties of the four arbitrary functions $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$. In the current paper we will adopt the aforementioned assumption and whenever the transformations (20)–(21) are recalled, equations (24) are also taken into account.

Sometimes it might be clearer to look at the transformations backwards as well. In order to keep the notation under better control we also introduce

$$\bar{g}_{\mu\nu} = \mathrm{e}^{2\gamma(\Phi)}g_{\mu\nu}, \tag{26}$$

$$\bar{\Phi} = f(\Phi), \tag{27}$$

such that $\gamma(\bar{f}(\bar{\Phi})) = -\bar{\gamma}(\bar{\Phi})$. If $\bar{f}$ is a bijection then the composition $\bar{f} \circ f$ is equal to the identity transformation, but we also want to include the possibility that either $\bar{f}$ or f or both are multivalued. When using the transformations (26)–(27) instead of (20)–(21), then for the transformation rules (24) of the four arbitrary functions the property of being 'barred' or not is interchanged. For an example compare (64) with (66).

In the literature, most of the calculations have been carried out in a specific parametrization, e.g. in the JF BDBW parametrization (2) or in the EF canonical parametrization (3). A specific parametrization is, in principle, equivalent to the general one [30, 38], but it turns out that for specific parametrizations the transformation from one to another may not be so unique at all since there are quantities that remain unseen in these parametrizations but nevertheless have complicated transformation rules [30, 41]. As an example let us consider $\mathcal{A}'$ in the JF BDBW parametrization. We obtain $\mathcal{A}'|_{\mathrm{J}} = 1$. Hence, an arbitrary power of the latter is also equal to one, and in this specific parametrization we cannot distinguish between $\mathcal{A}'|_{\mathrm{J}}$, $(\mathcal{A}'|_{\mathrm{J}})^2$, etc. However, all of these have different transformation properties. In the current paper, in order to overcome this shortcoming, we have adopted the notation by Flanagan [10], which has the following advantages: i) all four possible couplings (curvature ($\mathcal{A}$), kinetic ($\mathcal{B}$), self-interaction ($\mathcal{V}$) and matter (α)) of the scalar field are explicitly written out, ii) two transformation functions $\bar{\gamma}$ and $\bar{f}$ are kept separate from the coupling functions.

2.2.2. Transformation of the equations of motion. A straightforward calculation shows that under the local rescaling of the metric tensor (20) and the scalar field redefinition (21) the equation of motion (8) for the metric tensor $g_{\mu\nu}$, denoted in short as $E^{(g)}_{\mu\nu} = 0$, transforms as follows

$$E^{(g)}_{\mu\nu} = \mathrm{e}^{-2\bar{\gamma}} \bar{E}^{(\bar{g})}_{\mu\nu}. \tag{28}$$

Here we have made use of the fact that under the conformal transformation (20) the energy-momentum tensor $T_{\mu\nu}$ transforms as $T_{\mu\nu} = \mathrm{e}^{-2\bar{\gamma}} \bar{T}_{\mu\nu}$ and its contraction as $T = \mathrm{e}^{-4\bar{\gamma}} \bar{T}$.

Checking the transformation properties of the scalar field equation (9) that explicitly contains R gives

$$E^{(\Phi)} = \left(\bar{f}'\right)^{-1} \mathrm{e}^{-4\bar{\gamma}} \left\{ \bar{E}^{(\bar{\Phi})} + 2\bar{\gamma}' \bar{g}^{\mu\nu} \bar{E}^{(\bar{g})}_{\mu\nu} \right\}. \tag{29}$$

Therefore these transformations mix the scalar field equation (9) with the metric equation (8). The reason for this lies in the transformation properties of the variational derivatives

$$\begin{pmatrix} \dfrac{\delta}{\delta \Phi} \\ \dfrac{\delta}{\delta g^{\sigma\rho}} \end{pmatrix} = \begin{pmatrix} \dfrac{\delta \bar{\Phi}}{\delta \Phi} & \dfrac{\delta \bar{g}^{\mu\nu}}{\delta \Phi} \\ \dfrac{\delta \bar{\Phi}}{\delta g^{\sigma\rho}} & \dfrac{\delta \bar{g}^{\mu\nu}}{\delta g^{\sigma\rho}} \end{pmatrix} \begin{pmatrix} \dfrac{\delta}{\delta \bar{\Phi}} \\ \dfrac{\delta}{\delta \bar{g}^{\mu\nu}} \end{pmatrix} = \begin{pmatrix} \left(\bar{f}'\right)^{-1} & 2\bar{\gamma}' \left(\bar{f}'\right)^{-1} \bar{g}^{\,\mu\nu} \\ 0 & \mathrm{e}^{2\bar{\gamma}} \delta^{\mu}_{\sigma} \delta^{\nu}_{\rho} \end{pmatrix} \begin{pmatrix} \dfrac{\delta}{\delta \bar{\Phi}} \\ \dfrac{\delta}{\delta \bar{g}^{\mu\nu}} \end{pmatrix}. \tag{30}$$

In the context of the transformations (20)–(21) the prescription for using the contraction $g^{\mu\nu} E^{(g)}_{\mu\nu}$ to eliminate R from the scalar field equation of motion (9) can be seen as giving an unconfounded equation under the transformation: namely

$$E^{(\Phi)} + \frac{\mathcal{A}'}{\mathcal{A}} g^{\mu\nu} E^{(g)}_{\mu\nu} = \mathrm{e}^{-4\bar{\gamma}} \left(\bar{f}'\right)^{-1} \left\{ \bar{E}^{(\bar{\Phi})} + \frac{\bar{\mathcal{A}}'}{\bar{\mathcal{A}}} \bar{g}^{\mu\nu} \bar{E}^{(\bar{g})}_{\mu\nu} \right\}. \tag{31}$$

Note that as under transformations, $\dfrac{\delta}{\delta \Phi}$ gains an additive term, and $\delta g^{\mu\nu}$ also gains one which of course follows from (20). Since the action functional S_{m} for the matter fields χ^A functionally depends on $\mathrm{e}^{2\alpha} g_{\mu\nu}$, which is invariant under the transformations (20)–(21) in the sense of subsection 2.3 [10, 30], it follows that the equations of motion (10) for the matter fields are also invariant under these transformations. In order to sum up, let us take a look at the transformation of the varied action (5). A straightforward calculation reveals

$$
\begin{aligned}
\delta S =& \frac{1}{2\kappa^2}\int_{V_4} \mathrm{d}^4x\sqrt{-g}\left\{E^{(g)}_{\mu\nu}\delta g^{\mu\nu} + E^{(\Phi)}\delta\Phi + 2\kappa^2 \mathrm{e}^{4\alpha}E^{(\chi)}_A\delta\chi^A\right\}\\
&+\frac{1}{2\kappa^2}\int_{V_4} \mathrm{d}^4x\partial_\sigma\Big(\sqrt{-g}\left[\mathscr{B}^{\ \sigma}_{(g)} + \mathscr{B}^{\ \sigma}_{(\Phi)} + 2\kappa^2\mathrm{e}^{4\alpha}\mathscr{B}^{\ \sigma}_{(\chi)}\right]\Big)\\
=& \frac{1}{2\kappa^2}\int_{V_4} \mathrm{d}^4x\sqrt{-\bar{g}}\left\{\bar{E}^{(\bar{g})}_{\mu\nu}\left(\delta\bar{g}^{\mu\nu} - 2\bar{\gamma}'\bar{g}^{\mu\nu}\delta\bar{\Phi}\right)\right.\\
&\left.+\left(\bar{E}^{(\bar{\Phi})} + 2\bar{\gamma}'\bar{g}^{\mu\nu}\bar{E}^{(\bar{g})}_{\mu\nu}\right)\delta\bar{\Phi} + 2\kappa^2\mathrm{e}^{4\bar{\alpha}}\bar{E}^{(\chi)}_A\delta\chi^A\right\}\\
&+\frac{1}{2\kappa^2}\int_{V_4} \mathrm{d}^4x\partial_\sigma\Big(\sqrt{-\bar{g}}\left[\bar{\mathscr{B}}^{\ \sigma}_{(\bar{g})} + \bar{\mathscr{B}}^{\ \sigma}_{(\bar{\Phi})}\right] + \delta\Big(\sqrt{-\bar{g}}\ \bar{\mathscr{B}}^{\ \sigma}_{(\bar{S})}\Big) + \sqrt{-\bar{g}}\ 2\kappa^2\mathrm{e}^{4\bar{\alpha}}\bar{\mathscr{B}}^{\ \sigma}_{(\chi)}\Big)\\
=& \frac{1}{2\kappa^2}\int_{V_4} \mathrm{d}^4x\sqrt{-\bar{g}}\left\{\bar{E}^{(\bar{g})}_{\mu\nu}\delta\bar{g}^{\mu\nu} + \bar{E}^{(\bar{\Phi})}\delta\bar{\Phi} + 2\kappa^2\mathrm{e}^{4\bar{\alpha}}\bar{E}^{(\chi)}_A\delta\chi^A\right\} + \overline{\mathscr{B}\text{oundary terms}}.
\end{aligned}
\tag{32}
$$

The fourth line forms as follows: under the transformations (20)–(21) the boundary terms (6) and (7) mix with each other and some extra terms arise. The latter are exactly the ones obtained by varying the boundary term (23) which arose due to rewriting the action functional (1) in terms of $\bar{g}_{\mu\nu}$ and $\bar{\Phi}$. The boundary terms $\bar{\mathscr{B}}^{\ \sigma}_{(\chi)}$ that appear when the action functional (1) is varied w.r.t. the matter fields χ^A are invariant. As before, we have included the boundary terms for the sake of completeness although they do not contribute to the equations of motion. Indeed, from the viewpoint of the transformation properties they must also behave well.

One can think about the continuity equation (14) in the same spirit. Let us consider a symmetric second order tensor $E_{(\mu\nu)}$ having the following transformation properties: $E_{(\mu\nu)} = \mathrm{e}^{-2\bar{\gamma}}\bar{E}_{(\mu\nu)}$. For such a tensor

$$
\nabla^\mu E_{\mu\nu} = \mathrm{e}^{-4\bar{\gamma}}\bar{\nabla}^\mu\bar{E}_{\mu\nu} - \bar{\gamma}'\mathrm{e}^{-4\bar{\gamma}}\bar{g}^{\mu\lambda}\bar{E}_{\mu\lambda}\bar{\nabla}_\nu\bar{\Phi} \tag{33}
$$

holds. A straightforward calculation shows that previous knowledge is at least implicitly taken into account when the continuity equation is constructed. Indeed, by making use of (33), the transformation properties (28) of the tensor equations (8) and (29) covering the transformation properties of the scalar field equation (9), we obtain

$$
\begin{aligned}
E^{(c)}_\nu \equiv& \nabla^\mu E^{(g)}_{\mu\nu} + \frac{1}{2}E^{(\Phi)}\nabla_\nu\Phi\\
=& \mathrm{e}^{-4\bar{\gamma}}\left\{\bar{\nabla}^\mu\bar{E}^{(\bar{g})}_{\mu\nu} - \bar{\gamma}'\bar{g}^{\mu\lambda}\bar{E}^{(\bar{g})}_{\mu\lambda}\bar{\nabla}_\nu\bar{\Phi} + \frac{1}{2}\left(\bar{E}^{(\bar{\Phi})} + 2\bar{\gamma}'\bar{g}^{\mu\lambda}\bar{E}^{(\bar{g})}_{\mu\lambda}\right)\bar{\nabla}_\nu\bar{\Phi}\right\}\\
=& \mathrm{e}^{-4\bar{\gamma}}\left\{\bar{\nabla}^\mu\bar{E}^{(\bar{g})}_{\mu\nu} + \frac{1}{2}\bar{E}^{(\bar{\Phi})}\bar{\nabla}_\nu\bar{\Phi}\right\} = \mathrm{e}^{-4\bar{\gamma}}\bar{E}^{(c)}_\nu.
\end{aligned}
\tag{34}
$$

Hence, we have equations of motion given by (8), (10), (13) and (14) which only gain a common multiplier under the local Weyl rescaling of the metric tensor (20) and under the scalar field redefinition (21), but otherwise preserve their structure. An analogous conclusion was drawn in [42]. We deem that as these are general equations no problems arise when either the transformation (20) or (21) become singular at some isolated scalar field value.

2.2.3. Transformations in the Friedmann cosmology. Previously, the transformation properties of the field equations were discussed. The Friedmann cosmology is a particular case and the corresponding equations of motion (16)–(19) transform according to the rules (28), (31) and (34), of course. Nevertheless, there are some details that need to be mentioned. The line element in Friedmann cosmology has the form (15). In order to keep this form of the metric, each conformal transformation $g_{\mu\nu} = \mathrm{e}^{2\bar{\gamma}} \bar{g}_{\mu\nu}$ is followed by a time coordinate transformation and a redefinition of the scale factor

$$\mathrm{d}t \mapsto \mathrm{d}\bar{t}: \sqrt{\mathrm{e}^{2\bar{\gamma}}}\mathrm{d}\bar{t} = \mathrm{d}t; \qquad a(t) \mapsto \bar{a}(\bar{t}): \sqrt{\mathrm{e}^{2\bar{\gamma}}}\bar{a}(\bar{t}) = a(t). \tag{35}$$

Therefore as the cosmological time depends on the chosen parametrization we adopt the following notation

$$\dot{()} \equiv \frac{\mathrm{d}}{\mathrm{d}t}() \text{ and } \dot{\bar{()}} \equiv \frac{\mathrm{d}}{\mathrm{d}\bar{t}}\bar{()}. \tag{36}$$

Due to (35) the transformation of the Hubble parameter reads

$$H \equiv \frac{\dot{a}}{a} = \mathrm{e}^{-\bar{\gamma}}\left(\bar{H} + \bar{\gamma}'\dot{\bar{\Phi}}\right). \tag{37}$$

One can counter the additive term arising in (37), for example, by considering the quantity

$$\left(H + \frac{1}{2}\frac{\mathcal{A}'}{\mathcal{A}}\dot{\Phi}\right) = \mathrm{e}^{-\bar{\gamma}}\left(\bar{H} + \frac{1}{2}\frac{\bar{\mathcal{A}}'}{\bar{\mathcal{A}}}\dot{\bar{\Phi}}\right). \tag{38}$$

Also note that

$$\dot{\Phi} = \mathrm{e}^{-\bar{\gamma}}\bar{f}'\dot{\bar{\Phi}}, \qquad \ddot{\Phi} = \mathrm{e}^{-2\bar{\gamma}}\left(\bar{f}'\ddot{\bar{\Phi}} + \bar{f}''\dot{\bar{\Phi}}^2 - \bar{\gamma}'\bar{f}'\dot{\bar{\Phi}}^2\right) \tag{39}$$

are the transformations of the first and the second derivative of the scalar field w.r.t. the cosmological time respectively.

The transformation of ρ and p is determined by the transformation of the contraction T of the energy-momentum tensor $T_{\mu\nu}$. Using the transformation rule (30) of the variational derivative $\frac{\delta}{\delta g^{\mu\nu}}$ on the definition (11) of the matter energy-momentum tensor reveals that $T = \mathrm{e}^{-4\bar{\gamma}}\bar{T}$, as also mentioned after (28).

2.3. Invariants

A closer look at the transformation rules (24) of the four arbitrary functions $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$ allows us to write out objects that do not gain any additive or multiplicative terms under the local Weyl rescaling (20) and under the scalar field redefinition (21). Let us recall the three basic ones introduced in our recent paper [30]

$$\mathcal{I}_1(\Phi) \equiv \frac{\mathrm{e}^{2\alpha(\Phi)}}{\mathcal{A}(\Phi)}, \quad \mathcal{I}_2(\Phi) \equiv \frac{\mathcal{V}(\Phi)}{(\mathcal{A}(\Phi))^2}, \quad \mathcal{I}_3(\Phi) \equiv \pm\int\sqrt{\mathcal{F}(\Phi)}\,\mathrm{d}\Phi \tag{40}$$

where [10]

$$\mathcal{F} \equiv \frac{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}{4\mathcal{A}^2}, \quad \bar{\mathcal{F}} = \left(\bar{f}'\right)^2\mathcal{F}. \tag{41}$$

Under the scalar field redefinition (21) these quantities transform as scalar functions but their numerical value at some spacetime point $x^\mu \in V_4$ is nevertheless invariant. One can introduce further objects having the same transformation properties by making use of three operations: i) forming an arbitrary function of the invariants (40) etc, ii) introducing a quotient of derivatives $\mathcal{I}_i \equiv \mathcal{I}'_j/\mathcal{I}'_k$ or iii) integrating the previous result $\mathcal{I}_j \equiv \int\mathcal{I}_i(\Phi)\mathcal{I}'_k(\Phi)\mathrm{d}\Phi$ in the sense of an indefinite integral [30].

The basic quantities (40) were chosen since they are well known and used in the literature. For instance, in the JF BDBW parametrization $\mathcal{I}_2 = \mathcal{V}_\mathrm{J}/\Psi^2$ and

$\mathcal{I}_2' = \left(\Psi\mathcal{V}_J' - 2\mathcal{V}_J\right)/\Psi^3$ determine the fixed points in [43], while in [44] the term 'effective potential' refers to $\mathcal{I}_2$. The invariant $\mathcal{I}_3$ is essential in the Barrow and Parsons solution generating prescription [21]. Last but not least, in the JF BDBW parametrization $1/\mathcal{I}_1 = \Psi$ and in the EF canonical parametrization $\pm\mathcal{I}_3 = \varphi + const$. Therefore

$$\frac{\mathrm{d}\dfrac{1}{\mathcal{I}_1}}{\mathrm{d}\mathcal{I}_3} = -\frac{1}{\mathcal{I}_1^2}\frac{\mathcal{I}_1'}{\mathcal{I}_3'} = \pm\frac{\mathrm{d}\Psi}{\mathrm{d}\varphi} \tag{42}$$

can also be considered to be an invariant.

3. General relativity regime

In this section we first write down the conditions under which STG coincides with GR, i.e. we introduce the notion of the 'GR regime'. Second, we consider the GR limit, i.e. a dynamical approach to the GR regime.

3.1. Theory: part II

3.1.1. General relativity regime. GR is in rather good agreement with the experiments carried out in the solar system. Therefore whatever theory of gravitation we consider, its predictions —in order to be viable—must be close to those of GR, at least in the sufficient neighbourhood of the Sun. In the current paper we will bestow consideration upon STG in which the predictions are close to the ones obtained from GR because the field equations themselves are the same—at least in some regime. We shall use 'GR regime' to refer to such a situation.

In Einstein's GR, the tensor equation, a specific case of (8), does not contain the terms $\mathcal{B}\nabla_\mu\Phi\nabla_\nu\Phi$, $\mathcal{A}''\nabla_\mu\Phi\nabla_\nu\Phi$, $\mathcal{A}'\nabla_\mu\nabla_\nu\Phi$ or the contractions of these. Requiring that $\mathcal{B}$ and the derivatives of $\mathcal{A}$ are zero at the same value of the scalar field Φ in a generic theory needs finetuning, and therefore we instead impose that in the GR regime the scalar field is constant $\Phi = \Phi_0$, i.e.

$$\nabla_\mu\Phi\Big|_{\Phi_0} = 0 \tag{43}$$

and $\nabla_\mu\nabla_\nu\Phi = 0$. In this case (8) reduces to the Einstein equation in GR, with $\kappa^2/\mathcal{A}(\Phi_0)$ playing the role of the gravitational constant and $\ell^{-2}\mathcal{V}(\Phi_0)$ as the cosmological constant, both positive. The continuity equation (14) also reduces to $\nabla^\mu T_{\mu\nu} = 0$.

In order to maintain the constancy of the scalar field Φ the equation of motion (13) must become an identity $0 = 0$ at the scalar field value Φ_0. Let us divide (13) by $4\mathcal{A}\mathcal{F}$ and make use of the invariant objects (40) and $\mathcal{F}$, given by (41), in order to rewrite (13) in a more compact manner as follows

$$\Box\Phi + \left(\frac{\mathcal{A}'}{\mathcal{A}} - \frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'\mathcal{F}\right)g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi = \frac{\mathcal{A}}{2\ell^2}\frac{\mathcal{I}_2'}{\mathcal{F}} - \frac{\kappa^2}{4\mathcal{A}}\frac{\left(\ln\mathcal{I}_1\right)'}{\mathcal{F}}T. \tag{44}$$

The lhs of (44) contains derivatives and therefore for a constant scalar field value it vanishes. Hence, in order to avoid finetuning we impose that the source terms

$$\frac{\mathcal{A}}{2\ell^2}\frac{\mathcal{I}_2'}{\mathcal{F}}\Bigg|_{\Phi_0} = 2\ell^{-2}\frac{\left(\mathcal{A}\mathcal{V}' - 2\mathcal{V}\mathcal{A}'\right)}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}\Bigg|_{\Phi_0} = 0, \tag{45}$$

$$\left.\frac{\kappa^2}{4\mathcal{A}}\frac{\left(\ln \mathcal{I}_1\right)'}{\mathcal{F}}\right|_{\Phi_0} = \kappa^2 \left.\frac{\left(2\mathcal{A}\alpha' - \mathcal{A}'\right)}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}\right|_{\Phi_0} = 0 \tag{46}$$

in the regime where the predictions of the theory described by the action functional (1) are also close to those of GR. In the following we shall use 'vanishing source conditions' for referring to (45)–(46). In the JF BDBW parametrization (2) the second condition (46) can only be satisfied by letting $\omega(\Psi) \to \infty$ [41] and in this case the first condition is also satisfied.

Let us point out that one may also consider a situation where on the rhs of (44) the sum vanishes, but both additive terms separately are nonvanishing. In this case a so-called screening mechanism is operating, e.g. the chameleon effect [45] or the symmetron screening mechanism [46]. However, in these cases, the vanishing of the rhs of (44) depends on the matter contribution. If the latter changes, e.g. the energy density ρ in the Friedmann cosmology decreases as the Universe expands, then the scalar field must also evolve further. In the current paper, we are interested in basic cosmological scenarios where the scalar field dynamic ends once and for all, and therefore we do not focus upon the screening mechanisms.

For a specific matter content with $T \equiv 0$, e.g. radiation, the condition (46) is not needed [41]. If in addition to the latter $\mathcal{V} \equiv 0$ is also considered then the rhs of (44) vanishes automatically, and the GR regime can, in principle, be realized at any value of Φ. In the context of the original Brans–Dicke theory, with a constant parameter ω, i.e. a particular case of the JF BDBW parametrization (2), there is a discussion in the literature that in the case of $T \equiv 0$, $\mathcal{V} \equiv 0$ taking parametrically $\omega = \infty$ does not reduce the STG solutions to the ones of GR [47]. However, if $\nabla_\mu \Psi = 0$ is not imposed, only letting ω diverge is not sufficient for obtaining the GR regime indeed.

In addition to the vanishing source conditions (45)–(46), a little more is needed to achieve GR-like behaviour. Namely, for GR the well-known relation $-R + (4\Lambda) \propto T$ holds. Let us make use of the latter and obtain some restrictions from the contraction (12) of the tensor equation (8). First, a short glimpse reveals that vanishing or diverging $\mathcal{A}|_{\Phi_0}$ violates the mentioned condition. Second, as $\left.\frac{\kappa^2}{\mathcal{A}}\right|_{\Phi_0}$ is the effective gravitational 'constant' we impose $\mathcal{A}|_{\Phi_0} > 0$ in order to have an attractive gravity. Third, the same equation reveals that the potential $\mathcal{V}$, which at the constant scalar field value Φ_0 mimics the cosmological constant Λ, must be nondiverging as well. In the current paper we also assume it to be non-negative. Last but not least, we impose that α must be nondiverging, because otherwise the coupling of the matter fields to the geometry determined by $g_{\mu\nu}$ is unphysical. These assumptions are spelled out below in (56).

Analogously to the previous, let us point out that in the context of the GR regime the scalar field equation of motion (9) containing R might be a constraint equation. We start by assuming $|\mathcal{A}'|_{\Phi_0} < \infty$, because otherwise the behaviour of the effective gravitational 'constant' $\frac{\kappa^2}{\mathcal{A}}$ becomes unnatural if the scalar field Φ deviates from its constant value Φ_0. Under this assumption (9) also reveals that $|\alpha'|_{\Phi_0} < \infty$ and $|\mathcal{V}'|_{\Phi_0} < \infty$, because otherwise the constraint $-R + (4\Lambda) \propto T$ is violated. These conditions are captured below as (57). For the latter we have implicitly assumed that neither $\mathcal{B}$ nor its derivative diverges. In the current paper we restrict our analysis to the cases where only one out of the four arbitrary functions $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$ along with its derivatives might diverge. Hence if diverging $\mathcal{B}$ is under consideration then all other functions are assumed to be regular and (57) is therefore imposed as a general condition in the GR regime.

Let us analyze the possibilities of satisfying the vanishing source conditions (45)–(46) in more detail. The first and most obvious possibility is to demand that both numerators are zero at the same scalar field value Φ_0. The other possibility is to let the denominator diverge at the scalar field value Φ_0. In some sense this is more natural because no tuning is needed, i.e. if one of the conditions is satisfied then the other must be satisfied as well. Since the diverging $\mathcal{A}|_{\Phi_0}$ and $\mathcal{A}'|_{\Phi_0}$ cases have already been omitted, we are left with possibly diverging $\mathcal{B}|_{\Phi_0}$ (i.e. in essence JF BDBW $\omega(\Psi)$). In what follows we keep the latter in mind, but nevertheless make the most of the statements about $\mathcal{F}$ where $\mathcal{B}$ resides because the transformation property of $\mathcal{F}$, given by (41), is remarkably simpler than the rule for $\mathcal{B}$, given by (24b).

To sum up, we consider two possibilities for fulfilling the vanishing source conditions (45)–(46) [30]:

$$(i) \quad \{\Phi_\bullet\} \equiv \left\{ \Phi \,\middle|\, \mathcal{I}_2' = 0 = \mathcal{I}_1' \text{ and } \frac{1}{\left(\mathcal{I}_3'\right)^2} \equiv \frac{1}{\mathcal{F}} \neq 0 \right\}, \tag{47}$$

$$(ii) \quad \{\Phi_\star\} \equiv \left\{ \Phi \,\middle|\, \frac{1}{\left(\mathcal{I}_3'\right)^2} \equiv \frac{1}{\mathcal{F}} = 0 \right\}. \tag{48}$$

If $\mathcal{F}$ diverges then one must also demand that the last term on the lhs of (44) in the GR regime vanishes nevertheless. However, this is rather a question about the permitted behaviour of a specific solution, i.e. the order of magnitude of $\nabla_\mu\Phi \to 0$ w.r.t. $\Phi - \Phi_0 \to 0$, which we shall not yet discuss.

3.1.2. General relativity limit. Once we have a consistent notion of the GR regime it is of course important to find out whether a solution under consideration converges to that regime or repels from it. One useful tool for clarifying the question is provided by the dynamical systems method. In section 4 of the current paper we benefit from this method because the GR regime can be identified with a critical point in the $\left(\Phi, \nabla_\mu\Phi\right)$ space. More precisely, we linearize (44), i.e. the scalar field equation of motion. According to the Hartman–Grobman theorem, the linearized equation captures the qualitative behaviour of the full dynamics if and only if the critical point is hyperbolic, i.e. all eigenvalues have a nonzero real part. It can be shown that a necessary condition for the critical point to be hyperbolic is given by either of the conditions [35, 48]

$$0 < \left|C_2\right| < \infty: \quad C_2 \equiv -\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\frac{\mathcal{A}}{2\ell^2}\frac{\mathcal{I}_2'}{\mathcal{F}}\right)\Bigg|_{\Phi_0} = -\frac{\mathcal{A}}{2\ell^2}\left(\frac{\mathcal{I}_2''}{\mathcal{F}} + \left(\frac{1}{\mathcal{F}}\right)'\mathcal{I}_2'\right)\Bigg|_{\Phi_0}, \tag{49}$$

$$0 < \left|C_3\right| < \infty: \quad C_3 \equiv \frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\frac{\kappa^2}{4\mathcal{A}}\frac{\left(\ln \mathcal{I}_1\right)'}{\mathcal{F}}\right)\Bigg|_{\Phi_0} = \frac{\kappa^2}{4\mathcal{A}}\left(\frac{\left(\ln \mathcal{I}_1\right)''}{\mathcal{F}} + \left(\frac{1}{\mathcal{F}}\right)'\left(\ln \mathcal{I}_1\right)'\right)\Bigg|_{\Phi_0}. \tag{50}$$

In other words we assume that the leading term in the Taylor expansion of the rhs of (44) is linear w.r.t. $\Phi - \Phi_0$. In what follows we shall refer to (49)–(50) as 'first order small source conditions'. In (49)–(50) we have made use of the vanishing source conditions (45)–(46) in order to cancel some additive terms. Actually, due to the same conditions only one of the additive terms on the right hand sides of (49)–(50) can be nonvanishing. Perhaps it is also instructive to write out these conditions in the EF canonical parametrization (3) (cf [43, 49])

$$C_2\Big|_{\text{EF can.}} = -\frac{1}{2\ell^2}\mathcal{V}_E''\Big|_{\varphi_0}, \quad C_3\Big|_{\text{EF can.}} = \frac{\kappa^2}{2}\alpha_E''\Big|_{\varphi_0} \tag{51}$$

and in the JF BDBW parametrization (2) (see [33, 43])

$$C_2\,|_{\text{JF BDBW}} = -\,2\ell^{-2}\left(\frac{\left(\Psi\mathcal{V}_J' - 2\mathcal{V}_J\right)'}{2\omega+3} + \left(\frac{1}{2\omega+3}\right)'\left(\Psi\mathcal{V}_J' - 2\mathcal{V}_J\right)\right)\Bigg|_{\Psi_0},$$
$$C_3\,|_{\text{JF BDBW}} = -\,\kappa^2\left(\frac{1}{2\omega+3}\right)'\Bigg|_{\Psi_0}. \tag{52}$$

Let us make use of the first order small source conditions (49)–(50) in order to adopt the following three assumptions on $\mathcal{F}$ [30, 33]:

$$0 \leqslant \frac{1}{\mathcal{F}}\Bigg|_{\Phi_0} < \infty, \tag{53}$$

$$-\infty < \left(\frac{1}{\mathcal{F}}\right)^{\overbrace{\prime\ldots\prime}^{n-\text{times}}}\Bigg|_{\Phi_0} \equiv \frac{\mathrm{d}^n}{\mathrm{d}\Phi^n}\left(\frac{1}{\mathcal{F}}\right)\Bigg|_{\Phi_0} < \infty, \tag{54}$$

$$\text{if } \Phi_0 \equiv \Phi_\star, \text{ see (48), i.e.} \frac{1}{\mathcal{F}}\Bigg|_{\Phi_\star} = 0, \text{ then } \left(\frac{1}{\mathcal{F}}\right)'\Bigg|_{\Phi_\star} \neq 0. \tag{55}$$

The transformation rule for $\mathcal{F}$, given by (41), reveals that $\mathcal{F}$ preserves its sign under the local Weyl rescaling (20) and under the scalar field redefinition (21). Therefore, if we want to stay connected with the EF canonical parametrization (3), where $\mathcal{F}_E = 1$, then in any other parametrization $\mathcal{F}$ must be non-negative. Here we go one step further by imposing (53), i.e. assuming $\mathcal{F}$ to be strictly positive in order to avoid the possibility that in the vanishing source conditions (45)–(46) both the numerator and denominator vanish. In the following we shall refer to (53) as the 'positive $\mathcal{F}$ assumption'. Let us point out that in the JF BDBW parametrization (2) the limit $\mathcal{F}_J = 0$ corresponds to $\omega = -\frac{3}{2}$.

The 'differentiable $\frac{1}{\mathcal{F}}$ assumption' (54) guarantees that we can handle the possible singularity lying in $\mathcal{F}$. Last but not least, the 'nonvanishing $\left(\frac{1}{\mathcal{F}}\right)'$ assumption' (55) is a necessary condition for the critical point to be hyperbolic. The latter only applies if $\mathcal{F} \to \infty$. Therefore, e.g. in the EF canonical parametrization (3) the assumption (55) is automatically satisfied since $\mathcal{F}_E = 1$ never diverges. It can be shown that if the nonvanishing $\left(\frac{1}{\mathcal{F}}\right)'$ assumption (55) is not fulfilled then one cannot express the JF BDBW parametrization scalar field Ψ as a Taylor expansion of the EF canonical parametrization scalar field φ [30]. Let us point out that if the condition (55) holds, then the equation of motion (8) for the metric tensor and the equation of motion (13) for the scalar field converge to the GR regime at the same 'rate'. The latter is determined by $\mathcal{F}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi \to 0$ [41].

In order to sum up, let us gather the restrictions on the three arbitrary functions $\{\mathcal{A},\, \mathcal{V},\, \alpha\}$ while $\mathcal{B}$ is covered by assumptions (53)–(55):

$$0 < \mathcal{A}\,|_{\Phi_0} < \infty, \quad 0 \leqslant \mathcal{V}\,|_{\Phi_0} < \infty, \quad 0 < \mathrm{e}^{2\alpha}\,|_{\Phi_0} < \infty, \tag{56}$$

$$\left|\mathcal{A}'\right|_{\Phi_0} < \infty, \quad \left|\mathcal{V}'\right|_{\Phi_0} < \infty, \quad \left|\alpha'\right|_{\Phi_0} < \infty, \tag{57}$$

$$\left|\mathcal{A}''\right|_{\Phi_0} < \infty, \quad \left|\mathcal{V}''\right|_{\Phi_0} < \infty, \quad \left|\alpha''\right|_{\Phi_0} < \infty, \tag{58}$$

$$\text{if } \left.\frac{1}{\mathcal{F}}\right|_{\Phi_0} = 0 \text{ then either } \left.\mathcal{A}'\right|_{\Phi_0} \neq 0 \text{ or } \left.\mathcal{V}'\right|_{\Phi_0} \neq 0 \text{ or } \left.\alpha'\right|_{\Phi_0} \neq 0, \tag{59}$$

$$\text{if } \left.\frac{1}{\mathcal{F}}\right|_{\Phi_0} \neq 0 \text{ then either } \left.\mathcal{I}_1''\right|_{\Phi_0} \neq 0 \text{ or } \left.\mathcal{I}_2''\right|_{\Phi_0} \neq 0 \tag{60}$$

where (56)–(57) are necessary for a consistent notion of the GR regime and (58)–(60) complement the assumptions (53)–(55) on $\mathcal{F}$ in order to obtain a hyperbolic critical point when the dynamical systems method is used.

3.1.3. Two remarks. Two comments about the assumptions (53)–(55) on $\mathcal{F}$ are in order. First the nonvanishing $\left(\frac{1}{\mathcal{F}}\right)'$ assumption (55) imposes that $\left.\frac{1}{\mathcal{F}}\right|_{\Phi_\star} = 0$ is not an extrema of the same function. Therefore if the scalar field Φ evolves through the value $\Phi_\star$ then $\frac{1}{\mathcal{F}}$ becomes negative hence violating the positive $\mathcal{F}$ assumption (53). We would expect that a consistent theory be endowed with a mechanism that forbids the violation of the condition (53). In other words if $\left.\left(\frac{1}{\mathcal{F}}\right)'\right|_{\Phi_\star}$ is positive (negative) then the scalar field Φ values permitted by the field equations should not be allowed to be less (more) than the value $\Phi_\star$. Essentially, the same was pointed out in the context of the Friedmann cosmology where the argumentation was based on the field space dynamics [35].

Second, the differentiable $\frac{1}{\mathcal{F}}$ assumption (54) states that the limiting value

$$\lim_{\Phi\to\Phi_\star}\left[\left(\frac{1}{\mathcal{F}}\right)'\mathcal{F}\cdot\left(\Phi-\Phi_\star\right)\right] = M \tag{61}$$

holds. Here if $\mathcal{F}$ diverges, then M is the order of the first nonzero derivative $\left(\frac{1}{\mathcal{F}}\right)^{(M)} \neq 0$, otherwise $M = 0$. If the nonvanishing $\left(\frac{1}{\mathcal{F}}\right)'$ assumption (55) is also applied then we can always replace $\left(\frac{1}{\mathcal{F}}\right)'\mathcal{F}$ by $\frac{1}{\Phi-\Phi_\star}$ whenever calculating the limiting values in the process where $\mathcal{F}$ diverges. In the following we shall use the assumptions (53)–(55) on $\mathcal{F}$ and therefore in the current paper $M = 0$ or $M = 1$ are the two possibilities.

3.1.4. Barrow–Parsons classes. The assumptions (53)–(55) on $\mathcal{F}$ are restrictive, but there are many studies which consider a functional form of $\mathcal{F}$ obeying (53)–(55). A rather general classification of the possible functional forms of $\mathcal{F}$ in the JF BDBW parametrization (2) was given by Barrow and Parsons [21]. There they constrained the constant powers β_i so that the parametrized post-Newtonian conditions $\omega(\Psi) \to \infty$ and $\frac{\omega'(\Psi)}{\omega(\Psi)^3} \to 0$ were satisfied. Here, analogously to [28], we write out the further necessary restrictions on the Barrow–Parsons classes so that the assumptions (53)–(55) are satisfied.

1) $\dfrac{1}{\mathcal{F}_\mathrm{J}} \equiv \dfrac{4\Psi^2}{2\omega(\Psi)+3} \propto \Psi^2\left|1-\dfrac{\Psi}{\Psi_\star}\right|^{\beta_1}, \quad \beta_1 > \dfrac{1}{2}.$
   i) Assumption (53) is fulfilled if $|\Psi| \not\to \infty$. The latter is assured by assumption (56).
   ii) Assumption (54) is fulfilled if β_1 is an arbitrary positive integer power.

iii) Assumption (55) is fulfilled if $\beta_1 = 1$.

Hence we obtain $\dfrac{1}{\mathcal{F}_\mathrm{J}} \propto \Psi^2 \left|1 - \dfrac{\Psi}{\Psi_\star}\right|$ fulfilling the assumptions (53)–(55). Such functional forms have been considered e.g. in [24, 50].

2) $\dfrac{1}{\mathcal{F}_\mathrm{J}} \propto \Psi^2 \left|\ln\left(\dfrac{\Psi}{\Psi_\star}\right)\right|^{\beta_2}$, $\beta_2 > \frac{1}{2}$.

i) Assumption (53) is fulfilled if $|\Psi| \not\to \infty$ and $\Psi \not\to 0$.

ii) Assumption (54) is fulfilled if β_2 is an arbitrary positive integer power.

iii) Assumption (55) is fulfilled if $\beta_2 = 1$.

Hence we obtain $\frac{1}{\mathcal{F}_\mathrm{J}} \propto \Psi^2 \left|\ln\left(\frac{\Psi}{\Psi_\star}\right)\right|$ fulfilling the assumptions (53)–(55). Such functional forms have been considered e.g. in [17, 51].

3) $\dfrac{1}{\mathcal{F}_\mathrm{J}} \propto \Psi^2 \left|1 - \left(\dfrac{\Psi}{\Psi_\star}\right)^{\beta_3}\right|$, $\beta_3 > 0$.

i) Assumption (53) is fulfilled if $|\Psi| \not\to \infty$.

ii) Assumption (54) is fulfilled for arbitrary β_3.

iii) Assumption (55) is fulfilled for arbitrary β_3.

Hence we obtain $\frac{1}{\mathcal{F}_\mathrm{J}} \propto \Psi^2 \left|1 - \left(\frac{\Psi}{\Psi_\star}\right)^{\beta_3}\right|$ fulfilling the assumptions (53)–(55) for arbitrary β_3. Such theories have been studied e.g. by [23, 52, 53].

3.2. Transformations: part II

In the current subsection we analyze the transformation properties in the vicinity of the GR regime. In order to simplify the notation we drop an explicit reference to the point of evaluation $()|_0$. Let us start by studying the local Weyl rescaling of the metric tensor (20) and the scalar field redefinition (21). The former shall be restricted on mathematical grounds, but in order to impose conditions on the latter we make use of the assumptions (53)–(55) on $\mathcal{F}$. A preluding remark concerning the scalar field redefinition (21) is in order. Let us impose the function $\bar{f}(\bar{\Phi})$ to be at least directionally continuous but retain the possibility that the Jacobian $\bar{f}' \equiv \mathrm{d}\Phi/\mathrm{d}\bar{\Phi}$ of this coordinate transformation in the 1-dimensional field space may be singular or have zeros at some isolated value of the scalar field $\bar{\Phi}$. The latter is motivated by the observation that in the GR regime $\mathcal{F}$ can be singular in some parametrization.

Whenever the consistency between the constraints imposed on the transformation functions $\bar{\gamma}$ and $\bar{f}$, and on the four arbitrary functions $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$ is studied, we consider having two parametrizations where the assumptions on the four arbitrary functions hold. We then check whether the transformation between these two obeys the constraints on the transformation functions.

3.2.1. Constraints on $\bar{\gamma}$.

We hereby restrict the local Weyl rescaling of the metric tensor (20) by making mathematical assumptions, and analyze how the resulting constraints are related to the restrictions (56)–(58), which are imposed on the three arbitrary functions $\{\mathcal{A}, \mathcal{V}, \alpha\}$.

We start by assuming the local Weyl rescaling of the metric tensor (20) to be regular, i.e. the function $\bar{\gamma}(\bar{\Phi})$ and its first and second derivative, $\mathrm{d}\bar{\gamma}/\mathrm{d}\bar{\Phi}$ and $\mathrm{d}^2\bar{\gamma}/\mathrm{d}\bar{\Phi}^2$, respectively, do not diverge because otherwise we would introduce geometrical singularities via the local rescaling of the metric. Note that this excludes the interesting possibility of 'conformal continuation' [54]. However, here we are focused upon the GR regime which cannot be consistent with conformal continuation anyway. Let us proceed by pointing out a conclusion that follows from introducing the Weyl rescaling and the scalar field redefinition backwards, i.e. (26)–(27),

$$-\bar{\gamma}' \equiv -\frac{\mathrm{d}\bar{\gamma}\left(\bar{\Phi}\right)}{\mathrm{d}\bar{\Phi}} \equiv \frac{\mathrm{d}\gamma\left(\bar{f}\left(\bar{\Phi}\right)\right)}{\mathrm{d}\bar{\Phi}} = \frac{\mathrm{d}\gamma(\Phi)}{\mathrm{d}\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\bar{\Phi}} \equiv \gamma' \cdot \bar{f}'. \tag{62}$$

From assumption $|\bar{\gamma}'| < \infty$ we deduce that if $|\bar{f}'| \to \infty$ then in the same process $\gamma' \to 0$ because otherwise $\bar{\gamma}'$ would necessarily diverge. Hence as $|\bar{f}'| \to \infty$ implies $f' \to 0$ we conclude that for any transformation where $f' \to 0$ also $\gamma' \to 0$. The latter is a necessary condition. The actual value of the uncertainty $0 \cdot \infty$ depends on the limiting process which we have assumed to have a nondiverging result.

In order to show that the constraints on the transformation function $\bar{\gamma}$ are in accordance with the assumptions on the three arbitrary functions $\{\mathcal{A}, \mathcal{V}, \alpha\}$, given by (56)–(58), let us write out the following:

$$\bar{\mathcal{A}} = \mathrm{e}^{2\bar{\gamma}}\mathcal{A}, \tag{63}$$

$$\bar{\mathcal{A}}' = \mathrm{e}^{2\bar{\gamma}}\left(2\bar{\gamma}'\mathcal{A} + \bar{f}'\mathcal{A}'\right), \tag{64}$$

$$\bar{\mathcal{A}}'' = \mathrm{e}^{2\bar{\gamma}}\left(2\bar{\gamma}''\mathcal{A} + 4\left(\bar{\gamma}'\right)^2\mathcal{A} + 4\bar{\gamma}'\bar{f}'\mathcal{A}' + \left(\bar{f}'\right)^2\mathcal{A}'' + \bar{f}''\mathcal{A}'\right). \tag{65}$$

From (63) we see that a diverging $\bar{\gamma}$ would render $\bar{\mathcal{A}}$ infinite because we have assumed $0 < \mathcal{A} < \infty$. Due to the latter, without finetuning $|\bar{\gamma}'| \to \infty$ implies $|\bar{\mathcal{A}}'| \to \infty$ and analogously from (65) for the relation between $\bar{\gamma}''$ and $\bar{\mathcal{A}}''$. Hence, if we have two parametrizations where the restrictions (56)–(58) imposed on the three arbitrary functions $\{\mathcal{A}, \mathcal{V}, \alpha\}$ hold, then the local Weyl rescaling connecting these parametrizations must be regular. In the spirit of the discussion around (62) let us consider the case $\bar{f}' \to \infty$. From (64) we ascertain that in the same process $\mathcal{A}'$ must vanish for the limiting value $\lim \bar{f}'\mathcal{A}' < \infty$ to hold, because otherwise $\bar{\mathcal{A}}'$ would necessarily diverge. Again, let us make use of the backward transformations (26)–(27) in order to write the transformation (64) backwards

$$0 \stackrel{!}{=} \mathcal{A}' = \mathrm{e}^{2\gamma}\left(2\gamma'\bar{\mathcal{A}} + f'\bar{\mathcal{A}}'\right). \tag{66}$$

We hence conclude that in the process under consideration $f' \to 0$ implies $\gamma' \to 0$, and this is in perfect agreement with the discussion after (62).

Note that in the context of regular Weyl rescaling (20) the conditions (56), i.e. $0 < \mathcal{A} < \infty$ and $0 < \mathrm{e}^{2\alpha} < \infty$ are mathematical necessities for the existence of the transformations from an arbitrary frame to the Einstein frame (3) ($\mathcal{A}_{\mathrm{E}} = 1$) and to the Jordan frame (2) ($\alpha_{\mathrm{J}} = 0$), respectively.

3.2.2. Constraints on $\bar{f}$. Let us recall that the function $\bar{f}(\bar{\Phi})$ is imposed to be at least directionally continuous. However, it might be the case that $\bar{f}' = 0$ or $|\bar{f}'| \to \infty$. The latter has already been used implicitly because according to (41) in the EF canonical parametrization (3) $\bar{\mathcal{F}}_{\mathrm{E}} = 1$. Therefore, if $\mathcal{F}$ diverges in some other parametrization then also $\bar{f}' = 0 \Leftarrow \bar{\mathcal{F}}_{\mathrm{E}} = (\bar{f}')^2\mathcal{F}$.

Let us continue analyzing an analogous case more generically. We consider having $\mathcal{F} \to \infty$, $\bar{\mathcal{F}} < \infty$ and $\bar{f}' = 0$. We proceed under the differentiable $\frac{1}{\mathcal{F}}$ assumption (54). Due to the transformation properties of $\mathcal{F}$ itself, i.e. (41), the transformation of the first derivative reads

$$\left(\frac{1}{\mathcal{F}}\right)' = \frac{2\bar{f}''}{\bar{\mathcal{F}}} + \bar{f}'\left(\frac{1}{\bar{\mathcal{F}}}\right)'. \tag{67}$$

According to the nonvanishing $\left(\frac{1}{\mathcal{F}}\right)'$ assumption (55) the lhs of (67) is nonzero. The second term on the rhs is zero and hence the first term on the rhs must be nonzero. The positive $\mathcal{F}$ assumption (53) states that $\bar{\mathcal{F}}$ is nonvanishing and therefore also $\bar{f}'' \neq 0$ in the case under consideration.

Table 1 maps all the possibilities for transformations between $\mathcal{F}$ and $\bar{\mathcal{F}}$. Here six situations can be considered, but not all of them are distinct. The two possibilities v) and vi) for which $\bar{f}' \to \infty$ are taken into account by looking at the two possibilities i) and ii) for $\bar{f}' \to 0$ backwards. In order to examine the viability of the remaining four, let us analyze each case separately.

i) The case $\bar{f}' \to 0$ and $0 < |\bar{f}''| < \infty$.
This case does not have any pathologies so we keep it.
ii) The case $\bar{f}' \to 0$ and $|\bar{f}''| \to \infty$.
In order to reveal a pathology let us consider a transformation where the JF BDBW quantities are considered to be the 'unbarred' ones. Therefore $\mathcal{A}' = 1$ and the transformation (65) implies $\bar{\mathcal{A}}'' \to \infty$, which is something we want to avoid. Despite the fact that we used the JF BDBW parametrization, this behaviour is fairly general because of the assumption (59) arising from first order small source conditions (49)–(50). Due to such a pathology we neglect this possibility.
iii) The case $0 < |\bar{f}'| < \infty$ and $|\bar{f}''| < \infty$.
This transformation is also perfectly normal and we keep it.
iv)
  a) Almost the same as previous. Only that in this case both $\mathcal{F}$ and $\bar{\mathcal{F}}$ diverge. We keep it.
  b) The case $0 < |\bar{f}'| < \infty$ and $|\bar{f}''| \to \infty$.
  This case possesses the same pathology as case ii) and we therefore neglect it.

We will thus focus upon two possible cases. We shall refer to them according to the characteristics of the transformation function $\bar{f}$.

a) 'The regular case', based on cases iii) and iv) a) in table 1,

$$\begin{gathered} 0 < \left|\bar{f}'\right| < \infty, \qquad \left|\bar{f}''\right| < \infty; \\ \mathcal{F} < \infty, \quad \bar{\mathcal{F}} < \infty \quad \text{or} \quad \mathcal{F} \to \infty, \quad \bar{\mathcal{F}} \to \infty. \end{gathered} \tag{68}$$

b) 'The singular case', based on case i) in table 1,

$$\begin{gathered} \bar{f}' \to 0, \qquad 0 < \left|\bar{f}''\right| < \infty, \qquad \bar{\gamma}' \to 0; \\ \mathcal{F} \to \infty, \qquad \bar{\mathcal{F}} < \infty. \end{gathered} \tag{69}$$

We have also explicitly included knowledge of $\bar{\gamma}'$ given by discussion after (62) or equivalently after (66).

3.2.3. Transformation of the GR regime. In subsection 3.1 we gave a notion of the GR regime, and it is important to ascertain whether the given notion is invariant under the local rescaling of the metric tensor (20) and under the scalar field redefinition (21). Let us start by focusing upon the vanishing source conditions (45)–(46) for the GR regime at Φ_0. Due to the

Table 1. Conditions on the transformation function $\bar{f}$ based on definition (41), assumptions (53)—(55) and rule (67).

	$\bar{f}' \to 0$		$0 < \lvert\bar{f}'\rvert < \infty$		$\lvert\bar{f}'\rvert \to \infty$	
	$\bar{\mathcal{F}} < \infty$	$\bar{\mathcal{F}} \to \infty$	$\bar{\mathcal{F}} < \infty$	$\bar{\mathcal{F}} \to \infty$	$\bar{\mathcal{F}} < \infty$	$\bar{\mathcal{F}} \to \infty$
$\mathcal{F} < \infty$	—	—	iii) $0 \leqslant \lvert\bar{f}''\rvert < \infty$ $0 \leqslant \left\lvert\frac{\bar{f}''}{\bar{\mathcal{F}}}\right\rvert < \infty$	—	—	v) $\lvert\bar{f}''\rvert \to \infty$ $0 < \left\lvert\frac{\bar{f}''}{(\bar{f}')^3}\right\rvert < \infty$ $0 < \left\lvert\frac{\bar{f}''}{\bar{f}'\bar{\mathcal{F}}}\right\rvert < \infty$
$\mathcal{F} \to \infty$	i) $0 < \lvert\bar{f}''\rvert < \infty$ $0 < \left\lvert\frac{\bar{f}''}{\bar{\mathcal{F}}}\right\rvert < \infty$	ii) $\lvert\bar{f}''\rvert \to \infty$ $0 < \left\lvert\frac{\bar{f}''}{\bar{\mathcal{F}}}\right\rvert < \infty$	—	iv) a) $\lvert\bar{f}''\rvert < \infty$ $0 \leqslant \left\lvert\frac{\bar{f}''}{\bar{\mathcal{F}}}\right\rvert < \infty$ iv) b) $\bar{f}'' \to \infty$ $0 \leqslant \left\lvert\frac{\bar{f}''}{\bar{\mathcal{F}}}\right\rvert < \infty$	—	vi) $\lvert\bar{f}''\rvert \to \infty$ $0 < \left\lvert\frac{\bar{f}''}{\bar{f}'\bar{\mathcal{F}}}\right\rvert < \infty$

constraints (56)–(58) imposed on the three arbitrary functions $\{\mathcal{A}, \mathcal{V}, \alpha\}$ and the positive $\mathcal{F}$ assumption (53) the following holds [30, 41]

$$\frac{\mathcal{I}_2'}{2\mathcal{I}_3'} \equiv \pm \frac{\mathcal{I}_2'}{2\sqrt{\mathcal{F}}} \equiv \pm \frac{\left(\mathcal{A}\mathcal{V}' - 2\mathcal{V}\mathcal{A}'\right)}{\mathcal{A}^2\sqrt{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}} = 0 \quad \Leftrightarrow \quad \frac{\mathcal{A}\mathcal{I}_2'}{4\mathcal{F}} \equiv \frac{\left(\mathcal{A}\mathcal{V}' - 2\mathcal{V}\mathcal{A}'\right)}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2} = 0, \tag{70}$$

$$\frac{\left(\ln \mathcal{I}_1\right)'}{2\mathcal{I}_3'} \equiv \pm \frac{\left(\ln \mathcal{I}_1\right)'}{2\sqrt{\mathcal{F}}} \equiv \pm \frac{\left(2\mathcal{A}\alpha' - \mathcal{A}'\right)}{\sqrt{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2}} = 0$$
$$\Leftrightarrow \quad \frac{\left(\ln \mathcal{I}_1\right)'}{4\mathcal{A}\mathcal{F}} \equiv \frac{\left(2\mathcal{A}\alpha' - \mathcal{A}'\right)}{2\mathcal{A}\mathcal{B} + 3\left(\mathcal{A}'\right)^2} = 0. \tag{71}$$

The expressions on the left of both (70)–(71) are invariants in the spirit of subsection 2.3, and therefore their numerical value does not depend on the parametrization. On the right of (70)–(71) are the vanishing source conditions (45)–(46) respectively. Hence we see that if these conditions hold in one parametrization then they hold in any other. In what follows, the left hand sides of (70)–(71) are referred to as the 'invariant vanishing source conditions'.

The derivative of the scalar field Φ with respect to the spacetime coordinate transforms as follows: $\nabla_\mu \Phi = \bar{f}' \bar{\nabla}_\mu \bar{\Phi}$. Hence for the regular case (68) if $\nabla_\mu \Phi = 0$ then also $\bar{\nabla}_\mu \bar{\Phi} = 0$. For the singular case (69) the latter does not have to be zero since $\bar{f}' = 0$. Hence it might seem that the notion of the GR regime, i.e. $\Phi = \Phi_0$, $\nabla_\mu \Phi\big|_{\Phi_0} = 0$, is not invariant. Nevertheless, taking into account the equation of motion (44) for the scalar field allows us to overcome this problem. Namely, let us proceed by considering the transformation

$$\nabla_\mu \nabla_\nu \Phi = \bar{f}' \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\Phi} + \bar{f}'' \bar{\nabla}_\mu \bar{\Phi} \bar{\nabla}_\nu \bar{\Phi} + \bar{f}' \bar{\gamma}' \left(\bar{g}_{\mu\nu} \bar{g}^{\sigma\rho} \bar{\nabla}_\sigma \bar{\Phi} \bar{\nabla}_\rho \bar{\Phi} - 2 \bar{\nabla}_\mu \bar{\Phi} \bar{\nabla}_\nu \bar{\Phi} \right). \tag{72}$$

Again, for the regular case (68) if $\nabla_\mu \Phi = 0$ and $\nabla_\mu \nabla_\nu \Phi = 0$ then also $\bar{\nabla}_\mu \bar{\Phi} = 0$ and $\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\Phi} = 0$. However, in the singular case $\nabla_\mu \nabla_\nu \Phi = 0$ implies $\bar{\nabla}_\mu \bar{\Phi} = 0$ because $\bar{f}'' \neq 0$. Therefore $\nabla_\mu \Phi = 0$ is preserved for both the regular (68) and the singular (69) transformation. Last but not least, as all other terms in (44) are zero, $\bar{\Box} \bar{\Phi}$ must also vanish. The latter is automatically fulfilled if $\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\Phi} = 0$ and we conclude that the GR regime is preserved under the local Weyl rescaling (20) and under the scalar field redefinition (21).

Let us point out that in the singular case (69) the scalar field value $\Phi_\star$, defined by (48), is transformed into $\Phi_\bullet$, given by (47). However, the vanishing source conditions (45)–(46) are fulfilled for both cases. Also note that due to the invariant vanishing source conditions (70)–(71)

$$\left\{ \Phi_\star \right\} \cup \left\{ \Phi_\bullet \right\} = \bar{f} \left(\left\{ \bar{\Phi}_\star \right\} \cup \left\{ \bar{\Phi}_\bullet \right\} \right) \tag{73}$$

holds. In the following we will not distinguish between elements of the same set.

3.2.4. Transformation of the hyperbolic critical points. Let us consider the transformation of the first order small source conditions (49)–(50). Here we do not provide a thorough analysis but rather give an insightful explanation. The condition (49) is discussed in more detail in section 4.

We start by pointing out that the conditions under consideration are not given by invariants in the sense of subsection 2.3 but they are closely related to the following invariant objects

$$\frac{1}{\mathcal{I}_3'}\left(\frac{\mathcal{I}_2'}{\mathcal{I}_3'}\right)' \equiv \frac{1}{\pm\sqrt{\mathcal{F}}}\left(\frac{\mathcal{I}_2'}{\pm\sqrt{\mathcal{F}}}\right)' = \frac{\mathcal{I}_2''}{\mathcal{F}} + \frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'\mathcal{I}_2', \tag{74}$$

$$\frac{1}{\mathcal{I}_3'}\left(\frac{(\ln \mathcal{I}_1)'}{\mathcal{I}_3'}\right)' \equiv \frac{1}{\pm\mathcal{F}}\left(\frac{(\ln \mathcal{I}_1)'}{\pm\sqrt{\mathcal{F}}}\right)' = \frac{(\ln \mathcal{I}_1)''}{\mathcal{F}} + \frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'(\ln \mathcal{I}_1)'. \tag{75}$$

In principle we have taken the derivative of the invariant vanishing source conditions (70)–(71) w.r.t. the invariant $\mathcal{I}_3$ [30] and in the following we shall refer to (74)–(75) as the 'invariant first order small source conditions'. The quantities (74)–(75) differ in principle from the (noninvariant) first order small source conditions (49)–(50) only by a factor $\frac{1}{2}$ in front of the second additive term on the rhs. As mentioned in the discussion after (50) one of the additive terms must be zero in the GR regime. The same holds for (74)–(75). We hence conclude that (49)–(50) are nonvanishing (vanishing) if and only if the invariants (74)–(75) respectively are nonvanishing (vanishing). In other words, if the necessary conditions (49)–(50) are fulfilled in one parametrization then they are also satisfied in any other.

Therefore from (51)–(52) we obtain the following: if in the JF BDBW parametrization (2) $\left(\frac{1}{2\omega+3}\right)' \neq 0$ then also for the same theory written in the EF canonical parametrization (3) $\alpha'' \neq 0$.

4. Dynamical system in the Friedmann cosmology

The aim of this section is to work through a relatively simple example in the framework of the Friedmann cosmology (see subsubsection 2.1.3) in order to prove the following. Let us consider the GR regime as a hyperbolic critical point in the context of the dynamical systems approach. The qualitative behaviour of the critical point is determined by invariants, and therefore whether the theory under consideration converges to GR or repels from it does not depend on the chosen parametrization. In order to show the nontriviality of the transformations we provide a lot of calculational details.

4.1. Theory: part III

Let us start by using the notation of the current paper to rewrite the approach formulated in [35, 36]. Our focus is upon the transformation properties. Therefore, in order to make our calculations less lengthy and more transparent we truncate the physical side of the theory by considering flat Friedmann cosmology without matter, i.e., $k = 0$ and $T_{\mu\nu} = 0$. Note that this entails the dropping of the coupling function α from the theory. Due to the latter, the notion of the GR regime differs slightly from the one used in [35] because one of the vanishing source conditions, namely (46), is no longer needed.

4.1.1. Critical points for potential $\mathcal{V}$ dominated Universe with $\rho = 0, k = 0$. We want to study the scalar field equation (18) as a dynamical system in order to write out the critical points and study their properties. Let us follow the well-known prescription: solving the Friedmann constraint equation (16) as a quadratic equation for H and plugging the answer into the scalar field equation of motion (18). The resulting equation reads

$$\ddot{\Phi} = \frac{1}{2}\frac{\mathcal{A}'}{\mathcal{A}}\dot{\Phi}^2 + \frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'\mathcal{F}\dot{\Phi}^2 - \dot{\Phi}\varepsilon\sqrt{3\mathcal{F}\dot{\Phi}^2 + 3\ell^{-2}\mathcal{I}_2\mathcal{A}} - \frac{\mathcal{A}}{2\ell^2}\frac{\mathcal{I}_2'}{\mathcal{F}} \tag{76}$$

where $\varepsilon = +1$ ($\varepsilon = -1$) corresponds to the positive (negative) solution of (16) as a quadratic equation for H, i.e. in principle to the expanding (contracting) Universe. Analogically to (44) we made use of the invariants, defined by (40), and $\mathcal{F}$, given by (41), in order to write (76) in a more compact form. For a critical point one must impose $\dot{\Phi} = 0$. For the latter to be sustained $\ddot{\Phi} = 0$ must also hold, i.e. the rhs of (76) must vanish. Hence the critical point corresponds to the GR regime discussed in section 3. In the context of the latter, the scalar field equation (76) describes a process that may approach this regime. In the current case we have omitted the influence of α and therefore the condition (46) and the equivalent invariant condition (71) are no longer needed. Hence the GR regime ($\nabla_\mu \Phi = 0$ and the vanishing source conditions (45)–(46)) reduce to

$$\dot{\Phi} = 0, \qquad \frac{\mathcal{I}_2'}{\mathcal{F}} = 0. \tag{77}$$

Therefore, analogically to (47)–(48), while also taking into account the first order small source condition (49), we distinguish the scalar field values as

$$\Phi_\bullet: \quad \mathcal{I}_2'\big|_{\Phi_\bullet} = 0 \quad \text{and} \quad \mathcal{I}_2''\big|_{\Phi_\bullet} \neq 0 \quad \text{and} \quad \frac{1}{\mathcal{F}}\bigg|_{\Phi_\bullet} \neq 0, \tag{78}$$

$$\Phi_\star: \quad \frac{1}{\mathcal{F}}\bigg|_{\Phi_\star} = 0 \quad \text{and} \quad \left(\frac{1}{\mathcal{F}}\right)'\bigg|_{\Phi_\star} \neq 0 \quad \text{and} \quad \mathcal{I}_2'\big|_{\Phi_\star} \neq 0. \tag{79}$$

It is well known that for the scalar field value $\Phi_\bullet$ the equation (76) can be rewritten as an ordinary dynamical system [33, 43, 49], but the case corresponding to $\Phi_\star$ must be studied more carefully. Namely, if $\mathcal{F}$ diverges then (77) gives necessary but insufficient conditions. From (76) one can see that $\ddot{\Phi}|^{\Phi=\Phi_\star}_{\dot{\Phi}=0} = 0$ can only be achieved if for a trajectory of a specific solution under consideration the limiting value

$$\lim_{\substack{\Phi\to\Phi_\star \\ \dot{\Phi}\to 0}} \left\{ \left(\frac{1}{\mathcal{F}}\right)' \mathcal{F}\dot{\Phi}^2 \right\} = 0 \tag{80}$$

holds. In the case when the latter is violated, a trajectory passes the point in the phase space where the conditions given by (77) are fulfilled, i.e. the critical point. Hence, e.g. for an attractive critical point the limiting value (80) restricts the 'final' part of the trajectory—i.e. the order of the magnitude of $\dot{\Phi}$ relative to $\Phi - \Phi_\star$ in processes where the scalar field Φ evolves toward $\Phi_\star$—and stops there. Taking into account the knowledge of (61) we ascertain that the expression under the limiting value (80) is equivalent to $\frac{\dot{x}^2}{x}$. Therefore, it necessarily holds up to $\dot{\Phi}$ being the same or higher order small compared to $\Phi - \Phi_\star$ [33].

4.1.2. Perturbed equation. Let us introduce the following notation for small perturbations

$$x \equiv \Phi - \Phi_0, \tag{81}$$

$$\dot{x} \equiv \dot{\Phi} \tag{82}$$

where Φ_0 is defined by either (78) or by (79).

Let us first write out the first order perturbed approximation of (76) as follows

$$E^{(x)} \equiv -\ddot{x} + \frac{M}{2}\frac{\dot{x}^2}{x} - C_1^\varepsilon \cdot \dot{x} + C_2 \cdot x = 0. \tag{83}$$

While deriving the first order perturbed equation (83) for (76) we dropped the first term on the rhs of (76) because it is definitely a higher order term. For the first order approximation of the third and the fourth term on the rhs of (76) we use the Taylor expansion and the following constants

$$C_1^{\varepsilon} \equiv \frac{\varepsilon}{\ell}\sqrt{3\mathcal{I}_2\mathcal{A}}\Big|_{x=0}; \qquad C_2 \equiv -\left(\frac{\mathcal{A}}{2\ell^2}\frac{\mathcal{I}_2'}{\mathcal{F}}\right)'\Bigg|_{x=0} \equiv -\frac{\mathcal{A}}{2\ell^2}\Big|_{x=0}\lim_{x\to 0}\left[\frac{\mathcal{I}_2'}{\mathcal{F}}\Big/ x\right]. \tag{84}$$

The last equality for C_2 holds because if $x \to 0$, then due to the critical point condition (77) the numerator vanishes and we can make use of the l'Hospital rule. The constant C_2 is the same as that defined by the first order small source condition (49).

The second term on the rhs of (76) is a bit tricky. Namely, if $\mathcal{F}$ is finite then this is already a higher order term and we drop it, but if $\mathcal{F}$ diverges then calculating the Taylor expansion introduces coefficients that depend on the ratio $\frac{\dot{x}}{x}$. The latter is clearly something we want to avoid because the properties of a critical point should not depend on the choice of the trajectory. We instead make use of the knowledge obtained by the second remark (61) in subsubsection 3.1.3. Hence, due to the nonvanishing $\left(\frac{1}{\mathcal{F}}\right)'$ assumption (55), if $\mathcal{F}$ diverges then

$$\frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'\mathcal{F}\dot{\Phi}^2 \sim \frac{1}{2}\frac{\dot{x}^2}{x} \tag{85}$$

holds. In the diverging $\mathcal{F}$ case, this is the first order approximation to the limiting value (80). In order to capture these two possibilities ($\mathcal{F} < \infty$ and $\mathcal{F} \to \infty$) in one equation we substitute the second term on the rhs of the scalar field equation (76) by

$$\frac{M}{2}\frac{\dot{x}^2}{x}, \qquad M \equiv \lim_{x\to 0}\left\{\left(\frac{1}{\mathcal{F}}\right)'\mathcal{F}\cdot x\right\} \tag{86}$$

where M, introduced in (61), is a constant with the following properties. If $\mathcal{F} < \infty$ then $M = 0$ and the term vanishes but if $\mathcal{F} \to \infty$ then $M = 1$ and the term survives. Hence, we conclude that (83) is the first order approximation of (76).

4.1.3. Linearized equation. We have obtained a first order approximated equation (83), but in the case of the $\Phi_\star$ critical point (79), where $\mathcal{F}$ diverges, this is a nonlinear equation and we cannot apply the usual methods of dynamical systems directly. However, it turns out that (83) contains a hidden linearity. Let us make use of the coordinate transformation that was proposed in [35]

$$\tilde{x} \equiv \pm\frac{x}{|x|^{\frac{M}{2}}} \tag{87}$$

where the meaning of $\pm$ becomes clear later. The derivatives of $\tilde{x}$ with respect to cosmological time t read

$$\dot{\tilde{x}} = \pm\frac{\dot{x}}{|x|^{\frac{M}{2}}}\left(\frac{2-M}{2}\right), \qquad \ddot{\tilde{x}} = \pm\left(\ddot{x} - \frac{M}{2}\frac{\dot{x}^2}{x}\right)\frac{1}{|x|^{\frac{M}{2}}}\frac{2-M}{2}. \tag{88}$$

The obtained results can be used to rewrite (83) as

$$\mp\frac{2}{2-M}\,|x|^{\frac{M}{2}}\left\{\ddot{\tilde{x}} + C_1^{\varepsilon}\dot{\tilde{x}} - \frac{2-M}{2}C_2\tilde{x}\right\} = 0. \tag{89}$$

We are not interested in the trivial solution $x \equiv 0$. Therefore the expression in curly brackets must be equal to zero and this is a linear equation which can be written as an ordinary dynamical system

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{2-M}{2} C_2 & -C_1^{\varepsilon} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \tag{90}$$

where $\tilde{y} \equiv \dot{\tilde{x}}$. In the following we shall use 'linearized equation' to refer to the dynamical system (90) or equivalently to the underlying expression in the curly brackets of (89). The term 'linear equation' is used to denote the perturbed equation (83) in the case when $M = 0$. Note that for the latter the coordinate transformation (87) is actually an identity transformation up to a sign. Hence, for the case $M = 0$ the linear equation and the linearized equation coincide. The solutions and hence also the properties of the critical point are now determined by the eigenvalues of the matrix that contains the constant coefficients.

4.1.4. Solutions. The eigenvalues of the square matrix in (90) are

$$\lambda_{\pm}^{\varepsilon} = \frac{1}{2}\left(-C_1^{\varepsilon} \pm \sqrt{\left(C_1^{\varepsilon}\right)^2 + 2(2-M)C_2} \right). \tag{91}$$

The latter can be used to write out the solution for $\tilde{x}$. In order to determine the behaviour of $x \equiv \Phi - \Phi_0$, we invert the relation (87) as

$$x = \pm \left| \tilde{x} \right|^{\frac{2}{2-M}}. \tag{92}$$

If the eigenvalues λ_+ and λ_- are different then the solution for x reads

$$x(t) = \pm \left(K_1 \mathrm{e}^{\lambda_+^{\varepsilon} t} + K_2 \mathrm{e}^{\lambda_-^{\varepsilon} t} \right)^{\frac{2}{2-M}} \tag{93}$$

where K_1 and K_2 are integration constants. In principle the same result was obtained in [30]. Due to the power $\frac{2}{2-M}$ the theory under consideration is indeed endowed with a mechanism called for in the first remark of subsubsection 3.1.3. Namely, the diverging $\mathcal{F}$ implies $M = 1$ as mentioned in the discussion after (86), and in that case $x = \pm\tilde{x}^2$. Therefore we have an encoded mechanism, which due to the nonvanishing $\left(\frac{1}{\mathcal{F}}\right)'$ assumption (55) does not allow us to violate the positive $\mathcal{F}$ assumption (53). In the case $M = 0$ the power $\frac{2}{2-M} = 1$ because, as mentioned earlier, the coordinate transformation (87) is then an identity transformation up to a sign.

The $\pm$ in coordinate transformation (87) can now be reasoned as follows: if $M = 0$ then it does not matter whether $x = \tilde{x}$ or $x = -\tilde{x}$, but if $M = 1$ then in the spirit of the first remark in subsubsection 3.1.3 one must have $x \geqslant 0$ ($x \leqslant 0$) if $\left(\frac{1}{\mathcal{F}}\right)' > 0$ $\left(\left(\frac{1}{\mathcal{F}}\right)' < 0\right)$. The same applies to the solution (93) and there the sign '+' ('−') corresponds to $\left(\frac{1}{\mathcal{F}}\right)' > 0$ $\left(\left(\frac{1}{\mathcal{F}}\right)' < 0\right)$. Because of the previous reasoning we have also dropped the absolute value in (93).

Let us point out that in the case of the diverging $\mathcal{F}$ if the nonvanishing $\left(\frac{1}{\mathcal{F}}\right)'$ assumption (55) is not fulfilled then $C_2 = 0$, as can be read out from (84) or equivalently from (49). Therefore, one eigenvalue is zero and the critical point is nonhyperbolic. We conclude that the first order small source conditions (49)–(50) in subsubsection 3.1.2 are indeed well motivated.

4.2. Transformations: part III

4.2.1. Transformation of the perturbed equation. In order to study the transformation of the first order perturbed equation (83), let us first consider the transformation of x, $\dot{x}$ and $\ddot{x}$. For the latter, we take the definitions (81)–(82), write these in terms of the 'barred' quantities—i.e. make use of (21), (39), also taking into account the cosmological time transformation (35)

—and then use the Taylor expansion around $\bar{x} = 0$, $\dot{\bar{x}} = 0$ and $\ddot{\bar{x}} = 0$ in order to obtain polynomials with respect to $\bar{x}$, $\dot{\bar{x}}$ and $\ddot{\bar{x}}$. At the moment we shall keep the terms up to the second order for reasons that will become clear soon:

$$x \equiv \Phi - \Phi_0 = \bar{f}\left(\bar{\Phi}\right) - \bar{f}\left(\bar{\Phi}\right)|_0 \approx \left[\bar{f}'\right]_0 \bar{x} + \frac{1}{2}\left[\bar{f}''\right]_0 \bar{x}^2, \tag{94}$$

$$\dot{x} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi - \Phi_0\right) = \mathrm{e}^{-\bar{\gamma}}\bar{f}'\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}\bar{t}} \approx \left[\mathrm{e}^{-\bar{\gamma}}\bar{f}'\right]_0 \dot{\bar{x}} + \left[\mathrm{e}^{-\bar{\gamma}}\bar{f}''\right]_0 \bar{x}\dot{\bar{x}} - \left[\bar{\gamma}'\mathrm{e}^{-\bar{\gamma}}\bar{f}'\right]_0 \bar{x}\dot{\bar{x}}, \tag{95}$$

$$\begin{aligned} \ddot{x} &\equiv \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi - \Phi_0\right) = \mathrm{e}^{-2\bar{\gamma}}\bar{f}'\ddot{\bar{x}} + \mathrm{e}^{-2\bar{\gamma}}\bar{f}''\dot{\bar{x}}^2 - \mathrm{e}^{-2\bar{\gamma}}\bar{\gamma}'\bar{f}'\dot{\bar{x}}^2 \\ &\approx \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}'\right]_0 \ddot{\bar{x}} + \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}''\right]_0 \dot{\bar{x}}^2 - \left[\mathrm{e}^{-2\bar{\gamma}}\bar{\gamma}'\bar{f}'\right]_0 \dot{\bar{x}}^2 \\ &\quad + \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}''\right]_0 \bar{x}\ddot{\bar{x}} - 2\left[\mathrm{e}^{-2\bar{\gamma}}\bar{\gamma}'\bar{f}'\right]_0 \bar{x}\ddot{\bar{x}} \end{aligned} \tag{96}$$

where $\bar{\Phi}_0 : \Phi_0 = \bar{f}\left(\bar{\Phi}_0\right)$. When calculating the transformation of the first order perturbed equation (83) we only keep the leading order terms. Therefore, for the regular case (68) we substitute as follows:

$$x = \left[\bar{f}'\right]_0 \bar{x}, \qquad \dot{x} = \left[\mathrm{e}^{-\bar{\gamma}}\bar{f}'\right]_0 \dot{\bar{x}}, \qquad \frac{\dot{x}^2}{x} = \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}'\right]_0 \frac{\dot{\bar{x}}^2}{\bar{x}}, \qquad \ddot{x} = \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}'\right]_0 \ddot{\bar{x}}. \tag{97}$$

However, in the singular case (69) the linear order coefficients vanish due to $\bar{f}'|_0 = 0$ and the leading order is actually quadratic. Hence, in the singular case (69) we substitute as

$$\begin{aligned} &x = \frac{1}{2}\left[\bar{f}''\right]_0 \bar{x}\bar{x}, \qquad \dot{x} = \left[\mathrm{e}^{-\bar{\gamma}}\bar{f}''\right]_0 \bar{x}\dot{\bar{x}}, \qquad \frac{\dot{x}^2}{x} = 2\left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}''\right]_0 \bar{x}\frac{\dot{\bar{x}}^2}{\bar{x}}, \\ &\ddot{x} = \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}''\right]_0 \bar{x}\ddot{\bar{x}} + \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}''\right]_0 \bar{x}\frac{\dot{\bar{x}}^2}{\bar{x}}. \end{aligned} \tag{98}$$

Let us point out that in the case of the singular transformation (69) the order of magnitude of the small perturbation $\Phi - \Phi_0$ changes, i.e. x that is first order small in its own parametrization is actually second order small with respect to $\bar{x}$. Also note that $\dot{x} = 0$ whenever $\bar{x} = 0$.

One can show that the coefficients M, C_1^{ε}, C_2, defined by (86) and (84) respectively, transform as follows

$$M = \bar{Q}_1\bar{M} + \bar{Q}_2, \tag{99}$$

$$C_1^{\varepsilon} = \mathrm{e}^{-\bar{\gamma}}\big|_{\bar{\Phi}_0}\bar{C}_1^{\varepsilon}, \tag{100}$$

$$C_2 = \mathrm{e}^{-2\bar{\gamma}}\big|_{\bar{\Phi}_0}\bar{Q}_1^{-1}\bar{C}_2 \tag{101}$$

where $\bar{Q}_1$ and $\bar{Q}_2$ are the limiting values

$$\bar{Q}_1 \equiv \lim_{\bar{\Phi}\to\bar{\Phi}_0}\left\{\frac{\bar{f}\left(\bar{\Phi}\right) - \bar{f}\left(\bar{\Phi}\right)|_0}{\bar{f}' \cdot \left(\bar{\Phi} - \bar{\Phi}_0\right)}\right\}, \tag{102}$$

$$\bar{Q}_2 \equiv \lim_{\bar{\Phi}\to\bar{\Phi}_0}\left\{\frac{2\bar{f}''\left(\bar{f}\left(\bar{\Phi}\right)-\bar{f}\left(\bar{\Phi}\right)|_0\right)}{\left(\bar{f}'\right)^2}\right\}. \tag{103}$$

The assumption that $\bar{f}(\bar{\Phi})$ itself is at least directionally continuous implies that if $\Phi \to \Phi_0$ $(x \to 0)$ then also $\bar{\Phi} \to \bar{\Phi}_0$ $(\bar{x} \to 0)$. Hence, when calculating the limiting values we do not actually have to be concerned about the limiting process itself. The limiting values (102)–(103) can be calculated, for example, by making use of the l'Hospital rule. The results read

$$\bar{Q}_1 = \begin{cases} 1 & \text{for the regular case (68)} \\ \frac{1}{2} & \text{for the singular case (69)'} \end{cases} \tag{104}$$

$$\bar{Q}_2 = \begin{cases} 0 & \text{for the regular case (68)} \\ 1 & \text{for the singular case (69)}\end{cases}. \tag{105}$$

Let us point out that based on the definition of M, given by (86), one can determine from table 1 that

$$\begin{array}{llll} M = 0 \text{ and } \bar{M} = 0 & \text{or} \quad M = 1 \text{ and } \bar{M} = 1 & \text{for the regular case (68),} & \\ M = 1 \text{ and } \bar{M} = 0 & & \text{for the singular case (69).} & (106)\end{array}$$

This result is consistent with the transformation rule (99) for M and the results (104)–(105).

By making use of the transformation rules (97), (99)–(101) and results (104)–(105) we ascertain that in the regular case, the transformation of the perturbed equation (83) reads

$$\begin{aligned} E^{(x)} &= -\ddot{x} + \frac{M}{2}\frac{\dot{x}^2}{x} - C_1^{\varepsilon}\dot{x} + C_2 x \\ &= \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}'\right]_0\left\{-\ddot{\bar{x}} + \frac{1}{2}\left(1\cdot\bar{M} + 0\right)\frac{\dot{\bar{x}}^2}{\bar{x}} - \bar{C}_1^{\varepsilon}\dot{\bar{x}} + 1\cdot\bar{C}_2\bar{x}\right\} = \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}'\right]_0 \bar{E}^{(\bar{x})}. \end{aligned} \tag{107}$$

Analogously by using (98) etc for the singular case (69) the transformation reads

$$\begin{aligned} E^{(x)} = \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}''\right]_0 \bar{x}\left\{-\ddot{\bar{x}} + \frac{2}{2}\left(\frac{1}{2}\cdot\bar{M} + 1 - 1\right)\frac{\dot{\bar{x}}^2}{\bar{x}}\right. \\ \left. - \bar{C}_1^{\varepsilon}\dot{\bar{x}} + \frac{1}{2}\cdot 2\bar{C}_2\bar{x}\right\} = \left[\mathrm{e}^{-2\bar{\gamma}}\bar{f}''\right]_0\bar{x}\bar{E}^{(\bar{x})}. \end{aligned} \tag{108}$$

Note that in the case of the singular transformation (69) a nonlinear equation $(M = 1)$ is transformed into a linear one $(\bar{M} = 0)$, but its structure is nevertheless preserved.

4.2.2. Transformation of the linearized equation. The transformation of the quantity $\tilde{x}$, defined by (87), reads

$$\tilde{x} \equiv \pm\frac{x}{|x|^{\frac{M}{2}}} = \pm\frac{\bar{f}\left(\bar{\Phi}\right)-\bar{f}\left(\bar{\Phi}\right)\Big|_0}{\left|\bar{f}\left(\bar{\Phi}\right)-\bar{f}\left(\bar{\Phi}\right)\Big|_0\right|^{\frac{M}{2}}} \equiv \pm\sqrt{\left|\bar{Q}_3\right|}\,\tilde{\bar{x}} \tag{109}$$

where $\bar{Q}_3$ is the limiting value

$$\bar{Q}_3 = \lim_{\bar{\Phi}\to\bar{\Phi}_0}\left[\frac{\left(\bar{f}(\bar{\Phi}) - \bar{f}(\bar{\Phi})\big|_0\right)^{2-M}}{\left(\bar{\Phi} - \bar{\Phi}_0\right)^{2-\bar{M}}}\right]. \tag{110}$$

The latter can easily be calculated by using the knowledge of (106) and the l'Hospital rule. The result reads

$$\bar{Q}_3 = \begin{cases} \left(\bar{f}'\right)^2\Big|_0 \text{ or } \bar{f}'\Big|_0 & \text{for the regular case (68)} \\ \frac{1}{2}\bar{f}''\Big|_0 & \text{for the singular case (69)} \end{cases}. \tag{111}$$

Therefore for both the regular (68) and the singular (69) case $\bar{Q}_3$ is nonvanishing and nondiverging. Therefore the order of magnitude of $\tilde{x}$ is preserved under both transformations.

Combining the transformation rules of M and C_2, given by (99) and (101) respectively, reveals

$$(2 - M)C_2 = \mathrm{e}^{-2\bar{\gamma}}\Big|_0\left(\frac{2 - \bar{Q}_2}{\bar{Q}_1} - \bar{M}\right)\bar{C}_2 = \mathrm{e}^{-2\bar{\gamma}}\Big|_0\left(2 - \bar{M}\right)\bar{C}_2. \tag{112}$$

Note that in the context of the invariant first order small source condition (74) the constant M effectively plays the role of $\frac{1}{2}$ which makes the difference between the invariant and noninvariant first order small conditions, given by (74) and (49), respectively. Namely, due to the definition (49) of C_2 and the GR regime conditions (77) (see also (45)) in the Friedmann cosmology

$$\begin{aligned}(2 - M)C_2 &= -\frac{\mathcal{A}}{2\ell^2}\left\{(2 - M)\underbrace{\frac{\mathcal{I}_2''}{\mathcal{F}}}_{=0 \text{ if } M=1} + \underbrace{(2 - M)\left(\frac{1}{\mathcal{F}}\right)'\mathcal{I}_2'}_{=0 \text{ if } M=0}\right\} \\ &= -\frac{\mathcal{A}}{\ell^2}\left(\frac{\mathcal{I}_2''}{\mathcal{F}} + \frac{1}{2}\left(\frac{1}{\mathcal{F}}\right)'\mathcal{I}_2'\right). \end{aligned} \tag{113}$$

Let us stress that neither C_2 nor M are defined via invariants. However, combining these two gives us an expression that only gains a finite multiplier under the local Weyl rescaling (20) and under the scalar field redefinition (21). As suggested by (113) the expression $(2 - M)C_2$ is practically the one introduced by the invariant first order small source condition (74).

The results (100), (109) and (112) impose that the expression in curly brackets in (89), hence also the linearized equation transforms as follows

$$\ddot{\tilde{x}} + C_1^{\varepsilon}\dot{\tilde{x}} - \frac{2 - M}{2}C_2\tilde{x} = \pm\sqrt{|Q_3|}\,\mathrm{e}^{-2\bar{\gamma}}\Big|_0\left\{\ddot{\bar{\tilde{x}}} + \bar{C}_1^{\varepsilon}\dot{\bar{\tilde{x}}} - \frac{2 - \bar{M}}{2}\bar{C}_2\bar{\tilde{x}}\right\}. \tag{114}$$

Let us study the transformation of $(2 - M)C_2$ in more detail. Instead of considering the intermediate results (99) and (101) let us write the quantity $(2 - M)C_2$ using the definitions for C_2, M and invariants, given by (84), (86) and (40) respectively, as follows

$$(2-M)C_2 = -\frac{\mathcal{A}}{2\ell^2}\bigg|_0 \lim_{x\to 0}\left\{\left(2-\left(\frac{1}{\left(\mathcal{I}_3'\right)^2}\right)'\left(\mathcal{I}_3'\right)^2 x\right)\left(\frac{\frac{\mathcal{I}_2'}{\left(\mathcal{I}_3'\right)^2}}{x}\right)\right\}$$

$$= -\frac{\mathcal{A}}{\ell^2}\bigg|_0 \left\{\frac{\mathcal{I}_2''\mathcal{I}_3'}{\left(\mathcal{I}_3'\right)^3} - \frac{\mathcal{I}_2'\mathcal{I}_3''}{\left(\mathcal{I}_3'\right)^3}\right\}\bigg|_0 = -\frac{\mathcal{A}}{\ell^2}\bigg|_0 \left\{\frac{1}{\mathcal{I}_3'}\left(\frac{\mathcal{I}_2'}{\mathcal{I}_3'}\right)'\right\}\bigg|_0. \tag{115}$$

Hence $(2-M)C_2$ is indeed the invariant first order small source condition (74). Such analogous procedures can also be carried through in the case of condition (46). The previous results suggest that including the nonlinear term $\frac{M}{2}\frac{\dot{x}^2}{x}$ is an inevitable step.

4.2.3. Transformation of the solutions. A straightforward calculation reveals that due to the previously given transformation rules (100) and (112), the transformation of the eigenvalues (91) reads

$$\lambda_\pm^\varepsilon = \mathrm{e}^{-\bar{\gamma}}\big|_0 \bar{\lambda}_\pm^\varepsilon. \tag{116}$$

The result (115) allows us to write the eigenvalues (91) as [30]

$$\lambda_\pm^\varepsilon = \left[\frac{\sqrt{\mathcal{A}}}{2\ell}\left(-\varepsilon\sqrt{3\mathcal{I}_2} \pm \sqrt{3\mathcal{I}_2 - 2\frac{1}{\mathcal{I}_3'}\left(\frac{\mathcal{I}_2'}{\mathcal{I}_3'}\right)'}\right)\right]_0 \tag{117}$$

making the transformation properties obvious.

In order to obtain the transformation of the solution (93)

$$\bar{x}\left(\bar{t}\right) = \pm\left(\bar{K}_1 \mathrm{e}^{\bar{\lambda}_+^\varepsilon \bar{t}} + \bar{K}_2 \mathrm{e}^{\bar{\lambda}_-^\varepsilon \bar{t}}\right)^{\frac{2}{2-\bar{M}}} \tag{118}$$

in addition to the eigenvalues, one must also consider the transformation (35) of the cosmological time t. In (35) only the transformation of the time coordinate differential is given. Here we are interested in the transformations calculated at the critical point. Hence, in the lowest approximation level when considering the integral $\int \mathrm{e}^{\bar{\gamma}}\mathrm{d}\bar{t}$ we may assume the scalar field to be approximately constant. In other words $t = \mathrm{e}^{\bar{\gamma}}|_0\,\bar{t}$. Therefore

$$t\cdot\lambda_\pm^\varepsilon = \bar{t}\cdot\bar{\lambda}_\pm^\varepsilon. \tag{119}$$

Hence the quantity $t\cdot\lambda_\pm^\varepsilon$, i.e. the power of the exponent in the solution (93), is transformed into itself.

Last but not least, we have to consider the transformation of the power $\frac{2}{2-M}$ of the solution (93). In the regular case (68) $M = \bar{M}$ and the power does not change. However, in the singular case (69) $M = \frac{1}{2}\bar{M} + 1$ and hence

$$\frac{2}{2-M} = \frac{2}{2-\frac{1}{2}\bar{M}-1} = 2\frac{2}{2-\bar{M}}. \tag{120}$$

Therefore, the power of the solution (93) for x is twice the one for $\bar{x}$. The latter is in perfect agreement with the transformation of the small perturbation in the singular case (69), i.e. (98) where it was pointed out that $x \sim \bar{x}^2$. The difference of the power is a mathematical artefact due to the mapping between nonlinear and linear approximate equations, both covered by (83). One should keep in mind that in the generic parametrization, the value of the scalar field Φ itself is not measurable and the physical meaning of the scalar field is not the same for different parametrizations. For example the JF BDBW parametrization (2) scalar field Ψ

encodes the nonminimal coupling $\Psi = \frac{1}{\mathcal{I}_1}$ while the EF canonical parametrization (3) scalar field φ encodes the scalar field space volume [30].

However, leaving aside the power, the characteristic behaviour of a solution, i.e. convergence to the GR regime or divergence from it, is determined by the eigenvalues (91) and is therefore preserved under the local Weyl rescaling of the metric tensor (20), and under the scalar field redefinition (21), even if the latter is singular.

5. Summary

We investigated first generation scalar–tensor theories of gravity, characterized by four arbitrary coupling functions $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha\}$ and invariant under the local rescaling of the metric and scalar field redefinition (20)–(21), (24). Our main focus was upon the GR regime where the scalar field evolution has ceased and the remaining dynamical degrees of freedom are identical to those of GR. It is well known that in the GR regime the scalar field redefinition (21) connecting the JF BDBW (2) and EF canonical (3) parametrization is singular. As we pointed out in the introduction, this singularity is physically meaningful and not due to an unfortunate choice of coupling functions. Therefore, for showing the equivalence of the parametrizations it is also important to study the transformation properties in the case of a singular scalar field redefinition.

In section 2 we started with the general action functional (1) and derived the equations of motion for the metric tensor $g_{\mu\nu}$ (8), for the scalar field Φ (13) and the matter continuity equation (14). Specifying the FLRW line element gave the equations of motion (16)–(19) in the Friedmann cosmology. By (28), (31), (34) we showed how under the transformations these basic equations gain an overall multiplicative term containing the transformation functions of the metric rescaling and of the scalar field redefinition. To facilitate further discussion we also recalled the invariants (40) introduced in our earlier paper [30].

Section 3 concentrated on the GR regime, defined by (43), (45)–(46). This definition is supplemented by assumptions (56)–(57) that enforce the consistent notion of the GR regime and complementary restrictions (53)–(55), (58)–(60) necessary for making the corresponding critical point hyperbolic. To satisfy these conditions, the allowed transformation functions fall into two cases, regular (68) and singular (69). These results were used to show that the notion of the GR regime is invariant under the local Weyl rescaling and the scalar field redefinition.

In section 4 we considered small perturbations of the scalar field (81) in the neighbourhood of the GR regime in the context of potential dominated Friedmann cosmology. It turned out that the perturbed equations (83) in different parametrizations were in correspondence, despite the fact that this equation itself might be nonlinear in one parametrization and linear in some other, related by a singular transformation (69) giving relations (98). For instance, the perturbed equation in the JF BDBW parametrization in the case when ω diverges is nonlinear, while the corresponding perturbed equation in the EF canonical parametrization is linear. These results complement our recent paper [30] where a slightly different approach was used. Last but not least, we showed that the qualitative behaviour of the solutions, i.e. whether the theory converges to the general relativity regime or repels from it is independent of the parametrization.

To sum up, we demonstrated that if the general relativity regime as a hyperbolic critical point is under consideration then there is an exact correspondence between different parametrizations, even if the scalar field redefinition connecting them is singular. However, in the latter case it is rather important to note that the order of magnitude of the small perturbation of the scalar field around some constant value changes under the singular scalar field redefinition as in (98).

From a more general viewpoint we have developed a methodology which rather rigorously allows us to check whether the imposed conditions are sufficient for establishing the equivalence of parametrizations. It would be interesting to study whether the correspondence is preserved if the conditions (49)–(50) leading to the hyperbolic critical point are loosened.

As an outlook it would also be interesting to study the transformation properties in the context of second and third generation scalar–tensor theories [7, 8] while generalizing the local Weyl rescaling (conformal transformation) of the metric tensor to a disformal transformation [55].

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