Infinite graphic matroids

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Abstract

An infinite matroid is *graphic* if all of its finite minors are graphic and the intersection of any circuit with any cocircuit is finite. We show that a matroid is graphic if and only if it can be represented by a graph-like topological space: that is, a graph-like space in the sense of Thomassen and Vella. This extends Tutte's characterization of finite graphic matroids.

Working in the representing space, we prove that any circuit in a 3-connected graphic matroid is countable.

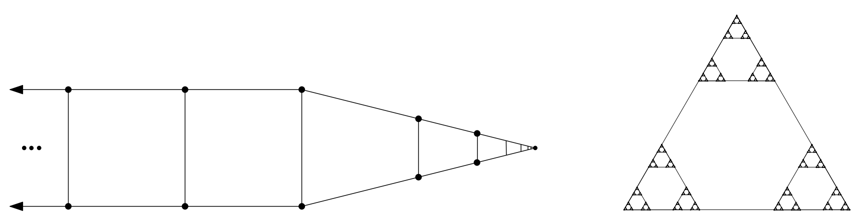
1 Introduction

There is a rich theory describing and employing the relationship between finite graphic matroids and finite graphs. In this paper, we will show how the foundations of this theory can be extended to infinite matroids [8]. A central result in the finite context is Tutte's characterisation by finitely many excluded minors of the class of matroids which can be represented by graphs [18].

Existing work with infinite graphic matroids has focused on a few possible constructions of matroids from infinite graphs, which generalise the construction of the cycle matroid of a finite graph. Most straightforwardly, for any infinite graph G we can consider the finite-cycle matroid, whose circuits are given by the finite cycles of G. We could also consider the algebraic-cycle matroid, whose circuits are given by finite cycles or double rays in G [14]. Alternatively, we can consider the topological cycle matroid, whose circuits are given by homeomorphic copies of the unit circle in the end-compactification of G [7]. Various ad-hoc extensions of these notions suggest themselves. For example, we could allow identification of ends with vertices in the definition of the topological cycle matroid [12].

Certain results about finite graphic matroids have been proved for these classes of infinite graphic matroids [6], [7], [9], [12], [16], and could also be proved about the ad-hoc extensions without too much trouble. But since all these notions fall far short of the natural boundary, namely the class of infinite matroids satisfying Tutte's excluded minor characterisation, in this paper we instead take the approach of isolating a notion of representation for which the representable matroids are precisely those satisfying Tutte's condition. Such matroids, and their representations, provide a natural context for the extension of results from finite to infinite graphic matroids.

That the existing approaches fall far short of providing representations of all graphic matroids is shown by examples like those depicted in Figure 1. Here the circuits of the matroids in question are again given by the (edge



(a) (b)

Figure 1: Subspaces of the plane inducing matroids

sets of) homeomorphic copies of the unit circle in the subspaces of the plane given in the pictures.

What these examples show is that infinite graphic matroids should, in general, be taken to be represented not by graphs but rather by graph-like topological spaces, in a sense akin to that of Thomassen and Vella [17]. This includes the existing approaches: the finite cycle matroid of a graph would be represented by its geometric realisation, the algebraic cycle matroid by a 1-point compactification and the topological cycle matroid by the end compactification.

We restrict our attention to tame matroids (those in which any intersection of a circuit with a cocircuit is finite) because this restriction has proved both natural and necessary in related representability problems [1], [2], [4]. We shall introduce a notion of representability of matroids over graph-like spaces for which we can prove the following:

Theorem 1.1. *A tame matroid satisfies Tutte's excluded minor characterisation if and only if it is representable over a graph-like space.*

We call matroids satisfying either of these equivalent conditions *graphic*.

At least for 3-connected matroids, the notion of representability is what you would hope: the circuits are given just as usual by homeomorphic copies of the unit circle. That this hope can be fulfilled is a little strange. After all, any circuit given in this way must be countable, and there is nothing in Tutte's excluded minor characterisation which appears to restrict the cardinality of circuits. We are saved by the following miraculous fact:

Theorem 1.2. *In any 3-connected tame matroid satisfying Tutte's excluded minor characterisation, all circuits are countable.*

In fact, in order to prove this we first introduce a notion of representability which doesn't entail any cardinality restrictions, then play the topological structure of the representing graph-like space off against the matroidal structure.

In an extended version of this work available at 'http://www.math.uni-hamburg.de/spag/dm/projects/matroids.html' and 'http://www.math.uni-hamburg.de/home/carmesin/', we show that the spaces in question are topologically well-behaved, and deduce essential desiderata, such as that the bases of the matroid correspond to minimal connected subspaces containing all vertices.

The structure of the paper is as follows. In Section 2, we recall some preliminary lemmas from the theory of infinite matroids. In Section 3 we introduce graph-like spaces and in Section 4 we introduce the subspaces which will play the role of topological circles. In Section 5 we introduce the notion of representation. In Section 6 we prove Theorem 1.1. In Section 7 we introduce a kind of forbidden substructure which we will make use of in our proof of Theorem 1.2 in Section 8. We conclude by discussing the notion of planarity for infinite matroids in Section 9.

2 Preliminaries

Throughout, notation and terminology for (infinite) graphs are those of [13], and for matroids those of [15, 8].

M always denotes a matroid and $E(M)$ (or just E), $\mathcal{I}(M)$ and $\mathcal{C}(M)$ denote its ground set and its sets of independent sets and circuits, respectively. For the remainder of this section we shall recall some basic facts about infinite matroids.

A set system $\mathcal{I} \subseteq \mathcal{P}(E)$ is the set of independent sets of a matroid if and only if it satisfies the following *independence axioms* [8].

(I1) $\varnothing \in \mathcal{I}(M)$.

(I2) $\mathcal{I}(M)$ is closed under taking subsets.

(I3) Whenever $I, I' \in \mathcal{I}(M)$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}(M)$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}(M)$, the set $\{I' \in \mathcal{I}(M) \mid I \subseteq I' \subseteq X\}$ has a maximal element.

A set system $\mathcal{C} \subseteq \mathcal{P}(E)$ is the set of circuits of a matroid if and only if it satisfies the following *circuit axioms* [8].

(C1) $\varnothing \notin \mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) (Circuit elimination) Whenever $X \subseteq o \in \mathcal{C}(M)$ and $\{o_x \mid x \in X\} \subseteq \mathcal{C}(M)$ satisfies $x \in o_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in o \setminus \left(\bigcup_{x \in X} o_x\right)$ there exists a $o' \in \mathcal{C}(M)$ such that $z \in o' \subseteq \left(o \cup \bigcup_{x \in X} o_x\right) \setminus X$.

(CM) $\mathcal{I}$ satisfies (IM), where $\mathcal{I}$ is the set of those subsets of E not including an element of $\mathcal{C}$.

For a base s of a matroid M, and $e \in E \setminus s$, there is a unique circuit o_e with $e \in o_e \subseteq s + e$. We call this circuit the *fundamental circuit* of e with respect to s. Similarly, for $f \in b$ we call the unique cocircuit b_f with $f \in b_f \subseteq (E \setminus s) + f$ the *fundamental cocircuit* of f with respect to s.

The following straightforward Lemmas can be proved as for finite matroids (see, for example, [3]).

Lemma 2.1. *Let M be a matroid and s be a base. Let o_e and b_f a fundamental circuit and a fundamental cocircuit with respect to s, then*

1. *$o_e \cap b_f$ is empty or $o_e \cap b_f = \{e, f\}$ and*

2. *$f \in o_e$ if and only if $e \in b_f$.*

Lemma 2.2. *For any circuit o containing two edges e and f, there is a cocircuit b such that $o \cap b = \{e, f\}$.*

Lemma 2.3. *Let I be some independent set in some matroid M. Then for each $e \in I$ there is a cocircuit b meeting I precisely in e*

Lemma 2.4. *Let M be a matroid with ground set $E = C\dot{\cup}X\dot{\cup}D$ and let o' be a circuit of $M' = M/C\backslash D$. Then there is an M-circuit o with $o' \subseteq o \subseteq o' \cup C$.*

Lemma 2.5. *Let M be a matroid, and let $w \subseteq E$. The following are equivalent:*

1. *w is a union of circuits of M.*
2. *w never meets a cocircuit of M just once.*

The basic theory of infinite binary matroids is introduced in [3]. One characterisation of such matroids given there is that every intersection of a circuit with a cocircuit is both finite and of even size.

Lemma 2.6. *Let M be a binary matroid and $X \subseteq E(M)$ with the property that it meets every circuit finitely and evenly. Then X is a disjoint union of cocircuits.*

Proof. By Zorn's Lemma, we can pick $Y \subseteq X$ maximal with the property that it is a disjoint union of cocircuits. As $Y \subseteq X$, the set Y meets every circuit finitely, and so meets every circuit evenly. By the choice of Y, the set $X \setminus Y$ does not include a circuit. But $X \setminus Y$ meets every circuit evenly, and so is empty by the dual of Lemma 2.5. This completes the proof. □

Lemma 2.7. *Suppose that M is a matroid, and $\mathcal{C}$, $\mathcal{C}^*$ are collections of subsets of $E(M)$ such that $\mathcal{C}$ contains every circuit of M, $\mathcal{C}^*$ contains every cocircuit of M, and for every $o \in \mathcal{C}$, $b \in \mathcal{C}^*$, $|o \cap b| \neq 1$. Then the set of minimal nonempty elements of $\mathcal{C}$ is the set of circuits of M and the set of minimal nonempty elements of $\mathcal{C}^*$ is the set of cocircuits of M.*

Proof. The conditions imply that no element of $\mathcal{C}$ ever meets a cocircuit of M just once, so every element of $\mathcal{C}$ is a union of circuits of M by Lemma 2.5. Since every circuit of M is in $\mathcal{C}$, the minimal nonempty elements of $\mathcal{C}$ are precisely the circuits of M. The other claim is obtained by a dual argument. □

A *switching sequence* for a base s in a matroid with ground set E is a finite sequence $(e_i | 1 \leq i \leq n)$ whose terms are alternately in s and not in s and where for $i < n$ if $e_i \in s$ then $e_{i+1} \in b_{e_i}$ and if $e_i \notin s$ then $e_{i+1} \in o_{e_i}$.

Lemma 2.8. *Let M be a connected matroid with a base s, and e and f be edges of M. Then there is a switching sequence with first term e and last term f.*

Proof. Let e be any edge of M, and let X be the set of those $f \in E(M)$ for which there is such a switching sequence. Then $s \cap X$ is a base for X, since for any $f \in X \setminus s$ we have $o_f \subseteq X$. Similarly, $s \setminus X$ is a base for $E(M) \setminus X$, since for any $f \in E(M) \setminus X \setminus s$ and any $g \in o_f$ we have $f \in b_g$ by Lemma 2.1 and so $g \notin X$. Thus X and $E(M) \setminus X$ form a separation of M, and since M is connected this means that X must be the whole of E, completing the proof. □

A k-separation of a matroid M is a partition (A, B) of the ground set of M such that each of A and B has size at least k and there are bases s_A and s_B of A and B and s of A such that $|s_A \cup s_B \setminus s| < k$. A 1-separation may also be called a *separation*. A matroid without l-separations for any $l < k$ is *k-connected*. A matroid is *connected* if it is 2-connected. Connected matroids can equivalently be characterised as those in which any 2 distinct edges lie on a common circuit [10].

3 Graph-like spaces

The key notion of this section is the following, which is based on a definition from [17]:

Definition 3.1. A *graph-like space* G is a topological space (also denoted G) together with a *vertex set* $V = V(G)$, an *edge set* $E = E(G)$ and for each $e \in E$ a continuous map $\iota_e^G \colon [0,1] \to G$ (the superscript may be omitted if G is clear from the context) such that:

- The underlying set of G is $V \sqcup [(0,1) \times E]$
- For any $x \in (0,1)$ and $e \in E$ we have $\iota_e(x) = (x, e)$.
- $\iota_e(0)$ and $\iota_e(1)$ are vertices (called the *endvertices* of e).
- $\iota_e \restriction_{(0,1)}$ is an open map.
- For any two distinct $v, v' \in V$, there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$.

The inner points of the edge e are the elements of $(0,1) \times \{e\}$.

Note that $V(G)$, considered as a subspace of G, is totally disconnected, and that G is Hausdorff.

Let e be an edge in a graph-like space with $\iota_e(0) \neq \iota_e(1)$. Then ι_e is a continuous injective map from a compact to a Hausdorff space and so it is a homeomorphism onto its image. The image is compact and so is closed, and therefore is the closure of $(0,1) \times \{e\}$ in G. So in this case ι_e is determined by the topology of G. The same is true if $\iota_e(0) = \iota_e(1)$: in this case we can lift ι_e to a continuous map from $S^1 = [0,1]/(0=1)$ to G, and argue as above that this map is a homeomorphism onto the closure of $(0,1) \times \{e\}$ in G. In this case, we say that e is a loop of G.

Next we shall define maps of graph-like spaces. Let G and G' be graph-like spaces. Two maps $\varphi_V : V(G) \to V(G')$ and $\varphi_E : E(G) \to (E(G') \times \{+,-\}) \sqcup V(G)$ induce a function φ sending points of G to points of G' as follows: a vertex v of G is mapped to $\varphi_V(v)$. Let e be an edge, and (r,e) one of its interior points. If $\varphi_E(e)$ is a vertex, then (r,e) is mapped to $\varphi_E(e)$. If $\varphi_E(e) = (f,+)$ for some $f \in E(G')$, then (r,e) is mapped to (r,f). Similarly, if $\varphi_E(e) = (f,-)$ for some $f \in E(G')$, then (r,e) is mapped to $(1-r,f)$. If a function arising in this way is continuous we call it a *map of graph-like spaces*. From this definition, it follows that if v is an endvertex of e, then $\varphi(v)$ is either an endvertex of or equal to the image of e.

Let us consider some examples of graph-like spaces. We shall write $[0,1]$ for the unique graph-like space without loops having precisely one edge and two vertices. There are exactly seven maps of graph-like spaces from $[0,1]$ to two copies of $[0,1]$ glued together at a vertex: four of these have one of the copies of $[0,1]$ as their image and the other three map the whole interval to a vertex. However, none of these maps is bijective nor has an inverse, even though the underlying topological spaces are homeomorphic.

Figures 1a and 1b from the introduction define graph-like spaces with vertices and edges as in the figures. In each case the topology is that induced by the embedding in the plane suggested by the figures. For a locally finite graph $G = (V,E)$, the topological space $|G|$ is a graph-like space with vertex set $V \cup \Omega(G)$ and edge set E (see [13] for the definition of $|G|$). Note that if G is finite, then $|G|$ is homeomorphic to the geometric realisation of G considered as a simplicial complex.

Lemma 3.2. *Let G be a graph-like space with only finitely many edges and finitely many vertices. Then G is homeomorphic to $|H|$ for some finite graph H.*

Proof. G is compact, since it is a union of finitely many compact subspaces. Let H be the graph with edge set $E(G)$ and vertex set $V(G)$, and in which v is an endpoint of e if and only if this is true in G. We now construct a map $\varphi \colon G \to |H|$ as follows: taking φ_V to be the identity and φ_E to be the function sending each edge e to $(e,+)$, we build φ as in the definition of a map of graph-like spaces.

It remains to show that the function φ is continuous: since it is a bijection from a compact to a Hausdorff space, it will then be a homeomorphism. We begin by noting that for any $e \in E(G)$, the restriction of φ to the image of ι_e^G is a homeomorphism, by the remarks following Definition 3.1. Now we need to show for any $x \in |H|$ that the inverse image of any open neighbourhood U of $\varphi(x)$ includes an open neighbourhood of x. If x is an interior point of an edge, this is clear. Otherwise, x is a vertex of $|H|$. Then there is an open neighbourhood $U' \subseteq U$ of x which only meets edges incident with x. For each such edge e, since the restriction of φ to the image of ι_e^G is a homeomorphism, there is an open set V_e of G with $V_e \cap \mathrm{Im}(\iota_e^G) = \varphi^{-1}(U') \cap \mathrm{Im}(\iota_e^G)$. Letting V be the intersection of the V_e, we obtain that V is an open neighbourhood of x included in $\varphi^{-1}(U)$, completing the proof that φ is continuous. □

All the above examples of graph-like spaces will turn out to induce matroids. Before we can make this more explicit, we must first introduce the notions of topological circuits and bonds in a graph-like space. The discussion of topological circuits will be delayed until the next section, but we will introduce topological bonds now.

Definition 3.3. Given a pair of disjoint open subsets of a graph-like space G partitioning the vertices, we call the set of those edges having an endvertex in both sets *a topological cut* of G. A *topological bond* of G is a minimal nonempty topological cut of G.

Given a graph-like space G and a set of edges $R \subseteq E(G)$, we define the graph-like space $G\restriction_R$, the *restriction* of G to R, to have the same vertex set as G and edge set R. Then the ground set of $G\restriction_R$ is a subset of that of G, and we give it the subspace topology. Evidently, for any topological cut b of G, $b \cap R$ is a topological cut of $G\restriction_R$. The *deletion* of D from G, denoted by $G\backslash D$, is $G\restriction_{(E\backslash D)}$. We abbreviate $G\backslash\{e\}$ by $G - e$. The inclusion map g_D from $G\backslash D$ to G is a map of graph-like spaces.

Note that $G\restriction R$ has the same vertex set as G, even though only the vertices in the closure of $(0, 1) \times R$ play an important role in the new space. By analogy to the notation of [13], we also introduce a notation for the graph-like space whose edges are those in R but whose vertices are those in the closure of $(0, 1) \times R$. We will call this subspace the *standard subspace with edge set* R, and denote it $\overline{R}$.

Given a graph-like space G and $C \subseteq E(G)$, we define the *contraction* G/C of G onto C as follows:

Let $\equiv_C$ be the relation on the vertices of G defined by $u \equiv_C v$ if every topological cut with u and v in different parts meets C. It is easy to check that $\equiv_C$ is an equivalence relation. The vertex set of G/C is the set of $\equiv_C$-equivalence classes, and the edge set is $E(G) \setminus C$.

It remains to define the topology of G/C. We shall obtain this as the quotient topology derived from a function $f_C \colon G \to G/C$, to be defined next.

The function f_C sends each vertex to its $\equiv_C$-equivalence class and is bijective on the interior points of edges of $E \setminus C$. The two endpoints of an edge in C are in the same equivalence class, and we send all of its interior points to that equivalence class.

Taking this quotient topology ensures that G/C is a graph-like space, and makes f_C a map of graph-like spaces. In G/C, the endpoints of an edge are the equivalence classes of its endpoints in G. For any topological cut b of G with $b \cap C = \emptyset$, the two sides of b are closed under $\equiv_C$ by definition, and so b is also a topological cut in G/C.

We define $G.X := G/(E \setminus X)$ and $G/e := G/\{e\}$. It is straightforward to check for disjoint sets C and D that $(G\backslash D)/C$ and $(G/C)\backslash D$ are equal and the following diagram commutes.

$$\begin{array}{ccc} G\backslash D & \xrightarrow{g_D} & G \\ {\scriptstyle f_C}\downarrow & & \downarrow{\scriptstyle f_C} \\ G/C\backslash D & \xrightarrow[g_D]{} & G/C \end{array}$$

Contraction behaves especially well when applied to one side of a topological cut [5].

4 Pseudoarcs and Pseudocircles

When investigating a topological space, it is common to consider arcs in that space, that is, continuous injections from the unit interval to that space. We must consider maps from a slightly more general kind of domain. These domains, which we will call *pseudo-lines*, will be graph-like spaces built from total orders in the following way:

Definition 4.1. Let P be a totally ordered set. To construct the pseudo-line $L(P)$, we take as our vertex set V the set of initial segments of P, and as our edge set P itself. Next, we take a subbasis of the topology to consist of the sets of the type $S(p,r)^+$ or $S(p,r)^-$ defined below.

For every $p \in P$ and $r \in (0,1)$, let $S(p,r)^-$ contain precisely those vertices which do not contain p. Furthermore, let $S(p,r)^-$ contain all interior points of edges x with $x < p$ together with $(0, r) \times \{p\}$.

Similarly, let $S(p, r)^+$ contain precisely those vertices which contain p. Furthermore, let $S(p, r)^+$ contain all interior points of edges x with $x > p$ together with $(r, 1) \times \{p\}$.

A *pseudo-path from v to w in a graph-like space G* is a map φ of graph-like spaces from a pseudo-line $L(P)$ to G with $\varphi(\emptyset) = v$ and $\varphi(P) = w$. The vertex v is called the *start-vertex* of the pseudo-path, and w is called the *end-vertex*.

A *pseudo-arc* is an injective pseudo-path. Any pseudo-arc is a homeomorphism onto its image since the domain is (as we shall soon show) compact, and the codomain is Hausdorff. Thus we will also refer to the images of pseudo-arcs as pseudo-arcs. In particular, a *pseudo-arc in* a graph-like space G is the image of such a map (in other words, it is a subspace of G which is also a pseudo-line).

Lemma 4.2. *The spaces $L(P)$ defined above are connected and compact.*

Proof. For the connectedness, let U be an open and closed set containing the start-vertex $\emptyset$. Since for any edge e the subspace topology of $\iota_e([0, 1])$ is that of $[0, 1]$, which is connected, the set $\iota_e([0, 1])$ is either completely included in U or disjoint from U. Let $v = \{p \in P | S(p, 1/2)^- \subseteq U\}$. Then the vertex v is in U since any neighbourhood of it meets U (even if $v = \emptyset$). So since U is open, it includes an open neighbourhood O of v. Since by our earlier remarks U includes all edges $p \in v$ and so also all vertices $w \subseteq v$, we may assume without loss of generality that either $v = P$ or else O has the form $S(p, r)^-$ for some $p \notin v$. In the second case we conclude that $p \in v$, which is impossible. Hence $v = P$. Since the closure of $\bigcup_{p \in P} \iota_p((0, 1))$ is the whole of $L(P)$, the closed set U is the whole of $L(P)$. Hence $L(P)$ is connected, as desired.

It remains to show that $L(P)$ is compact. By Alexander's theorem, it suffices to check that any open cover by subbasic open elements has a finite subcover. Let $L(P) = \bigcup_{i \in I^+} S(p_i, r_i)^+ \cup \bigcup_{i \in I^-} S(p_i, r_i)^-$ be an open cover by subbasic open sets. Let $v = \{p \in P | \exists i \in I^- : p < p_i\}$.

First we consider the case where there is some $i \in I^+$ with $v \in S(p_i, r_i)^+$. Then $p_i \in v$, so there is some $j \in I^-$ such that $p_i < p_j$. This means that $S(p_i, r_i)^+$ and $S(p_j, r_j)^-$ cover $L(P)$.

Otherwise there is some $i \in I^-$ with $v \in S(p_i, r_i)^-$. Then $p_i \notin v$ and so p_i is maximal amongst the p_j with $j \in I^-$. Thus $v + p_i$ is contained in some $S(p_k, r_k)^+$ with $k \in I^+$. Then $S(p_i, r_i)^-$ and $S(p_k, r_k)^+$, together with some finite collection of sets from our cover covering the compact subspace $\iota_{p_i}([0, 1])$, form a finite subcover, completing the proof. □

Example 4.3. If $P = \omega_1$, then $L(P)$ is the *long line*, which is not homeomorphic to $[0,1]$.

Remark 4.4. *Any nontrivial pseudo-line is the closure of the set of interior points of its edges. Any nontrivial pseudo-arc in a graph-like space is the standard subspace corresponding to its set of edges.*

Remark 4.5. *Contracting a set of edges of a pseudo-line $L(P)$ corresponds to removing that set of edges from the associated poset P.*

Corollary 4.6. *Any contraction of a pseudo-line is a pseudo-line.* $\square$

Lemma 4.7. *Any nontrivial pseudo-line $L(P)$ with only countably many edges is homeomorphic to the unit interval.*

Proof. Let $\bar{\mathbb{Q}} = \mathbb{Q} \cap (0,1)$. Consider the lexicographic linear order on $P \times \bar{\mathbb{Q}}$. This is dense, countable and has neither a largest nor a smallest element. Since the theory of such linear orders is countably categorical, this order is isomorphic to the order of $\bar{\mathbb{Q}}$. Pick an isomorphism ϕ from $P \times \bar{\mathbb{Q}}$ to $\bar{\mathbb{Q}}$.

For any $x \in [0,1]$ such that there are $p \in P$ and $q, r \in \bar{\mathbb{Q}}$ with $\phi(p,q) < x < \phi(p,r)$ we set $f(x) = (p, \sup\{q \in \bar{\mathbb{Q}} | \phi(p,q) < x\})$ (in such cases, p is clearly uniquely determined). Otherwise we set $f(x) = \{p \in P | (\forall q \in \bar{\mathbb{Q}}) \phi(p,q) < x\}$. This gives an injection f from $[0,1]$ to $L(P)$. It is continuous by the definition of the topology on $L(P)$, and so is a homeomorphism since $[0,1]$ is compact and $L(P)$ is Hausdorff. $\square$

Lemma 4.8. *Let $s_1 <_L \ldots <_L s_n$ be finitely many edges of a pseudo-line L. Let $S = \bigcup_{i=1}^{n} \iota_{s_i}((0,1))$. Then $L \setminus S$ has $n+1$ components each of which is a pseudo-line. These are $S(s_1, 1/2)^- \setminus S$, and $S(s_{i+1}, 1/2)^- \cap S(s_i, 1/2)^+) \setminus S$ for $1 \le i \le n-1$ and $S(s_n, 1/2)^+ \setminus S$.*

Proof. The assertion follows by induction from the following. Let $e \in L$. Then $L - e$ has two components that are both pseudo-arcs. These are $S(e, 1/2)^- \setminus ((0,1) \times \{e\})$ and $S(e, 1/2)^+ \setminus ((0,1) \times \{e\})$. $\square$

We get a total order $\leqq$ on the set of points of the space $L(P)$ as follows: if v and w are vertices, we set $v \leqq w$ when $v \subseteq w$. If v is a vertex and (p,q) an interior point of an edge, we set $v \leqq (p,q)$ when $p \notin v$ and $(p,q) \leqq v$ when $p \in v$. Finally, we order the interior points of edges by the lexicographic order on $P \times (0,1)$.

Lemma 4.9. *Let X be a nonempty closed subset of a pseudo-line $L(P)$. Then X contains a $\leqq$-smallest and a $\leqq$-biggest element.*

Proof. First we show that X contains a $\leqq$-biggest element.

Let $v = \{p \in P | (\exists x \in X)(\exists r \in (0,1))(p,r) \leqq x\}$. If $v \in X$ then it is evidently the $\leqq$-biggest element of X. Otherwise, since X is closed, there must be some basic open set containing v but avoiding X. Without loss of generality this set is of the form $S(e,r)^+$. Then $e \in v$, and so there must be some $r' \in (0,1)$ with $(e,r') \in X$. Since X is closed there is a maximal such r'. Then (e,r') is the maximal element of X.

The proof that X contains a $\leqq_L$-smallest element is analogous. □

The *concatenation of two pseudo-lines* L *and* M is obtained from the disjoint union of L and M by identifying the end-vertex of L with the start-vertex of M.

Remark 4.10. *The concatenation of two pseudo-lines is a pseudo-line.* □

Remark 4.11. *Taking the concatenation of 2 pseudo-lines corresponds to taking the disjoint union of the two corresponding posets, where in the new ordering we take all elements of the second poset to be greater than all elements of the first.*

Let $P : L \to G$ and $Q : M \to G$ be two pseudo-arcs such that the end-vertex t_P of P is the start-vertex s_Q of Q. Then *their concatenation* is the function $f : (L \sqcup M)/(t_P = s_Q) \to G$ which restricted to L is just P and restricted to M is just Q. For a pseudo-arc $Q : M \to G$ and vertices x and y in the image of Q, we write xQy for the restriction of Q to those points of M that are both $\leqq_L$-bigger than $Q^{-1}(x)$ and $\leqq_L$-smaller than $Q^{-1}(y)$. Note that xQy is a pseudo-arc from x to y. If Q is a pseudo-arc from v to w and x and y are vertices in the image of Q, we abbreviate xQw by xQ and vQy by Qy.

Lemma 4.12. *Let $P : L \to G$ be a pseudo-arc from x to y and $Q : M \to G$ be a pseudo-arc from y to z. Then the concatenation of P and Q includes a pseudo-arc from x to z*

The corresponding Lemma about arcs needs the requirement that $x \neq z$. However, we avoid this requirement because there is a pseudo-line whose start- and end-vertex are equal, namely the trivial pseudo-line.

Proof. Let I be the intersection of the image of P with the image of Q, which is closed, being the intersection of two closed sets. Then $P^{-1}(I)$ is closed as P is continuous, and contains a $\leqq_L$-minimal element w by Lemma 4.9.

If w is not a vertex, then $P(w)$ is not a vertex and thus is contained in $\iota_e((0,1))$ for some edge e. Since P and Q both contain the whole of $\iota_e([0,1])$

if they contain some point from $\iota_e((0,1))$, the same is true for I. But then $\iota_e([0,1]) \subseteq I$, which contradicts the choice of w. Hence w is a vertex. Let $w' = P(w)$

Thus $w'Q$ is a pseudo-arc. By Remark 4.10, the concatenation of Pw' and $w'Q$ is the desired pseudo-arc since their images meet precisely in w'. □

A *pseudo-circle* is a graph-like space obtained by identifying the endvertices of a nontrivial pseudo line.

We have the following relation between pseudo-lines and pseudo-circles. Every pseudo-circle C with one edge removed is a pseudo-line with endvertices the endvertices of the removed edge.

Conversely, let P and Q be pseudo-lines where P has endvertices s_P and t_P and Q has endvertices s_Q and t_Q. Then the graph-like space obtained from the disjoint union of P and Q by identifying s_P with t_Q and t_P with s_Q is a pseudo-circle or else is the trivial graph-like space.

So from Corollary 4.6 we obtain the following:

Corollary 4.13. *Any contraction of a pseudo-circle in which not all edges are contracted is a pseudo-circle.* □

Using Lemma 4.7 we get:

Corollary 4.14. *Any countable pseudo-circle is homeomorphic to S^1.* □

Definition 4.15. A *cyclic order on a set* X is a relation $R \subseteq X^3$, written $[a,b,c]_R$, that satisfies the following axioms:

1. Cyclicity: If $[a,b,c]_R$ then $[b,c,a]_R$.
2. Asymmetry: If $[a,b,c]_R$ then not $[c,b,a]_R$.
3. Transitivity: If $[a,b,c]_R$ and $[a,c,d]_R$ then $[a,b,d]_R$.
4. Totality: If a, b, and c are distinct, then either $[a,b,c]_R$ or $[c,b,a]_R$.

Remark 4.16. *The edge set of a pseudo-circle C has a canonical cyclic order R_C (up to choosing an orientation). Conversely, for any nonempty cyclic order there exists a pseudo-circle (unique up to isomorphism) such that its edge set has the same cyclic order.* □

We also get a cyclic order R'_C on the set of all points of a pseudo-circle C, corresponding to the order $\leqq$ on the set of points of a pseudo-line. Once more there are two canonical choices of cyclic order on C, one for each orientation of C; in fact, we shall take this as our definition of an orientation of C. For us, an orientation of a pseudo-circle C is a choice of one of the two canonical cyclic orders of the points of C.

Let $s \subseteq o$ and let $R \subseteq o^3$ be a cyclic order. The *cyclic order of s inherited from R* is R restricted to s^3. We say that e, g are *clockwise adjacent* in the cyclic order R if $[e, g, f]_R$ for any other f in o. In a finite cyclic order, for each e there is a unique g clockwise adjacent to e, which we denote by $n(e)$.

From Lemma 4.8 we obtain the following.

Corollary 4.17. *Let s be a finite nonempty set of edges of a pseudo-circle C. Let $S = \bigcup_{e \in s} \iota_e((0,1))$. Then $L \setminus S$ has $|s|$ components each of which is a pseudo-line.*

For each such component there is a unique $e \in s$ such that the component contains precisely those edges f with $[e, f, n(e)]_{R_C}$, where $n(e)$ is taken with respect to the induced cyclic order on s. □

For a graph-like space G, we also use the term pseudo-circle to describe an injective map of graph-like spaces from a pseudo-circle to G, as well as the image of such a map. In particular, a *pseudo-circle in G* is the image of such a map (or, in other words, it is a subspace of G which is also a pseudo-circle). If G is a graph-like space and C is a pseudo-circle in G, the set of edges of C is called a *topological circuit* of G. Thus the pseudo-circles in G are precisely the standard subspaces of G corresponding to the topological circuits.

Lemma 4.18. *The intersection of a topological circuit with a topological cut is never only one edge.*

Proof. Suppose for a contradiction that there are a topological circuit o and a topological cut b that intersect in only one edge f. In the graph-like space $\overline{o}$, the set $b \cap o$ is a topological cut consisting of a single edge f. This contradicts the fact that removing any edge does not disconnect the pseudo-circle $\overline{o}$, which completes the proof. □

We can also show that the intersection of topological circuits with topological cuts is finite. In fact, we can prove something a little more general.

Lemma 4.19. *Let o be a set of edges in a graph-like space G such that $\overline{o}$ is compact. The the intersection of o with any topological cut b is finite.*

Proof. Let b be induced by the open sets U and U'. The sets $U \cap \overline{o}$ and $U' \cap \overline{o}$, together with all the sets $(0,1) \times \{e\}$ with $e \in o$, comprise an open cover of $\overline{o}$. So there is a finite subcover, which can only contain $(0,1) \times \{e\}$ for finitely many edges e. For any other edge f of o we must have $(0,1) \times \{f\} \subseteq U \cup U'$, and it must be a subset either of U or of V since it is connected: in particular, no such f can be in b. □

5 Graph-like spaces inducing matroids

In this section we will explain what it means for a graph-like space to induce a matroid and prove some fundamental facts about graph-like spaces inducing matroids which we will need in Section 6 and Section 8.

If for a graph-like space G there is a matroid M on $E(G)$ whose circuits are precisely the topological circuits of G and whose cocircuits are precisely the topological bonds of G, then we say that G *induces* M, and we may denote M by $M(G)$. Note that there can only be one such matroid since a matroid is uniquely defined by its set of circuits.

Example 5.1. For any finitely separable graph G the space $|G|$ induces the topological cycle matroid $M_C(G)$. The one-point compactification of a locally finite graph G induces the algebraic cycle matroid $M_A(G)$; if G is not locally finite and does not include a subdivision of the Bean graph, a similar construction can be used to construct a noncompact graph-like space that induces $M_A(G)$. Finally, the geometric realisation of G induces the finite cycle matroid $M_{FC}(G)$.

Lemma 5.2. *Let G be a graph-like space, and suppose G induces a matroid M. Then for any $C, D \subseteq E(M)$, the graph-like space $G/C\backslash D$ induces $M/C\backslash D$.*

Proof. Let $\mathcal{C}$ and $\mathcal{C}^*$ be respectively the collection of topological circuits and the collection of topological cuts of $G/C\backslash D$. We will show that every circuit of $M/C\backslash D$ is in $\mathcal{C}$, and that every cocircuit of $M/C\backslash D$ is in $\mathcal{C}^*$. Lemma 4.18 states that for every $o \in \mathcal{C}$, $b \in \mathcal{C}^*$, $|o \cap b| \neq 1$, so it will follow by Lemma 2.7 that the topological circuits of $G/C\backslash D$ are the circuits of $M/C\backslash D$ and that the minimal topological cuts (i.e. the topological bonds) of $G/C\backslash D$ are the cocircuits of $M/C\backslash D$, completing the proof.

Let o be a circuit of $M/C\backslash D$. By Lemma 2.4 there is a circuit o' of M such that $o \subseteq o' \subseteq o \cup C$. Since o' is a circuit of M, there is a pseudo-circle O in G with edge-set o'. Let $f_C : G \to G/C$ be as in the definition of the contraction G/C. Then $f_C\restriction_O$ is a map of graph-like spaces from O to a subspace of $G/C\backslash D$ that has edge-set o. If it describes a contraction of $O \cap C$, then Lemma 4.13 implies that o is a circuit of $G/C\backslash D$ as required. Otherwise, some vertex of $G/C\backslash D$ must contain two vertices p and q of O

such that their deletion from the pseudo-circle O leaves two elements e and f of o in different components of $O - p - q$. Then by Lemma 2.2 there is a cocircuit b of $M/C\backslash D$ with $o \cap b = \{e, f\}$. Using the dual of Lemma 2.4, there is a cocircuit b' of M with $b \subseteq b' \subseteq b \cup D$, so that $o' \cap b' = \{e, f\}$. b' is a topological bond of G not meeting C and with p and q on opposite sides, contradicting the assumption that they are identified when we contract C.

Let b be a cocircuit of $M/C\backslash D$. It follows by the dual of Lemma 2.4 that there is a cocircuit b' of M (hence also a topological cut of G) such that $b \subseteq b' \subseteq b \cup D$. Let U, V be the disjoint open sets in G that partition $V(G)$ so that the set of edges with an end in each of U and V is b'. Let $f_C : G \mapsto G/C$ be the map of graph-like spaces describing the contraction of C from G. Since b' is disjoint from C, f_C does not identify any element of U with any element of V. Thus $f_C(U), f_C(V)$ are open sets in $G/C\backslash D$, and b is the set of edges with an end in each, showing that b is a topological cut of $G/C\backslash D$, as required.

□

Proposition 5.3. *Let G be a graph-like space inducing a connected matroid M with a base s. Then for any edges e and f of M, and any endvertices v of e and w of f, there is a unique pseudo-arc from v to w that uses only edges in s.*

Proof. By Lemma 2.8, we can find a switching sequence $(e_i | 1 \leq i \leq n)$ for s with first term e and last term f. Pick a sequence $(v_i | 1 \leq i \leq n)$, with first term v and last term w, where for each i the vertex v_i is an endvertex of e_i. Then for any $i < n$ we can find a pseudo-arc from v_i to v_{i+1} using only edges of s: if $e_i \in s$ then we take an interval of the pseudo-arc $\overline{o_{e_{i+1}}} \setminus e_{i+1}$, and if $e_i \notin s$ then we take an interval of the pseudo-arc $\overline{o_{e_i}} \setminus e_i$. Repeatedly applying Lemma 4.12 we find the desired pseudo-arc from v to w.

To show uniqueness, we suppose for a contradiction that there are 2 distinct such pseudo-arcs R_1 and R_2. Then without loss of generality there is an edge e_0 in $R_1 \setminus R_2$.

Let $a \in R_1 \cap R_2$ be the $\leqq_{R_1}$-smallest point that is still $\leqq_{R_1}$-bigger than any point on e_0; such a point exists as the intersection of the two pseudo-arcs is closed. Similarly, let $b \in R_1 \cap R_2$ be the $\leqq_{R_1}$-biggest point that is still $\leqq_{R_1}$-smaller than any point on e_0. Then aR_1b and bR_2a are internally disjoint. Therefore aR_1bR_2a is a pseudo-circle all of whose edges are in s, a contradiction. □

Remark 5.4. *The proof of uniqueness above does not make use of the assumption that v and w are endvertices of edges.*

Let us call the pseudo-arc whose uniqueness is noted above vsw by analogy to the special case where s is a pseudo-arc. Next, we give a precise description of vsw.

Proposition 5.5. *The pseudo-arc vsw contains precisely those edges of s whose fundamental cocircuit with respect to s separates v from w. Its linear order is given by $e \leq f$ if and only if e lies on the same side as v of the fundamental cocircuit b_f of f.*

Proof. Let R be the pseudo-arc from v to w using edges in s only. Since R is connected, it must contain all edges whose fundamental cocircuit with respect to s separates v from w.

On the other hand let e be an edge on R. Let z_1 and z_2 be the endvertices of e, with $z_1 \leqq_R z_2$. Then by the above we can join v to z_1 by the pseudo-arc vRz_1 and w to z_2 by the pseudo-arc wRz_2. In G with the fundamental cocircuit of e removed, z_1 and z_2 lie on different sides, which we will call A_1 and A_2. Since $vRz_1 \subseteq A_1$ and $wRz_2 \subseteq A_2$, the fundamental cocircuit of e separates v from w, which completes the proof of the first part.

The second part is immediate from the definitions. □

6 Existence

Let G be a graph-like space inducing a matroid M. Then every finite minor of M is induced by a finite minor of G (finite in the sense that it only has finitely many edges) by Lemma 5.2. But this finite minor must consist simply of a graph, together with a (possibly infinite) collection of spurious vertices, by Lemma 3.2 applied to the closure of the set of edges. In particular, every finite minor of M is graphic. We also know that M has to be tame, by Lemma 4.19. The aim of this section is to prove that these conditions are also sufficient to show that M is induced by some graph-like space. More precisely, we wish to show:

Theorem 6.1. *Let M be a matroid. The following are equivalent.*

1. *There is a graph-like space G inducing M.*
2. *M is tame and every finite minor of M is the cycle matroid of some graph.*

The forward implication was proved above. The rest of this section will be devoted to proving the reverse implication. The strategy is as follows: we consider an extra structure that can be placed on certain matroids, with the following properties:

- There is such a structure on any matroid induced by a graph-like space (in particular, there is such a structure on any finite graphic matroid).
- Given such a structure on a matroid M, we can obtain a graph-like space inducing M.
- The structure is finitary.

Then we proceed as follows: given a tame matroid all of whose finite minors are graphic, we obtain a graph framework on each finite minor. Then the finitariness of the structure, together with the tameness of the matroid, allows us to show by a compactness argument that there is a graph framework on the whole matroid. From this graph framework, we build the graph-like space we need.

6.1 Graph frameworks

A *signing* for a tame matroid M is a choice of functions $c_o\colon o \to \{-1, 1\}$ for each circuit o of M and $d_b\colon b \to \{-1, 1\}$ for each cocircuit b of M such that for any circuit o and cocircuit b we have

$$\sum_{e \in o \cap b} c_o(e) d_b(e) = 0\,,$$

where the sums are evaluated over $\mathbb{Z}$. The sums are all finite since M is tame. A tame matroid is *signable* if it has a signing.

Signings for finite matroids were introduced in [19], where it was shown that a finite matroid is signable if and only if it is regular, i.e. representable over any field. This result was extended to tame infinite matroids, for a suitable infinitary notion of representability, in [2]. In [1] it is shown that the standard matroids associated to graphs are all signable. The construction for a graph G is as follows: we begin by choosing some orientation for each edge of G (equivalently, we choose some digraph whose underlying graph is G). We also choose a cyclic orientation of each circuit of the matroid and an orientation of each bond used as a cocircuit of the matroid. Then $c_o(e)$ is 1 if the orientation of e agrees with the orientation of o and -1 otherwise. Similarly, $d_b(e)$ is 1 if the orientation of e agrees with that of b and -1 otherwise. Then the terms $c_o(e)d_b(e)$ are independent of the orientation of e: such a term is 1 if o traverses b at e in a forward direction, and -1 if o traverses b at e in the reverse direction. Since o must traverse b the same number of times in each direction, all the sums in the definition evaluate to 0.

We therefore think of a signing, in a graphic context, as providing information about the cyclic orderings of the circuits and about the direction in which each edge in a given bond points relative to that bond. In order to reach the notion of a graph framework, we need to modify the notion of a signing in two ways. Firstly, we need to add some extra information specifying on which side of a bond b each edge not in b lies. Secondly, we need to add some conditions saying that these data induce well-behaved cyclic orderings on the circuits.

Recall that if s has a cyclic order R, then we say that $p, q \in s$ are *clockwise adjacent* in R if $[p, q, g]_R$ is in the cyclic order for all $g \in s - p - q$.

Definition 6.2. A *graph framework* on a matroid M consists of a signing of M and a map $\sigma_b : E \setminus b \to \{-1, 1\}$ for every cocircuit b, which we think of as telling us which side of the bond b each edge lies on, satisfying certain conditions. First, we require that these data induce a cyclic order R_o for each circuit o of M: For distinct elements e, f and g of M, we take $[e, f, g]_{R_o}$ if and only if both $e, f, g \in o$ and there exists a cocircuit b of M such that $b \cap o = \{e, f\}$ and $\sigma_b(g) = c_o(f)d_b(f)$. That is, we require that each such relation R_o satisfies the axioms for a cyclic order given in Definition 4.15. In particular, by asymmetry and totality, we require that this condition is independent from the choice of b: if o is a circuit with distinct elements e, f and g, and b and b' are cocircuits such that $o \cap b = o \cap b' = \{e, f\}$, then $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $\sigma_{b'}(g) = c_o(f)d_{b'}(f)$. Let o be a circuit, b be a cocircuit and s be a finite set with $b \cap o \subseteq s \subseteq o$. Then $s \subseteq o$ inherits a cyclic order $R_o\restriction_s$from o. Our final conditions are as follows: for any two $p, q \in s$ clockwise adjacent in $R_o\restriction_s$ we require:

1. If $p, q \in b$, then $c_o(p)d_b(p) = -c_o(q)d_b(q)$.
2. If $p, q \notin b$, then $\sigma_b(p) = \sigma_b(q)$.
3. If $p \in b$ and $q \notin b$, then $c_o(p)d_b(p) = \sigma_b(q)$.
4. If $p \notin b$ and $q \in b$, then $c_o(q)d_b(q) = -\sigma_b(p)$.

Graph frameworks behave well with respect to the taking of minors. Let M be a matroid with a graph framework, and let $N = M/C\backslash D$ be a minor of M. For any circuit o of N we may choose by Lemma 2.4 a circuit o' of M with $o \subseteq o' \subseteq o \cup C$. This induces a function $c_{o'}\restriction_o \colon o \to \{-1, 1\}$. Similarly for any cocircuit b of N we may choose a cocircuit b' of N with $b \subseteq b' \subseteq b \cup D$, and this induces functions $d_{b'}\restriction_b \colon b \to \{-1, 1\}$ and $\sigma_{b'}\restriction_{E(N)\setminus b} \colon E(N) \setminus b \to$

$\{-1, 1\}$. Then these choices comprise a graph framework on N, with R_o given by the restriction of $R_{o'}$ to o.

Next we show that every matroid induced by a graph-like space has a graph framework. Let M be a matroid induced by a graph-like space G. Fix for each topological bond of G a pair (U_b, V_b) of disjoint open sets in G inducing b, and fix an orientation $R'_{\overline{o}}$ of the pseudo-circle $\overline{o}$ inducing each topological circle o (recall from Section 4 that an orientation of a pseudo-circle is a choice of one of the two canonical cyclic orders of the set of points). For each topological circuit o, let the function $c_o \colon o \to \{-1, 1\}$ send e to 1 if $[\iota_e(0), \iota_e(0.5), \iota_e(1)]_{R'_{\overline{o}}}$, and to -1 otherwise. For each topological bond d_b, let the function $d_b \colon b \to \{-1, 1\}$ send e to 1 if $\iota_e(0) \in U_e$ and to -1 if $\iota_e(0) \in V_e$. Finally, for each topological bond d_b, let the function $\sigma_b \colon E \setminus b \to \{-1, 1\}$ send e to -1 if the end-vertices of e are both in U_b and to 1 if they are both in V_b.

Lemma 6.3. *The c_o, d_b and σ_b defined above give a graph framework on M.*

Proof. The key point will be that the cyclic ordering R_o we obtain on each circuit o will be that induced by the chosen orientation $R'_{\overline{o}}$. So let o be a topological circuit of G. First we show that for any distinct edges e, f and g in o and any topological bond b with $o \cap b = \{e, f\}$ we have $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R_{\overline{o}}}$. For any edge $e \in b$ we define $\iota_e^b \colon [0,1] \to G$ to be like ι_e but with the orientation changed to match b. That is, we set $\iota_e^b(r) = \iota_e(r)$ if $\iota_e(0) \in U_b$ and $\iota_e^b(r) = \iota_e(1-r)$ if $\iota_e(0) \in V_b$.

Since the pseudo-circle $\overline{o}$ with edge set o is compact, there can only be finitely many edges in o with both endpoints in U_b but some interior point not in U_b, so by adding the interiors of those edges to U_b if necessary we may assume without loss of generality that there are no such edges, and similarly we may assume that if an edge of o has both endpoints in V_b then all its interior points are also in V_b. Thus the two pseudo-arcs obtained by removing the interior points of e and f from $\overline{o}$ are both entirely contained in $U_b \cup V_b$. Since each of these two pseudo-arcs is connected and precisely one endvertex of e is in U_b, we must have that one of these pseudo-arcs, which we will call R^U is included in U_b. And the other, which we will call R^V, is included in V_b. The end-vertices of R^U must be $\iota_e^b(0)$ and $\iota_f^b(0)$, and those of R^V must be $\iota_e^b(1)$ and $\iota_f^b(1)$.

Suppose first of all that $\sigma_b(g) = 1$. Let R be the pseudo-arc $\iota_f^b(0) f \iota_f^b(1) R^V \iota_e^b(1)$. Then $c_o(f)d_b(f) = 1$ if and only if the ordering along R agrees with the orientation of $\overline{o}$, which happens if and only if $[\iota_f(0.5), \iota_g(0.5), \iota_e(0.5)]_{R'_{\overline{o}}}$, which is equivalent to $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R'_{\overline{o}}}$. The case that $\sigma_b(g) = -1$ is similar. This completes the proof that for any distinct edges e, f and g in o and any topological bond b with $o \cap b = \{e, f\}$ we have $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R'_{\overline{o}}}$.

In particular, the construction of Definition 6.2 really does induce cyclic orders on all the circuits. We now show that these cyclic orders satisfy (1)-(4). Let o, b, s, p and q be as in Definition 6.2. Without loss of generality $\overline{o}$ is the whole of G. We may also assume without loss of generality that all edges e are oriented so that $c_o(e) = 1$. Since $\overline{o}$ is compact we may as before assume that all interior points of edges not in s are in either U_b or V_b. Thus each of the pseudo-arcs obtained by removing the interior points of the edges in s, as in Corollary 4.17, is entirely included in U_b or V_b. Since they both lie on one of these pseudo-arcs, $\iota_p(1)$ and $\iota_q(0)$ are either both in U_b or both in V_b. We shall deal with the case that both are in V_b: the other is similar. In case (1), we get $d_b(p) = 1$ and $d_b(q) = -1$. In case (2), we get $\sigma_b(p) = \sigma_b(q) = 1$. In case (3), we get $d_b(p) = 1$ and $\sigma_b(q) = 1$. Finally in case (4) we get $\sigma_b(p) = 1$ and $d_b(q) = -1$. Since we are assuming that $c_o(p) = c_o(q) = 1$, in each case the desired equation is satisfied. This completes the proof. □

Since a graph framework is a finitary structure, we can lift it from finite minors to the whole matroid.

Lemma 6.4. *Let M be a tame matroid such that every finite minor is a cycle matroid of a finite graph. Then M has a graph framework.*

Proof. By Lemma 6.3 we get a graph framework on each finite minor of M. We will construct a graph framework for M from these graph frameworks by a compactness argument. Let $\mathcal{C}$ and $\mathcal{C}^*$ be the sets of circuits and of cocircuits of M. Let $H = \bigcup_{o \in \mathcal{C}} o \times \{o\} \sqcup \bigcup_{b \in \mathcal{C}^*} b \times \{b\} \sqcup \bigcup_{\tilde{b} \in \mathcal{C}^*} (E \setminus \tilde{b}) \times \{\tilde{b}\} \sqcup \bigcup_{o \in \mathcal{C}} o \times o^3$. Endow $X = \{-1, 1\}^H$ with the product topology. Any element in X encodes a choice of functions $c_o \colon e \mapsto x(o, e)$ for every circuit o, functions $d_b \colon e \mapsto x(b, e)$ and $\sigma_b \colon e \mapsto x(\tilde{b}, e)$ for every cocircuit $\tilde{b}$, and ternary relations $R_o = \{(e, f, g) \in o^3 | x(e, f, g) = 1\}$ for each circuit o.

To comprise a graph framework, these function have to satisfy several properties. These will be encoded by the following six types of closed sets.

For any circuit o and cocircuit b, let $C_{o,b} = \{x \in X | \sum_{e \in o \cap b} x(o, e)x(b, e) = 0\}$. Note that the functions c_o and d_b corresponding to any x in the intersection of all these closed sets will form a signing.

Secondly, for every circuit o, distinct edges $e, f, g \in o$ and cocircuit b such that $o \cap b = \{e, f\}$, let $C_{o,b,g} = \{x \in X | x(o, e, f, g) = x(\tilde{b}, g)x(o, f)x(b, f)\}$.

So x is in the intersection of these closed sets if and only if the cyclic orders encoded by x are given as in Definition 6.2.

Thirdly any circuit o and distinct elements e, f, g of o we set $C_{o,e,f,g,\mathrm{Cyc}} = \{x \in X | x(o,e,f,g) = x(o,f,g,e)\}$. Note that for any x and o in the intersection of all these closed sets the relation R_o derived from x will satisfy the Cyclicity axiom. Similarly we get sets $C_{o,e,f,g,\mathrm{AT}}$ encoding the Asymmetry and Totality axioms and $C_{o,e,f,g,h,\mathrm{Trn}}$ encoding the Transitivity axiom.

Finally, for every circuit o, cocircuit b, finite set s with $o \cap b \subseteq s$, and $p, q \in s$ distinct, let $C_{b,o,s,p,q}$ denote the set of those x such that, if p and q are clockwise adjacent with respect to $R_o\restriction_s$, then the appropriate condition of (1)-(4) from Definition 6.2 is satisfied.

By construction, any x in the intersection of all those closed sets gives rise to a graph framework. As X has the finite intersection property, it remains to show that any finite intersection of those closed sets is nonempty. Given a finite family of those closed sets, let B and O be the set of all those cocircuits and circuits, respectively, that appear in the index of these sets. Let F be the set of those edges that either appear in the index of one of those sets or are contained in some set s or appear as the intersection of a circuit in O and a cocircuit in B. As the family is finite and M is tame, the sets B,O and F are finite.

By Lemma 4.6 from [2] we find a finite minor M' of M satisfying the following.

> For every M-circuit $o \in O$ and every M-cocircuit $b \in B$, there are M'-circuits o' and M'-cocircuits b' with $o' \cap F = o \cap F$ and $b' \cap F = b \cap F$ and $o' \cap b' = o \cap b$.

By Lemma 6.3 M' has a graph framework $((c'_o | o \in \mathcal{C}(M')), (d'_b | b \in \mathcal{C}^*(M')), (\sigma'_b | b \in \mathcal{C}^*(M')))$, giving cyclic orders $R'_{o'}$ on the circuits o'. Now by definition any x with $c_o\restriction_F = c'_o\restriction_F$ and $d_b\restriction_F = d'_b\restriction_F$ and $\sigma_b\restriction_F = \sigma'_b\restriction_F$ and $R_o\restriction_{o'} = R_{o'}$ for $o \in O$ and $b \in B$ will lie in the intersection of all the closed sets in the finite family, as required. This completes the proof. □

6.2 From graph frameworks to graph-like spaces

In this subsection, we prove the following lemma, which, together with Lemma 6.4, gives the reverse implication of Theorem 6.1.

Lemma 6.5. *Let M be a tame matroid with a graph framework $\mathcal{F}$. Then there exists a graph-like space $G = G(M, \mathcal{F})$ inducing M.*

We take our notation for the graph framework as in Definition 6.2.

We begin by defining G. The vertex set will be $V = \{-1, 1\}^{\mathcal{C}^*(M)}$, and of course the edge set will be $E(M)$. As in Definition 3.1, the underlying set of the topological space G will be $V \sqcup ((0,1) \times E)$.

Next we give a subbasis for the topology of G. First of all, for any open subset U of $(0,1)$ and any edge $e \in E(M)$ we take the set $U \times \{e\}$ to be open. The other sets in the subbasis will be denoted $U_b^i(\epsilon_b)$ where $i \in \{-1, 1\}$, $b \in \mathcal{C}^*(M)$ and $\epsilon_b : b \to (0,1)$. Roughly, $U_b^1(\epsilon_b)$ should contain everything that is above b and $U_b^{-1}(\epsilon_b)$ should contain everything that is below b, so that removing the edges of b from G disconnects G. In other words, $G \setminus (\bigcup_{e \in b}(0,1) \times \{e\})$ should be disconnected because the open sets $U_b^1(\epsilon_b)$ and $U_b^{-1}(\epsilon_b)$ should partition it (for every ϵ_b). Formally, we define $U_b^i(\epsilon_b)$ as follows.

$$\begin{aligned} U_b^i(\epsilon_b) &= \{v \in V | v(b) = i\} \cup \bigcup_{e \in E \setminus b, \sigma_b(e) = i} (0,1) \times \{e\} \\ &\cup \bigcup_{e \in b, d_b(e) = i} (1 - \epsilon_b(e), 1) \times \{e\} \cup \bigcup_{e \in b, d_b(e) = -i} (0, \epsilon_b(e)) \times \{e\} \end{aligned}$$

To complete the definition of G, it remains to define the maps ι_e for every $e \in E(M)$. For each $r \in (0,1)$, we must set $\iota_e(r) = (r, e)$. For $r \in \{0, 1\}$, we let:

$$\iota_e(0)(b) = \begin{cases} \sigma_b(e) & \text{if } e \notin b \\ -d_b(e) & \text{if } e \in b \end{cases}; \iota_e(1)(b) = \begin{cases} \sigma_b(e) & \text{if } e \notin b \\ d_b(e) & \text{if } e \in b \end{cases};$$

Note that ι_e is continuous and $\iota_e\restriction_{(0,1)}$ is open. This completes the definition of G. Next, we check the following.

Lemma 6.6. *G is a graph-like space.*

Proof. The only nontrivial thing to check is that for any distinct $v, v' \in V$, there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$. Indeed, if $v \neq v'$, there is some $b \in \mathcal{C}^*$ such that $v(b) \neq v'(b)$, and then for any ϵ_b with $\epsilon_b(e) \leq 1/2$ for each $e \in E(M)$, the sets $U_b^1(\epsilon_b)$ and $U_b^{-1}(\epsilon_b)$ have all the necessary properties.

□

Having proved that G is a graph-like space, it remains to show that G induces M. This will be shown in the next few lemmas.

Lemma 6.7. *Any circuit o of M is a topological circuit of G.*

The proof, though long, is simply a matter of unwinding the above definitions, and may be skipped.

Proof. By the symmetry of the construction of G, we may assume without loss of generality that $c_o(e) = 1$ for all $e \in o$. The graph framework of M induces a cyclic order R_o on o. From this cyclic order we get a corresponding pseudo-circle C with edge set o by Remark 4.16. We begin by defining a map f of graph-like spaces from C to G as follows. First we define $f(v)$ for a vertex v by specifying $f(v)(b)$ for each cocircuit b of M.

If $b \cap o = \emptyset$, then $(f(v))(b) = \sigma_b(e)$ for some $e \in o$. This is independent of the choice of e by condition (2) in the definition of graph frameworks. This ensures that $f^{-1}(U_b^i(\epsilon_b)) = C$ if $i = \sigma_b(e)$, and $f^{-1}(U_b^i(\epsilon_b)) = \emptyset$ if $i = -\sigma_b(e)$.

If $b \cap o =: s$ is nonempty, then s is finite as M is tame. The cyclic order of o induces a cyclic order on $s \cup \{v\}$: choose $p_{v,b}$ so that $p_{v,b}$ and v are clockwise adjacent in this cyclic order. We take $(f(v))(b) = d_b(p_{v,b})$.

Finally, we define the action of f on interior points of edges by $f(\iota_e^C(r)) = \iota_e^G(r)$ for $r \in (0, 1)$. We may check from the definitions above that this formula also holds at $r = 0$ and $r = 1$. First we deal with the case that $r = 0$. We check the formula pointwise at each cocircuit b of M. In the case that $b \cap o = \emptyset$, we have $f(\iota_e^C(0))(b) = \sigma_b(e) = \iota_e^G(0)(b)$. Next we consider those b with $e \in b$. Let $s = o \cap b$, so that $p_{\iota_e^C(0),b}$ and e are clockwise adjacent in s. Thus $f(\iota_e^C(0))(b) = d_b(p_{\iota_e^C(0),b}) = -d_b(e) = \iota_e^G(0)(b)$ by condition (1) in the definition of graph frameworks and our assumption that $c_o(f) = 1$ for any $f \in o$. The other possibility is that $b \cap o$ is nonempty but $e \notin b$. In this case, let $s = b \cap o + e$, so that $p_{\iota_e^C(0),b}$ and e are clockwise adjacent in s. Thus $f(\iota_e^C(0))(b) = d_b(p_{\iota_e^C(0)}) = \sigma_b(e) = \iota_e^G(0)$ by condition (3) in the definition of graph frameworks and our assumption on c_o. The equality $f(\iota_e^C(1)) = \iota_e^G(1)$ may also be checked pointwise. The cases with $e \notin b$ are dealt with as before, but the case $e \in b$ needs a slightly different treatment: we note that in this case $p_{\iota_e^C(1),b} = e$, so that $f(\iota_e^C(1))(b) = d_b(e) = \iota_e^G(1)$.

It is clear by definition that f is injective on interior points of edges. To see that f is injective on vertices, let v and w be vertices of C such that $f(v) = f(w)$ and suppose for a contradiction that $v \neq w$. Since C is a pseudo-circle, there are two edges e and f in C such that v and w lie in different components of $C \backslash \{e, f\}$. By Lemma 2.2, there is a cocircuit b of M with $o \cap b = \{e, f\}$. Without loss of generality we have $e = p_{v,b}$. It follows that $f = p_{w,b}$. Since e and f are clockwise adjacent in the induced cyclic order on $\{e, f\}$, we have $f(v)(b) = d_b(e) = -d_b(f) = -f(w)(b)$ by condition (1) in the definition of graph frameworks and our assumption that $c_o(f) = 1$

for any $f \in o$. This is the desired contradiction. So f is injective.

To see that f is continuous, we consider the inverse images of subbasic open sets of G. It is clear that for any edge e and any open subset U of $(0,1)$, $f^{-1}(\{e\} \times U) = \{e\} \times U$ is open in C, so it remains to check that each set of the form $f^{-1}(U^i_b(\epsilon_b))$ is open in C. If $b \cap o = \emptyset$ then this set is either empty or the whole of C. So suppose that $b \cap o \neq \emptyset$, and let $x \in f^{-1}(U^i_b(\epsilon_b))$. If x is an interior point of an edge e then it is clear that some open neighborhood of x of the form $\{e\} \times U$ is included in $f^{-1}(U^i_b(\epsilon_b))$.

We are left with the case that x is a vertex and $s = b \cap o \neq \emptyset$. By Corollary 4.17, the component of $C \backslash s$ containing x is the pseudo-arc A consisting of all points y on C with $[a, y, b]_{R_C}$, together with a and b, for some vertices $a = \iota^C_p(1)$ and $b = \iota^C_q(0)$, where for any vertex v of A we have $p_{v,b} = p$ and where p and q are clockwise adjacent in the restriction of R_o to s. Since $f(x) \in U^i_b(\epsilon_b)$, we have $i = f(x)(b) = d_b(p)$ and so for any other vertex v of A we also have $f(v)(b) = d_b(p) = i$, so that $f(v) \in U^i_b(\epsilon_b)$. For any edge e of A, applying condition (3) in the definition of graph frameworks to p and e in the set $s+e$ gives $\sigma_b(e) = d_b(p) = i$, so that $f''(0,1) \times e = (0,1) \times e \subseteq U^i_b(\epsilon_b)$. By definition, we have $(1 - \epsilon_b(p), 1) \times \{p\} \subseteq U^i_b(\epsilon_b)$, and using condition (1) in the definition of graph frameworks we get $d_b(q) = -d_b(p) = -i$, so that $(0, \epsilon_b(q)) \times \{q\} \subseteq U^i_b(\epsilon_b)$. We have now shown that every point y of C with $[\iota^C_p(1 - \epsilon_b(p)), y, \iota^C_q(\epsilon_b(q))]_{R_C}$ is in $f^{-1}(U^i_b(\epsilon_b))$. But the set of such points is open in C, which completes the proof of the continuity of f.

We have shown that the map f is a map of graph-like spaces from the pseudo-circle C to G and that the edges in its image are exactly those in o, so that o is a topological circuit of G as required.

□

It is clear that any cocircuit of M is a topological cut of G, as witnessed by the sets $U^{-1}_b(\frac{1}{2})$ and $U^1_b(\frac{1}{2})$. Combining this with Lemmas 6.7 and 4.18, we are in a position to apply Lemma 2.7 with $\mathcal{C}$ the set of topological circuits and $\mathcal{D}$ the set of topological cuts in G. The conclusion is Lemma 6.5, which together with Lemma 6.4 gives us Theorem 6.1.

7 A forbidden substructure

The next lemma gives a useful forbidden substructure for graph-like spaces inducing matroids.

Lemma 7.1. *Let G be a graph-like space, and let v be a vertex in it. Let $\{Q_n | n \in \mathbb{N}\}$ be a set of pseudo-arcs starting at v, and vertex-disjoint apart* *from that. Suppose also that the union of the edge sets of the Q_n is independent. Let y be a point in the closure of the set of their endvertices. Assume there is a nontrivial v-y-pseudo-arc P that is vertex-disjoint from all the $Q_n - v$.*

Then G does not induce a matroid.

Proof. First, we shall show that $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ does not include a pseudo-circle. Suppose for a contradiction that it includes a pseudo-circle K. Then K must include some edge e from P and some edge f from Q_m for some $m \in \mathbb{N}$. Going along K starting from f until we first hit the closed set P, we get two disjoint pseudo-arcs L_1 and L_2, one for each cyclic order of K. Formally, we consider the pseudo-arc $K - f$ endowed with the linear order $\leqq_{K-f}$. Let s be its start vertex and t be its endvertex. Let l_1 be the first point of $K - f$ in P, and let l_2 be the last point of $K - f$ in P. Then $L_1 = s(K - f)l_1$ and $L_2 = l_2(K - f)t$.

We shall show that each of these pseudo-arcs contains v. Since f and $P - v$ are in different components of $(P \cup Q_m) - v$, each L_i contains either v or some edge f' in some Q_l with $l \in \mathbb{N} - m$. Note that fL_if' is included in $\bigcup_{n\in\mathbb{N}} Q_n$ and is an f-f'-pseudo-arc. By the independence of $\bigcup_{n\in\mathbb{N}} Q_n$ and Remark 5.4, it must be that $fL_if' = fQ_mvQ_lf'$. In particular, $v \in L_i$, as desired. This contradicts that L_1 and L_2 are disjoint. Thus $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ does not include a pseudo-circle.

Now suppose for a contradiction that G induces a matroid M. We pick $e \in P$ arbitrarily. Since $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ is M-independent as shown above, by Lemma 2.3 there must be a cocircuit meeting $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ precisely in e. This cocircuit defines a topological cut of G with the two endvertices of e on different sides. This contradicts that $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup (P - e)$ is connected. □

Figure 2:

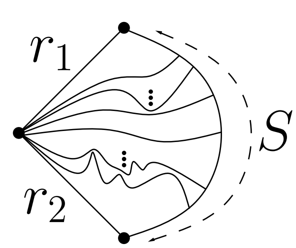


Figure 2: The situation of Lemma 7.2.

Lemma 7.2. *Let G be a graph-like space in which there is a pseudo-circle C with a vertex v of C that is indicent with two edges r_1 and r_2 of C. Let S be the pseudo-arc with edge set $E(C) - r_1 - r_2$. Assume there are infinitely many pseudo-arcs Q_n starting at v to points in S that are vertex-disjoint aside from v.*

If $\bigcup_{n \in \mathbb{N}} Q_n$ does not include a pseudo-circle, then G does not induce a matroid.

Proof. Without loss of generality, we may assume that the pseudo-arcs Q_n only meet S in their end-vertices. By Ramsey's theorem there is an infinite subset N of $\mathbb{N}$ such that the endpoints in S of the Q_n for $n \in N$ form a sequence that is either increasing or decreasing with respect to the linear order $\leqq_S$ of the pseudo-arc S. Let y be their limit point. Let P be the v-y-pseudo-arc included in C that avoids all the endpoints of those Q_n with $n \in N$. Note that P is nontrivial since it has to include either r_1 or r_2. Applying Lemma 7.1 now gives the desired result. □

Corollary 7.3. *Let G be a graph-like space, C a pseudo-circle of G, and r_1 and r_2 distinct edges of C. Let S_1 and S_2 be the two components of $C \setminus \{r_1, r_2\}$. If there is an infinite set W of edges of G each with one end-vertex in S_1 and the other in S_2 and with all of their end-vertices in S_2 distinct, then G does not induce a matroid.*

Proof. Let G' be the graph-like space obtained from G by contracting all edges of S_1. Then in G', there is a vertex v that is endvertex of all edges in W. On the other hand, the other endvertices are distinct for any two edges in W. Indeed, let b be the cocircuit meeting C in precisely r_1 and r_2. Then $W \subseteq b$ and no two endvertices in S_2 are identified.

The set $\overline{W}$ cannot include a pseudo-circle with at least 3 edges since then v would be an endvertex of at least 3 edges of that pseudo-circle, which is impossible. So by Lemma 7.2 with each of the Q_n given by a single edge of W, we obtain that G' does not induce a matroid. By Lemma 5.2, nor does G. □

8 Countability of circuits in the 3-connected case

Our aim in this section is to prove the following:

Theorem 8.1. *Any topological circuit in a graph-like space inducing a 3-connected matroid is countable.*

For the remainder of the section we fix such a graph-like space G, inducing a 3-connected matroid M, and we also fix a pseudo-circle C of G, whose edge set gives a circuit o of M.

We begin by taking a base s of M/o, and letting $G' = G/s$. Thus by Lemma 5.2 G' induces the matroid $M' = M/s$ in which o is a spanning circuit. For any $e \in o$, $o - e$ is a base of o and so $s \cup o - e$ is a base of M, which we shall denote s^e. We shall call the edges of $E(M') \setminus o$ which are not loops *bridges*. We denote the set of bridges by Br. The endpoints of each bridge lie on the pseudo-circle C' corresponding to o in G'. The edges of C' are the same as those of C, but the vertices are different: recall that the vertices of the contraction $G' = G/s$ were defined to be equivalence classes of vertices of G. Each of these can contain at most one vertex of C, since o is a circuit of M'. Thus each vertex of C' contains a unique vertex of C.

Lemma 8.2. *Let $g \in o$ and let f be a bridge with endpoints v' and w' in G'. Let v be the vertex of C contained in v', and w the vertex of C contained in w'. Let x be the endvertex of f in G contained in v', and y the endvertex of f in G in contained in w'. Then the fundamental circuit o_f of f with respect to the base s^g of M is given by concatenating 4 pseudo-arcs: the first, from x to y, consists of only f. The second, from y to w, contains only edges of s. The third, from w to v contains only edges of o - it is the interval of $C - g$ from w to v. The fourth, from v to x, contains only edges of s.*

Proof. $o_f \cap o$ must consist of the fundamental circuit of f with respect to the base $o - g$ of M' - that is, of the interval of $C' - g$ from w' to v'. So the pseudo-arc $v(C - g)w$, which is the closure of this set of edges, lies on the pseudo-circle $\bar{o}_f$. So $(\bar{o}_f - f) \setminus v(C - g)w$ consists of two pseudo-arcs joining v and w to x and y. These two pseudo-arcs use edges from s only. Since v and y lie in different connected components of $G\restriction_s$, we must have that the first goes from v to x, and the second goes from w to y. This completes the proof. □

Lemma 8.3. *For any distinct edges e and f of C, there is a bridge whose endvertices separate e from f in C.*

Proof. Since M is 3-connected, $\{e, f\}$ is not a bond of M, so we can pick some $g \notin \{e, f\}$ in the fundamental bond of f with respect to the base s^e. Then f lies in the fundamental circuit o_g of g, which is therefore not a subset of $s + g$. Thus g is a bridge, and since the fundamental circuit of g with respect to the base $o - e$ of M' contains f but not e the endpoints of g separate e from f. □

Given that we are aiming to prove Theorem 8.1, we may as well assume that o has at least 2 elements, and by Lemma 8.3 we obtain that there is at least one bridge. We now fix a particular bridge e_0, and make use of the 3-connectedness of M to build a tree structure capturing the way the endpoints of the bridges divide up C'. We will call this tree the *partition tree*, and define it in terms of certain auxiliary sequences $(I_n \subseteq \mathrm{Br})$, $(J_n \subseteq V(C'))$ and (K_n) indexed by natural numbers, given recursively as follows:

We always construct J_n from I_n as the set of endvertices of elements of I_n, and K_n as the set of components of $C' \setminus J_n$. We take I_0 to be $\{e_0\}$, and I_{n+1} to be the set of bridges that have endvertices in different elements of K_n or at least one endvertex in J_n.

Then the nodes of the tree at depth n will be the elements of K_n, with p a child of q if and only if it is a subset of q.

Lemma 8.4. *Every bridge is in some I_n.*

Proof. Suppose not, for a contradiction, and let e be any bridge which is in no I_n. In particular, the endpoints of e both lie in the same component of $C - J_0$, so there is a pseudo-arc joining them in C that meets neither endvertex of e_0. Let f be any edge of this pseudo-arc. Let v_0' be any endvertex of e_0, and let v_0 be the unique vertex of C contained in v_0'.

For each n, let B_n be the element of K_n of which f is an edge, and let $B = \bigcap_{n \in \mathbb{N}} B_n$ and $A = C \setminus B$. Note that any 2 vertices in B are joined by a unique pseudo-arc in B, and that A has the same property. Since the two endvertices of e_0 (in G') avoid B_1, they are both in A. Since e is in no I_n, its two endvertices lie in B.

Let A_V be the set of endvertices v of edges of G such that the first point of $vs^f v_0$ on C is contained in a vertex in A. Let A_E be the set of edges of G that have both endvertices in A_V, and let $B_E = E(M) \setminus A_E$. Note that for any vertex $v \in A_V$, all edges of the unique v-C-path included in s_f lie in A_E. And for any $v \notin A_V$, all edges of the unique v-C-path included in s_f lie in B_E.

We shall show that (A_E, B_E) is a 2-separation of M, which will give the desired contradiction since we are assuming that M is 3-connected.

First, we show that $s^f \cap A_E$ is a base of A_E. It is clearly independent. Let g be any edge in $A_E \setminus s^f$. Suppose first of all that g is a bridge. We decompose the fundamental circuit of g as in Lemma 8.2, taking the notation from that lemma. Then since each of the endpoints x and y of g is in A_V, every edge of this fundamental circuit is in A_E, as required.

So suppose instead that g isn't a bridge, that is, g is a loop in M'. Let R_1 and R_2 be the pseudo-arcs from the endpoints x and y of g to v_0 which use only edges from s^f. Let z be the first point of R_1 to lie on R_2. Then zR_1v_0 and zR_2v_0 must be identical, as both are pseudo-arcs from z to v_0 using only edges of s^f. Let k be the first point on this pseudo-arc that is in C. By assumption, $k \in A$. Also, xR_1zR_2y is a pseudo-arc from x to y using only edges from s^f, so must form (with g) the fundamental circuit of g with respect to s^f, so can meet C at most in a single vertex (since g is a loop in M'). Thus all edges in this fundamental circuit lie on either xR_1k or yR_2k, and so are in A_E, as required.

Next, we show that $(s_f \cap B_E) + f$ is a base of B_E. It is independent since A includes some edge as e_0 is a bridge. Let g be any edge in $B_E \setminus s^f - f$. If g isn't a bridge we can proceed as before, so we suppose it is a bridge. We decompose the fundamental circuit of g as in Lemma 8.2, taking the notation from that Lemma. At least one of v' and w' lies in B: without loss of generality it is v'. Suppose for a contradiction that w' is in A. Then either w' is in some J_n or it is an element of some K_n not containing f. In either case, $g \in I_{n+1}$ and so $v' \in J_{n+1}$, giving the desired contradiction since we are assuming $v' \in B$. Thus w' is also in B. Let R be the pseudo-arc from v to w in B. Then g is spanned by the pseudo-arc $xs^f vRws^f y$, which uses only edges of $s_f \cap B_E + f$. To see this we apply Lemma 8.2 with some edge not in B_1 in place of f of that lemma.

Since each of A_E and B_E has at least 2 elements, and the union of the bases for them given above only contains one more element than the base s^f of M, this gives a 2-separation of M, completing the proof. □

Lemma 8.5. *Every node of the Partition-tree has at most countably many children.*

Proof. Let $x \in K_n$ be a node of the Partition-tree. Then the closure $\bar{x}$ of the set of interior points of edges of x is a pseudo-arc. Let $\hat{x}$ be the set obtained from this pseudo-arc by removing its end-vertices. An x-bridge is a bridge with one endvertex in $\hat{x}$ and one in its complement. Thus every element of $J_{n+1} \cap x$ must be an endvertex of an x-bridge or of $\bar{x}$.

Let v_1 and v_2 be vertices of $\hat{x}$ with $v_1 \leqq_{\bar{x}} v_2$. Suppose for a contradiction that there are infinitely many elements of J_{n+1} between v_1 and v_2. Pick a corresponding set W of infinitely many x-bridges with different attachment points between v_1 and v_2. Since neither of v_1 and v_2 is an endpoint of $\bar{x}$, there are edges e_1 and e_2 in x such that all points of e_1 are $\leqq_{\bar{x}}$-smaller than v_1, and similarly all points of e_2 are $\leqq_{\bar{x}}$-bigger than v_2. Then by Corollary 7.3 with $r_1 = e_1$ and $r_2 = e_2$, G' does not induce a matroid, which gives the desired contradiction.

We have established that between any two elements of $J_{n+1} \cap \hat{x}$ there are only finitely many others. Hence $J_{n+1} \cap \hat{x}$ is finite or has the order type of $\mathbb{N}$, $-\mathbb{N}$ or $\mathbb{Z}$. In all these cases there are only countably many children of x, since these children are the connected components of $x \setminus (J_{n+1} \cap x)$. □

We now consider rays in the partition tree: a *ray* consists of a sequence $(k_n \in K_n | n \in \mathbb{N})$ such that for each n the node k_{n+1} is a child of k_n. Given such a ray, we call the set $\bigcap_{n \in \mathbb{N}} k_n$ its *partition class*.

Lemma 8.6. *The partition class of any ray includes at most one edge.*

Proof. Suppose for a contradiction that there is some ray (k_n) whose partition class includes 2 different edges e and f. Then by Lemma 8.3 there is a bridge g whose endvertices separate e from f in C. By Lemma 8.4, g lies in some I_n. But then e and f lie in different elements of K_n, so can't both lie in k_n, which is the desired contradiction. □

For any element k of K_n with $n \geq 1$, the *parent* $p(k)$ is the unique element of K_{n-1} including k.

An element k of K_n with $n \geq 2$ is *good* if no bridge in I_n has endvertices in two different components of $\overline{p(p(k)) \setminus k}$. Note that $\overline{p(p(k)) \setminus k}$ has at most two components. Note that if k is not good, there have to be two vertices in different components of not only $\overline{p(p(k)) \setminus k}$ but also $p(p(k)) \setminus k$.

Lemma 8.7. *Every node of the Partition-tree has at most one good child.*

Proof. Suppose for a contradiction that some $x \in K_n$ with $n \geq 1$ has two good children y_1 and y_2. Since they are different, there is an element i of J_{n+1} separating them, and a bridge e in I_{n+1} of which i is an endvertex. Since $i \notin J_n$, $e \notin I_n$ and so the other endvertex j of e must lie in $p(x) = p(p(y_1)) = p(p(y_2))$. Now the two endvertices of e have to be in different components of $\overline{p(p(y_1)) \setminus y_1}$ or $\overline{p(p(y_2)) \setminus y_2}$. Hence y_1 and y_2 cannot both be good at the same time, a contradiction. □

Lemma 8.8. *Let (k_n) be a ray whose partition class includes an edge. Then all but finitely many nodes on it are good.*

Proof. Let e be the edge in the partition class of this ray. Let f be any edge of $C \setminus k_0$.

Suppose for a contradiction that there is an infinite set N of natural numbers such that k_n is not good for any $n \in N$. Let N' be an infinite subset of N that does not contain 0, 1 or any pair of consecutive natural numbers. For each $n \in N'$, pick a bridge e_n in I_n with endvertices in both components of $\overline{p(p(k_n)) \setminus k_n}$, which is possible since k_n is not good. The endvertices of e_n are in J_n but not J_{n-2} and so we cannot find $m \neq n \in N'$ such that e_m and e_n share an endvertex. Applying Corollary 7.3 with $r_1 = e$, $r_2 = f$ and $W = \{e_n | n \in \mathbb{N}\}$ yields that G' does not induce a matroid, a contradiction. This completes the proof. □

Proof of Theorem 8.1. For each edge of C there is a unique ray whose partition class contains that edge. By Lemma 8.8, we can find a first node on that ray such that it and all successive nodes are good. This gives a map from the edges of C to the nodes of the partition tree. By Lemma 8.7 and Lemma 8.6, this map is injective. By Lemma 8.5 the partition tree has only countably many nodes. □

9 Planar graph-like spaces

A nice consequence of Theorem 8.1 is the following.

Corollary 9.1. *Let M be a tame 3-connected matroid such that all finite minors are planar. Then $E(M)$ is at most countable.*

Proof. Let e be some edge. By Lemma 2.8, there is a switching sequence from e to any other edge. Hence it suffices to show that there are only countably many different switching sequences starting at e. We show by induction that there are only countably many switching sequences of length n for each n. The case $n = 1$ is obvious. The first $n - 1$ elements of a switching sequence of length n form a switching sequence of length $n - 1$. On the other hand, there are only countably many ways to extend a given switching sequence of length $n - 1$ to one of length n since all circuits and cocircuits of M are countable by Theorem 8.1. Hence there are only countably many switching sequences of length n. This completes the proof. □

This raises the question how to embed the graph-like space constructed from a tame matroid all of whose finite minors are planar in the plane. However, we shall construct such a matroid that does not seem to be embeddable in this sense the plane. Let N be the matroid whose circuits are the edge sets of topological circles in the topological space depicted in Figure 3. We omit the proof that this gives a matroid - it can be found in [11]. However, much of the complication of this matroid was introduced to make it 3-connected, and if we do not require 3-connectedness then it is easy to construct other simpler examples sharing the essential property of this matroid: it is tame and all finite minors are planar, but the topology of the graph-like space it induces has no countable basis of neighbourhoods for the vertex at the apex, so it cannot be embedded into the plane.

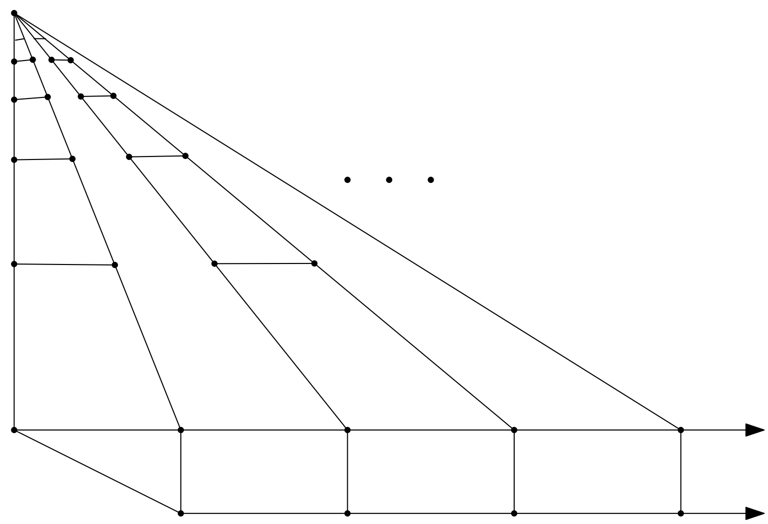


Figure 3: The matroid N.

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