On automorphisms and focal subgroups of blocks

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Abstract

Given a p-block B of a finite group with defect group P and fusion system $\mathcal{F}$ on P we show that the rank of the group $P/\mathfrak{foc}(\mathcal{F})$ is invariant under stable equivalences of Morita type. The main ingredients are the $*$-construction, due to Broué and Puig, a theorem of Weiss on linear source modules, arguments of Hertweck and Kimmerle applying Weiss' theorem to blocks, and connections with integrable derivations in the Hochschild cohomology of block algebras.

1 Introduction

Throughout this paper, p is a prime, and $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal $J(\mathcal{O}) = \pi\mathcal{O}$ for some $\pi \in \mathcal{O}$, residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p, and field of fractions K of characteristic zero. For any $\mathcal{O}$-algebra A which is free of finite rank as an $\mathcal{O}$-module and for any positive integer r denote by $\mathrm{Aut}_r(A)$ the group of $\mathcal{O}$-algebra automorphisms α with the property that α induces the identity on $A/\pi^r A$, and denote by $\mathrm{Out}_r(A)$ the image of $\mathrm{Aut}_r(A)$ in the outer automorphism group $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ of A.

Given a finite group G, a *block of* $\mathcal{O}G$ is an indecomposable direct factor B of $\mathcal{O}G$ as an algebra. Any such block B determines a p-subgroup P of G, called a *defect group of* B. A primitive idempotent i in B^P such that $\mathrm{Br}_P(i) \neq 0$ is called a *source idempotent*; the choice of a source idempotent determines a fusion system $\mathcal{F}$ on P. We denote by $\mathfrak{foc}(\mathcal{F})$ the $\mathcal{F}$-focal subgroup of P; this is the subgroup of P generated by all elements of the form $\varphi(u)u^{-1}$, where $u \in P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle u\rangle, P)$. Clearly $\mathfrak{foc}(\mathcal{F})$ is a normal subgroup of P containing the derived subgroup of P.

If $\mathcal{O}$ is large enough, then the Broué-Puig $*$-construction in [5] induces an action of the group $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ on the set $\mathrm{Irr}_K(B)$ of irreducible K-valued characters of G associated with B, sending $\zeta \in \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ and $\chi \in \mathrm{Irr}_K(B)$ to $\zeta * \chi \in \mathrm{Irr}_K(B)$. The group $\mathrm{Out}(B)$ acts in the obvious way on $\mathrm{Irr}_K(B)$ by precomposing characters with automorphisms; that is, for $\alpha \in \mathrm{Aut}(B)$ and $\chi \in \mathrm{Irr}_K(B)$, viewed as a central function on B, the assignment $\chi^\alpha(x) = \chi(\alpha(x))$ for all $x \in G$ defines a character $\chi^\alpha \in \mathrm{Irr}_K(B)$ which depends only on the image of α in $\mathrm{Out}(B)$. See §2 below for more details and references.

Theorem 1.1. *Let G be a finite group. Let B be a block algebra of $\mathcal{O}G$ with a nontrivial defect group P, source idempotent $i \in B^P$ and associated fusion system $\mathcal{F}$ on P. Suppose that $\mathcal{O}$ contains a primitive $|G|$-th root of unity. Let τ_p be a primitive p-th root of unity in $\mathcal{O}$ and let m be the positive integer such that $\pi^m\mathcal{O} = (1-\tau_p)\mathcal{O}$. Let μ be the subgroup of $\mathcal{O}^\times$ generated by τ_p. There is a unique injective group homomorphism*

$$\Phi : \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times) \to \mathrm{Out}_1(B)$$

such that for any $\zeta \in \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ *the class* $\Phi(\zeta)$ *in* $\mathrm{Out}_1(B)$ *has a representative in* $\mathrm{Aut}_1(B)$ *which sends* ui *to* $\zeta(u)ui$ *for all* $u \in P$*. Moreover,* Φ *has the following properties.*

(i) For any $\zeta \in \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ *and any* $\chi \in \mathrm{Irr}_K(B)$ *we have* $\chi^{\Phi(\zeta)} = \zeta * \chi$*.*

(ii) If $\mathcal{O}$ *is finitely generated as a module over the ring of* p*-adic integers, then the group homomorphism* Φ *restricts to an isomorphism* $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mu) \cong \mathrm{Out}_m(B)$*.*

Remark 1.2. The group homomorphism Φ lifts the well-known action of $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ on $\mathrm{Irr}_K(B)$ via the $*$-construction. The existence of Φ as stated is a straightforward consequence of the hyperfocal subalgebra of a block. We will give in addition a proof which does not require the hyperfocal subalgebra, based on some more general statements on automorphisms of source algebras in section 3. The point of statement (ii) is that the left side in the isomorphism depends on the fusion system of B and the right side on the $\mathcal{O}$-algebra structure of B. The extent of the connections between these two aspects of block theory remains mysterious. Numerous 'local to global' conjectures predict that invariants of the fusion system of a block B should essentially determine invariants of the $\mathcal{O}$-algebra B, if not outright then up to finitely many possibilities. The 'global to local' direction is perhaps even less understood: does the $\mathcal{O}$-algebra structure of a block algebra determine the key invariants on the local side, such as defect groups, fusion systems, and possibly Külshammer-Puig classes?

Remark 1.3. With the notation above, the group $P/\mathfrak{foc}(\mathcal{F})$ has a topological interpretation: by [3, Theorem 2.5] this group is the abelianisation of the fundamental group of the p-completed nerve of a centric linking system of $\mathcal{F}$.

The subgroup $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mu)$ of $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ is isomorphic to the quotient of $P/\mathfrak{foc}(\mathcal{F})$ by its Frattini subgroup. Since $P/\mathfrak{foc}(\mathcal{F})$ is abelian, it follows that the rank of $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mu)$ is equal to the rank of $P/\mathfrak{foc}(\mathcal{F})$. Thus Theorem 1.1 has the following consequence.

Corollary 1.4. *Suppose that* $\mathcal{O}$ *is finitely generated as a module over the ring of* p*-adic integers. With the notation from 1.1, the group* $\mathrm{Out}_m(B)$ *is a finite elementary abelian* p*-group of rank equal to the rank of the abelian* p*-group* $P/\mathfrak{foc}(\mathcal{F})$*. In particular, if* $P/\mathfrak{foc}(\mathcal{F})$ *is elementary abelian, then* $\mathrm{Out}_m(B) \cong P/\mathfrak{foc}(\mathcal{F})$*.*

Combining Theorem 1.1 with invariance statements on the subgroups $\mathrm{Out}_m(B)$ from [13] yields the following statement.

Corollary 1.5. *Suppose that* $\mathcal{O}$ *is finitely generated as a module over the ring of* p*-adic integers. Let* G, G' *be finite groups, and let* B, B' *be block algebras of* $\mathcal{O}G$, $\mathcal{O}G'$ *with nontrivial defect groups* P, P' *and fusion systems* $\mathcal{F}$, $\mathcal{F}'$ *on* P, P'*, respectively. If there is a stable equivalence of Morita type between* B *and* B'*, then the ranks of the abelian* p*-groups* $P/\mathfrak{foc}(\mathcal{F})$ *and* $P'/\mathfrak{foc}(\mathcal{F}')$ *are equal.*

It remains an open question whether there is in fact an isomorphism $P/\mathfrak{foc}(\mathcal{F}) \cong P'/\mathfrak{foc}(\mathcal{F}')$ in the situation of this corollary. If P and P' are elementary abelian, this follows trivially from the above. In that case one can be slightly more precise, making use of the following well-known facts. The Hochschild cohomology in positive degrees of a block algebra is invariant under stable equivalences of Morita type. In particular, a stable equivalence of Morita type between two block algebras preserves the Krull dimensions of their Hochschild cohomology algebras over k, and these dimensions are equal to the rank of the defect groups. A stable equivalence of Morita type between two block algebras preserves also the order of the defect groups. A finite p-group which has the same order and rank as an elementary abelian p-group is necessarily elementary abelian as well.

Corollary 1.6. *Suppose that $\mathcal{O}$ is finitely generated as a module over the ring of p-adic integers. With the notation of 1.5, if there is a stable equivalence of Morita type between B and B' and if one of P, P' is elementary abelian, then there is an isomorphism $P \cong P'$ which induces isomorphisms $\mathfrak{foc}(\mathcal{F}) \cong \mathfrak{foc}(\mathcal{F}')$ and $P/\mathfrak{foc}(\mathcal{F}) \cong P'/\mathfrak{foc}(\mathcal{F}')$.*

The main ingredients for the proof of Theorem 1.1 are results of Puig on source algebras of blocks, a theorem of Weiss [20, Theorem 3], and results from Hertweck and Kimmerle [9].

Theorem 1.1 (ii) can be formulated in terms of integrable derivations, a concept due to Gerstenhaber [8], adapted to unequal characteristic in [13]. Let A be an $\mathcal{O}$-algebra such that A is free of finite rank as an $\mathcal{O}$-module. Let r be a positive integer and let $\alpha \in \mathrm{Aut}_r(A)$. Then $\alpha(a) = a + \pi^r\mu(a)$ for all $a \in A$ and some linear endomorphism μ of A. The endomorphism of $A/\pi^r A$ induced by μ is a derivation on $A/\pi^r A$. Any derivation on $A/\pi^r A$ which arises in this way is called *A-integrable.* The set of A-integrable derivations of $A/\pi^r A$ is an abelian group containing all inner derivations, hence determines a subgroup of $HH^1(A/\pi^r A)$, denoted $HH^1_A(A/\pi^r A)$. Note that π^r annihilates $HH^1(A/\pi^r A)$. Thus, if $p \in \pi^r A$, then $HH^1_A(A/\pi^r A)$ is an elementary abelian quotient of $\mathrm{Out}_r(A)$. See [13, §3] for more details.

Theorem 1.7. *Let G be a finite group. Let B a block of $\mathcal{O}G$ with a nontrivial defect group P and a fusion system $\mathcal{F}$ on P. Suppose that $\mathcal{O}$ contains a primitive $|G|$-th root of unity and that $\mathcal{O}$ is finitely generated as a module over the ring of p-adic integers. Denote by τ_p a primitive p-th root of unity in $\mathcal{O}$, and let m be the positive integer such that $\pi^m\mathcal{O} = (1-\tau_p)\mathcal{O}$. We have a canonical group isomorphism*

$$\mathrm{Out}_m(B) \cong HH^1_B(B/\pi^m B)\ .$$

In particular, $HH^1_B(B/\pi^m B)$ is a finite elementary abelian p-group of rank equal to the rank of $P/\mathfrak{foc}(\mathcal{F})$.

Remark 1.8. The group homomorphism Φ in Theorem 1.1 depends on the choice of P and i, but it is easy to describe the impact on Φ for a different choice. By [14, Theorem 1.2], if P' is another defect group of B and $i' \in B^{P'}$ a source idempotent, then there is $x \in G$ such that ${}^xP = P'$ and such that ${}^xi$ belongs to the same local point of P' on B as i'. Thus we may assume that $i' = {}^xi$. Conjugation by x sends the fusion system $\mathcal{F}$ on P determined by i to the fusion system $\mathcal{F}'$ on P' determined by the choice of i', hence induces group isomorphisms $\mathfrak{foc}(\mathcal{F}) \cong \mathfrak{foc}(\mathcal{F}')$ and $P/\mathfrak{foc}(\mathcal{F}) \cong P'/\mathfrak{foc}(\mathcal{F}')$. This in turn induces a group isomorphism $\nu : \mathrm{Hom}(P'/\mathfrak{foc}(\mathcal{F}'), \mathcal{O}^\times) \cong \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$. Precomposing Φ with ν yields the group homomorphism Φ' as in Theorem 1.1 for P', i', $\mathcal{F}'$ instead of P, i, $\mathcal{F}$, respectively. Indeed, if an automorphism β of B in $\mathrm{Aut}_1(B)$ sends ui to $\zeta(u)ui$, then conjugating β by x yields an automorphism β' in $\mathrm{Aut}_1(B)$ belonging to the same class as β in $\mathrm{Out}_1(B)$, and by construction, β' sends $u'i'$ to $\zeta'(u')u'i'$, where $u' \in P'$ and ζ' corresponds to ζ via ν.

2 Background material

2.1. The terminology on fusion systems required in this paper can be found in [2, Part I]. By a fusion system we always mean a saturated fusion system in the sense of [2, I.2.2]. For a broader treatment on fusion systems, see [6]. Given a fusion system $\mathcal{F}$ on a finite p-group P, the *focal subgroup of $\mathcal{F}$ in P* is the subgroup, denoted $\mathfrak{foc}(\mathcal{F})$, generated by all elements of the form $u^{-1}\varphi(u)$, where u is an element of a subgroup Q of P and where $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. The focal subgroup $\mathfrak{foc}(\mathcal{F})$ is normal in P and contains the derived subgroup $[P,P]$ of P; thus $P/\mathfrak{foc}(\mathcal{F})$ is abelian. The focal subgroup contains the *hyperfocal subgroup* $\mathfrak{hyp}(\mathcal{F})$ generated by all elements $u^{-1}\varphi(u)$ as above with the additional condition that φ has p'-order. We have $\mathfrak{foc}(\mathcal{F}) = \mathfrak{hyp}(\mathcal{F})[P,P]$. Both $\mathfrak{foc}(\mathcal{F})$ and $\mathfrak{hyp}(\mathcal{F})$ are not only normal in P but in fact stable under $\mathrm{Aut}_{\mathcal{F}}(P)$. A subgroup Q of P is called *$\mathcal{F}$-centric* if for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ we have $C_P(\varphi(Q)) = Z(\varphi(Q))$. As a consequence of Alperin's fusion theorem, $\mathfrak{foc}(\mathcal{F})$ is generated by all elements of the form $u^{-1}\varphi(u)$, where u is an element of an $\mathcal{F}$-centric subgroup Q of P and where $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. See [2, I, §7] for more details on focal and hyperfocal subgroups.

2.2. We describe in this paragraph the definition and properties of source algebras which we will need in this paper. For introductions to some of the required block theoretic background material, see for instance [19] and [2, Part IV]. We assume that k is large enough for the finite groups in this paragraph (this is to ensure that the fusion systems associated with blocks are indeed saturated and that hyperfocal subalgebras of source algebras exist).

Given a finite group G, a block of $\mathcal{O}G$ is an indecomposable direct factor B of $\mathcal{O}G$ as an $\mathcal{O}$-algebra. The unit element $b = 1_B$ of B is then a primitive idempotent in $Z(\mathcal{O}G)$, called the *block idempotent of* B. For P a p-subgroup of G we denote by $\mathrm{Br}_P : (\mathcal{O}G)^P \to kC_G(P)$ the *Brauer homomorphism* induced by the map sending $x \in C_G(P)$ to its image in $kC_G(P)$ and sending $x \in G \setminus C_G(P)$ to zero. This is a surjective algebra homomorphism. Thus $\mathrm{Br}_P(b)$ is either zero, or an idempotent in $kC_G(P)^{N_G(P)}$. If P is maximal subject to the condition $\mathrm{Br}_P(b) \neq 0$, then P is called a *defect group* of B. The defect groups of B are conjugate in G.

The condition $\mathrm{Br}_P(b) \neq 0$ implies that there is a primitive idempotent i in B^P such that $\mathrm{Br}_P(i) \neq 0$. The idempotent i is then called a *source idempotent of* B and the algebra $A = iBi = i\mathcal{O}Gi$ is then called a *source algebra of* B. We view A as an *interior P-algebra*; that is, we keep track of the image iP of P in A via the group homomorphism $P \to A^\times$ sending $u \in P$ to $ui = iu = iui$. This group homomorphism is injective and induces an injective algebra homomorphism $\mathcal{O}P$ which has a complement as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule, because A is projective as a left or right $\mathcal{O}P$-module. As an $\mathcal{O}(P \times P)$-module, A is a direct summand of $\mathcal{O}G$, and hence $i\mathcal{O}Gi$ is a permutation $\mathcal{O}(P \times P)$-module. The isomorphism class of A as an interior P-algebra is unique up to conjugation by elements in $N_G(P)$. By [14, 3.6] the source algebra A and the block algebra B are Morita equivalent via the bimodules $Bi = \mathcal{O}Gi$ and $iB = i\mathcal{O}G$. This Morita equivalence induces an isomorphism $\mathrm{Out}_1(A) \cong \mathrm{Out}_1(B)$; see Lemma 3.10 below for a more precise statement. The strategy to prove Theorem 1.1 is to construct a group homomorphism $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times) \to \mathrm{Out}_1(A)$ and then show that its composition with the isomorphism $\mathrm{Out}_1(A) \cong \mathrm{Out}_1(B)$ satisfies the conclusions of Theorem 1.1.

It follows from work of Alperin and Broué [1] that B determines a fusion system on any defect group P, uniquely up to conjugation. By work of Puig [15], every choice of a source algebra A determines a fusion system $\mathcal{F}$ on P. More precisely, the fusion system $\mathcal{F}$ is determined by the $\mathcal{O}P$-$\mathcal{O}P$-bimodule structure of A: every indecomposable direct summand of A as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule is isomorphic to $\mathcal{O}P_\varphi \otimes_{\mathcal{O}Q} \mathcal{O}P$ for some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$, and the morphisms in $\mathcal{F}$ which arise in this way generate $\mathcal{F}$. Here $\mathcal{O}P_\varphi$ is the $\mathcal{O}P$-$\mathcal{O}Q$-bimodule which is equal to $\mathcal{O}P$ as a left $\mathcal{O}P$-module, and on which $u \in Q$ acts on the right by multiplication with $\varphi(u)$. See [11, §7] for an expository account of this material. Fusion systems on a defect group P of B obtained from different choices of source idempotents are $N_G(P)$-conjugate.

By [17, Theorem 1.8], the source algebra A has, up to conjugation by P-stable invertible elements in A, a unique unitary P-stable subalgebra D, called the *hyperfocal subalgebra of $i\mathcal{O}Gi$*, such that $D \cap Pi = \mathfrak{hyp}(\mathcal{F})i$ and such that $A = \oplus_u Du$, with u running over a set of representatives in P of $P/\mathfrak{hyp}(\mathcal{F})$.

2.3. Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. In what follows the use of automorphisms as subscripts to modules is as in [10]. That is, for any $\alpha \in \mathrm{Aut}(A)$ and any A-module U we denote by ${}_\alpha U$ the A-module which is equal to U as an $\mathcal{O}$-module, with $a \in A$ acting as $\alpha(a)$ on U. If α is inner, then ${}_\alpha U \cong U$. We use the analogous notation for right modules and bimodules. If U and V are A-A-bimodules and $\alpha \in \mathrm{Aut}(A)$, then we have an obvious isomorphism of A-A-bimodules $(U_\alpha) \otimes_A V \cong U \otimes_A ({}_{\alpha^{-1}}V)$. We need the following standard fact (we sketch a proof for the convenience of the reader).

Lemma 2.4. *Let A be an $\mathcal{O}$-algebra and B a subalgebra of A. Let $\alpha \in \mathrm{Aut}(A)$ and let $\beta : B \to A$ be an $\mathcal{O}$-algebra homomorphism. The following are equivalent.*

(i) There is an automorphism α' of A which extends the map β such that α and α' have the same image in $\mathrm{Out}(A)$.

(ii) There is an isomorphism of A-B-bimodules $A_\beta \cong A_\alpha$.

(iii) There is an isomorphism of B-A-bimodules ${}_\beta A \cong {}_\alpha A$.

Proof. Clearly (i) implies (ii) and (iii). Suppose that (ii) holds. An A-B-bimodule isomorphism $\Phi : A_\beta \cong A_\alpha$ is in particular a left A-module automorphism of A, hence induced by right multiplication with an element $c \in A^\times$. The fact that Φ is also a homomorphism of right B-modules implies that $\beta(b)c = c\alpha(b)$ for all $b \in B$. Thus α' defined by $\alpha'(a) = c\alpha(a)c^{-1}$ for all $a \in A$ defines an automorphism of A which extends β and whose class in $\mathrm{Out}(A)$ coincides with that of α. Thus (ii) implies (i). A similar argument shows that (iii) implies (i). □

A frequently used special case of Lemma 2.4 (with $B = A$ and $\beta = \mathrm{Id}$) is that $A \cong A_\alpha$ as A-A-bimodules if and only if α is inner. Note that besides being an algebra automorphism, α is also an isomorphism of A-A-bimodules $A_{\alpha^{-1}} \cong {}_\alpha A$. Any A-A-bimodule of the form A_α for some $\alpha \in \mathrm{Aut}(A)$ induces a Morita equivalence on A, with inverse equivalence induced by $A_{\alpha^{-1}}$. An A-A-bimodule M which induces a Morita equivalence on $\mathrm{mod}(A)$ is of the form A_α for some $\alpha \in \mathrm{Aut}(M)$ if and only if $M \cong A$ as left A-modules, which is also equivalent to $M \cong A$ as right A-modules. This embeds $\mathrm{Out}(A)$ as a subgroup of $\mathrm{Pic}(A)$. This embedding identifies $\mathrm{Out}_r(A)$ with the kernel of the canonical homomorphism of Picard groups $\mathrm{Pic}(A) \to \mathrm{Pic}(A/\pi^r A)$, where r is a positive integer. See e. g. [7, §55 A] for more details.

2.5. Let A and B be $\mathcal{O}$-algebras which are free of finite rank as $\mathcal{O}$-modules. Let M be an A-B-bimodule such that M is finitely generated projective as a left A-module and as a right B-module. Let N be a B-A-bimodule which is finitely generated as a left B-module and as a right A-module. Following Broué [4] we say that *M and N induce a stable equivalence of Morita type between A and B* if we have isomorphisms $M \otimes_B N \cong B \oplus Y$ and $N \otimes_A M \cong A \oplus X$ as $B \otimes_\mathcal{O} B^{\mathrm{op}}$-modules and $A \otimes_\mathcal{O} A^{\mathrm{op}}$-modules, respectively, such that Y is a projective $B \otimes_\mathcal{O} B^{\mathrm{op}}$-module and X is a projective $A \otimes_\mathcal{O} A^{\mathrm{op}}$-module.

Theorem 2.6 ([10, Theorem 4.2], [13, Lemma 5.2])**.** *Let A, B be $\mathcal{O}$-algebras which are free of finite rank as $\mathcal{O}$-modules, such that the k-algebras $k \otimes_\mathcal{O} A$ and $k \otimes_\mathcal{O} B$ are indecomposable nonsimple* *selfinjective with separable semisimple quotients. Let r be a positive integer. Suppose that the canonical maps $Z(A) \to Z(A/\pi^r A)$ and $Z(B) \to Z(B/\pi^r B)$ are surjective. Let M be an A-B-bimodule and N a B-A-bimodule inducing a stable equivalence of Morita type between A and B. For any $\alpha \in \mathrm{Aut}_r(A)$ there is $\beta \in \mathrm{Aut}_r(B)$ such that ${}_{\alpha^{-1}}M \cong M_\beta$ as A-B-bimodules, and the correspondence $\alpha \mapsto \beta$ induces a group isomorphism $\mathrm{Out}_r(A) \cong \mathrm{Out}_r(B)$.*

If M and N induce a Morita equivalence, then the hypothesis on $k \otimes_{\mathcal{O}} A$ and $k \otimes_{\mathcal{O}} B$ being selfinjective with separable semisimple quotients is not needed; see the [13, Remark 5.4] for the necessary adjustments. The proof is a variation of [10, Theorem 4.2]; details can be found in [13, Lemma 5.2]. The surjectivity hypothesis for the map $Z(A) \to Z(A/\pi^r A)$ ensures that two automorphisms in $\mathrm{Aut}_r(A)$ represent the same class in $\mathrm{Out}(A)$ if and only if they differ by conjugation by an element in $1 + \pi^r A$; equivalently, we have $\mathrm{Inn}_r(A) = \mathrm{Inn}(A) \cap \mathrm{Aut}_1(A)$, where $\mathrm{Inn}_r(A)$ is the subgroup of $\mathrm{Inn}(A)$ consisting of automorphisms given by conjugation by elements in $1 + \pi^r A$; this follows from [13, 3.2]. We will use this fact without further reference.

Given two finite groups G, H, we consider any $\mathcal{O}G$-$\mathcal{O}H$-bimodule M as an $\mathcal{O}(G \times H)$-module via $(x, y) \cdot m = xmy^{-1}$, where $x \in G$, $y \in H$, $m \in M$. The easy proof of the following well-known lemma is left to the reader.

Lemma 2.7. *Let G be a finite group and $\zeta : G \to \mathcal{O}^\times$ a group homomorphism. Denote by $\mathcal{O}_\zeta$ the $\mathcal{O}G$-module which is equal to $\mathcal{O}$ as an $\mathcal{O}$-module and on which any $x \in G$ acts as multiplication by $\zeta(x)$. Set $\Delta G = \{(x, x) \mid x \in G\}$ and consider $\mathcal{O}_\zeta$ as a module over $\mathcal{O}\Delta G$ via the canonical group isomorphism $\Delta G \cong G$.*

(i) The $\mathcal{O}$-linear endomorphism η of $\mathcal{O}G$ defined by $\eta(x) = \zeta(x)x$ for all $x \in G$ is an $\mathcal{O}$-algebra automorphism of $\mathcal{O}G$, and the map $\zeta \mapsto \eta$ induces an injective group homomorphism $\mathrm{Hom}(G, \mathcal{O}^\times) \to \mathrm{Out}(\mathcal{O}G)$.

(ii) There is an isomorphism of $\mathcal{O}(G \times G)$-modules ${}_\eta\mathcal{O}G \cong \mathrm{Ind}_{\Delta G}^{G \times G}(\mathcal{O}_\zeta)$ which sends $x \in G$ to $\zeta(x^{-1})(x, 1) \otimes 1$.

3 On automorphisms of source algebras

Theorem 3.1. *Let G be a finite group and B a block of $\mathcal{O}G$ with a nontrivial defect group. Let i be a source idempotent in B^P and $A = iBi$ the corresponding source algebra of B with associated fusion system $\mathcal{F}$ on P. Assume that $\mathcal{O}$ contains a primitive $|G|$-th root of unity. Identify $\mathcal{O}P$ with its image in A. Let $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$.*

There is $\alpha \in \mathrm{Aut}_1(A)$ satisfying $\alpha(u) = \zeta(u)u$ for all $u \in P$ if and only if $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$. In that case the class of α in $\mathrm{Out}_1(A)$ is uniquely determined by ζ, and the correspondence $\zeta \mapsto \alpha$ induces an injective group homomorphism $\Psi : \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times) \to \mathrm{Out}_1(A)$.

We denote by $\mathrm{Aut}_P(A)$ the group of $\mathcal{O}$-algebra automorphisms of A which preserve the image of P in A elementwise; that is, $\mathrm{Aut}_P(A)$ is the group of automorphisms of A as an interior P-algebra. By a result of Puig [16, 14.9], the group $\mathrm{Aut}_P(A)$ is canonically isomorphic to a subgroup of the p'-group $\mathrm{Hom}(E, k^\times)$, where E is the inertial quotient of B associated with A.

Whenever convenient, we identify the elements in $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ with the subgroup of all $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$ satisfying $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$. Note that if $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$ and if η is the automorphism of $\mathcal{O}P$ defined by $\eta(u) = \zeta(u)u$ for all $u \in P$, then $\eta \in \mathrm{Aut}_1(\mathcal{O}P)$ because the image in k of any p-power root of unity in $\mathcal{O}$ is equal to 1_k.

The fastest way to show the existence of α as stated uses the hyperfocal subalgebra. We state this as a separate lemma, since we will give at the end of this section a second proof of this fact which does not require the hyperfocal subalgebra.

Lemma 3.2. *Let $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$ such that $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$. Then there is $\alpha \in \mathrm{Aut}_1(A)$ such that $\alpha(u) = \zeta(u)u$ for all $u \in P$.*

Proof. Let D be a hyperfocal subalgebra of the source algebra A of B. That is, D is a P-stable subalgebra of A such that $D \cap Pi = \mathfrak{hyp}(\mathcal{F})$ and $A = \oplus_u Du$, where u runs over a set of representatives of $P/\mathfrak{hyp}(\mathcal{F})$ in P. For $d \in D$ and $u \in P$ define $\alpha(du) = \zeta(u)du$. Since $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$, this is well-defined, and extends linearly to A. A trivial verification shows that this is an $\mathcal{O}$-algebra automorphism of A which acts as the identity on D. The image of the p-power root of unity $\zeta(u)$ in k is 1, and hence $\alpha \in \mathrm{Aut}_1(A)$. □

Lemma 3.3. *Let $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$ such that there exists $\alpha \in \mathrm{Aut}(A)$ satisfying $\alpha(u) = \zeta(u)u$ for all $u \in P$. Then $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$.*

Proof. Let Q be an $\mathcal{F}$-centric subgroup of P and $\varphi \in \mathrm{Aut}(Q)$. We need to show that $u^{-1}\varphi(u) \in \ker(\zeta)$ for any $u \in Q$. Since Q is $\mathcal{F}$-centric, it follows from [19, (41.1)] that Q has a unique local point δ on A. Let $\alpha \in \mathrm{Aut}(A)$ such that $\alpha(u) = \zeta(u)u$ for all $u \in P$. Note that u and $\alpha(u)$ act in the same way by conjugation on A because they differ by a scalar. Thus α induces an automorphism on A^Q which preserves the ideals of relative traces A_R^Q, where R is a subgroup of Q. It follows that α permutes the points of Q on A preserving the property of being local. The uniqueness of δ implies that $\alpha(\delta) = \delta$. Let $j \in \delta$. Thus $\alpha(j) = d^{-1}jd$ for some $d \in (A^Q)^\times$. After replacing α, if necessary, we may assume that α fixes j and still satisfies $\alpha(u) = \zeta(u)u$ for all $u \in Q$. By [15, 2.12, 3.1] there exists $c \in A^\times$ such that ${}^c(uj) = \varphi(u)j$ for all $u \in Q$. Applying α to this equation yields

$$ {}^{\alpha(c)}\zeta(u)uj = \zeta(\varphi(u))\varphi(u)j $$

for all $u \in Q$. Conjugating by c^{-1} and multiplying by $\zeta(u^{-1})$ yields

$$ {}^{c^{-1}\alpha(c)}uj = \zeta(u^{-1}\varphi(u))uj $$

for all $u \in Q$. Set $\kappa(u) = \zeta(u^{-1}\varphi(u))$. This defines a group homomorphism $\kappa : Q \to \mathcal{O}^\times$. We need to show that κ is the trivial group homomorphism. Since $c^{-1}\alpha(c)$ centralises j, it follows that the element $w = c^{-1}\alpha(c)j$ belongs to $(jAj)^\times$. Moreover, by the above, conjugation by w on jAj induces an inner automorphism σ of jAj whose restriction to $\mathcal{O}Q$ (identified with its image $\mathcal{O}Qj$ in jAj) is the automorphism θ of $\mathcal{O}Q$ defined by $\theta(u) = \kappa(u)u$ for all $u \in Q$. Since σ is inner, we have $jAj_\sigma \cong jAj$ as jAj-jAj-bimodules. Thus we have $jAj_\theta \cong jAj$ as $\mathcal{O}Q$-$\mathcal{O}Q$-bimodules. In particular, jAj_θ is a permutation $\mathcal{O}(Q \times Q)$-module. Since $\mathrm{Br}_Q(j) \neq 0$, it follows that $\mathcal{O}Q$ is a direct summand of jAj as an $\mathcal{O}(Q \times Q)$-module. Thus $\mathcal{O}Q_\theta$ is a direct summand of jAj_θ as an $\mathcal{O}(Q \times Q)$-module. But $\mathcal{O}Q_\theta$ is a permutation $\mathcal{O}(Q \times Q)$-module if and only if $\kappa = 1$. The result follows. □

The two Lemmas 3.2 and 3.3 prove the first statement of Theorem 3.1. The following lemmas collect the technicalities needed for proving the remaining statements of Theorem 3.1.

Lemma 3.4. *Let $\alpha \in \mathrm{Aut}(A)$. Then A_α is isomorphic to a direct summand of $A_\alpha \otimes_{\mathcal{O}P} A$ as an A-A-bimodule.*

Proof. By [12, 4.2], A is isomorphic to a direct summand of $A \otimes_{\mathcal{O}P} A$. Tensoring by $A_\alpha \otimes_A -$ yields the result. □

Lemma 3.5. *Let $\alpha \in \mathrm{Aut}_1(A)\cdot\mathrm{Inn}(A)$ such that A_α is isomorphic to a direct summand of $A\otimes_{\mathcal{O}P} A$ as an A-A-bimodule. Then $\alpha \in \mathrm{Inn}(A)$.*

Proof. Tensoring A_α and $A \otimes_{\mathcal{O}P} A$ by $Bi \otimes_A - \otimes_A iB$ implies that $Bi_\alpha \otimes_A iB$ is isomorphic to a direct summand of $Bi \otimes_{\mathcal{O}P} iB$. This shows that $Bi_\alpha \otimes_A iB$ is a p-permutation $\mathcal{O}(G \times G)$-module. Since $\alpha \in \mathrm{Aut}_1(A) \cdot \mathrm{Inn}(A)$, it follows that α induces an inner automorphism on $k \otimes_{\mathcal{O}} A$. Thus $k \otimes_{\mathcal{O}} Bi_\alpha \otimes_A iB \cong k \otimes_{\mathcal{O}} Bi \otimes_A iB$. The fact that p-permutation $k(G \times G)$-modules lift uniquely, up to isomorphism, to p-permutation $\mathcal{O}(G \times G)$-modules implies that $Bi_\alpha \otimes_A iB \cong Bi \otimes_A iB$ as $\mathcal{O}(G \times G)$-modules. Mutliplying both modules on the left and on the right by i implies that $A_\alpha \cong A$ as A-A-bimodules, and hence α is inner. □

Lemma 3.6. *We have* $\mathrm{Aut}_P(A) \cap (\mathrm{Aut}_1(A) \cdot \mathrm{Inn}(A)) \leq \mathrm{Inn}(A)$.

Proof. Let $\alpha \in \mathrm{Aut}_P(A) \cap (\mathrm{Aut}_1(A) \cdot \mathrm{Inn}(A))$. By 3.4, A_α is a isomorphic to a direct summand of $A_\alpha \otimes_{\mathcal{O}P} A \cong A \otimes_{\mathcal{O}P} A$, where last isomorphism uses the fact that α fixes the image of P in A. Thus α is inner by 3.5. □

The following lemma shows that there is a well-defined group homomorphism $\Psi : \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times) \to \mathrm{Out}_1(A)$ as stated in Theorem 3.1.

Lemma 3.7. *Let α, $\alpha' \in \mathrm{Aut}_1(A) \cdot \mathrm{Inn}(A)$ such that $\alpha(u) = \alpha'(u)$ for all $u \in P$. Then α and α' have the same image in* $\mathrm{Out}(A)$.

Proof. The automorphism $\alpha^{-1} \circ \alpha'$ belongs to $\mathrm{Aut}_1(A) \cdot \mathrm{Inn}(A)$ and acts as the identity on P. Thus this automorphism is inner by 3.6. The result follows. □

Remark 3.8. The assumption in the previous lemma that both α, α' belong to $\mathrm{Aut}_1(A) \cdot \mathrm{Inn}(A)$ is necessary. For instance, if p is odd, P is cyclic of order p, and $E \leq \mathrm{Aut}(P)$ is the subgroup of order 2, then $\mathrm{Id}_{\mathcal{O}P}$ extends to the automorphism β in $\mathrm{Aut}_P(\mathcal{O}P \rtimes E)$ sending the nontrivial element t of E to $-t$. Clearly β does not induce an inner automorphism on $kP \rtimes E$.

The next lemma shows that the group homomorphism Ψ in the statement of Theorem 3.1 is injective; this completes the proof of Theorem 3.1.

Lemma 3.9. *Let $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$. Suppose that there is $\alpha \in \mathrm{Aut}(A)$ such that $\alpha(u) = \zeta(u)u$ for all $u \in P$. If α is inner, then $\zeta = 1$.*

Proof. Denote by η the automorphism of $\mathcal{O}P$ defined by $\eta(u) = \zeta(u)u$ for all $u \in P$. Suppose that α is inner. Then $A_\alpha \cong A$ as A-A-bimodules. Since α extends η, it follows that $A_\eta \cong A$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodules. In particular, A_η is a permutation $\mathcal{O}(P \times P)$-module. Since $\mathcal{O}P$ is a direct summand of A as an $\mathcal{O}(P \times P)$-module, it follows that A_η has $\mathcal{O}P_\eta$ as an indecomposable direct summand. This is a trivial source $\mathcal{O}(P \times P)$-module if and only if $\zeta = 1$, whence the result. □

The connection with automorphisms of B is described in the following observation, combining some of the standard facts on automorphisms mentioned previously.

Lemma 3.10. *Every $\alpha \in \mathrm{Aut}_1(A)$ extends to an automorphism $\beta \in \mathrm{Aut}_1(B)$, and the correspondence $\alpha \mapsto \beta$ induces a group isomorphism $\mathrm{Out}_1(A) \cong \mathrm{Out}_1(B)$.*

Proof. The algebras A and B are Morita equivalent via the bimodules Bi and iB. Let $\alpha \in \mathrm{Aut}_1(A)$. By Theorem 2.6 there is $\beta \in \mathrm{Aut}_1(B)$ such that ${}_{\beta^{-1}}Bi \cong Bi_\alpha$ as B-A-bimodules, and the correspondence $\alpha \mapsto \beta$ induces a group isomorphism $\mathrm{Out}_1(A) \cong \mathrm{Out}_1(B)$. We need to show that β can be chosen in such a way that it extends α. Since β induces the identity on $B/\pi B$, it follows from standard lifting idempotent theorems that i and $\beta(i)$ are conjugate in $B^\times$ via an element in $1 + \pi B$. Thus, after replacing β in its class if necessary, we may assume that $\beta(i) = i$. It follows that β restricts to an automorphism α' in $\mathrm{Aut}_1(A)$ representing the same class as α in $\mathrm{Out}_1(A)$. Thus α is equal to $\gamma \circ \alpha'$ for some inner automorphism γ of A given by conjugation with an element $c \in i + \pi A \subseteq A^\times$. Therefore, denoting by δ the inner automorphism of B given by conjugation with $1 - i + c \in 1 + \pi B$, it follows that $\delta \circ \beta$ extends α. □

In the remainder of this section, we give a proof of Lemma 3.2 which does not require the hyperfocal subalgebra. We start by showing that certain automorphisms in $\mathrm{Aut}_1(A) \cdot \mathrm{Inn}(A)$ can be conjugated into $\mathrm{Aut}_1(A)$ via a P-stable invertible element, and deduce that $\mathrm{Im}(\Psi)$ acts trivially on pointed groups on A.

Lemma 3.11. *Let $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$ such that there exists $\alpha \in \mathrm{Aut}_1(A)\mathrm{Inn}(A)$ satisfying $\alpha(u) = \zeta(u)u$ for all $u \in P$.*

(i) There is $c \in (A^P)^\times$ such that the automorphism α' defined by $\alpha'(a) = c^{-1}\alpha(a)c$ for all $a \in A$ satisfies $\alpha' \in \mathrm{Aut}_1(A)$.

(ii) We have $\alpha'(u) = \zeta(u)u$ for all $u \in P$, and the classes of α' and of α in $\mathrm{Out}(A)$ are equal.

(iii) For any pointed group Q_ϵ on A we have $\alpha(\epsilon) = \epsilon$.

Proof. By the assumptions on α, the automorphism $\bar{\alpha}$ induced by α on $\bar{A} = k \otimes_{\mathcal{O}} A$ is inner and fixes the image of P in $\bar{A}$. Thus $\bar{\alpha}$ is induced by conjugation with an invertible element $\bar{c} \in (\bar{A}^P)^\times$. The map $A^P \to \bar{A}^P$ is surjective, hence so is the induced map $(A^P)^\times \to (\bar{A}^P)^\times$. Choose an inverse image $c \in (A^P)^\times$ of $\bar{c}$. Then α' as defined satisfies (i). Since conjugation by c fixes the image of P in A, statement (ii) follows from (i). Conjugation by c fixes any subgroup Q of P and hence any point ϵ of Q on A. Thus, in order to prove (iii), we may replace α by α'; that is, we may assume that $\alpha \in \mathrm{Aut}_1(A)$. The hypotheses on α imply that $\alpha(\mathcal{O}Q) = \mathcal{O}Q$. Since A^Q is the centraliser in A of $\mathcal{O}Q$ it follows that α restricts to an automorphism of A^Q. The canonical map $A^Q \to (k \otimes_{\mathcal{O}} A)^Q$ is surjective, hence induces a bijection between the points of Q on A and on $k \otimes_{\mathcal{O}} A$. Since α induces the identity on $k \otimes_{\mathcal{O}} A$, it follows that A fixes all points of Q on A. □

Lemma 3.12. *Let $\zeta : P \to \mathcal{O}^\times$ a group homomorphism such that $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$. Denote by η the $\mathcal{O}$-algebra automorphism of $\mathcal{O}P$ sending $u \in P$ to $\zeta(u)u$. Let Q be a subgroup of P, let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$, and set $W = \mathcal{O}P_\varphi \otimes_{\mathcal{O}Q} \mathcal{O}P$. There is a unique isomorphism of $\mathcal{O}P$-$\mathcal{O}P$-bimodules $W \cong {}_\eta W_\eta$ induced by the map sending a tensor $u \otimes v$ to $\zeta(uv)u \otimes v$, where u, $v \in P$.*

Proof. We need to show that the assignment $u \otimes v \mapsto \zeta(uv)u \otimes v$ is well-defined. Let $u, v \in P$ and $w \in Q$. By the definition of W, the images of $u\varphi(w) \otimes v$ and $u \otimes wv$ in W are equal. Thus we need to show that $u\varphi(w) \otimes v$ and $u \otimes wv$ have the same image under this assignment. The image of $u\varphi(w) \otimes v$ is $\zeta(u\varphi(w)v)u\varphi(w) \otimes v = \zeta(u\varphi(w)v)u \otimes wv$. The image of $u \otimes wv$ is $\zeta(uwv)u \otimes wv$. Since ζ is a group homomorphism satisfying $\zeta(w) = \zeta(\varphi(w))$, it follows that $\zeta(uwv) = \zeta(u\varphi(w)v)$. This shows that the map $u \otimes v \mapsto \zeta(uv)u \otimes v$ is an $\mathcal{O}$-linear isomorphism. A trivial verification shows that this map is also a homomorphism of $\mathcal{O}P$-$\mathcal{O}P$-bimodules. □

Lemma 3.13. *Let $\zeta : P \to \mathcal{O}^\times$ a group homomorphism such that $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$. Denote by η the $\mathcal{O}$-algebra automorphism of $\mathcal{O}P$ sending $u \in P$ to $\zeta(u)u$. We have an isomorphism of $\mathcal{O}P$-$\mathcal{O}P$-bimodules $A \cong {}_\eta A_\eta$ which induces the identity on $k \otimes_\mathcal{O} A$.*

Proof. Note that η induces the identity on kP because the image in k of any p-power root of unity is 1. It follows from the main result in [15] (see also [11, Appendix] for an account of this material) that every indecomposable direct summand of A as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule is isomorphic to a bimodule of the form $\mathcal{O}P_\varphi \otimes_{\mathcal{O}Q} \mathcal{O}P$ for some subgroup Q of P and some $\varphi \in \mathrm{Hom}_\mathcal{F}(Q, P)$. Thus 3.13 follows from 3.12. □

Lemma 3.14. *Let $\zeta : P \to \mathcal{O}^\times$ a group homomorphism such that $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$. Denote by η the $\mathcal{O}$-algebra automorphism of $\mathcal{O}P$ sending $u \in P$ to $\zeta(u)u$. Set $A^e = A \otimes_\mathcal{O} A^{\mathrm{op}}$. Set $\bar{A} = k \otimes_\mathcal{O} A$ and $\bar{A}^e = \bar{A} \otimes_k \bar{A}^{\mathrm{op}}$. The canonical algebra homomorphism*

$$\mathrm{End}_{A^e}(A_\eta \otimes_{\mathcal{O}P} A) \to \mathrm{End}_{\bar{A}^e}(\bar{A} \otimes_{kP} \bar{A})$$

is surjective.

Proof. A standard adjunction yields a canonical linear isomorphism

$$\mathrm{End}_{A^e}(A\eta \otimes_{\mathcal{O}P} A) \cong \mathrm{Hom}_{A\otimes_\mathcal{O}\mathcal{O}P^{\mathrm{op}}}(A_\eta, A_\eta \otimes_{\mathcal{O}P} A)\ .$$

Using $A_\eta \cong A \otimes_{\mathcal{O}P} \mathcal{O}P_\eta$, another standard adjunction implies that this is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}P\otimes\mathcal{O}P^{\mathrm{op}}}(\mathcal{O}P_\eta, A_\eta \otimes_{\mathcal{O}P} A)\ .$$

Using $\mathcal{O}P_\eta \cong {}_{\eta^{-1}}\mathcal{O}P$ and 'twisting' by η on the left side of both arguments, the previous expression is canonically isomorphic to

$$\mathrm{Hom}_{\mathcal{O}P\otimes_\mathcal{O}\mathcal{O}P^{\mathrm{op}}}(\mathcal{O}P, {}_\eta A_\eta \otimes_{\mathcal{O}P} A)\ .$$

Using 3.13, this is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}P\otimes_\mathcal{O}\mathcal{O}P^{\mathrm{op}}}(\mathcal{O}P, A \otimes_{\mathcal{O}P} A)\ .$$

Since $A \otimes_{\mathcal{O}P} A$ is a permutation $\mathcal{O}(P \times P)$-module, it follows that the canonical map

$$\mathrm{Hom}_{\mathcal{O}P\otimes_\mathcal{O}\mathcal{O}P^{\mathrm{op}}}(\mathcal{O}P, A \otimes_{\mathcal{O}P} A) \to \mathrm{Hom}_{kP\otimes_k kP^{\mathrm{op}}}(kP, \bar{A} \otimes_{kP} \bar{A})$$

is surjective. Since the previous isomorphisms commute with the canonical surjections modulo $J(\mathcal{O})$, the result follows. □

Lemma 3.15. *Let $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$. Denote by η the automorphism of $\mathcal{O}P$ defined by $\eta(u) = \zeta(u)u$ for all $u \in P$. Let $\alpha \in \mathrm{Aut}_1(A)$. The following are equivalent.*

(i) The class of α in $\mathrm{Out}(A)$ has a representative α' in $\mathrm{Aut}_1(A)$ which extends η.

(ii) There is an A-$\mathcal{O}P$-bimodule isomorphism $A_\eta \cong A_\alpha$.

(iii) There is an $\mathcal{O}P$-A-bimodule isomorphism ${}_\eta A \cong {}_\alpha A$.

(iv) As an A-A-bimodule, A_α is isomorphic to a direct summand of $A_\eta \otimes_{\mathcal{O}P} A$.

Proof. By 2.4, (i) implies (ii) and (iii). If (ii) holds, then by 2.4 there is $\alpha' \in \mathrm{Aut}(A)$ which extends η and which represents the same class as α in $\mathrm{Out}(A)$. In particular, $\alpha' \in \mathrm{Aut}_1(A)\cdot\mathrm{Inn}(A)$. It follows from 3.11 that we may choose α' in $\mathrm{Aut}_1(A)$. Thus (ii) implies (i). The analogous argument shows that (iii) implies (i). Suppose again that (ii) holds. It follows from 3.4 that A_α is isomorphic to a direct summand of $A_\alpha \otimes_{\mathcal{O}P} A$, hence of $A_\eta \otimes_{\mathcal{O}P} A$. Thus (ii) implies (iv). Note that A_α remains indecomposable as an A-$\mathcal{O}P$-bimodule. If (iv) holds, then in particular, A_α is isomorphic to a direct summand of $A_\eta \otimes_{\mathcal{O}P} A$ as an A-$\mathcal{O}P$-bimodule, hence of $A_\eta \otimes_{\mathcal{O}P} W$ for some indecomposable $\mathcal{O}P$-$\mathcal{O}P$-bimodule summand W of A. As before, any such summand is of the form $OP_\varphi \otimes_{\mathcal{O}P} \mathcal{O}P$ for some subgroup Q of P and some $\varphi \in \mathrm{Hom}_\mathcal{F}(Q, P)$. Since A, hence A_η, is projective as a right $\mathcal{O}P$-module, it follows that every indecomposable A-$\mathcal{O}P$-bimodule summand of $A_\eta \otimes_{\mathcal{O}P} W$ has $\mathcal{O}$-rank divisible by $|P| \cdot |P : Q|$. Now $|P|$ is the highest power of p which divides the $\mathcal{O}$-rank of A. Thus, as an A-$\mathcal{O}P$-bimodule, A_α is isomorphic to a direct summand of $A_\eta \otimes_{\mathcal{O}P} \mathcal{O}P_\varphi$ for some φ belonging to $\mathrm{Aut}_\mathcal{F}(P)$. Any such φ is induced by conjugation by some element in $A^\times$, and hence A_α is isomorphic to a direct summand of A_η, as an A-$\mathcal{O}P$-bimodule. Since both have the same $\mathcal{O}$-rank, they are isomorphic. This shows that (iv) implies (ii) and concludes the proof. □

Second proof of Lemma 3.2. Let $\zeta : P \to \mathcal{O}^\times$ be a group homomorphism such that $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$. Denote by η the $\mathcal{O}$-algebra automorphism of $\mathcal{O}P$ sending $u \in P$ to $\zeta(u)u$. Set $\bar{A} = k \otimes_\mathcal{O} A$. The $\bar{A}$-$\bar{A}$-bimodule $\bar{A} \otimes_{kP} \bar{A}$ has a direct summand isomorphic to $\bar{A}$. It follows from standard lifting theorems of idempotents and 3.14 that the A-A-bimodule $A_\eta \otimes_{\mathcal{O}P} A$ has an indecomposable direct summand N satisfying $k \otimes_\mathcal{O} N \cong \bar{A}$. Then N induces a Morita equivalence on $\mathrm{mod}(A)$ which induces the identity on $\mathrm{mod}(k \otimes_\mathcal{O} A)$. Thus $N \cong A_\alpha$ for some $\alpha \in \mathrm{Aut}_1(A)$. It follows from 3.15 that we may choose α in $\mathrm{Aut}_1(A)$ in such a way that α extends η. □

4 Proof of Theorem 1.1

We need the interpretation from [14, §5] of the $*$-construction at the source algebra level. For χ a class function on G and u_ϵ a pointed element on $\mathcal{O}G$ we set $\chi(u_\epsilon) = \chi(uj)$ for some, and hence any, $j \in \epsilon$. By [14, 4.4] the matrix of values $\chi(u_\epsilon)$, with $\chi \in \mathrm{Irr}_K(B)$ and u_ϵ running over a set of representatives of the conjugacy classes of local pointed elements contained in P_γ is the matrix of generalised decomposition numbers of B. This matrix is nondegenerate, and hence a character $\chi \in \mathrm{Irr}_K(B)$ is determined by the values $\chi(u_\epsilon)$, with u_ϵ as before. By the description of the $*$-construction in [14, §5], for any local pointed element u_ϵ contained in P_γ we have

$$(\zeta * \chi)(u_\epsilon) = \zeta(u)\chi(u_\epsilon) \ .$$

The source algebra $A = iBi$ and the block algebra B are Morita equivalent via the bimodule iB and its dual, which is isomorphic to Bi. Through this Morita equivalence, χ corresponds to an irreducible character of $K \otimes_{\mathcal{O}} A$, obtained from restricting χ from B to A. Let u_ϵ be a local pointed element on $\mathcal{O}G$ contained in P_γ. Then there is $j \in \epsilon$ such that $j = ij = ji$, hence such that $j \in A$. In other words, the formula $(\zeta * \chi)(u_\epsilon) = \zeta(u)\chi(u_\epsilon)$ indeed describes the $*$-construction at the source algebra level.

By Lemma 3.10, the above Morita equivalence between A and B induces a group isomorphism $\mathrm{Out}_1(A) \cong \mathrm{Out}_1(B)$ obtained from extending automorphisms of A in $\mathrm{Aut}_1(A)$ to automorphisms of B in $\mathrm{Aut}_1(B)$. Composed with the group homomorphism Ψ from Theorem 3.1, this yields an injective group homomorphism $\Phi : \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times) \to \mathrm{Out}_1(B)$. The uniqueness statement in Theorem 1.1 follows from the uniqueness statement in Theorem 3.1.

In order to show that $\chi^{\Phi(\zeta)} = \zeta * \chi$, it suffices to prove that an automorphism α of A representing the class $\Psi(\zeta)$ sends the character of $K \otimes_{\mathcal{O}} A$ corresponding to χ to that corresponding to $\zeta * \chi$. By the above, and using the same letter χ for the restriction of χ to A, it suffices to show that $\chi^\alpha(u_\epsilon) = \zeta(u)\chi(u_\epsilon)$. By 3.1, we may choose α such that $\alpha(ui) = \zeta(u)ui$. By 3.11, α fixes the point ϵ; that is, $\alpha(j) \in \epsilon$. It follows that $\chi^\alpha(u_\epsilon) = \chi(\alpha(uj)) = \chi(\zeta(u)u\alpha(j)) = \zeta(u)\chi(u_\epsilon)$. This proves the statement (i) of Theorem 1.1.

Remark 4.1. The fact that the $*$-construction on characters lifts to the action of automorphisms can be used to give an alternative proof of the fact that Φ (or equivalently, Ψ) is injective. If α as defined in the above proof is inner, then an automorphism of B corresponding to α fixes any $\chi \in \mathrm{Irr}_K(B)$, or equivalently, $\zeta * \chi = \chi$ for any $\chi \in \mathrm{Irr}_K(B)$. This however forces $\zeta = 1$ as the group $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ acts faithfully on $\mathrm{Irr}_K(B)$ via the $*$-construction; in fact, it acts freely on the subset of height zero characters in $\mathrm{Irr}_K(B)$ by [18, §1].

For the proof of Theorem 1.1 (ii), assume that $\mathcal{O}$ is finitely generated over the p-adic integers (this assumption is needed in order to quote results from [9] and [20]). Let $\alpha \in \mathrm{Aut}_m(A)$. Since $(1 - \tau_p)\mathcal{O} = \pi^m\mathcal{O}$, it follows that α induces the identity on $B/(1 - \tau_p)B$. Thus $B_\alpha/(1 - \tau_p)B_\alpha \cong B/(1-\tau_p)B$. By Weiss' theorem as stated in [9, Theorem 3.2], B_α is a monomial $\mathcal{O}(P \times P)$-module, hence so is iB_α. Since iB_α is indecomposable as an $\mathcal{O}(P \times G)$-module and relatively $\mathcal{O}(P \times P)$-projective, it follows that iB_α is a linear source module. Since ΔP is a vertex of $k \otimes_{\mathcal{O}} iB$, this is also a vertex of iB_α. Thus there is $\zeta : P \to \mathcal{O}^\times$ such that iB_α is isomorphic to a direct summand of $\mathrm{Ind}_{\Delta P}^{P\times G}(\mathcal{O}_\zeta)$, where $\mathcal{O}_\zeta = \mathcal{O}$ with (u, u) acting as multiplication by $\zeta(u)$ for all $u \in P$. By 2.7 we have $\mathrm{Ind}_{\Delta P}^{P\times P}(\mathcal{O}_\zeta) \cong {}_\eta\mathcal{O}P$, where $\eta \in \mathrm{Aut}_1(\mathcal{O}P)$ is defined by $\eta(u) = \zeta(u)u$ for all $u \in P$. Thus iB_α is isomorphic to a direct summand of ${}_\eta B$, hence isomorphic to ${}_\eta jB$ for some primitive idempotent $j \in B^P$. Then j is necessarily a source idempotent because $k \otimes_{\mathcal{O}} iB_\alpha \cong k \otimes_{\mathcal{O}} iB$ and $\mathrm{Br}_P(i) \neq 0$. Since the local points of P on B are $N_G(P)$-conjugate, we may assume that j is $N_G(P)$-conjugate to i, so after replacing both j and ζ by an $N_G(P)$-conjugate we may assume that ${}_\eta iB \cong iB_\alpha$. We also may assume that α fixes i. Then multiplication on the right by i implies that ${}_\eta A \cong A_\alpha$, where we abusively use the same letter α for the automorphism of A obtained from restricting the automorphism α on B. Since $A_\alpha \cong {}_{\alpha^{-1}}A$, it follows from 3.15 that α can be chosen to extend η^{-1}. It follows from 3.3 that $\mathfrak{foc}(\mathcal{F}) \leq \ker(\zeta)$. This shows that the class of α is equal to $\Psi(\zeta^{-1})$. It remains to show that ζ has values in the subgroup μ of order p of $\mathcal{O}^\times$. Since Φ is injective, it suffices to show that the class of α has order at most p in $\mathrm{Out}(A)$, or equivalently, that α^p is inner. Since $\alpha \in \mathrm{Aut}_m(A)$, an easy calculation shows that $\alpha^p \in \mathrm{Aut}_{m+1}(A)$. It follows from [9, 3.13] that $\mathrm{Out}_{m+1}(A)$ is trivial, thus α^p is indeed inner. This concludes the proof of Theorem 1.1.

Acknowledgement. The present work is partially funded by the EPSRC grant EP/M02525X/1.

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