Dynkin isomorphism and Mermin–Wagner theorems for hyperbolic sigma models and recurrence of the two-dimensional vertex-reinforced jump process

Roland Bauerschmidt* Tyler Helmuth† Andrew Swan‡

March 5, 2018

Abstract

We prove the vertex-reinforced jump process (VRJP) is recurrent in two dimensions for any translation invariant finite range initial rates. Our proof has two main ingredients. The first is a direct connection between the VRJP and sigma models whose target space is a hyperbolic space $\mathbb{H}^n$ or its supersymmetric counterpart $\mathbb{H}^{2|2}$. These results are analogues of well-known relations between the Gaussian free field and the local times of simple random walk. The second ingredient is a Mermin–Wagner theorem for these sigma models. This result is of intrinsic interest for the sigma models and also implies our main theorem on the VRJP. Surprisingly, our Mermin–Wagner theorem applies even though the symmetry groups of $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$ are non-amenable.

1 Introduction and results

1.1. Introduction. Our results have motivation from two different perspectives, that of sigma models with hyperbolic symmetry and their relevance for the Anderson transition, and that of a model of reinforced random walks known as the *vertex-reinforced jump process* (VRJP).

The VRJP was originally introduced by Werner and has attracted a great deal of attention recently [5–7,21,22]. The VRJP on a vertex set Λ is a continuous-time random walk that jumps from a vertex i to a neighbouring vertex j at time t with rate proportional to $\beta_{ij}(1+L_t^j)$, where L_t^j is the local time of j at time t and $\beta_{ij} \geqslant 0$ are the initial rates. This interaction leads the VRJP to be effectively attracted to itself.

One of our new results is the following theorem.

Theorem 1.1. *Consider the vertex-reinforced jump process* (X_t) *on* $\mathbb{Z}^d$ *when the initial rates* β *are finite range and translation invariant. If* $d = 1, 2$ *then* (X_t) *is recurrent in the sense that the expected time* (X_t) *spends at the origin is infinite.*

For sufficiently small initial rates recurrence results for the VRJP have previously been established [1,8,21]. See [1] for a discussion and precise statements. It has also been shown that the linearly edge-reinforced random walk (ERRW) with constant initial weights is recurrent in

*University of Cambridge, Statistical Laboratory, DPMMS. Email: `rb812@cam.ac.uk`.
†University of Bristol, School of Mathematics. Email: `th17948@bristol.ac.uk`.
‡University of Cambridge, Statistical Laboratory, DPMMS. Email: `acks2@cam.ac.uk`.

two dimensions [19,22], but the recurrence of the VRJP for all initial rates was an open problem until the present work. The relation between the ERRW and VRJP is discussed below.

Theorem 1.1 is in fact a consequence of our proof of a Mermin-Wagner theorem for hyperbolic sigma models and a new and very direct relation between VRJPs and hyperbolic sigma models that parallels the well-known relationship between simple random walks and Gaussian free fields (the BFS–Dynkin isomorphism theorem).

Before giving precise definitions of our models and stating our results, we briefly indicate the motivations behind hyperbolic sigma models, and their relations with reinforced random walks. We also explain some consequences of our results for hyperbolic sigma models. Readers primarily interested in the VRJP may wish to skip ahead to Section 1.2.

Hyperbolic sigma models were introduced as effective models to understand the Anderson transition [9, 25–27, 30]. In Efetov's supersymmetric method [12] the expected absolute value squared of the resolvent of random band matrices, i.e., $\mathbb{E}|(H-z)^{-1}(i,j)|^2$ where $z \in \mathbb{C}_+$ and H is a random band matrix, can be expressed as a correlation function of a supersymmetric spin model. The spins of this model are invariant under the hyperbolic symmetry $OSp(2,1|2)$. Extended states correspond to spontaneous breaking of this non-compact symmetry. The supersymmetric hyperbolic sigma model, or $\mathbb{H}^{2|2}$ model, was introduced by Zirnbauer [30] and first studied by Disertori, Spencer and Zirnbauer [9]. It is an approximation of the random band matrix model above where radial fluctuations are neglected. This is similar to how the $O(n)$ model is an approximation of models of $\mathbb{R}^n$-valued spins with rotational symmetry such as $|\varphi|^4$-theories. More detailed motivation for hyperbolic spin models is given in [25, 27].

The $\mathbb{H}^{2|2}$ model is believed to capture the physics of the Anderson transition. As is expected for the Anderson model, it was proved in [9] that the $OSp(2,1|2)$ symmetry of the $\mathbb{H}^{2|2}$ model is spontaneously broken in $d \geqslant 3$ for sufficiently small disorder — consistent with the existence of extended states. Furthermore, it was proved [8] that for sufficiently large disorder this is not the case — consistent with Anderson localisation. In dimension $d \leqslant 2$, it is conjectured that extended states do not exist for any disorder strength. Equation (1.16) below is the corresponding statement for the $\mathbb{H}^{2|2}$ model, and we have thus completed the expected qualitative picture for the phase diagram of the $\mathbb{H}^{2|2}$ model. This can be considered as a version of the Mermin–Wagner theorem. For recent and extremely precise results in dimension one, see [24].

Based on the similarity of certain explicit formulas, it was suggested that there is a connection between the $\mathbb{H}^{2|2}$ model and linearly edge-reinforced random walks [9]. This connection was first confirmed in [21] by relating marginals of the $\mathbb{H}^{2|2}$ model to the limiting local time profile of the VRJP. It was also shown there that the linearly edge reinforced walk is obtained from the VRJP when averaging over random initial rates. Further marginals of the $\mathbb{H}^{2|2}$ model were explored in [6]. For a discussion of the history of the VRJP, see [21].

Our hyperbolic analogue of the BFS–Dynkin isomorphism theorem, Theorem 1.2 below, is a different relation between the $\mathbb{H}^{2|2}$ model and the VRJP than was found in [21], and it provides a more direct relation between the correlation structures of the models. Moreover, our statement also applies without supersymmetry, i.e., when the spins take values in $\mathbb{H}^n$. We will explain further extensions of Theorem 1.2 in the case of $\mathbb{H}^n$, e.g., to multipoint correlations and loop measures, in a forthcoming publication.

1.2. Model definitions. We now define the VRJP and the hyperbolic sigma models. The walk and the sigma models are both defined in terms of a set Λ of vertices and non-negative edge weights $\beta = (\beta_{ij})_{i,j\in\Lambda}$, where by edge weights we mean that $\beta_{ij} = \beta_{ji}$. For our Mermin-Wagner theorem we will make use of two assumptions on β. We call β *finite range* if for each $i \in \Lambda$ we have $\beta_{ij} = 0$ for all but finitely many j. If $\Lambda = \mathbb{Z}^d$ we call β *translation invariant* if $\beta_{ij} = \beta_{T(i)T(j)}$ for all translations T of $\mathbb{Z}^d$.

1.2.1. Vertex-reinforced jump process. Let Λ be a finite or countable set. The VRJP is a history-dependent continuous-time random walk (X_t) on Λ that takes jumps from vertex i to vertex j with rate $\beta_{ij}(1 + L_t^j)$, where

$$L_t^j \equiv \int_0^t 1_{X_s = j}\, ds. \tag{1.1}$$

L_t^j is called the *local time* of the walk at vertex j up to time t. We will write $L_t \equiv (L_t^i)_{i\in\Lambda}$ for the collection of local times. It will also be useful to consider the joint process (X_t, L_t), which is a Markov process with generator $\mathcal{L}$ acting on sufficiently nice functions $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ by

$$\mathcal{L}^\beta g(i,\ell) = \sum_j \beta_{ij}(1+\ell_j)(g(j,\ell) - g(i,\ell)) + \frac{\partial}{\partial \ell_i} g(i,\ell), \qquad i \in \Lambda,\ \ \ell \in \mathbb{R}^\Lambda. \tag{1.2}$$

We denote by $\mathbb{E}_{i,\ell}^\beta$ the expectation of the process (X_t, L_t) with initial condition $X_0 = i$ and $L_0 = \ell$. The VRJP is the marginal of X_t in the special case $L_0 = 0$; by a slight abuse of terminology we will call (X_t, L_t) the VRJP as well.

1.2.2. Hyperbolic sigma models. Let $\mathbb{R}^{n,1}$ denote $(n+1)$-dimensional Minkowski space. Its elements are vectors $u = (x, y^1, \dots, y^{n-1}, z)$, and it is equipped with the indefinite inner product $u \cdot u = x^2 + (y^1)^2 + \cdots + (y^{n-1})^2 - z^2$. Note that although x plays the same role as the y^i, we distinguish it in our notation for later convenience. Recall that n-dimensional hyperbolic space $\mathbb{H}^n$ can be realized as

$$\mathbb{H}^n \equiv \{u \in \mathbb{R}^{n,1} \mid u \cdot u = -1, z > 0\}. \tag{1.3}$$

Suppose Λ is finite and $h > 0$. To each vertex $i \in \Lambda$ we associate a spin $u_i \in \mathbb{H}^n$. The energy of a spin configuration $u = (u_i)_{i\in\Lambda} \in (\mathbb{H}^n)^\Lambda$ is

$$H(u) = H_{\beta,h}(u) \equiv \sum_{i,j} \beta_{ij}(-u_i \cdot u_j - 1) + h \sum_j (z_j - 1). \tag{1.4}$$

The $\mathbb{H}^n$ sigma model is the measure with density proportional to $e^{-H(u)}$ with respect to the $|\Lambda|$-fold product of the measure μ on $\mathbb{H}^n$ induced by the Minkowski metric (see (2.4) for an explicit expression), and we let $\langle\cdot\rangle_{\mathbb{H}^n}$ denote the expectation associated to this model:

$$\langle F(u)\rangle_{\mathbb{H}^n} \equiv \frac{\int_{(\mathbb{H}^n)^\Lambda} F(u)\, e^{-H(u)}\, \mu^{\otimes\Lambda}(du)}{\int_{(\mathbb{H}^n)^\Lambda} e^{-H(u)}\, \mu^{\otimes\Lambda}(du)}. \tag{1.5}$$

The energy (1.4) favours spin alignment because $u \cdot v \leqslant -1$ for $u, v \in \mathbb{H}^n$ with equality if and only if $u = v$.

1.2.3. Supersymmetric hyperbolic sigma model. In this section we will introduce a probability measure which enables the computation of a special class of observables of the full supersymmetric $\mathbb{H}^{2|2}$ model. These restricted observables will suffice for a description of a special, but interesting, case of our results. Our most general results use the full supersymmetric formalism.

As explained in Section 2, at each vertex $i \in \Lambda$ there is a superspin $u_i = (x_i, y_i, z_i, \xi_i, \eta_i) \in \mathbb{H}^{2|2}$ where ξ_i and η_i are Grassmann variables. For the moment all that is needed is that the expectation of a function $F(y)$ of the $y \equiv (y_i)_{i\in\Lambda}$ coordinates can be written as

$$\langle F(y)\rangle_{\mathbb{H}^{2|2}} = \int_{(\mathbb{R}^2)^\Lambda} F(e^t s)\, e^{-\widetilde{H}(s,t)}\, dt\, ds, \tag{1.6}$$

where $dt\,ds \equiv \prod_i dt_i\,ds_i$, $e^t s \equiv (e^{t_i} s_i)_{i\in\Lambda}$,

$$\widetilde{H}(s,t) = \widetilde{H}_{\beta,h}(s,t) \equiv \sum_{i,j} \beta_{ij} \left(\cosh(t_i - t_j) - 1 + \frac{1}{2}(s_i - s_j)^2 e^{t_i + t_j} \right) + h \sum_i \left(\cosh(t_i) - 1 + \frac{1}{2} s_i^2 e^{t_i} \right) + \sum_i (t_i + \log(2\pi)) - \log\det D_{\beta,h}(t), \quad (1.7)$$

and the matrix $D_{\beta,h}(t)$ on $\mathbb{R}^\Lambda$ is defined by the quadratic form

$$(v, D_{\beta,h}(t)v) \equiv \sum_{i,j} \beta_{ij} e^{t_i+t_j} (v_i - v_j)^2 + h \sum_i e^{t_i} v_i^2, \quad v \in \mathbb{R}^\Lambda. \quad (1.8)$$

The determinant $\det D_{\beta,h}(t)$ does not depend on the s variables and it is positive since $D_{\beta,h}(t)$ is positive definite. Thus $e^{-\widetilde{H}(s,t)}dt\,ds$ is a positive measure, and we will show in Section 2 that it is in fact a probability measure, i.e., $\langle 1\rangle_{\mathbb{H}^{2|2}} = 1$.

1.3. Results. We now state our main results and show how Theorem 1.1 is a consequence.

1.3.1. Hyperbolic BFS–Dynkin Isomorphism. The following theorem is a hyperbolic analogue of the Dynkin isomorphism theorem. The Dynkin isomorphism theorem relates the local times of a simple random walk to the square of a Gaussian free field. As the Dynkin isomorphism theorem was proved by Brydges–Fröhlich–Spencer in [3, Theorem 2.2], and later expressed in a better form by Dynkin [11], we prefer to call it the BFS–Dynkin isomorphism. The general idea of relating Gaussian fields to simple random walks is due to [28]. For recent discussions of these ideas see [16, 29]. Supersymmetric versions of these results for simple random walks go back to Luttinger and Le Jan [15, 17].

Note that while we have not yet defined the meaning of $\langle g\rangle_{\mathbb{H}^{2|2}}$ for a general function g, we have given a meaning in the case that g is identically one by (1.6).

Theorem 1.2. *Suppose Λ is finite and β is a collection of non-negative edge weights. Let $h > 0$, let $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be any bounded smooth function, and let $a, b \in \Lambda$. Consider the $\mathbb{H}^n$ model, $n \geqslant 2$, and let $y = (y_i)_{i\in\Lambda} = (y_i^r)_{i\in\Lambda}$ for some $r = 1, \dots, n-1$. Then*

$$\sum_b \langle y_a y_b g(b, z-1)\rangle_{\mathbb{H}^n} = \langle z_a \int_0^\infty \mathbb{E}^\beta_{a,z-1}(g(X_t, L_t))\, e^{-ht}\, dt\rangle_{\mathbb{H}^n}. \quad (1.9)$$

For the $\mathbb{H}^{2|2}$ model, we have

$$\sum_b \langle y_a y_b g(b, z-1)\rangle_{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{E}^\beta_{a,0}(g(X_t, L_t))\, e^{-ht}\, dt. \quad (1.10)$$

Remark 1.3. *Theorem 1.2 also holds for the $\mathbb{H}^1$ model, but as the proof requires slightly different considerations we have not included it here.*

Taking the function g to be identically one in (1.10) implies that

$$\langle y_a y_b\rangle_{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{E}^\beta_{a,0}(1_{X_t=b})\, e^{-ht}\, dt. \quad (1.11)$$

The right-hand side can be interpreted as the two-point function of the VRJP with a uniform killing rate h. We remark that the extension of the theorem to non-constant h is straightforward.

1.3.2. Hyperbolic Mermin–Wagner Theorem. In this section we assume that $\Lambda = \Lambda_L$ is the discrete d-dimensional torus $\mathbb{Z}^d/(L\mathbb{Z})^d$ of side length $L \in \mathbb{N}$, and that β is translation invariant and finite range. We will write $\langle\cdot\rangle = \langle\cdot\rangle_{\beta,h}$ in place of $\langle\cdot\rangle_{\mathbb{H}^n}$ and $\langle\cdot\rangle_{\mathbb{H}^{2|2}}$. Denote

$$\lambda(p) \equiv \sum_{j\in\Lambda} \beta_{0j}(1-\cos(p\cdot j)), \qquad p \in \Lambda^\star, \tag{1.12}$$

where here $\cdot$ is the Euclidean inner product on $\mathbb{R}^d$ and $\Lambda^\star$ is the Fourier dual of the discrete torus Λ. Denote the two-point function and its Fourier transform by

$$G_{\beta,h}(j) = G^L_{\beta,h}(j) \equiv \langle y_0 y_j\rangle_{\beta,h}, \qquad \hat{G}_{\beta,h}(p) = \hat{G}^L_{\beta,h}(p) = \sum_{j\in\Lambda} G_{\beta,h}(j) e^{i(p\cdot j)}. \tag{1.13}$$

The following theorem is an analogue of the Mermin–Wagner Theorem for the $O(n)$ model, in the form presented in [13].

Theorem 1.4. *Let $\Lambda = \mathbb{Z}^d/(L\mathbb{Z})^d$, $L \in \mathbb{N}$. For the $\mathbb{H}^n$ model, $n \geqslant 2$, with magnetic field $h > 0$,*

$$\hat{G}_{\beta,h}(p) \geqslant \frac{1}{(1+(n+1)G_{\beta,h}(0))\lambda(p)+h}. \tag{1.14}$$

Similarly, for the $\mathbb{H}^{2|2}$ model with $h > 0$,

$$\hat{G}_{\beta,h}(p) \geqslant \frac{1}{(1+G_{\beta,h}(0))\lambda(p)+h}. \tag{1.15}$$

Remark 1.5. *By* (1.11) *the two-point function $G_{\beta,h}$ equals that of the VRJP in the case $\mathbb{H}^{2|2}$, and hence the two-point function of the VRJP satisfies* (1.15) *as well.*

Remark 1.6. *For $d \geqslant 3$, the bound* (1.15) *shows that $\tilde{f}$ can be replaced by f in [9, Theorem 3] using the upper bound proved there for $G_{\beta,h}(0)$.*

Corollary 1.7. *Under the assumptions of Theorem 1.4, for $d = 1, 2$,*

$$\lim_{h\downarrow 0} \lim_{L\to\infty} G_{\beta,h}(0) = \infty. \tag{1.16}$$

Proof. Since $(2\pi L)^{-d}\sum_{p\in\Lambda^*} e^{i(p\cdot j)} = 1_{j=0}$, summing the bounds (1.14) and (1.15) over $p \in \Lambda^\star$ and interchanging sums implies (with $n = 0$ for $\mathbb{H}^{2|2}$)

$$G_{\beta,h}(0) \geqslant \frac{1}{(2\pi L)^d} \sum_{p\in\Lambda^\star} \frac{1}{(1+(n+1)G_{\beta,h}(0))\lambda(p)+h}. \tag{1.17}$$

The assumption of β being finite range and non-negative implies $\lambda(p) \leqslant C(\beta)|p|^2$. If $d \leqslant 2$ it follows that

$$\lim_{L\to\infty} \frac{1}{(2\pi L)^d} \sum_{p\in\Lambda^\star} \frac{1}{\lambda(p)+h} \uparrow \infty \quad \text{as } h \downarrow 0, \tag{1.18}$$

and, as $G_{\beta,h} \geqslant 0$, this implies (1.16). □

1.3.3. Consequences for the vertex-reinforced jump process. In contrast to Corollary 1.7, it has been proved [9, 27] that when $d \geqslant 3$ and $\beta_{ij} = \beta 1_{|i-j|=1}$,

$$\lim_{h\downarrow 0} \lim_{L\to\infty} G_{\beta,h}(0) < \infty \tag{1.19}$$

for all $\beta > 0$ in the case of $\mathbb{H}^2$ and for all sufficiently large $\beta > 0$ for $\mathbb{H}^{2|2}$. In the $\mathbb{H}^{2|2}$ case (1.19) corresponds to transience of the VRJP (see Corollary 1.8 below) and to the uniform boundedness (in the spectral parameter $z \in \mathbb{C}_+$) of the expected square of the absolute value of the resolvent for random band matrices in the sigma model approximation [25] when $d \geqslant 3$ (recall Section 1.1). It also implies that the hyperbolic symmetry is spontaneously broken.

Due to the non-amenability of hyperbolic group actions, the question of spontaneous symmetry breaking for hyperbolic sigma models is, in general, subtle. The usual formulations of the Mermin–Wagner theorem for models with compact symmetries cannot hold in the non-amenable case [23], and, in fact, spontaneous symmetry breaking appears to occur in all dimensions [10,20]. Nonetheless, (1.16) and (1.19) show that the two-point function — the observable of interest for the VRJP and the random matrix problem — does undergo a transition analogous to that occurring in systems with compact symmetries.

Proof of Theorem 1.1. We must prove that for any translation invariant finite range β

$$\int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})\, dt = \infty, \tag{1.20}$$

where the expectation refers to that of the VRJP on $\mathbb{Z}^d$ and $d = 1, 2$. This is true since, for any finite range β, one has

$$\begin{aligned} \int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})\, dt &= \lim_{h\downarrow 0} \int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})\, e^{-ht}\, dt \\ &= \lim_{h\downarrow 0} \lim_{L\to\infty} \int_0^\infty \mathbb{E}_{0,0}^{\beta,\Lambda_L}(1_{X_t=0})\, e^{-ht}\, dt = \infty. \end{aligned} \tag{1.21}$$

The first equality is by monotone convergence, and the final equality is obtained by combining (1.16) for the $\mathbb{H}^{2|2}$ model and (1.11).

For the second equality it suffices, by using the tail of the exponential e^{-ht}, to verify that the integrand converges for $t \leqslant T$ for any bounded T. Since the jump rate $1 + L_t^i$ is bounded by $1 + T$, the walk is exponentially unlikely to take more than $O(T^3)$ jumps to new vertices up to time T. VRJPs on Λ_L and $\mathbb{Z}^d$ can be coupled to be the same until they exit a ball of radius less than $\frac{1}{2}L$, an event which requires at least L/R jumps to occur, where R is the radius of the finite range step distribution. This completes the proof. □

The analogue of Theorem 1.1 for the ERRW with constant initial weights was established in [19, 22], but not for the VRJP. Mermin-Wagner type theorems have also been proven for the ERRW in one and two dimensions [18, 19]. The techniques used deal directly with ERRWs, and hence are rather different from those employed in this paper.

Our relation between the two-point functions of the $\mathbb{H}^{2|2}$ model and the VRJP also gives a new proof of the following Corollary 1.8, previously obtained in [21]. Both proofs rely on [9].

Corollary 1.8. *The vertex-reinforced jump process on $\mathbb{Z}^d$, $d \geqslant 3$, $\beta_{ij} = \beta 1_{|i-j|=1}$ and β sufficiently large, is transient, i.e., the expected time spent at the origin is finite.*

Proof. The argument mirrors the proof of Theorem 1.1, using (1.19) in place of (1.16). □

2 Supersymmetry and horospherical coordinates

In this section we define horospherical coordinates for $\mathbb{H}^n$ and then define the supersymmetric $\mathbb{H}^{2|2}$ model precisely. We also collect Ward identities and relations between derivatives that will be used in the proofs of Theorems 1.2 and 1.4.

2.1. Horospherical coordinates. As observed in [27, 30], the hyperbolic spaces $\mathbb{H}^n$ are naturally parametrised by horospherical coordinates that are useful for the analysis of the corresponding sigma models. For $\mathbb{H}^n$, these are global coordinates $t \in \mathbb{R}$, $\tilde{s} \in \mathbb{R}^{n-1}$, in terms of which

$$x = \sinh t - \frac{1}{2}|\tilde{s}|^2 e^t, \quad y^i = e^t s^i \quad (i = 1, \dots, n-1), \quad z = \cosh t + \frac{1}{2}|\tilde{s}|^2 e^t. \tag{2.1}$$

Both x, z are scalars while $\tilde{y} = (y^1, \dots, y^{n-1})$ and $\tilde{s} = (s^1, \dots, s^{n-1}) \in \mathbb{R}^{n-1}$ are $n-1$ dimensional vectors and $|\tilde{s}|^2 = \sum_{i=1}^{n-1}(s^i)^2$. By this change of variables one has [14]

$$\int_{(\mathbb{H}^n)^\Lambda} F(u)\, \mu^{\otimes\Lambda}(du) = \int_{(\mathbb{R}^n)^\Lambda} F(u(\tilde{s}, t)) \prod_i e^{(n-1)t_i}\, dt_i\, d\tilde{s}_i. \tag{2.2}$$

By a short calculation,

$$-u_i \cdot u_j = \cosh(t_i - t_j) + \frac{1}{2}|\tilde{s}_i - \tilde{s}_j|^2 e^{t_i + t_j}, \qquad z_i = \cosh t_i + \frac{1}{2}|\tilde{s}_i|^2 e^{t_i}. \tag{2.3}$$

Thus in horospherical coordinates,

$$H(\tilde{s}, t) = \sum_{i,j} \beta_{ij} \left(\cosh(t_i - t_j) - 1 + \frac{1}{2}|\tilde{s}_i - \tilde{s}_j|^2 e^{t_i + t_j} \right) + h \sum_i \left(\cosh(t_i) - 1 + \frac{1}{2}|\tilde{s}_i|^2 e^{t_i} \right), \tag{2.4}$$

where by a slight abuse of notation we have re-used the symbol H. Moreover, the following relations, in which we set $s_i = s_i^r$ and $y_i = y_i^r$ for some fixed $r = 1, \dots, n-1$, hold:

$$\frac{\partial z_i}{\partial s_i} = y_i, \qquad \frac{\partial y_i}{\partial s_i} = x_i + z_i, \qquad \frac{\partial (u_i \cdot u_j)}{\partial s_i} = y_j(x_i + z_i) - y_i(x_j + z_j). \tag{2.5}$$

Furthermore,

$$\frac{\partial^2}{\partial s_j^2} z_j = e^{t_j} = x_j + z_j, \qquad \frac{\partial^2}{\partial s_i \partial s_l}(-1 - u_j \cdot u_l) = \begin{cases} -e^{t_j + t_l} = -(x_j + z_j)(x_l + z_l), & i = j, \\ +e^{t_j + t_l} = +(x_j + z_j)(x_l + z_l), & i = l, \\ 0, & \text{else}. \end{cases} \tag{2.6}$$

2.2. Supersymmetry. Let Λ be a finite set. We will define an algebra Ω_Λ of forms (which generalise random variables) that constitute the observables on the super-space $(\mathbb{R}^{2|2})^\Lambda$. The super-space itself only has meaning through this algebra of observables. We also define an integral associated to this algebra. We then introduce the supersymmetry generator and the localisation lemma. For a more detailed introduction to the mathematics of supersymmetry, see, e.g., [2, 4, 9].

2.2.1. Supersymmetric integration. For each vertex $i \in \Lambda$, let x_i, y_i be real variables and ξ_i, η_i be two Grassmann variables. Thus by definition all of the x_i and y_i commute with each other and with all of the ξ_i and η_i and all of the ξ_i and η_i anticommute. The way in which the anticommutation relations are realized is unimportant, but concretely, we can define an algebra of $4^{|\Lambda|} \times 4^{|\Lambda|}$ matrices ξ_i and η_i realising the required anticommutation relations for the Grassmann variables. To fix signs in forthcoming expressions, fix an arbitrary order $i_1, \dots, i_{|\Lambda|}$ of the vertices in Λ.

We define the algebra Ω_Λ to be the algebra of smooth functions on $(\mathbb{R}^2)^\Lambda$ with values in the algebra of $4^{|\Lambda|} \times 4^{|\Lambda|}$ matrices that have the form

$$F = \sum_{I,J \subset \Lambda} F_{I,J}(x,y)(\eta\xi)_{I,J}, \tag{2.7}$$

where the coefficients $F_{I,J}$ are smooth functions on $(\mathbb{R}^2)^\Lambda$, and $(\eta\xi)_{I,J}$ is given by the ordered product $\prod_{i \in I \cap J} \eta_i \xi_i \prod_{i \in I \setminus J} \xi_i \prod_{j \in J \setminus I} \eta_j$. This ordering has been chosen so that $(\eta\xi)_{\Lambda,\Lambda}$ is $\eta_1\xi_1 \dots \eta_\Lambda \xi_\Lambda$. We call elements of Ω_Λ forms because the forms of differential geometry are instances [4,15]. The integral (sometimes called a superintegral) of a form $F \in \Omega_\Lambda$ is defined by

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^2)^\Lambda} F_{\Lambda,\Lambda}(x,y) \prod_{i \in \Lambda} \frac{dx_i \, dy_i}{2\pi}, \tag{2.8}$$

where $\mathbb{R}^{2|2}$ refers to the number of commuting and anticommuting variables.

The *degree* of a coefficient $F_{I,J}$ is $|I| + |J|$. Thus the integral of a form F is a constant multiple of the usual Lebesgue integral of the top degree part of F. A form $F \in \Omega_\Lambda$ is *even* if the degree of all non-vanishing coefficients $F_{I,J}$ is even in (2.7). Even forms commute. For even forms $F^1, \dots, F^p$ and a smooth function $g \in C^\infty(\mathbb{R}^p)$, the form $g(F^1, \dots, F^p) \in \Omega_\Lambda$ is defined by formally Taylor expanding g about the degree-0 part $(F^1_{\varnothing,\varnothing}(x,y), \dots, F^p_{\varnothing,\varnothing}(x,y))$. This is well-defined as there is no ambiguity in the ordering if the F^i are all even, and the anticommutation relations satisfied by the ξ_i and η_i imply the expansion is finite.

2.2.2. Localisation. Temporarily set $x = x_i, y = y_i, \xi = \xi_i$, and $\eta = \eta_i$. Define an operator $\partial_\eta \colon \Omega_\Lambda \to \Omega_\Lambda$ by linearity, $\partial_\eta(\eta F) = F$, and $\partial_\eta F = 0$ if F does not contain a factor η. Define ∂_ξ in the same manner. Define Q_i by its action on forms F by

$$Q_i F \equiv \xi \partial_x F + \eta \partial_y F + x \partial_\eta F - y \partial_\xi F. \tag{2.9}$$

The *supersymmetry generator* Q acts on a form $F \in \Omega_\Lambda$ by $QF \equiv \sum_{i \in \Lambda} Q_i F$.

Definition 2.1. *$F \in \Omega_\Lambda$ is* supersymmetric *if $QF = 0$.*

The supersymmetry generator acts as a derivation on the algebra of forms, see, e.g., [4]. This implies that any form that can be written as a function of the collection of forms

$$\tau_{ij} = x_i x_j + y_i y_j + \xi_i \eta_j - \eta_i \xi_j, \qquad i,j \in \Lambda, \tag{2.10}$$

is supersymmetric. The following *localisation lemma* is fundamental; proofs can be found in [4,9].

Lemma 2.2 (Localisation lemma)**.** *Let $F \in \Omega_\Lambda$ be a smooth form with sufficient decay that can be written as a function of the $(\tau_{ij})_{i,j \in \Lambda}$. Then*

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F = F_{\varnothing,\varnothing}(0,0). \tag{2.11}$$

2.3. The $\mathbb{H}^{2|2}$ model. We can now define the $\mathbb{H}^{2|2}$ sigma model and justify our earlier claim that its y marginal is the probability measure (1.6). Given $(x_i, y_i, \xi_i, \eta_i)$ as above define an even variable z_i by

$$z_i \equiv \sqrt{1 + x_i^2 + y_i^2 + 2\xi_i\eta_i} = \sqrt{1 + x_i^2 + y_i^2} + \frac{\xi_i\eta_i}{\sqrt{1 + x_i^2 + y_i^2}}, \tag{2.12}$$

where the equality is by the definition of a function of a form. We will write $u_i = (x_i, y_i, z_i, \xi_i, \eta_i)$. Define the "inner product"

$$u_i \cdot u_j \equiv x_i x_j + y_i y_j - z_i z_j + \xi_i \eta_j - \eta_i \xi_j, \tag{2.13}$$

generalising the Minkowski inner product above (1.3); we have written "inner product" as this is only terminology, since (2.13) is not a quadratic form in the classical sense. Then by a short calculation

$$u_i \cdot u_i = -1, \tag{2.14}$$

which we interpret as meaning that u_i is in the supermanifold $\mathbb{H}^{2|2}$. Since $z_i = \sqrt{1 + \tau_{ii}}$ and $u_i \cdot u_j = \tau_{ij} - z_i z_j$, the forms $u_i \cdot u_j$ and z_i are supersymmetric for all $i, j \in \Lambda$.

The $\mathbb{H}^{2|2}$ integral of a form $F \in \Omega_\Lambda$ is defined by

$$\int_{(\mathbb{H}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^{2|2})^\Lambda} F \prod_{i\in\Lambda} \frac{1}{z_i}, \tag{2.15}$$

and the $\mathbb{H}^{2|2}$ model is defined by the following action (which is now a form in Ω_Λ)

$$H \equiv H_{\beta,h} = \sum_{i,j} \beta_{ij}(-u_i \cdot u_j - 1) + h \sum_i (z_i - 1) \in \Omega_\Lambda. \tag{2.16}$$

Lastly, we define the super-expectation of an observable $F \in \Omega_\Lambda$ in the $\mathbb{H}^{2|2}$ model by

$$\langle F \rangle_{\mathbb{H}^{2|2}} \equiv \int_{(\mathbb{H}^{2|2})^\Lambda} F e^{-H}. \tag{2.17}$$

Lemma 2.2 implies that $\langle 1 \rangle_{\mathbb{H}^{2|2}} = 1$, as promised in Section 1.2.3.

2.4. Supersymmetric horospherical coordinates. The $\mathbb{H}^{2|2}$ model can also be reparametrised in a supersymmetric version of horospherical coordinates [9]. In this parametrisation, t and s are two real variables and $\bar\psi$ and ψ are two Grassmann variables. As in the previous section, we denote the algebra of such forms by $\widetilde{\Omega}_\Lambda$. The tilde refers to horospherical coordinates. We write

$$x = \sinh t - e^t(\frac{1}{2}s^2 + \bar\psi\psi), \quad y = e^t s, \quad z = \cosh t + e^t(\frac{1}{2}s^2 + \bar\psi\psi), \quad \xi = e^t\bar\psi, \quad \eta = e^t\psi. \tag{2.18}$$

There is a generalisation of the change of variables formula from standard integration to superintegration. We only require the following special case given in [9]. Forms $F \in \Omega_\Lambda$ are in correspondence with forms $\widetilde{F} \in \widetilde{\Omega}_\Lambda$ obtained by substituting the relations (2.18) into (2.7) using the definition of functions of forms. Moreover, expanding

$$\widetilde{F} = \sum_{I,J\subset\Lambda} \widetilde{F}_{I,J}(t,s)(\bar\psi\psi)_{I,J} \tag{2.19}$$

the superintegral over F can expressed as

$$\int_{(\mathbb{H}^{2|2})^\Lambda} F = \int_{(\mathbb{R}^2)^\Lambda} \widetilde{F}_{\Lambda,\Lambda}(t,s) \prod_i e^{-t_i} \frac{dt_i\, ds_i}{2\pi}. \tag{2.20}$$

If a function $F(y)$ depends only on the y coordinates then F has degree 0, and a computation (see [9, Sec. 2.2]) shows that

$$\begin{aligned} \langle F(y)\rangle_{\mathbb{H}^{2|2}} = \int_{(\mathbb{H}^{2|2})^\Lambda} F(y) e^{-H} &= \int_{(\mathbb{R}^2)^\Lambda} F(e^t s)(e^{-H})_{\Lambda,\Lambda} \prod_i e^{-t_i} \frac{dt_i\, ds_i}{2\pi} \\ &= \int_{(\mathbb{R}^2)^\Lambda} F(e^t s) e^{-\widetilde{H}(t,s)} \prod_i dt_i\, ds_i, \end{aligned} \tag{2.21}$$

with the function $\widetilde{H}$ given by (1.6).

Analogously to (2.3) a calculation gives the expressions

$$-u_i \cdot u_j = \cosh(t_i - t_j) + \frac{1}{2}(s_i - s_j)^2 e^{t_i + t_j} + (\bar\psi_i - \bar\psi_j)(\psi_i - \psi_j) e^{t_i + t_j} \tag{2.22}$$

$$z_i = \cosh t_i + (\frac{1}{2} s_i^2 + \bar\psi_i \psi_i) e^{t_i}. \tag{2.23}$$

We again check that

$$\frac{\partial z_i}{\partial s_i} = y_i, \qquad \frac{\partial y_i}{\partial s_i} = x_i + z_i, \qquad \frac{\partial (u_i \cdot u_j)}{\partial s_i} = y_j(x_i + z_i) - y_i(x_j + z_j) \tag{2.24}$$

and

$$\frac{\partial^2}{\partial s_j^2} z_j = e^{t_j} = x_j + z_j, \quad \frac{\partial^2}{\partial s_i \partial s_l}(-1 - u_j \cdot u_l) = \begin{cases} -e^{t_j + t_l} = -(x_j + z_j)(x_l + z_l), & i = j, \\ +e^{t_j + t_l} = +(x_j + z_j)(x_l + z_l), & i = l, \\ 0, & \text{else.} \end{cases} \tag{2.25}$$

2.5. Ward identities. In this section we record useful Ward identities. These identities are a reflection of the underlying symmetries of the target spaces $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$, see [9].

2.5.1. $\mathbb{H}^n$*.* For the $\mathbb{H}^n$ model we have the identities

$$\langle x_j g(z)\rangle_{\mathbb{H}^n} = 0. \tag{2.26}$$

for any smooth function g. Moreover, by symmetry, $\langle g(y^r)\rangle_{\mathbb{H}^n} = \langle g(x)\rangle_{\mathbb{H}^n}$ for r in $1, \dots, n-1$.

2.5.2. $\mathbb{H}^{2|2}$*.* For the $\mathbb{H}^{2|2}$ model we have identities analogous to (2.26):

$$\langle x_j g(z)\rangle_{\mathbb{H}^{2|2}} = 0 \tag{2.27}$$

for any smooth function g. We also have $\langle g(x)\rangle_{\mathbb{H}^{2|2}} = \langle g(y)\rangle_{\mathbb{H}^{2|2}}$ by symmetry. The following identities arise from combining (2.27) with supersymmetry:

$$\begin{aligned} \langle e^{t_j + t_l}\rangle_{\mathbb{H}^{2|2}} &= \langle (x_j + z_j)(x_l + z_l)\rangle_{\mathbb{H}^{2|2}} = \langle x_j x_l + z_j z_l\rangle_{\mathbb{H}^{2|2}} = 1 + \langle y_j y_l\rangle_{\mathbb{H}^{2|2}}, \\ \langle e^{t_j}\rangle_{\mathbb{H}^{2|2}} &= \langle x_j + z_j\rangle_{\mathbb{H}^{2|2}} = 1. \end{aligned} \tag{2.28}$$

The evaluations $\langle z_i z_j\rangle_{\mathbb{H}^{2|2}} = \langle z_i\rangle_{\mathbb{H}^{2|2}} = 1$ are by Lemma 2.2, which implies more generally that for any smooth function g with rapid decay,

$$\int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,0}} g(z) = g(1). \tag{2.29}$$

3 Proof of Theorem 1.2

In this section, for the $\mathbb{H}^n$ model, we will let y_a denote the component y_a^1 of $u_a \in \mathbb{H}^n$ and s_a the corresponding component s_a^1 in horospherical coordinates. By symmetry (recall Section 2.5), the results of this section are valid if we replace y_a^1 by any of the first $n-1$ components of u_a.

We will prove that for the $\mathbb{H}^n$ model, $n \geqslant 2$,

$$\sum_b \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,h}} y_a y_b g(b, z-1) = \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,h}} z_a \int_0^\infty \mathbb{E}^\beta_{a,z-1}(g(X_t, L_t))\, e^{-ht}\, dt. \tag{3.1}$$

In (3.1), and in the rest of this section, we omit the measure $\mu^{\otimes\Lambda}(du)$ for integrals over $(\mathbb{H}^n)^\Lambda$ from the notation. For the $\mathbb{H}^{2|2}$ model we prove that

$$\sum_b \int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,h}} y_a y_b g(b, z-1) = \int_0^\infty \mathbb{E}^\beta_{a,0}(g(X_t, L_t))\, e^{-ht}\, dt. \tag{3.2}$$

Theorem 1.2 in the case of $\mathbb{H}^{2|2}$ is precisely (3.2), and follows by normalising (3.1) for $\mathbb{H}^n$. The identities (3.1) and (3.2) are a result of the following integration by parts identities. Recall that $\mathcal{L}^\beta$ denotes the generator (1.2) of the joint position and local time process (X_t, L_t) of the VRJP.

Lemma 3.1. *Let Λ be finite, let $a \in \Lambda$, and let $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay. For the $\mathbb{H}^n$ model, $n \geqslant 2$,*

$$-\sum_b \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,0}} y_a y_b \mathcal{L}^\beta g(b, z-1) = \int e^{-H_{\beta,0}} z_a g(a, z-1). \tag{3.3}$$

For the $\mathbb{H}^{2|2}$ model,

$$-\sum_b \int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,0}} y_a y_b \mathcal{L}^\beta g(b, z-1) = g(a, 0). \tag{3.4}$$

Proof. The proofs are essentially the same for $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$, so we carry them out in parallel.

We write $\mathcal{L}$ for $\mathcal{L}^\beta$, H for $H_{\beta,0}$, and the integral stands for $\int_{(\mathbb{H}^n)^\Lambda}$ respectively $\int_{(\mathbb{H}^{2|2})^\Lambda}$. By (2.5) (resp. (2.24)) we have $y_b \frac{\partial}{\partial \ell_b} g(b, z-1) = \frac{\partial}{\partial s_b} g(b, z-1)$ where $\frac{\partial}{\partial \ell_b}$ denotes the derivative with respect to the b-th component of the second argument. Therefore

$$\begin{aligned}\sum_b \int e^{-H} y_a y_b \mathcal{L} g(b, z-1)& \\ &= \int e^{-H} y_a \Big(\sum_{b,c} \beta_{bc} y_b z_c (g(c, z-1) - g(b, z-1)) + \sum_b \frac{\partial}{\partial s_b} g(b, z-1) \Big).\end{aligned} \tag{3.5}$$

Recall (2.2) (resp. (2.20)) and integrate the second term in the equation above by parts. This produces two terms; by the rapid decay of g there are no boundary terms. For the first term produced by the integration by parts, using (2.5) (resp. (2.24)) again,

$$\sum_b \int e^{-H} y_a \left(-\frac{\partial H}{\partial s_b} \right) g(b, z-1) = \sum_b \int e^{-H} y_a \left(\sum_c \beta_{bc} \frac{\partial (u_b \cdot u_c)}{\partial s_b} \right) g(b, z-1) \tag{3.6}$$

$$= \sum_{b,c} \int e^{-H} y_a \beta_{bc} y_b z_c (g(c, z-1) - g(b, z-1)). \tag{3.7}$$

This term cancels the first term on the right-hand side of (3.5). For the second term produced by the integration by parts, we use that $\int x_a e^{-H} g(b,z) = 0$ by (2.26) (resp. (2.27))

$$\int e^{-H} \frac{\partial y_a}{\partial s_b} g(b, z-1) = \delta_{ab} \int e^{-H} (x_a + z_a) g(b, z-1) = \delta_{ab} \int e^{-H} z_a g(a, z-1). \tag{3.8}$$

In the supersymmetric case, the localisation lemma in the special case (2.29) further implies that the last right-hand side can be evaluated as

$$\delta_{ab} \int e^{-H} z_a g(a, z-1) = \delta_{ab} g(a, 0). \tag{3.9}$$

Altogether, we have shown (3.3) (resp. (3.4)). □

Proof of Theorem 1.2. It suffices to show (3.1) and (3.2) with $h = 0$, by replacing $g(b, z-1)$ by $g(b, z-1)e^{-h(z-1)}$. Therefore from now on assume $h = 0$. To get (3.2) from (3.4), we apply (3.4) with $g(i,\ell)$ replaced by $g_t(i,\ell) = \mathbb{E}_{i,\ell}(g(X_t, L_t))$. By the definition of the generator we have $\mathcal{L} g_t(i,\ell) = \frac{\partial}{\partial t} g_t(i,\ell)$, so (3.4) gives

$$\mathbb{E}_{a,0}(g(X_t, L_t)) = -\frac{\partial}{\partial t} \left(\sum_b \int e^{-H} y_a y_b g_t(b, z-1) \right). \tag{3.10}$$

Note that the process (X_t, L_t) is transient even if the marginal (X_t) is recurrent because $\sum_i L_t^i \to \infty$ as $t \to \infty$. Therefore, integrating both sides over t and using that $g_t(x,\ell) \to 0$ as $t \to \infty$, which follows from the transience of (X_t, L_t) and the rapid decay of $g = g_0$, we get

$$\int_0^\infty \mathbb{E}_{a,0}(g(X_t, L_t))\, dt = \sum_b \int e^{-H} y_a y_b g(b, z-1). \tag{3.11}$$

The proof of (3.1) from (3.3) is entirely analogous. □

4 Proof of Theorem 1.4

The proof of the hyperbolic Mermin–Wagner follows that of the usual Mermin–Wagner theorem as presented in [13] closely. We begin with the non-supersymmetric case. Due to the non-compact target space, differences occur in the bound of the term $\langle |DH|^2 \rangle$ and in the role of the coordinate in the direction of the magnetic field. As in the previous section we write H for $H_{\beta,h}$. We will write $\bar{A}$ to denote the complex conjugate of A.

Proof of (1.14). As in the previous section we write y_j for y_j^1. We also write $\langle \cdot \rangle$ for $\langle \cdot \rangle_{\mathbb{H}^n}$, and we use horospherical coordinates throughout the proof. Throughout the proof H will denote the energy of a spin configuration in horospherical coordinates, recall (2.4).

Let

$$S(p) = \frac{1}{\sqrt{|\Lambda|}} \sum_j e^{i(p \cdot j)} y_j, \quad D = \frac{1}{\sqrt{|\Lambda|}} \sum_j e^{-i(p \cdot j)} \frac{\partial}{\partial s_j}. \tag{4.1}$$

By the Cauchy–Schwarz inequality,

$$\langle |S(p)|^2 \rangle \geqslant \frac{|\langle S(p) DH \rangle|^2}{\langle |DH|^2 \rangle}. \tag{4.2}$$

In the following, we compute the terms on the left- and right-hand sides of the above inequality. Note that we have the integration by parts identity $\langle FDH\rangle = \langle DF\rangle$ for any smooth $F\colon (\mathbb{H}^n)^\Lambda \to \mathbb{R}$ that does not grow too fast; the vanishing of boundary terms can be seen by looking at the expression for H (i.e., by (2.4)).

By the assumed translation invariance of β,

$$\langle |S(p)|^2\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_j y_l\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_0 y_{j-l}\rangle = \sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle, \tag{4.3}$$

$$\langle S(p)DH\rangle = \langle DS(p)\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle\frac{\partial y_j}{\partial s_l}\rangle = \frac{1}{|\Lambda|}\sum_j\langle x_j + z_j\rangle = \langle z_0\rangle, \tag{4.4}$$

$$\langle |DH|^2\rangle = \langle D\bar{D}H\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\left\langle\frac{\partial^2 H}{\partial s_j\partial s_l}\right\rangle. \tag{4.5}$$

In (4.4) we have used $\langle x_j\rangle = 0$; recall Section 2.5. By $\langle x_j z_k\rangle = 0$, Cauchy–Schwarz, translation invariance, that $\langle x_0^2\rangle = \langle y_0^2\rangle$ (recall the symmetries from Section 2.5.1), and the constraint $u_0\cdot u_0 = -1$, observe that

$$\langle (x_j+z_j)(x_l+z_l)\rangle = \langle x_j x_l + z_j z_l\rangle \leqslant \langle x_0^2\rangle + \langle z_0^2\rangle = 1 + (n+1)\langle y_0^2\rangle. \tag{4.6}$$

Thus, using (2.6) and $\langle x_j\rangle = 0$ once more, (4.5) can be rewritten and bounded above by

$$\langle |DH|^2\rangle = \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}\langle (x_j+z_j)(x_l+z_l)\rangle(1-e^{ip\cdot(j-l)}) + \frac{h}{|\Lambda|}\sum_j\langle x_j+z_j\rangle \tag{4.7}$$

$$\leqslant \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}(1+(n+1)\langle y_0^2\rangle)(1-\cos(p\cdot(j-l))) + h\langle z_0\rangle. \tag{4.8}$$

In summary, we have shown (recall (1.12))

$$\langle |DH|^2\rangle \leqslant (1+(n+1)\langle y_0^2\rangle)\lambda(p) + h\langle z_0\rangle. \tag{4.9}$$

Substituting the above bounds into (4.2) gives

$$\sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle = \langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)DH\rangle|^2}{\langle |DH|^2\rangle} \geqslant \frac{\langle z_0\rangle^2}{(1+(n+1)\langle y_0^2\rangle)\lambda(p)+h\langle z_0\rangle} \tag{4.10}$$

$$\geqslant \frac{1}{(1+(n+1)\langle y_0^2\rangle)\lambda(p)+h}. \tag{4.11}$$

The last inequality follows from $h\geqslant 0$ and $1\leqslant\langle z_0\rangle$, which holds by the definition of $\mathbb{H}^n$. □

Proof of (1.15). We use that the expectation of a function $F(y)$ can be written using horospherical coordinates in terms of the *probability measure* (1.6). Throughout this proof, we denote the expectation with respect to this probability measure by $\langle\cdot\rangle$. By the Cauchy–Schwarz inequality, and since $S(p)$ is a function of the y,

$$\langle |S(p)|^2\rangle_{\mathbb{H}^{2|2}} = \langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)D\widetilde{H}\rangle|^2}{\langle |D\widetilde{H}|^2\rangle}. \tag{4.12}$$

The probability measure $\langle\cdot\rangle$ obeys the integration by parts $\langle F D\widetilde{H}\rangle = \langle DF\rangle$ identity for any function $F = F(s,t)$ that does not grow too fast. Therefore by translation invariance we find that, as in the case of $\mathbb{H}^n$,

$$\langle |S(p)|^2\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_j y_l\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_0 y_{j-l}\rangle = \sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle, \tag{4.13}$$

$$\langle S(p) D\widetilde{H}\rangle = \langle DS(p)\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle \frac{\partial y_j}{\partial s_l}\rangle = \frac{1}{|\Lambda|}\sum_j \langle e^{t_j}\rangle = 1, \tag{4.14}$$

where the last identity uses (2.28). By (2.28), Cauchy–Schwarz, and translation invariance we have

$$\langle e^{t_j+t_l}\rangle = 1 + \langle y_j y_l\rangle \leqslant 1 + \langle y_0^2\rangle. \tag{4.15}$$

Using (4.15) and the integration by parts identity it follows that

$$\begin{aligned}\langle |D\widetilde{H}|^2\rangle = \langle D\bar{D}\widetilde{H}\rangle &= \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}\langle e^{t_j+t_l}\rangle(1-\cos(p\cdot(j-l))) + \frac{h}{|\Lambda|}\sum_j\langle e^{t_j}\rangle \\ &\leqslant \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}(1+\langle y_0^2\rangle)(1-\cos(p\cdot(j-l))) + h \\ &= (1+\langle y_0^2\rangle)\lambda(p) + h. \end{aligned}\tag{4.16}$$

In summary, we have proved

$$\sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle = \langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)D\widetilde{H}\rangle|^2}{\langle |D\widetilde{H}|^2\rangle} \geqslant \frac{1}{(1+\langle y_0^2\rangle)\lambda(p)+h} \tag{4.17}$$

as claimed. □

Acknowledgments

We would like to thank D. Brydges, M. Disertori, M. Niedermaier, T. Spencer, and B. Tóth for helpful discussions on the topics of this paper. T.H. is supported by EPSRC grant no. EP/P003656/1. A.S. is supported by EPSRC grant no. 1648831.

References

[1] O. Angel, N. Crawford, and G. Kozma. Localization for linearly edge reinforced random walks. *Duke Mathematical Journal*, 163(5):889–921, 2014.

[2] D. Brydges, S.N. Evans, and J.Z. Imbrie. Self-avoiding walk on a hierarchical lattice in four dimensions. *Ann. Probab.*, 20(1):82–124, 1992.

[3] D. Brydges, J. Fröhlich, and T. Spencer. The random walk representation of classical spin systems and correlation inequalities. *Comm. Math. Phys.*, 83(1):123–150, 1982.

[4] D.C. Brydges, J.Z. Imbrie, and G. Slade. Functional integral representations for self-avoiding walk. *Probab. Surv.*, 6:34–61, 2009.

[5] B. Davis and S. Volkov. Continuous time vertex-reinforced jump processes. *Probab. Theory Related Fields*, 123(2):281–300, 2002.

[6] M. Disertori, F. Merkl, and S.W.W. Rolles. A supersymmetric approach to martingales related to the vertex-reinforced jump process. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(1):529–555, 2017.

[7] M. Disertori, C. Sabot, and P. Tarrès. Transience of edge-reinforced random walk. *Comm. Math. Phys.*, 339(1):121–148, 2015.

[8] M. Disertori and T. Spencer. Anderson localization for a supersymmetric sigma model. *Comm. Math. Phys.*, 300(3):659–671, 2010.

[9] M. Disertori, T. Spencer, and M.R. Zirnbauer. Quasi-diffusion in a 3D supersymmetric hyperbolic sigma model. *Comm. Math. Phys.*, 300(2):435–486, 2010.

[10] A. Duncan, M. Niedermaier, and E. Seiler. Vacuum orbit and spontaneous symmetry breaking in hyperbolic sigma-models. *Nuclear Phys. B*, 720(3):235–288, 2005.

[11] E.B. Dynkin. Markov processes as a tool in field theory. *J. Funct. Anal.*, 50(2):167–187, 1983.

[12] K.B. Efetov. Supersymmetry and theory of disordered metals. *Adv. in Phys.*, 32(1):53–127, 1983.

[13] J. Fröhlich and T. Spencer. On the statistical mechanics of classical Coulomb and dipole gases. *J. Statist. Phys.*, 24(4):617–701, 1981.

[14] Ġ. Horváth. Formulas on hyperbolic volume. *Aequationes Math.*, 83(1-2):97–116, 2012.

[15] Y. Le Jan. Temps local et superchamp. In *Séminaire de Probabilités, XXI*, volume 1247 of *Lecture Notes in Math.*, pages 176–190. Springer, Berlin, 1987.

[16] Y. Le Jan. *Markov paths, loops and fields*, volume 2026 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 38th Probability Summer School held in Saint-Flour, 2008, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[17] J.M. Luttinger. The asymptotic evaluation of a class of path integrals. II. *J. Math. Phys.*, 24(8):2070–2073, 1983.

[18] F. Merkl and S.W.W. Rolles. Edge-reinforced random walk on one-dimensional periodic graphs. *Probab. Theory Related Fields*, 145(3-4):323–349, 2009.

[19] F. Merkl and S.W.W. Rolles. Recurrence of edge-reinforced random walk on a two-dimensional graph. *Ann. Probab.*, 37(5):1679–1714, 2009.

[20] M. Niedermaier and E. Seiler. Non-amenability and spontaneous symmetry breaking—the hyperbolic spin-chain. *Ann. Henri Poincaré*, 6(6):1025–1090, 2005.

[21] C. Sabot and P. Tarrès. Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model. *Journal of the European Mathematical Society (JEMS)*, 17(9):2353–2378, 2015.

[22] C. Sabot and X. Zeng. A random Schrödinger operator associated with the Vertex Reinforced Jump Process on infinite graphs. Preprint, arXiv:1507.07944.

[23] E. Seiler. The strange world of non-amenable symmetries. In *Mathematical quantum field theory and renormalization theory*, volume 30 of *COE Lect. Note*, pages 66–77. Kyushu Univ. Fac. Math., Fukuoka, 2011.

[24] M. Shcherbina and T. Shcherbina. Universality for 1d random band matrices: sigma-model approximation. Preprint, arXiv:1802.03813.

[25] T. Spencer. SUSY statistical mechanics and random band matrices. In *Quantum many body systems*, volume 2051 of *Lecture Notes in Math.*, pages 125–177. Springer, Heidelberg, 2012.

[26] T. Spencer. Duality, statistical mechanics, and random matrices. In *Current developments in mathematics 2012*, pages 229–260. Int. Press, Somerville, MA, 2013.

[27] T. Spencer and M.R. Zirnbauer. Spontaneous symmetry breaking of a hyperbolic sigma model in three dimensions. *Comm. Math. Phys.*, 252(1-3):167–187, 2004.

[28] K. Symanzik. Euclidean quantum field theory. In R. Jost, editor, *Local Quantum Field Theory*, New York, 1969. Academic Press.

[29] A.-S. Sznitman. *Topics in occupation times and Gaussian free fields*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[30] M.R. Zirnbauer. Fourier analysis on a hyperbolic supermanifold with constant curvature. *Comm. Math. Phys.*, 141(3):503–522, 1991.