Theorems and Unawareness

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Abstract

This paper provides a set-theoretic model of knowledge and unawareness, in which reasoning through theorems is employed. A new property called Awareness Leads to Knowledge shows that unawareness of theorems not only constrains an agent's knowledge, but also, can impair his reasoning about what other agents know. For example, in contrast to Li (2006), Heifetz, Meier, and Schipper (2006) and the standard model of knowledge, it is possible that two agents disagree on whether another agent knows a particular event.

1 INTRODUCTION

A common assumption in economics is that agents who participate in a model perceive the "world" the same way the analyst does. This means that they understand how the model works, they know all the relevant theorems and they do not miss any dimension of the problem they are facing. In essence, agents are as educated and as intelligent as the analyst and they can make the best decision, given their information and preferences. Modeling unawareness aims at relaxing this assumption, so that agents may perceive a more simplified version of the world.

The standard model of knowledge without unawareness was introduced into economics by Aumann (1976). Its simplicity and the fact that it was purely set-theoretic led to many economic applications. One of the first attempts to model unawareness is by Geanakoplos (1989), using non-partitional information structures. However, Dekel, Lipman, and Rustichini (1998) propose three intuitive properties for unawareness and show that they are incompatible with the use of a standard state space.[1] On the other hand, Fagin and Halpern (1988), Halpern (2001), Modica and Rustichini (1994, 1999) and Halpern and Rêgo (2005) construct syntactic models. Two papers that try to circumvent the problem and provide a set-theoretic generalization of the standard model are Li (2006) and Heifetz, Meier, and Schipper (2006). They depart from the standard model in that they use multiple state spaces, one for each state of awareness. Feinberg (2004, 2005), Sadzik (2005), Copic and Galeotti (2006), Li (2006b), Heifetz, Meier, and Schipper (2007), Filiz (2006) and Ozbay (2006) model unawareness in the context of games.

This paper provides a model of knowledge and awareness, using multiple state spaces. In order to illustrate its main difference with the models of Heifetz, Meier, and Schipper (2006) and Li (2006), consider the following example, depicted in the figure below. There are two agents, Holmes and Watson, and two relevant dimensions or questions: "Did the dog bark?" and "Was there an intruder?". Holmes is always aware of both questions, so his subjective state space is the full state space, containing the four states $(\omega_1, \omega_2, \omega_3, \omega_4)$ on the plane. At state ω_4 which specifies that there was no intruder and no barking, Holmes knows that there is no intruder because he knows that the dog did not bark and he is also aware of and knows the

[1] Ely (1998) argues against one of these properties and suggests a one-agent model which employs a standard state space. Xiong (2007) proposes using the "knowing whether" rather than the "knowing that" operator and suggests two different unawareness operators that circumvent the impossibility result of Dekel, Lipman, and Rustichini (1998).

theorem “no barking implies no intruder”.[2] Hence, $P^H(\omega_4) = \omega_4$.

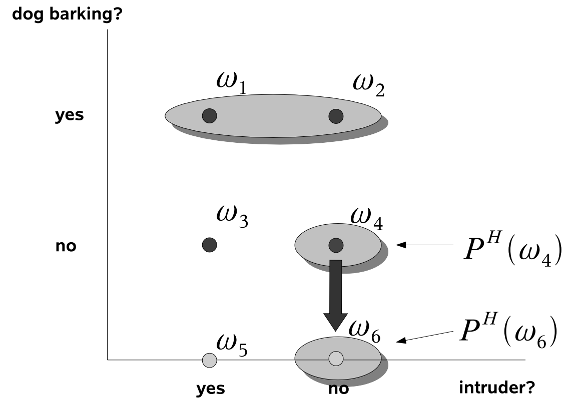

Figure 1

Watson is aware only of the question “Was there an intruder?” and he is unaware of the theorem “no barking implies no intruder”. His subjective state space consists of states ω_5 and ω_6 on the horizontal axis. The property “Projections Preserve Knowledge” in Heifetz, Meier, and Schipper (2006) and the construction in Li (2006) prescribe that when Watson reasons at ω_6 about Holmes’ knowledge, he projects $P^H(\omega_4) = \omega_4$ to his state space. Therefore, he reasons that $P^H(\omega_6) = \omega_6$ and that Holmes knows at ω_6 that there is no intruder. We argue that this is restrictive. Since Watson is unaware of any theorem that could lead someone to know whether there is an intruder, he should not be able to correctly deduce that Holmes knows at ω_6.

In order to accommodate the example so that Watson reasons that Holmes does not know whether there is an intruder, we have to abandon projections. Modeling reasoning through theorems does exactly that. When Watson reasons about Holmes at ω_6, he is unaware of the theorem “no barking implies no intruder” and therefore he cannot reason that Holmes is aware of it. As a result, $P^H(\omega_6) = \{\omega_5, \omega_6\}$ and Watson reasons that Holmes does not know.[3]

The example shows that unawareness can restrict Watson’s reasoning about Holmes’ knowledge about an event that both are aware of. This is not captured in other papers that model unawareness. Moreover, Watson formally makes no mistake. It is true that with Watson’s awareness, Holmes would not know that there is no intruder and Watson can reason only up to his awareness. Essentially, there are two different views of Holmes’ knowledge. This is formally captured in this model by creating one knowledge operator for each state of awareness. If Watson is aware of questions V_1 then his view of Holmes’ knowledge is K_{V_1}. But Holmes is aware of more questions, V_2, so his view of Holmes’ knowledge is K_{V_2}.[4] The relationship between the two is given by the property Awareness Leads to Knowledge which states that if V_2 contains V_1 then K_{V_2} will contain (even strictly) K_{V_1} when both are projected to the same state space. That is, higher states of awareness give a more complete description of one’s knowledge. Heifetz, Meier, and Schipper (2006) specify one knowledge operator K so that there is always one objective view of Holmes’ knowledge.

One can argue that another way of accommodating the example is with a model that allows false beliefs. Such a (syntactic) model is provided by Halpern and Rêgo (2005), who extend that of Heifetz, Meier, and Schipper (2006). However, allowing false beliefs would have stronger implications - that agents may make mistakes about any event, not just events which describe other agents’ knowledge. In order to allow for unawareness of theorems without allowing for agents to have false beliefs in general, we retain the truth property but index knowledge, K_V, with a set of questions V.

The paper proceeds as follows. Section 2 introduces the basic single-agent model, while its main results are presented in Section 3. We conclude in Section 4. The Appendix contains the proofs and the construction of

[2] That is, he considers ω_3 to be impossible. Specifying $P^H(\omega_3) = \{\omega_3\}$ is irrelevant for the example since this state never occurs.

[3] One argument against this reasoning is that Watson could be aware that Holmes is smarter than him, so that he could always think that Holmes could know, even though Watson cannot describe exactly how this can happen. But this argument also suggests that Watson can never be certain that Holmes does not know an event. The reason is that Watson can always think that Holmes is smarter, more aware and therefore *could* know. But this suggests that in an environment with unawareness an agent can never be certain that another agent does not know an event, which is clearly not true.

[4] In other words, Watson is only aware of the formula “Holmes, up to awareness V_1, knows that there is no intruder”. He is unaware of the respective formula when V_2 is substituted for V_1. More importantly, the formula “Holmes knows that there is no intruder” is not expressed in this model because knowledge, K_V, is always indexed with a set of questions V.

the state space for the multi-agent model.

2 THE MODEL

2.1 PRELIMINARIES

Consider a set of questions Q and denote by A_q the set of possible answers for question $q \in Q$. The set A_q can contain one, two, or more answers. The notion of awareness that will be defined in the following sections requires that if an agent is aware of a question, then he is aware of all possible answers. The *full state space* Ω^* is a subset of the Cartesian product $\underset{q\in Q}{\times} A_q$. In the example, the full state space consists of the four states on the plane, ω_1, ω_2, ω_3 and ω_4. Given any set of questions $V \subseteq Q$, a *subjective state space* Ω is the projection of Ω^* to the Cartesian product $\underset{q\in V}{\times} A_q$. For instance, Watson's subjective state space consists of ω_5 and ω_6. It is the projection of the full state space to the question he is aware of. An *event* E is a subset of a subjective state space Ω (and given Ω^*, there is a unique subjective state space Ω satisfying this inclusion). Define $\mathcal{V}_E$ to be the unique set of questions such that $E \subseteq \Omega \subseteq \underset{q\in \mathcal{V}_E}{\times} A_q$. If the event is $\{\omega_5\}$, then $\mathcal{V}_{\{\omega_5\}}$ is the question "Was there an intruder?". Define the *negation of* E to be the complement of E with respect to the subjective state space Ω of which it is a subset. Denote the complement of Ω by the empty set associated with it, $\emptyset_{\mathcal{V}_\Omega}$.

Take two sets of questions, $V' \subseteq V \subseteq Q$, and let Ω to be the subjective state space generated by V, and Ω' to be the subjective state space generated by V'. There exists a surjective projection $\Pi^V_{V'} : \Omega \to \Omega'$. For any subjective state $\omega \in \Omega$, $\Pi^V_{V'}(\omega)$ is the *restriction of* ω to the smaller set of questions V'. For instance, the restriction of ω_2 to Watson's state space is ω_6. For an event $E \subseteq \Omega$, its restriction to V' is $\Pi^V_{V'}(E)$. Its enlargement to the bigger set of questions V'' is denoted by $(\Pi^{V''}_V)^{-1}(E)$. The restriction of $\{\omega_2, \omega_3\}$ is the event $\{\omega_5, \omega_6\}$ while the enlargement of $\{\omega_5, \omega_6\}$ is $\{\omega_1, \omega_2, \omega_3, \omega_4\}$. To save on notation and only when it is unambiguous of which state space E is a subset, we abbreviate restrictions and enlargements by $E_{V'}$ and $E_{V''}$, respectively.

2.2 THE FULL STATE SPACE

For a subset of questions $V \subseteq Q_0$, where Q_0 is the set of basic questions, the resulting Cartesian product of their answers is $\underset{q\in V}{\times} A_q$.[5] Define mV to be the question "What subjective states in $\underset{q\in V}{\times} A_q$ does the agent consider impossible?". The collection of possible answers for question mV is the collection of all proper subsets of $\underset{q\in V}{\times} A_q$. The questions mV capture the agent's knowledge of theorems, as shown in Section 2.4.

For each question q, where $q \in Q_0$, or $q = mV$, for $V \subseteq Q_0$, define aq to be the question "Is the agent aware of question q?". This question captures the agent's awareness of questions, as shown in Section 2.3. In a multi-agent model, it will also capture the agent's knowledge about each agent's awareness. The possible answers for this type of questions are just two: "yes" and "no". Questions of the type aaq, $aaaq$, $aa \dots aq$ are not defined. Justification for this restriction will be given in Section 2.3, where awareness of questions will be defined.

The set of all questions Q contains the basic questions Q_0, together with the *epistemic* questions of the type mV, where $V \subseteq Q_0$, and of the type aq, where $q \in Q_0$, or $q = mV$, for $V \subseteq Q_0$. The full state space Ω^* is a subset of the Cartesian product of the answers of all questions in Q: $\Omega^* \subseteq \underset{q\in Q}{\times} A_q$. Define $\mathcal{S}$ to be the union of all state spaces: $\mathcal{S} = \bigcup\{\Pi^Q_V(\Omega^*) : \emptyset \neq V \subseteq Q\}$. The construction of the full state space in the multi-agent case is more complicated, as an agent has to reason about other agents' reasoning as well. The details are given in the appendix.

2.3 AWARENESS

The awareness of an agent is given by W, which is a mapping from $\mathcal{S}$ to sets of questions. For any state $\omega \in \mathcal{S}$,

$$W(\omega) = \bigcup \{\{q, aq\} \subseteq \mathcal{V}_{\{\omega\}} : \omega_{aq} = \text{“yes”}\}$$

denotes the questions, of which the agent is aware if $\omega \in \mathcal{S}$ occurs. If ω specifies "yes" to question aq, then the agent is aware of question q at ω. We then assume that he is also aware of question aq. Questions

[5] The basic questions describe the physical world but not the agents' knowledge or awareness.

of the type aaq, $aaaq$, $aa\dots aq$ are not permitted by the model. The first reason for this restriction stems for the definition of W, which specifies that the agent is aware of q and aq if ω specifies "yes" to question aq. Therefore, question aaq, which would also specify whether the agent is aware of question aq, is not needed. Another reason why these higher orders of questions would seem necessary is to express that if an agent is aware of something, then he is aware that he is aware of it. One of the results of Theorem 2 is exactly this property and it does not require these higher order questions. In the multi-agent case however, questions of the type $a^i a^j a^k q$ where $i \neq j$ and $j \neq k$ arise naturally when common knowledge is defined and thus will be included in the formal model. The agent's subjective state space at $\omega \in \mathcal{S}$ is $\Omega(\omega) = \Omega^*_{W(\omega)}$, which is the projection of the full state space Ω^* to the set of questions he is aware of at ω.[6]

Take an event E and define $U(E)$ to be the set of states $\omega \in \mathcal{S}$ that describe that the agent is unaware of it:

$$U(E) = \{\omega \in \mathcal{S} : \mathcal{V}_E \nsubseteq W(\omega)\}\,.$$

The agent is unaware of event E if he is not aware of all questions $\mathcal{V}_E$ that generate this event.

Given a set of questions V that generate the state space Ω^*_V, we define $U_V(E) = \Omega^*_V \cap U(E)$ to be the states of that particular state space, which describe that the agent is unaware of E. Hence, $U_V(E) \subseteq \Omega^*_V$ is an event. Denote the complement of $U_V(E)$ by $A_V(E)$. It is natural to require that V be big enough so that the generated state space Ω^*_V can adequately express E and the agent's awareness of it. Hence, we first require that V should contain all questions in $\mathcal{V}_E$. Secondly, we require that for each question $q \in \mathcal{V}_E$, V contains its respective counterpart aq. Denote this set of questions by $\alpha(\mathcal{V}_E)$.[7] Then, the condition is that $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq V$.

2.4 THEOREMS AND IMPOSSIBLE STATES

A theorem of the form "A implies B" can equivalently be expressed as the impossibility of the state that specifies "A is true but B is false". The agent's knowledge of theorems is given by the function M, which maps $\mathcal{S}$ to subsets of $\mathcal{S}$. For any $\omega \in \mathcal{S}$,

$$M(\omega) = \{\omega' \in \Omega(\omega) :$$

$$\{\omega'\}_V \in \omega_{mV}, \{mV\} \cup V \subseteq W(\omega)\}$$

denotes the set of subjective states that the agent considers impossible at ω, and expresses what theorems he knows at that state. An element $\omega' \in \Omega(\omega)$ of the agent's state space at ω is considered impossible if two conditions are met. Firstly, at ω the agent is aware of question mV and all questions in V. That is, he can formulate the Cartesian product $\times_{q\in V} A_q$ and ask the question mV: "What states in $\times_{q\in V} A_q$ does the agent consider impossible?". Secondly, the projection of ω' to the set of questions V is contained in ω_{mV}, which is the answer that ω specifies for question mV. This answer, ω_{mV}, is an event, a subset of the Cartesian product $\times_{q\in V} A_q$.

2.5 IMMEDIATE PERCEPTION

It is assumed that for some questions $q \in Q$ that the agent is aware of, he always knows the answer. For example, questions that describe what the agent sees or hears. Denote by X the set of all such questions. The following axiom is assumed throughout the paper. Define $\mathbb{E}$ to be the set that contains all epistemic questions $aq \in Q$ for $q \in Q$ and any $mV \in Q$, for $V \subseteq Q$: $\mathbb{E} = \{aq \in Q : q \in Q\} \cup \{mV \in Q : V \subseteq Q\}$.

Axiom 1. $\mathbb{E} \subseteq X$.

The axiom states that X contains at least all the epistemic questions that belong to Q.

2.6 POSSIBILITY AND KNOWLEDGE

For any $\omega \in \mathcal{S}$,

$$P(\omega) = \{\omega' \in \Omega(\omega) :$$

$$\omega'_q = \omega_q, q \in W(\omega) \cap X\} \setminus M(\omega)$$

denotes the subjective states the agent considers possible if ω occurs. More specifically, at ω the agent is aware of questions that belong to $W(\omega)$ and his subjective state space is $\Omega(\omega)$. For the questions in $W(\omega)$ that also belong to X, he knows the answer. This is the answer that ω specifies for that question. For all other

[6] If $W(\omega) = \emptyset$, then define $\Omega(\omega) = \emptyset$. In that case, $\Omega(\omega)$ is not an event and carries no awareness.

[7] The respective counterpart of aq is aq itself, since question aaq is not allowed in the model. Formally, for any $V \subseteq Q$, $\alpha(V) = \{aq : q \in V, q \neq aq' \text{ for all } q' \in Q\} \bigcup \{q \in V : q = aq', q' \in Q\}$.

questions in $W(\omega)$ he does not know the answer, but he can utilize his knowledge of theorems by excluding the impossible states $M(\omega)$. The following axiom states that the agent never excludes the true state.

Axiom 2. *For all $\omega^* \in \Omega^*$, $\{\omega^*\}_{W(\omega^*)} \in P(\omega^*)$.*

Axiom 2 implies that for all $\omega \in \mathcal{S}$ such that $W(\omega) \neq \emptyset$, $\{\omega\}_{W(\omega)} \in P(\omega)$.

Take an event E and define $K(E)$ to be the set of states $\omega \in \mathcal{S}$ that describe that the agent knows E:

$$K(E) = \{\omega \in \mathcal{S} : \mathcal{V}_E \subseteq W(\omega) \text{ and } (P(\omega))_{\mathcal{V}_E} \subseteq E\}.$$

The agent knows E if he is aware of it and in all the states he considers possible, it obtains. Given a set of questions V that generate state space Ω^*_V, we define $K_V(E) = \Omega^*_V \cap K(E)$ to be the event of that particular state space, which describes that the agent knows E.[8]

3 RESULTS

The following Theorem generalizes properties $P1, P2$ and $P3$ of the standard model without unawareness. All the results of this section are valid for the multi-agent case as well.

Theorem 1.

1. $\{\omega\}_{W(\omega)} \notin M(\omega) \iff \{\omega\}_{W(\omega)} \in P(\omega)$.[9]
2. $\omega' \in P(\omega)$ *implies* $P(\omega') = P(\omega)$.

The next property is the most important departure from other models dealing with unawareness, and stems from the explicit use of reasoning through theorems in the construction.

Property 1. Awareness Leads to Knowledge *Suppose axiom 2 holds. For any event E, if $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq V_2 \subseteq V_1$, then*

- $K_{V_2}(E) \subseteq (K_{V_1}(E))_{V_2}$ *and*
- $K_{V_2}(E) \supseteq (K_{V_1}(E))_{V_2}$ *is not necessarily true.*

[8] As with the unawareness operator $U_V(E)$, we impose the restriction $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq V$.

[9] The following property is also true. Suppose $M \subseteq \Omega^*_V$ is a set of impossible states, $\omega \in \Omega^*_V$, $\omega \notin M$ and $M(\omega)_V \subseteq M$. Then $\{\omega\}_{W(\omega)} \in P(\omega)$.

The condition $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq V_2, V_1$ ensures that the state spaces generated by V_1 and V_2 are rich enough to describe the agent's knowledge of E, so that $K_{V_2}(E)$, $K_{V_1}(E)$ are well defined, as explained in Section 2.6. The condition $V_2 \subseteq V_1$ says that the state space generated by questions V_1 is richer than that generated by questions V_2. The property then states that state spaces which are generated by more questions give a more complete description of the agent's knowledge of an event E. In other words, if a more complete description of the world ω specifies that the agent knows event E, ($\omega \in K_{V_1}(E)$), the less complete description $\{\omega\}_{V_2}$ may specify that he does not know it ($\{\omega\}_{V_2} \notin K_{V_2}(E)$).

The next theorem verifies properties that have been proposed in the literature, or are generalizations of properties of the standard model.

Theorem 2. *Suppose $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \cup \mathcal{V}_F \cup \alpha(\mathcal{V}_F) \cup \alpha(V) \subseteq V$. Then:*

1. **Subjective Necessitation** *Suppose axiom 2 holds. Then, for all $\omega \in \Omega^*_V$, $\omega \in K_V(\Omega(\omega))$.*
2. **Generalized Monotonicity** $E_{\mathcal{V}_E \cup \mathcal{V}_F} \subseteq F_{\mathcal{V}_E \cup \mathcal{V}_F}, \mathcal{V}_F \subseteq \mathcal{V}_E \implies K_V(E) \subseteq K_V(F)$.
3. **Conjunction** $K_V(E) \cap K_V(F) = K_V(E_{\mathcal{V}_E \cup \mathcal{V}_F} \cap F_{\mathcal{V}_E \cup \mathcal{V}_F})$.
4. **The Axiom of Knowledge** *Suppose axiom 2 holds. Then, $K_V(E) \subseteq E_V$.*
5. **The Axiom of Transparency** $\omega \in K_V(E) \iff \omega \in K_V(K_{W(\omega)}(E))$.
6. **The Axiom of Wisdom** *Suppose axiom 2 holds. Then, $\omega \in A_V(E) \cap \neg K_V(E) \iff \omega \in K_V(A_{W(\omega)}(E) \cap \neg K_{W(\omega)}(E))$.*
7. **Plausibility** $U_V(E) \subseteq \neg K_V(E) \cap \neg K_V(\neg K_V(E))$.
8. **Strong Plausibility** $U_V(E) \subseteq \neg K_V(E) \cap \neg K_V(\neg K_V(E)) \cap \ldots \cap \neg K_V(\neg K_V(\ldots \neg K_V(E)))$.
9. **AU Introspection** $U_V(E) \subseteq U_V(U_V(E))$.
10. **KU Introspection** $K_V(U_V(E)) = \emptyset_V$.
11. **Symmetry** $U_V(E) = U_V(\neg E)$.

12. **AA-Self Reflection** $\omega \in A_V(E) \iff \omega \in A_V(A_{W(\omega)}(E))$.

13. **AK-Self Reflection** $\omega \in A_V(E) \iff \omega \in A_V(K_{W(\omega)}(E))$.

14. **A-Introspection** *Suppose axiom 2 holds. Then,* $\omega \in A_V(E) \iff \omega \in K_V(A_{W(\omega)}(E))$.

The condition $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \cup \mathcal{V}_F \cup \alpha(\mathcal{V}_F) \cup \alpha(V) \subseteq V$ only ensures that the events $U_V(E)$, $K_V(E)$, $U_V(F)$, $K_V(F)$, and $U_V(U_V(E))$ are well defined. The first six properties are generalizations of the six properties of the standard model. Some of these generalizations are proposed by Li (2006). Plausibility, Strong Plausibility, AU Introspection and KU Introspection are the properties used by Dekel, Lipman, and Rustichini (1998) to show that unawareness precludes the use of a standard state space. Symmetry, AA-Self Reflection, AK-Self Reflection and A-Introspection have been proposed by Modica and Rustichini (1999) and Halpern (2001).

4 CONCLUDING REMARKS

In this paper we argue that with unawareness of theorems it is possible that two agents disagree on whether a third agent knows a particular event. This disagreement does not arise because agents make logical mistakes or have false beliefs but because they have different awareness, which implies that they reason differently about the knowledge of others. The idea that differences in awareness may specify different views of one's knowledge is captured by formulating, for each state of awareness V, a knowledge operator K_V. The relation between knowledge expressed with awareness V and knowledge expressed with awareness V' is captured by the property Awareness Leads to Knowledge. These connections between awareness and knowledge are not accommodated in Heifetz, Meier, and Schipper (2006) and Li (2006). In particular, Heifetz, Meier, and Schipper (2006) specify an objective knowledge operator K, so that there can never be two different views of one's knowledge, because of differences in awareness.

In a companion work we show that unawareness of theorems has interesting implications. In particular, one of the results of the standard model of knowledge is that asymmetric information alone cannot explain trade. Using the multi-agent version of this model we show that asymmetric information due to asymmetric awareness *can* allow for trade. The literature on no-trade theorems stems from the result of Aumann (1976) that if agents have common priors and their posteriors about an event are common knowledge, then these posteriors must be identical. It is shown that in an environment with unawareness the same result is true only for common priors and posteriors which are defined on a "common" state space, which is the state space that not only everyone is aware of, but it is also common knowledge that everyone is aware of. However, as the property Awareness Leads to Knowledge suggests, state spaces which carry more awareness give a more complete description of one's knowledge and posteriors. In an example with two agents we show that although the posteriors defined on this "common" state space are common knowledge and therefore identical, there still can be trade because one agent's higher awareness implies that his actual posterior is different and beyond the other agent's reasoning. Heifetz, Meier, and Schipper (2006, 2007) also provide examples where trade takes place. Comparison between the different approaches is provided in the companion work.

A Appendix

Proof of Theorem 1.

1(a). The proof is immediate from the definition of $P(\omega)$.

1(b). For footnote 9 we have that $\omega \notin M \implies \omega \notin (\Pi^V_{W(\omega)})^{-1}(M(\omega)) \implies \Pi^V_{W(\omega)}(\omega) \notin M(\omega) \implies \Pi^V_{W(\omega)}(\omega) \in P(\omega)$.

2. First, we prove the following proposition.

Proposition 1. $\omega \in P(\omega_1)$ *implies*

i) $W(\omega_1) = W(\omega)$.
ii) $M(\omega_1) = M(\omega)$.

Proof.

i) Suppose $q \in W(\omega_1)$. There are two cases. Either $q \neq aq'$ for any $q' \in Q$, or $q = aq'$ for some $q' \in Q$. In the first case, we have that ω_{1aq} = "yes" and $aq \in W(\omega_1)$. In the second case, $\omega_{1aq'}$ = "yes" and

$aq' \in W(\omega_1)$. The proof is identical in both cases, so we just illustrate the first case. From axiom 1, $aq \in X \cap W(\omega_1)$. Since $\omega \in P(\omega_1)$, we have $\omega_{aq} =$ "yes", which, together with $\{q, aq\} \subseteq W(\omega_1) = \mathcal{V}_\omega$ implies $\{q, aq\} \subseteq W(\omega)$. The other direction is immediate since $\mathcal{V}_\omega = W(\omega_1)$.

ii) Suppose $\omega_2 \in M(\omega_1)$. Then, there exist $\{mV\}$, V such that $\{mV\} \cup V \subseteq W(\omega_1)$ and $\Pi_V^{W(\omega_1)}(\omega_2) \in \omega_{1mV}$. From $i)$ we have $W(\omega_1) = W(\omega)$, which implies $\{mV\} \cup V \subseteq W(\omega)$. Moreover, from axiom 1 we have that $mV \in X \cap W(\omega_1)$. Thus, $\omega \in P(\omega_1)$ implies $\omega_{mV} = \omega_{1mV}$ and therefore $\omega_2 \in M(\omega)$. The other direction is identical.

□

Sets $P(\omega_1)$ and $P(\omega)$ are repeated below:

$$P(\omega_1) = \{\omega_2 \in \Omega(\omega_1) :$$

$$\omega_{2q} = \omega_{1q}, q \in W(\omega_1) \cap X\} \setminus M(\omega_1),$$

$$P(\omega) = \{\omega_2 \in \Omega(\omega) :$$

$$\omega_{2q} = \omega_q, q \in W(\omega) \cap X\} \setminus M(\omega).$$

From Proposition 1 we have $W(\omega_1) = W(\omega)$ and $M(\omega_1) = M(\omega)$. Since $\omega \in P(\omega_1)$ implies that $\omega_q = \omega_{1q}$ for all $q \in W(\omega_1) \cap X = W(\omega) \cap X$, we have that $P(\omega_1) = P(\omega)$.

□

Proof of Property 1.

First we prove that if $V_2 \subseteq V_1$, then $(K_{V_2}(E))_{V_1} \subseteq K_{V_1}(E)$. Suppose $\omega \in (K_{V_2}(E))_{V_1}$. Then, $\{\omega\}_{V_2} \in K_{V_2}(E)$, which implies that $\emptyset \neq P(\{\omega\}_{V_2}) \subseteq E_{W(\{\omega\}_{V_2})}$ and $\mathcal{V}_E \subseteq W(\{\omega\}_{V_2})$. We have to show that $\mathcal{V}_E \subseteq W(\omega)$ and $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$. Firstly, since $V_2 \subseteq V_1$ we also have $W(\{\omega\}_{V_2}) \subseteq W(\omega)$. Therefore, $\mathcal{V}_E \subseteq W(\omega)$. Non emptiness of $P(\omega)$ is guaranteed by axiom 2.

We next show that $(P(\omega))_{W(\{\omega\}_{V_2})} \subseteq P(\{\omega\}_{V_2})$. Suppose that $\omega' \in (P(\omega))_{W(\{\omega\}_{V_2})}$. Then, there exists $\omega_1 \in P(\omega)$ such that $\{\omega_1\}_{W(\{\omega\}_{V_2})} = \omega'$. Moreover, $\omega_1 \in P(\omega)$ implies that $\omega_{1q} = \omega_q$ for all $q \in W(\omega) \cap X$, hence $\omega'_q = \omega_q$ for all $q \in W(\{\omega\}_{V_2}) \cap X$. Next, we need to show that $\omega' \notin M(\{\omega\}_{V_2})$. Suppose that $\omega' \in M(\{\omega\}_{V_2})$. Then, there exist V and mV such that $V \cup \{mV\} \subseteq W(\{\omega\}_{V_2})$ and $\{\omega'\}_V \in \omega_{mV}$. Since $\{\omega_1\}_{W(\{\omega\}_{V_2})} = \omega'$ and $V \cup \{mV\} \subseteq W(\{\omega\}_{V_2}) \subseteq W(\omega)$, we have that $\{\omega_1\}_V \in \omega_{mV}$, which implies that $\omega_1 \in M(\omega)$ and $\omega_1 \notin P(\omega)$, a contradiction. Hence, $\omega' \notin M(\{\omega\}_{V_2})$ and $\omega' \in P(\{\omega\}_{V_2})$.

We have shown that $(P(\omega))_{W(\{\omega\}_{V_2})} \subseteq P(\{\omega\}_{V_2}) \subseteq E_{W(\{\omega\}_{V_2})}$, and $\mathcal{V}_E \subseteq W(\{\omega\}_{V_2}) \subseteq W(\omega)$. Therefore, $P(\omega) \subseteq E_{W(\omega)}$, which implies that $(K_{V_2}(E))_{V_1} \subseteq K_{V_1}(E)$. Finally, since $V_2 \subseteq V_1$, we also have that $K_{V_2}(E) \subseteq (K_{V_1}(E))_{V_2}$. For the second bullet, a counter example is provided in the Holmes and Watson example.

□

Proof of Theorem 2.

1. **Subjective Necessitation** First, note that $K_V(\Omega(\omega))$ is well defined becasue $W(\omega) \cup \alpha(W(\omega)) \subseteq V$. Subjective necessitation then follows from $\mathcal{V}_{\Omega(\omega)} = W(\omega)$ and $\emptyset \neq P(\omega) \subseteq \Omega(\omega)$.

2. **Generalized Monotonicity** Suppose $\omega \in K_V(E)$. Then, $\mathcal{V}_E \subseteq W(\omega)$ and $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$. Also, $\mathcal{V}_F \subseteq W(\omega)$ which implies that $E_{W(\omega)} \subseteq F_{W(\omega)}$. Therefore, $\omega \in K_V(F)$.

3. **Conjunction** We have that $\mathcal{V}_E \subseteq W(\omega)$ and $\mathcal{V}_F \subseteq W(\omega)$ if and only if $\mathcal{V}_E \cup \mathcal{V}_F \subseteq W(\omega)$. Also, $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$ and $\emptyset \neq P(\omega) \subseteq F_{W(\omega)}$ if and only if $\emptyset \neq P(\omega) \subseteq E_{W(\omega)} \cap F_{W(\omega)} = (E_{\mathcal{V}_E \cup \mathcal{V}_F} \cap F_{\mathcal{V}_E \cup \mathcal{V}_F})_{W(\omega)}$. The latter equality follows because $\omega_1 \in (E_{\mathcal{V}_E \cup \mathcal{V}_F} \cap F_{\mathcal{V}_E \cup \mathcal{V}_F})_{W(\omega)} \iff \{\omega_1\}_{\mathcal{V}_E \cup \mathcal{V}_F} \in E_{\mathcal{V}_E \cup \mathcal{V}_F} \cap F_{\mathcal{V}_E \cup \mathcal{V}_F} \iff \omega_1 \in E_{W(\omega)} \cap F_{W(\omega)}$.

4. **The Axiom of Knowledge** $\omega \in K_V(E)$ implies $\mathcal{V}_E \subseteq W(\omega)$ and $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$. Axiom 2 implies $\{\omega\}_{W(\omega)} \in P(\omega)$. Hence, $\{\omega\}_{W(\omega)} \in E_{W(\omega)}$, which implies $\omega \in E_V$.

5. **The Axiom of Transparency** Suppose $\omega \in K_V(E)$. Then, $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$ and $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$. We have to show that $\emptyset \neq P(\omega) \subseteq K_{W(\omega)}(E)$, or that $\omega_1 \in P(\omega)$ implies $\mathcal{V}_E \subseteq W(\omega_1)$ and $\emptyset \neq P(\omega_1) \subseteq E_{W(\omega_1)}$.

From Proposition 1, we have that $\omega_1 \in P(\omega)$ implies $W(\omega_1) = W(\omega)$. Hence, $\mathcal{V}_E \subseteq W(\omega_1) = W(\omega)$. From Theorem 1 we have that $\omega_1 \in P(\omega)$ implies $P(\omega) = P(\omega_1)$. Thus, $\emptyset \neq P(\omega_1) \subseteq E_{W(\omega)} = E_{W(\omega_1)}$.

Suppose $\omega \in K_V K_{W(\omega)}(E)$, which implies that $\emptyset \neq P(\omega) \subseteq K_{W(\omega)}(E)$. Hence, for all $\omega_1 \in P(\omega)$, we have that $\omega_1 \in K_{W(\omega)}(E)$, $W(\omega) = W(\omega_1)$, $P(\omega) = P(\omega_1)$ and $\emptyset \neq P(\omega_1) \subseteq E_{W(\omega)}$. Therefore, $\emptyset \neq P(\omega) \subseteq E_{W(\omega)}$ and $\omega \in K_V(E)$.

6. **The Axiom of Wisdom** Suppose $\omega \in A_V(E) \cap \neg K_V(E)$. Then, $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$ and either $P(\omega) = \emptyset$ or $\emptyset \neq P(\omega) \nsubseteq E_{W(\omega)}$. Axiom 2 implies that $P(\omega) \neq \emptyset$, so we just have to show that $P(\omega) \subseteq A_{W(\omega)}(E) \cap \neg K_{W(\omega)}(E)$. Suppose $\omega_1 \in P(\omega)$. Proposition 1 implies that $W(\omega_1) = W(\omega)$. Hence, $\mathcal{V}_E \subseteq W(\omega_1)$ and $\omega_1 \in A_{W(\omega)}(E)$. Theorem 1 implies that $P(\omega) = P(\omega_1)$. Thus $P(\omega_1) \nsubseteq E_{W(\omega)} = E_{W(\omega_1)}$ and $\omega_1 \in \neg K_{W(\omega)}(E)$.

   Suppose $\omega \in K_V(A_{W(\omega)} \cap \neg K_{W(\omega)}(E))$. Then, $\emptyset \neq P(\omega) \subseteq A_{W(\omega)} \cap \neg K_{W(\omega)}(E)$. Since $A_{W(\omega)}(E)$ is defined only if $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$, we have that $\omega \in A_V(E)$. It remains to show that $\omega \in \neg K_V(E)$, or that $P(\omega) \nsubseteq E_{W(\omega)}$. We know that for all $\omega_1 \in P(\omega)$, $\omega_1 \in \neg K_{W(\omega)}(E)$, which implies that $P(\omega_1) \nsubseteq E_{W(\omega)}$. Since $P(\omega) = P(\omega_1)$, we have that $P(\omega) \nsubseteq E_{W(\omega)}$.

8. **Strong Plausibility** By assumption, $\mathcal{V}_E \subseteq V = \mathcal{V}_{\neg K_V(E)} = \mathcal{V}_{\neg K_V(\neg K_V(E))} = \mathcal{V}_{\neg K_V(\neg K_V(\ldots \neg K_V(E)))}$. Suppose $\omega \in U_V(E)$. Then, $\mathcal{V}_E \nsubseteq W(\omega)$ and therefore $V \nsubseteq W(\omega)$. Hence, $\omega \in \neg K_V(E) \cap \neg K_V(\neg K_V(E)) \cap \ldots \cap \neg K_V(\neg K_V(\ldots \neg K_V(E)))$.

9. **AU Introspection** Suppose $\omega \in U_V(E)$, Then, $\mathcal{V}_E \nsubseteq W(\omega)$ and since $\mathcal{V}_E \subseteq V = \mathcal{V}_{U_V(E)}$, we have $\mathcal{V}_{U_V(E)} \nsubseteq W(\omega)$, which implies $\omega \in U_V(U_V(E))$.

10. **KU Introspection** Suppose $\omega \in K_V(U_V(E))$. Then, $W(\omega) = V$ and there exists $\omega_1 \in P(\omega) \subseteq U_V(E)$. From Proposition 1 we have that $W(\omega_1) = W(\omega) = V$. Moreover, the definition of $U_V(E)$ implies that $\mathcal{V}_E \subseteq V$. Therefore, $\mathcal{V}_E \subseteq W(\omega_1)$. But $\omega_1 \in U_V(E)$ implies that $\mathcal{V}_E \nsubseteq W(\omega_1)$, a contradiction.

11. **Symmetry** Follows from $\mathcal{V}_E = \mathcal{V}_{\neg E}$.

12. **AA-Self Reflection** $\omega \in A_V(E)$ implies $W(\omega) \cup \alpha(W(\omega)) \subseteq V$ and $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$. Therefore, $A_V(A_{W(\omega)}(E))$ is well defined and $\omega \in A_V(A_{W(\omega)}(E))$. For the other direction, suppose that $\omega \in A_V(A_{W(\omega)}(E))$. Since $A_{W(\omega)}(E)$ is defined only if $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega)$, we have that $\omega \in A_V(E)$.

13. **AK-Self Reflection** The proof is similar.

14. **A-Introspection** Suppose $\omega \in A_V(E)$. Then, $\mathcal{V}_E \cup \alpha(\mathcal{V}_E) \subseteq W(\omega) \subseteq V$ and $W(\omega) = \mathcal{V}_{A_{W(\omega)}(E)}$, so we just have to show that $\emptyset \neq P(\omega) \subseteq A_{W(\omega)}(E)$. That $P(\omega)$ is non empty follows from axiom 2. Suppose that $\omega_1 \in P(\omega)$. From Proposition 1, we have $W(\omega) = W(\omega_1)$ which implies $\mathcal{V}_E \subseteq W(\omega_1)$ and $\omega_1 \in A_{W(\omega)}(E)$. For the other direction, suppose that $\omega \in K_V(A_{W(\omega)}(E))$. This implies that $\omega \in A_V(A_{W(\omega)}(E))$ and $\omega \in A_V(E)$ follows from AA-Self Reflection.

□

A.1 THE FULL STATE SPACE

This section gives a detailed construction of the full state space, which is the state space of the analyst or of a fully aware agent. The construction is similar to that of a beliefs space: starting from an initial state space S, define each player's first order beliefs on S, then each player's second order beliefs on S and all other players first order beliefs, and so on. The difference with this formulation is that instead of beliefs we have the epistemic questions $a^i q$ and $m^i V$, that describe the awareness of questions and knowledge of theorems for each agent i.

For any state space $\Omega = \underset{q \in V}{\times} A_q$, let $\mathcal{E}^i(\Omega)$ be the set of epistemic questions of agent i about Ω. This set will consist of questions of the type $a^i q$ and $m^i V$. In particular, suppose $\Omega = \underset{q \in V}{\times} A_q$ is generated from a set of questions V. The set of all questions of the type $m^i V_1$, for all nonempty subsets V_1 of V is

$$\left\{ m^i V_1 : \emptyset \neq V_1 \subseteq V \right\}. \tag{1}$$

These questions represent all the theorems that agent i can potentially have about state space Ω.

The set

$$\left\{a^i q : q \in V \cup \left\{m^i V_1 : \emptyset \neq V_1 \subseteq V\right\}\right\} \quad (2)$$

contains all the $a^i q$ questions, for all questions in V and in $\{m^i V_1 : \emptyset \neq V_1 \subseteq V\}$. Denote the union of the two sets of questions in (1) and (2) by $\mathcal{E}^i(\Omega)$. An element that gives an answer to all questions in $\mathcal{E}^i(\Omega)$ describes agent i's awareness of questions and knowledge of theorems, about state space Ω.

To construct the full state space Ω^*, we begin with an initial state space $S = \underset{q\in Q_0}{\times} A_q$, which is generated from a finite or countably infinite set of basic questions Q_0. A state of nature $s \in S$ gives a detailed description of the world, but not what agents are aware of or know. Let $\Omega_1^i = S$ be agent i's first order state space. Questions in $\mathcal{E}^i(\Omega_1^i)$ describe agent i's awareness of questions and knowledge of theorems about state space Ω_1^i. Define the set of all combinations of answers for these questions to be T_1^i:

$$T_1^i = \underset{q\in\mathcal{E}^i(\Omega_1)}{\times} A_q,$$

which we interpret as the first order type of agent i. The second order state space for agent i is

$$\Omega_2^i = S \underset{j\neq i}{\times} T_1^j.$$

An element in Ω_2^i describes the state of nature $s \in S$, together with the awareness of questions and knowledge of theorems about S, for all agents besides i. The set $\mathcal{E}^i(\Omega_2^i)$ contains all the epistemic questions of agent i about state space Ω_2^i. Note that there are some questions in $\mathcal{E}^i(\Omega_2^i)$ that also belong to $\mathcal{E}^i(\Omega_1^i)$. For example, if q is a basic question and belongs to Q_0, then $a^i q$ belongs to $\mathcal{E}^i(\Omega_1^i) \cap \mathcal{E}^i(\Omega_2^i)$. To avoid any duplication of questions, we define the second order type of agent i to be

$$T_2^i = \underset{q\in\mathcal{E}^i(\Omega_2^i)\setminus\mathcal{E}^i(\Omega_1^i)}{\times} A_q.$$

An element in $T_1^i \times T_2^i$ specifies the questions the agent is aware of and the theorems he knows in state space Ω_2^i. Accordingly, the third order state space of agent i is

$$\Omega_3^i = \Omega_2^i \underset{j\neq i}{\times} T_2^j.$$

Continuing inductively, we define for all $k \geq 1$,

$$\Omega_{k+1}^i = \Omega_k^i \underset{j\neq i}{\times} T_k^j,$$

$$T_{k+1}^i = \underset{q\in\mathcal{E}^i(\Omega_{k+1}^i)\setminus\mathcal{E}^i(\Omega_k^i)}{\times} A_q.$$

Note that T_{k+1}^i is non-empty for all k, as new epistemic questions are created in each step. Define T^i to be the Cartesian product $\overset{\infty}{\underset{n=1}{\times}} T_n^i$. An element in T^i contains an answer for all epistemic questions about agent i. In particular, it gives an answer to only questions of the type $a^i q$, or of the type $m^i V$, where q can be either a basic question or an epistemic question about another agent (e.g. $q = a^j a^k a^i q'$), while V can contain both basic and epistemic questions for all other agents. Note that questions of the type $a^i a^i \ldots a^i q$ are not created. Summarizing, an element in T^i describes agent i's awareness of questions and knowledge of theorems for each successively bigger state space Ω_k, where $k \geq 1$.

Interpreting T^i as the set of all types for agent i, we can define a full state to specify a state of nature $s \in S$, together with a type for each player $i \in I$. The full state space Ω^* is then a subset of the Cartesian product $S \underset{i\in I}{\times} T^i$:

$$\Omega^* \subseteq S \underset{i\in I}{\times} T^i.$$

The set of all questions that generate the full state space Ω^* is denoted by Q. Formally, $\mathcal{V}_{\Omega^*} = Q$ and $\Omega^* \subseteq \underset{q\in Q}{\times} A_q$.

Acknowledgments

I am grateful to Larry Epstein and Paulo Barelli for their continuous guidance and encouragement throughout this project. I would also like to thank Jan Eeckhout, Jeffrey Ely, Yossi Feinberg, Dino Gerardi, Aviad Heifetz, Jing Li, Bart Lipman, Alessandro Pavan, Burkhard Schipper, William Thomson, Gábor Virág, the seminar participants at British Columbia, Colorado at Boulder, Maastricht, Queen Mary, Rochester, Southampton, Western Ontario, the 2006 NBER/NSF/CEME Conference on General Equilibrium Theory and the 2006 Midwest Economic Theory Conference. All remaining errors are mine.

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