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A detection algorithm for the first jump time in sample trajectories of jump-diffusions driven by α-stable white noise

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Abstract

The purpose of this paper is to develop a detection algorithm for the first jump point in sampling trajectories of jump-diffusions which are described as solutions of stochastic differential equations driven by α-stable white noise. This is done by a multivariate Lagrange interpolation approach. To this end, we utilise computer simulation algorithm in MATLAB to visualise the sampling trajectories of the jump-diffusions for various combinations of parameters arising in the modelling structure of stochastic differential equations.

Key words: Stochastic differential equations, α-stable processes, simulation, multivariate Lagrange interpolation.

AMS Subject Classification(2010): 65C99; 68U20; 60E07; 60G17

1 Introduction

With the passage of time, modelling time evolution uncertainty by stochastic differential equations (SDEs) appears in many diverse areas such as studies of dynamical particle systems in physics, biological and medical studies, engineering and industrial studies, as well as most recently micro analytic studies in mathematical finance and social sciences. Beyond modelling uncertainty by Gaussian or normal distributions, there is a large amount of sample data featured with heavy-tailed distributions. On the other side, it is necessary to admit symmetry for the mean (average) by using Gaussian models while asymmetry and/or skewness are accepted by non-Gaussian models, for instance, the generalised hyperbolic distribution (in particular, the normal-inverse Gaussian distribution) discovered by Barndorff-Nielsen, see Barndorff-Nielsen (1997) and references therein. In some applications, asymmetric or heavy-tailed models are needed or even inevitable, in which a model using stable distributions could be a viable candidate. Another important feature of such non-Gaussian models is the use of probability distributions with infinite moments which turns to be more realistic than Gaussian models from the view point of heavy tail type data (cf. e.g. (Samorodnitsky and Taqqu, 1994)). The research on modelling uncertainty using stable distributions and stable stochastic processes have been increased dramatically, see e.g. (Giacometti, Bertocch, Rachev, and Fabozzi, 2007), (Zopounidis and Pardalos, 2013), (Dror, L'Ecuyer, and Szidarovszky, 2002) and (Fiche, Cexus, Martin, and Khenchaf, 2013). The self-similarity property of stable distributions has drawn more and more attention from both theoretical and practical view points, i.e (Campbell, Lo, and MacKinlay, 1997; Mandelbrot, 1960) and (Zolotarev, 1986; Leland, Taqqu, Willinger, and Wilson, 1993; Shlesinger, Zaslavsky, and Frisch, 1995). We refer the reader to (Du, Wu, and Yang, 2010) for discussions of utilising α-stable distributions to model the mechanism of Collateralised Debt Obligations (CDOs) in mathematical finance.

Historically, probability distributions with infinite moments are also encountered in the study of critical phenomena. For instance, at the critical point one finds clusters of all sizes while the mean of the distribution of clusters sizes diverges. Thus, analysis from the earlier intuition about moments had to be shifted to newer notions involving calculations of exponents, like e.g. Lyapunov, spectral, fractal etc., and topics such as strange kinetics and strange attractors have to be investigated. It was Paul Lévy who first grappled in-depth with probability distributions with infinite moments. Such distributions are now called Lévy distributions. Today, Lévy distributions have been expanded into diverse areas including turbulent diffusion, polymer transport and Hamiltonian chaos, just to mention a few. Although Lévy's ideas and algebra of random variables with infinite moments appeared in the 1920s and the 1930s (cf. (Lévy, 1925, 1937)), it is only from the 1990s that the greatness of Lévy's theory became much more appreciated as a foundation for probabilistic aspects of chaotic dynamics with high entropy in statistical analysis in mathematical modelling (cf. (Samorodnitsky and Taqqu, 1994; Shlesinger, Zaslavsky, and Frisch, 1995), see also (Mandelbrot, 1960; Zolotarev, 1986)). Indeed, in statistical analysis, systems with highly complexity and (nonlinear) chaotic dynamics became a vast area for the application of Lévy processes and the phenomenon of dynamical chaos became a real laboratory for developing generalisations of Lévy processes to create new tools to study nonlinear dynamics and kinetics. Following up this point, SDEs driven by Lévy type processes, in particular α-stable processes or α-stable white noise, and their influence on long time statistical asymptotic will be unavoidably encountered. Comparing to the continuity feature of trajectories of diffusions – solutions of SDEs driven by Brownian motion or Gaussian white noise, jump-diffusions possess a feature that trajectories are with jumps which seem to be more natural when volatile noise influence becomes extremely high. The nature of trajectories of jump-diffusions is that there are countable jump times and there are diffusion trajectories between any two jump times. For SDEs driven by α-stable noise, the solution trajectories enjoy certain self similar property. Therefore, from modelling aspect, to detect the first jump time is crucial, as one can treat the model as a diffusion model before that time. With self similar property, one can further infer the structure of trajectories of jump-diffusions driven by α-stable noise. Due to high uncertainty, the first jump time is of course a random time (also called stopping times). Theoretically, it is not possible to get the first jump time analytically, but one could try to simulate sampling trajectories to get an algorithm towards statistical detection of the (random) first jump time. The motivation of this paper is to obtain a critical link among the parameters in the SDEs driven by α-stable white noises to develop a detection algorithm for the first jump time. This can be further linked to sampling data analysis after model identifications (i.e., through certain specification of the parameters in the equations). We mainly focus on testing two typical SDEs in modelling, one class is the SDEs with linear drift coefficient and additive α-stable white noise and the solutions are called α-stable Ornstein-Uhlenbeck processes and the other class is the linear SDEs (i.e., SDEs with linear drift and diffusion coefficients or the linear SDEs with multiplicative α-stable noise) and the solutions are called α-stable geometric Lévy motion. As the chaotic structure of sample trajectories of α-stable processes are varying for α in the different intervals $(0, 1)$ and $(1, 2)$ with $\alpha = 1$ being critical (see, e.g., (Janicki and

Weron, 1994)), respectively, we have performed our simulations of the sample solution trajectories with the sample size of $2^9 = 512$, which yields a clear picture to identify successfully the first jump time for each simulated trajectory. Furthermore, we use such sample data to find the critical link of the parameters arising in the coefficients of the SDEs. We hope that our results obtained in this paper would lead to further investigations for more general models, such as those determined by SDEs with affine coefficients or with periodic coefficients (treated as bounded coefficients over the whole spaces), as well as higher order representations of the first jump time in terms of the parameters and rigorous estimates of the first jump time. We will carry out these studies in our forthcoming papers. To the best of our knowledge, there is not any work in the literature addressing such problem. To end up our introduction, we would like to mention that the study of SDEs driven by Lévy processes is well presented in the monograph (Applebaum, 2009). Numerical solutions and simulations of α-stable stochastic processes were carried out in (Janicki and Weron, 1994).

2 Preliminaries

Given a probability space $(\Omega, \mathcal{F}, P)$ endowed with a complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We start with recalling the definition of Lévy process and some theorems, see (Applebaum, 2009), (Janicki and Weron, 1994) and (Samorodnitsky and Taqqu, 1994) for details.

Definition 2.1. A stochastic process $\{X_t\}_{t\geq 0}$ is called Lévy process if:

1. X_0=0 almost surely;

2. X has independent and stationary increments;

3. X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} P(|X_t - X_s| > a) = 0.$$

Every Lévy process has a *cádlág* (i.e., right continuous with left limits) modification. The associated jump process $\{\Delta X_t\}_{t \geq 0}$ is defined as

$$\Delta X_t = X_t - X_{t-}$$

where X_{t-} stands for the left limit of X_t at the point t. Fix $t \in [0, \infty)$ and a Borel measurable set $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, set

$$N(t, A) = \#\{0 \leq s \leq t; \Delta X_s \in A\} = \sum_{0 \leq s \leq t} \chi_A(\Delta X_s)$$

where $\#\{...\}$ stands for the cardinal number of set $\{...\}$ and χ_A denotes the indicator function of A. If A is bounded Borel set, then $N(t, A) < \infty$ almost surely for all $t \geq 0$. N is a Poisson random measure with intensity measure $\nu(A) = \mathbb{E}(N(1, A))$ and $\tilde{N}$ is a compensated Poisson martingale measure

$$\tilde{N}(t, A) = N(t, A) - t\nu(A).$$

Lévy processes enjoy the celebrated Lévy-Itô decomposition, see e.g. (Applebaum, 2009), which we state as follows. For any (real-valued) Lévy process X, there exist a constant $b \in \mathbb{R}$, a Brownian motion B and a Poisson random measure N on $[0, \infty) \times (\mathbb{R} \setminus \{0\})$ which is independent of B such that, for each $t \geq 0$,

$$X_t = bt + B_t + \int_{0<|x|<1} x\tilde{N}(t, dx) + \int_{|x| \geq 1} xN(t, dx).$$

Next, we introduce α-stable Lévy processes. We have first the following

Definition 2.2. A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, and $\mu \in \mathbb{R}$ such that its characteristic function has the following form

$$E \exp i\theta X = \begin{cases} \exp\Big\{ -\sigma^\alpha |\theta|^\alpha \big(1 - i\beta (\mathrm{sign}\theta) \tan \frac{\pi\alpha}{2}\big) + i\mu\theta \Big\} & \text{if } \alpha \neq 1 \\ \exp\Big\{ -\sigma|\theta| \big(1 + i\beta \frac{\pi}{2} (\mathrm{sign}\theta) \ln|\theta| + i\mu\theta \Big\} & \text{if } \alpha = 1 \end{cases}$$

where

$$\mathrm{sign}\theta = \begin{cases} 1 & \text{if } \theta > 0, \\ 0 & \text{if } \theta = 0, \\ -1 & \text{if } \theta < 0. \end{cases}$$

We denote $X \sim S_\alpha(\sigma, \beta, \mu)$.

The parameter α is the index of stability, β is the skewness parameter, σ is the scale parameter and μ is shift. β is irrelevant when α =2. When $\beta = \mu = 0$, X is a symmetric α-stable random variable and is denoted by $X \sim S\alpha S$. We focus our attention on symmetric case in this paper.

Definition 2.3. A stochastic process $\{L_t\}_{t\geq 0}$ is called the (standard) α-stable Lévy motion if

1. L_0=0 almost surely;

2. L has independent increments;

3. $L_t - L_s \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t < \infty$.

In this paper, we are concerned with the following SDE driven by α-stable Lévy motion

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t + c(X_{t-})dL_t$$

where $b, \sigma, c : \mathbb{R} \to \mathbb{R}$ are measurable coefficients, $\{B_t\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}$-Brownian motion, and $\{L_t\}_{t\geq 0}$ is an α-stable $\{\mathcal{F}_t\}$-Lévy process with the following Lévy-Ito representation

$$L_t = \int_0^{t+} dL_s = \int_o^{t+} \int_{0<|z|<1} z\tilde{N}(ds, dz) + \int_0^{t+} \int_{|z|\geq 1} zN(ds, dz)$$

with $N : \mathcal{B}([0, \infty) \times \mathbb{R} \setminus \{0\}) \to \mathbb{N} \cup \{0\}$ being the Poisson random (counting) measure on $(\Omega, \mathcal{F}, P)$ and

$$\tilde{N}(dt, dz) := N(dt, dz) - \frac{dtdz}{|z|^{1+\alpha}}$$

the associated compensated martingale measure with density $\mathbb{E}N(dtdz) = \frac{dtdz}{|z|^{1+\alpha}}$, where $\alpha \in (0, 2)$ is fixed.

Under the usual conditions, like linear growth and local Lipschitz conditions, for the coefficients b, σ, c, there is a unique solution to the above SDE with initial data X_0 (see, e.g., (Applebaum, 2009)). We then apply Itô formula to obtain the solutions for this type of SDEs. In what follows, we introduce two typical SDEs fulfilling the usual conditions.

2.1 The α-stable Ornstein-Uhlenbeck processes

The α-stable Ornstein-Uhlenbeck processes are solutions of the following type SDEs

$$dX_t = -\lambda X_t dt + \mu dL_t \tag{1}$$

for $\lambda > 0$, where the α-stable white noise dL_t is formulated as follows

$$dL_t = \int_{0<|z|<1} z\tilde{N}(dt, dz) + \int_{|z|\geq 1} zN(dt, dz).$$

By Itô formula (cf., e.g., (Applebaum, 2009)), the solution is explicitly given as follows

$$\begin{aligned} X_t =& e^{-\lambda t} X_0 + \mu e^{-\lambda t} \int_0^{t+} \int_{0<|z|<1} e^{\lambda t} z \tilde{N}(ds, dz) \\ &+ \mu e^{-\lambda t} \int_0^{t+} \int_{|z|\geq 1} e^{\lambda t} z N(ds, dz). \end{aligned} \tag{2}$$

2.2 The α-stable geometric Lévy motion

Consider the following linear SDE

$$dX_t = \lambda X_t dt + m X_t dB_t + \mu X_{t-} dL_t \tag{3}$$

where $\lambda > 0$, $m > 0$ and $\mu > 0$. Then by Itô formula, one can derive the following explicit solution

$$\begin{aligned} X_t =& X_0 \exp\{(\lambda - \frac{1}{2} m^2) t + m B_t + \mu \int_0^{t+} \int_{|z|\geq 1} \ln|1+z| N(ds, dz) \\ &+ \mu \int_0^{t+} \int_{0<|z|<1} \ln|1+z| \tilde{N}(ds, dz) \\ &+ \mu \int_0^{t} \int_{0<|z|<1} [\ln|1+z| - z] \frac{dz}{|z|^{1+\alpha}} ds\}. \end{aligned} \tag{4}$$

Due to the above expression, the solution is called an α-stable geometric Lévy motion.

2.3 Sample trajectories

By applying simulation methods in MATLAB, sample trajectories can be generated and codes are listed in Appendix A. Following graphs illustrate a few sample trajectories of α-stable Ornstein-Uhlenbeck processes and α-stable geometric L*évy* motions respectively with a number of parameters combinations. Also, a few first jump points are marked in both graphs.

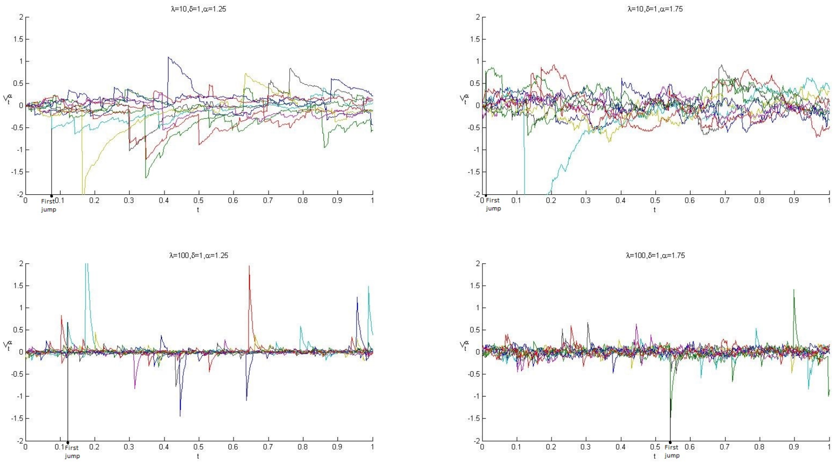

Figure 1: α-stable Ornstein-Uhlenbeck processes

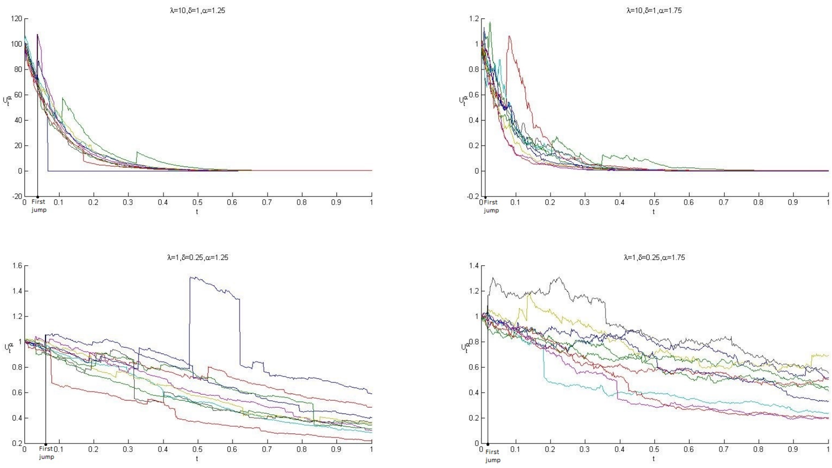

Figure 2: α-stable geometric *Lévy* motions

Remark 2.1. Given interval $[0, t]$, a mesh $\{m_i = i\tau, i = 0, 1, \ldots, n\}$ on $[0, t]$ with fixed natural number n and $\tau = t/n$. Simulation results presented in this paper are based on $t = 1$, $n = 512$ and $\tau = 1/512$.

Our aim is to develop an algorithm of detecting first jump point in the sample trajectories. Links among parameters in SDEs driven by α-stable Lévy motions will be given. These links could then be adopted for and implemented in model/data fitting purpose in future.

3 Simulations and examples

In this section, by utilising a multivariate Lagrange interpolation method (see for example references (De Boor and Ron, 1990), (Saniee, 2008), (Calvi, 2005),(Liang, Cui, and Zhang, 2006) and (Sauer and Xu, 1995)), links will be obtained for α-stable Ornstein-Uhlenbeck process and α-stable geometric Lévy motion respectively to detect first jump point from simulations by interpolation method introduced in the previous section. Sample trajectories of α-stable Ornstein-Uhlenbeck process described by Equation (1) with different combinations of parameters in its SDE are included in Appendix B. And the case for α-stable geometricLévy motion can be found in Appendix C. We could clarify the model into different perspectives by observations and general characteristics of trajectories are summarised as follows,

1. Fix λ and μ, sample trajectories $\{X_t\}_{t\geq 0}$ become more tempered as the stability index α increases, but the jump size becomes smaller and smaller so that the trajectories become less and less volatile. In other words, for smaller stability index α, the trajectories of $\{X_t\}_{t\geq 0}$ are generally more tough than those of bigger stability index α.

2. Fix μ and α, trajectories look more likely deterministic exponential paths along with the increase of λ. As for bigger α, the trajectories are chaotic more sharply.

3. Fix λ and α, increasing the volatility parameter μ indicates higher

chaoticity.

3.1 α-stable Ornstein-Uhlenbeck process

For the triple (λ, μ, α), there is a link among the three parameters λ, μ and α towards first jump point detection of the sample trajectories. By substantial amount of simulations, we randomly choose the situations and keep records of values of the parameters λ, μ and α when the first jump appears. Especially, the degree 1 linear relationship among these three parameters is useful in data modelling for uncertainty targeted problems in reality.

λ	μ	α	t	X_t^α
1	0.25	1	0.06055	0.4198
1	1	1.75	0.003906	-0.1551
1	100	0.75	0.03125	18.82
10	0.25	0.5	0.02148	0.4561
1000	0.25	1.75	0.001952	0.0374

We have degrees $n = 1$, variables $m = 4$, so terms$= \begin{pmatrix} 1+4 \\ 1 \end{pmatrix} = 5$. If we have $g = f(a, b, c, d)$ which is a degree 1 function with 4 parameters, and

$$g_i = \beta_1 a_i + \beta_2 b_i + \beta_3 c_i + \beta_4 d_i + \beta_5$$

where $\beta_1, \beta_2, \cdots, \beta_5$ are coefficients, $1 \leq i \leq 5$.

$$0.4198 = \beta_1 + 0.25\beta_2 + \beta_3 + 0.06055\beta_4 + \beta_5$$

$$-0.1551 = \beta_1 + \beta_2 + 1.75\beta_3 + 0.003906\beta_4 + \beta_5$$

$$18.82 = \beta_1 + 100\beta_2 + 0.75\beta_3 + 0.03125\beta_4 + \beta_5$$

$$0.4561 = 10\beta_1 + 0.25\beta_2 + 0.5\beta_3 + 0.02148\beta_4 + \beta_5$$

$$0.0374 = 1000\beta_1 + 0.25\beta_2 + 1.75\beta_3 + 0.001952\beta_4 + \beta_5$$

By calculation

$$g = 0.00034a + 0.18b - 0.52c + 5.76d + 0.54.$$

Then

$$X_t^\alpha = 0.00034\lambda + 0.18\mu - 0.52\alpha + 5.76t + 0.54.$$

If we take the average value of t, we have

$$\bar{t} = 0.0238276$$

and average value of X_t^α, we have

$$\overline{X_t^\alpha} = 3.91564.$$

Therefore

$$0.00034\lambda + 0.18\mu - 0.52\alpha = 3.24.$$

We summarise our deviation as

Proposition 3.1. *The link among parameters for first jump point detection of the sample trajectories of α-stable Ornstein-Uhlenbeck process is given by the following liner equation*

$$0.00034\lambda + 0.18\mu - 0.52\alpha = 3.24.$$

3.2 α-stable geometricLévy motion

Similarly, for the triple (λ, μ, α) described in Equation (3), we are working on determining a link among these three parameters towards first jump time detection. The data and calculations have been processed to obtain the degree 1 linear relationship are as follows.

λ	μ	α	t	X_t^α
1	0.5	1.25	0.001952	1.043
1	1	1	0.007813	1.372
100	0.5	1.75	0.001953	0.9523
100	10	1.25	0.005859	0.5114
1000	1	0.75	0.001796	-0.7903

We have degrees $n = 1$, variables $m = 4$, so terms= $\begin{pmatrix} 1+4 \\ 1 \end{pmatrix} = 5$. If we have $g = f(a, b, c, d)$ which is a degree 1 function with 4 parameters, and

$$g_i = \beta_1 a_i + \beta_2 b_i + \beta_3 c_i + \beta_4 d_i + \beta_5$$

where $\beta_1, \beta_2, \cdots, \beta_5$ are coefficients, $1 \leq i \leq 5$. We have

$$\begin{aligned} 1.043 &= \beta_1 + 0.5\beta_2 + 1.25\beta_3 + 0.001952\beta_4 + \beta_5 \\ 1.372 &= \beta_1 + \beta_2 + \beta_3 + 0.007813\beta_4 + \beta_5 \\ 0.9523 &= 100\beta_1 + 0.5\beta_2 + 1.75\beta_3 + 0.001953\beta_4 + \beta_5 \\ 0.5114 &= 100\beta_1 + 10\beta_2 + 1.25\beta_3 + 0.005859\beta_4 + \beta_5 \\ -0.7903 &= 1000\beta_1 + \beta_2 + 0.75\beta_3 + 0.001796\beta_4 + \beta_5 \end{aligned}$$

By calculation

$$g = -0.0017124a - 0.066287b + 0.15752c + 68.508d + 0.74723.$$

Then

$$X_t^\alpha = -0.0017124\lambda - 0.066287\mu + 0.15752\alpha + 68.508t + 0.74723.$$

If we take the average value of t, we have

$$\bar{t} = 0.0038746$$

and average value of X_t^α, we have

$$\overline{X_t^\alpha} = 0.61768.$$

Therefore

$$-0.0017124\lambda - 0.066287\mu + 0.15752\alpha = -0.3949911.$$

Proposition 3.2. *The link among parameters for first jump point detection of the sample trajectories of α-stable geometricLévy motion is given by the following liner equation*

$$-0.0017124\lambda - 0.066287\mu + 0.15752\alpha = -0.3949911.$$

Remark 3.1. Here we only consider linear Lagrange interpolation. One can extend to higher order polynomial interpolation in which more computation is needed. Our consideration gives a simple yet efficient calculation.

Appendices

A α-stable random variable generator

Following codes are used to generate sample trajectories (Veillette, 2014).

```
function r = stblrnd(alpha,beta,gamma,delta,varargin)


if nargin < 4
    error('stats:stblrnd:TooFewInputs','Requires at least four
    input arguments.');
end
```

```
if alpha <= 0 || alpha > 2 || ~isscalar(alpha)
    error('stats:stblrnd:BadInputs',' "alpha" must be a scalar
    which lies in the interval (0,2]');
end
if abs(beta) > 1 || ~isscalar(beta)
    error('stats:stblrnd:BadInputs',' "beta" must be a scalar
    which lies in the interval [-1,1]');
end
if gamma < 0 || ~isscalar(gamma)
    error('stats:stblrnd:BadInputs',' "gamma" must be a
    non-negative scalar');
end
if ~isscalar(delta)
    error('stats:stblrnd:BadInputs',' "delta" must be a scalar');
end

[err, sizeOut] = genOutsize(4,alpha,beta,gamma,delta,varargin{:});
if err > 0
    error('stats:stblrnd:InputSizeMismatch','Size information is
    inconsistent.');
end

if alpha == 2
    r = sqrt(2) * randn(sizeOut);

elseif alpha==1 && beta == 0
    r = tan( pi/2 * (2*rand(sizeOut) - 1) );
```

```
elseif alpha == .5 && abs(beta) == 1
    r = beta ./ randn(sizeOut).^2;

elseif beta == 0
    V = pi/2 * (2*rand(sizeOut) - 1);
    W = -log(rand(sizeOut));
    r = sin(alpha * V) ./ ( cos(V).^(1/alpha) ) .* ...
        ( cos( V.*(1-alpha) ) ./ W ).^( (1-alpha)/alpha );

elseif alpha ~= 1
    V = pi/2 * (2*rand(sizeOut) - 1);
    W = - log( rand(sizeOut) );
    const = beta * tan(pi*alpha/2);
    B = atan( const );
    S = (1 + const * const).^(1/(2*alpha));
    r = S * sin( alpha*V + B ) ./ ( cos(V) ).^(1/alpha) .* ...
       ( cos( (1-alpha) * V - B ) ./ W ).^((1-alpha)/alpha);

else
    V = pi/2 * (2*rand(sizeOut) - 1);
    W = - log( rand(sizeOut) );
    piover2 = pi/2;
    sclshftV =  piover2 + beta * V ;
    r = 1/piover2 * ( sclshftV .* tan(V) - ...
        beta * log( (piover2 * W .* cos(V) ) ./ sclshftV ) );
```

```
end

if alpha ~= 1
    r = gamma * r + delta;
else
    r = gamma * r + (2/pi) * beta * gamma * log(gamma) + delta;
end

end
```

B Sample trajectories of α-stable Ornstein-Uhlenbeck process

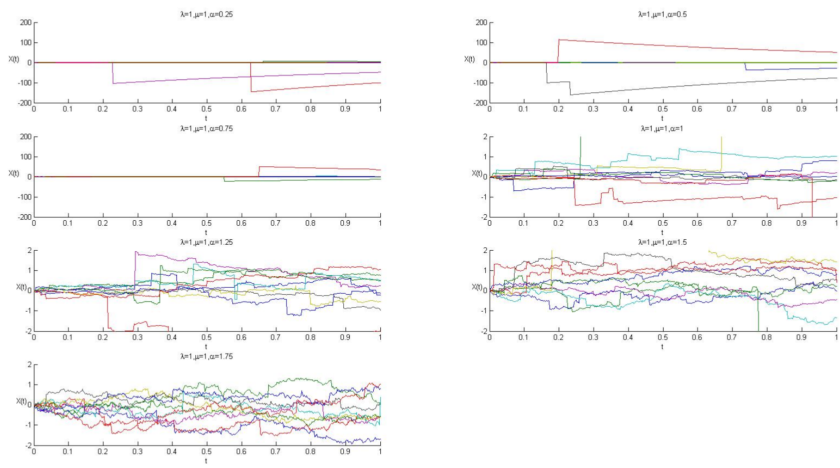

Figure 3: Fix λ=1 and μ=1 with α increases

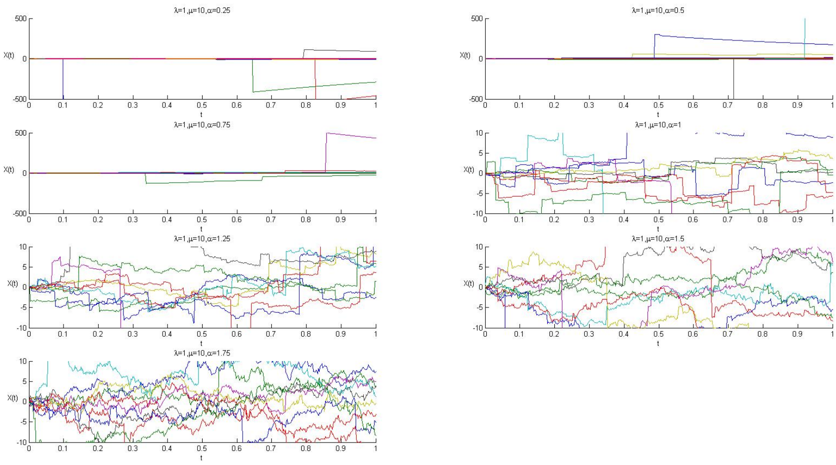

Figure 4: Fix λ=1 and μ=10 with α increases

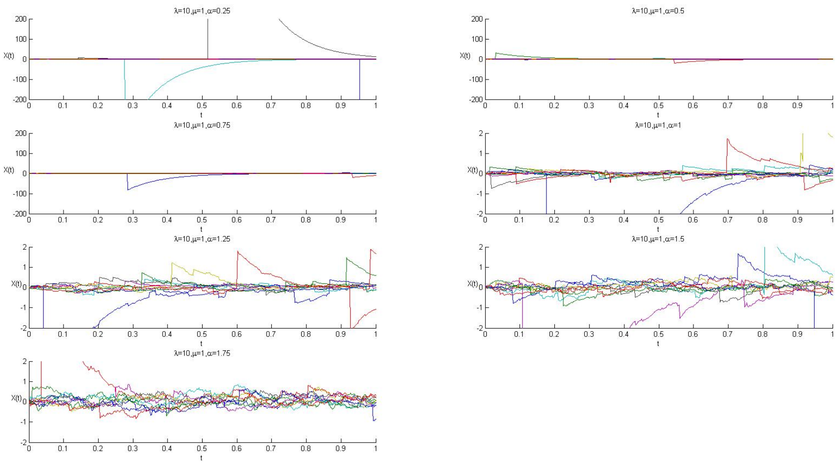

Figure 5: Fix λ=10 and μ=1 with α increases

C Sample trajectories of α-stable geometricLévy motion

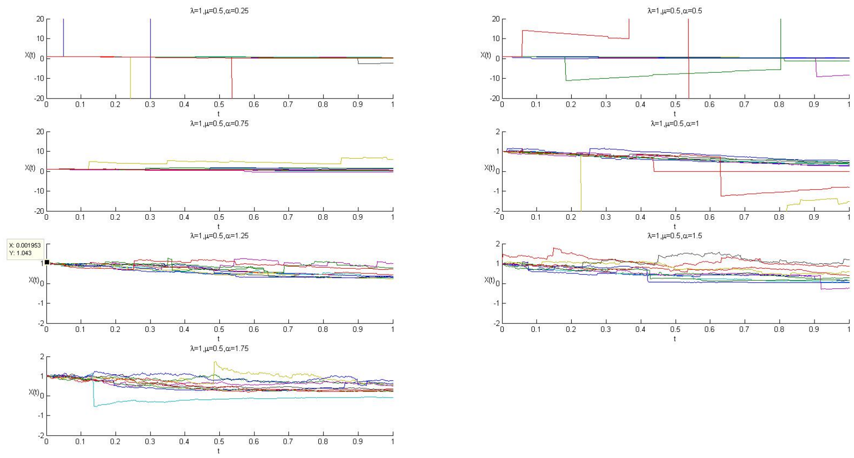

Figure 6: Fix λ=1 and μ=0.5 with α increases

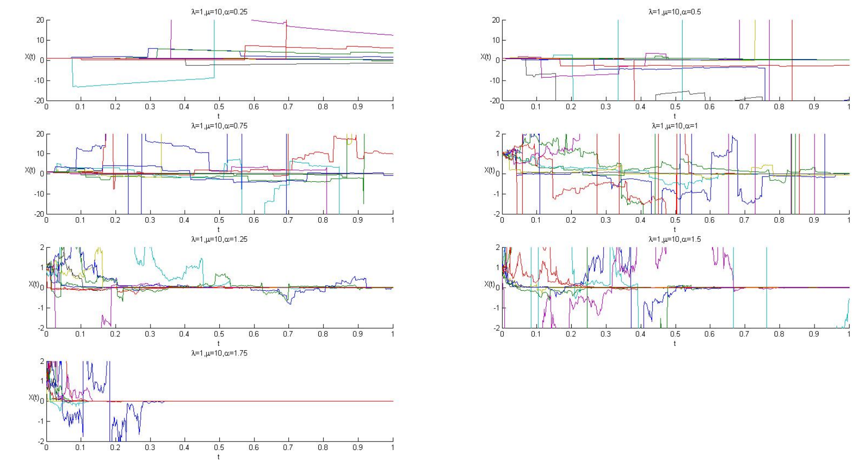

Figure 7: Fix λ=1 and μ=10 with α increases

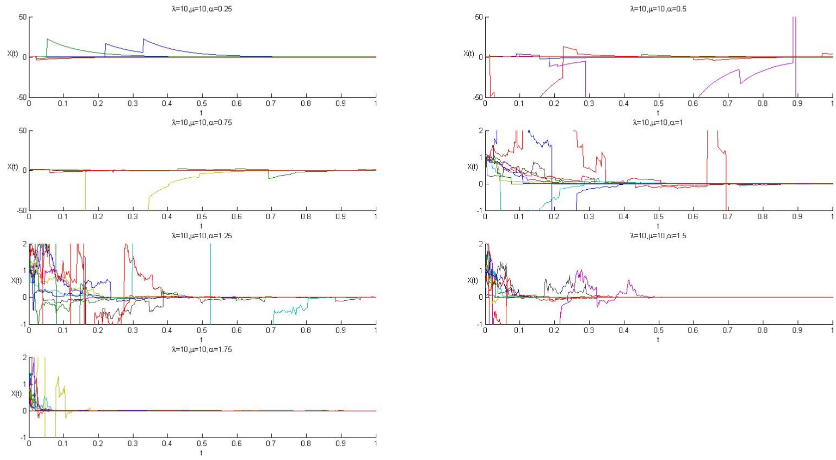

Figure 8: Fix λ=10 and μ=10 with α increases

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References

Applebaum, D. (2009). *Lévy processes and stochastic calculus*. Cambridge university press.

Barndorff-Nielsen, O. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics* 24 (1): 1-13.

Campbell, J. Y., Lo, A. W. C., & MacKinlay, A. C. (1997). *The econometrics of financial markets*. Princeton University press.

De Boor, C., & Ron, A. (1990). On multivariate polynomial interpolation. *Constructive Approximation* 6(3): 287-302.

Calvi, J. P. (2005). Lectures on multivariate polynomial interpolation.

Dror, M., L'ecuyer, P., & Szidarovszky, F. (Eds.). (2002). *Modeling Uncertainty: An Examination of Stochastic Theory, Methods, and Applications* (Vol. 46). Springer Science & Business Media.

Du, H., Wu, J.-L., & Yang, W. (2010). On the mechanism of CDOs behind the current financial crisis and mathematical modeling with Lévy distributions. *Intelligent Information Management* 2(02): 149-158.

Fiche, A., Cexus, J.C., Martin, A., & Khenchaf, A. (2013) Features modeling with an α-stable distribution: Application to pattern recognition based on continuous belief functions. *Information Fusion* 14(4): 504-520.

Giacometti, R., Bertocchi, M., Rachev, S. T., & Fabozzi, F. J. (2007) Stable distributions in the Black-Litterman approach to asset allocation. *Quantitative Finance* 7(4): 423-433.

Janicki, A., & Weron, A. (1993) *Simulation and Chaotic Behavior of α-Stable Stochastic Processes* (Vol.178). CRC Press.

Leland, W. E., Taqqu, M. S., Willinger, W., & Wilson, D. W. (1993) On the self-similar nature of Ethernet traffic. *ACM SIGCOMM Computer Communication Review* 23(4): 183-193.

Lévy P. (1925) *Calcul des probabilités. Gauther-Villars.*

Lévy P. (1937) Théorie de l'addition des variables aléatoires. Gauther-Villars.

Liang, X. Z., Cui, L. H., & Zhang, J. L. (2004). The application of Cayley–Bacharach theorem to bivariate Lagrange interpolation. *Journal of computational and applied mathematics* 163(1): 177-187.

Mandelbrot, B. (1960). The Pareto-Lévy Law and the Distribution of Income. *International Economic Review* 1(2): 79-106.

Samorodnitsky, G., & Taqqu, M.S. (1994) *Stable Random Processes: Stochastic Models with Infinite Variance*. CRC Press.

Saniee, K.(2008) A simple expression for multivariate Lagrange interpolation. Copyright@SIAM.
https://www.siam.org/students/siuro/vol1issue1/S01002.pdf

Sauer, T.; Xu, Y. (1995). A case study in multivariate Lagrange interpolation. *NATO ASI Series C Mathematical and Physical Sciences-Advanced Study Institute* 454: 443-452.

Shlesinger, M. F., Zaslavsky, G. M., & Frisch, U. (1995). Lévy flights and related topics in physics. *Lecture notes in physics* 450: 52.

Zolotarev, V. M. (1986). *One-dimensional stable distributions* (Vol. 65). American Mathematical Soc..

Zopounidis, C., & Pardalos, P. M. (2013). *Managing in Uncertainty: Theory and Practice* (Vol. 19). Springer Science & Business Media.

Veillette, M. (2014) `https://github.com/markveillette/stbl`.