

Research Article

Dreibein as Prepotential for Three-Dimensional Yang-Mills Theory

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We advocate and develop the use of the dreibein (and the metric) as prepotential for three-dimensional SO(3) Yang-Mills theory. Since the dreibein transforms homogeneously under gauge transformation, the metric is gauge invariant. For a generic gauge potential, there is a unique dreibein on fixing the boundary condition. Topologically nontrivial monopole configurations are given by conformally flat metrics, with scalar fields capturing the monopole centres. Our approach also provides an ansatz for the gauge potential covering the topological aspects.

1. Introduction

In this paper we advocate and develop the use of the dreibein or triad (the 3D version of the vielbein) as the basic variable or prepotential for Yang-Mills theory. The first clear proposal to use the dreibein came from Haagensen and Johnson [1] in the context of Hamiltonian formalism in $3 + 1$ dimensions. (References [2–8] are some other works on Yang-Mills theory involving various constructions of the metric.) The defining equation for the dreibein is the condition for the dreibein to be torsion-free with respect to a connection one-form. The authors of [1] claimed that there are zero modes associated with this equation due to nonuniqueness of the dreibein e_a^i corresponding to a given gauge potential A_i^a but also gave arguments that the zero modes would not affect their discussion. Then Haagensen et al. [9] followed it up with a deformation of the defining equation for the dreibein, removing the deformation at the end. References [5–8] have used the original defining equation for the dreibein as in [1] and not the deformation of it. In this work also, we use the original defining equation of [1]. We analyse the case of a generic Yang-Mills potential, satisfying $\det B_i^a \neq 0$ (B_i^a being the nonabelian magnetic field) and argue that a unique dreibein is obtained by fixing the boundary condition. Consequently, the torsion-free dreibein is a useful prepotential. The calculation

of the Jacobian will be affected by the zero modes present for vanishing nonabelian magnetic field, but the Jacobian will not be needed in the present work.

Our formulation is useful in several ways:

- (1) The metric arising out of the dreibein is gauge invariant, and so the topological properties of the field configurations which we link to it are also gauge invariant. The Yang-Mills action is like R^2 theory, but without the diffeomorphism invariance. It should be noted that *our interest in the present work is about Yang-Mills theory, not gravity*.
- (2) Our formulation can be useful for a nonperturbative understanding of Yang-Mills theory. We find that the topological, monopole configurations (which may drive confinement) correspond to conformally flat metrics. In our earlier works [5, 10], we showed how to locate such a configuration to an internal point (the “centre”) in a gauge invariant way. In this work, the “centres” are identified as certain points at which the conformal mode (a scalar field) is extremum.
- (3) There has been extensive interest in obtaining an ansatz for the Yang-Mills potential that exhibits the topological aspects [11–16]. We propose an ansatz

which has a scalar and a spin-two part in addition to a pure gauge. The scalar part comes from the conformally flat metric which captures the monopole configurations.

Our techniques can be extended in a straightforward way to SO(4) Yang-Mills theory in 4-Euclidean dimensions. Haagensen and Johnson [1] have also addressed generalization to other gauge groups. Our approach can also be applied to the 3 + 1-dimensional Yang-Mills theory in the Hamiltonian formulation, with the physical states described as wave functionals of the metric.

Ever since the proposal of nonabelian gauge theory, the similarity with Einstein gravity has led to extensive work exploring the relationship between the two. Examples of this on the gravity side include the Chern-Simons formulation of 2 + 1 gravity [17], Ashtekar and loop gravity formulation [18], and the pure connection formulation of general relativity [19]. We have already cited several works which attack gauge theory using gravity [1–9]; a recent work is [20]. In spite of all this, our approach gives a new way to attack the exotic features of Yang-Mills theory.

The paper is arranged as follows. In Section 2, the dreibein e_i^a is defined through the torsion-free condition, and the gauge invariance of the metric is emphasized. In Section 3, we analyse the existence and uniqueness of e_i^a for a given gauge potential A_i^a . Section 4 deals with topological (monopole) field configurations and Section 5 presents the gauge potential ansatz. In Section 6, we discuss our results.

2. Dreibein as Prepotential

Consider the set of nine first-order partial differential equations (see, e.g., [1])

$$\epsilon_{ijk} (\partial_j e_k^a + A_j^{ab} e_k^b) = 0, \quad (1)$$

which constitute the ‘‘torsion-free condition’’ for the dreibein $e_i^a(x)$ ($i = 1, 2, 3$ are the space indices and $a = 1, 2, 3$ are the group indices) with respect to a connection one-form $A_i^{ab}(x)$. The dreibein e_i^a is viewed as the square root of a metric

$$g_{ij}(x) = e_i^a(x) e_j^a(x). \quad (2)$$

If $\det e_i^a(x) \neq 0$, we can expand the LHS of (1) in the e_i^a basis and write

$$\partial_j e_k^a + A_j^{ab} e_k^b = \Gamma_{jk}^l e_l^a. \quad (3)$$

Imposing the symmetry

$$\Gamma_{jk}^l = \Gamma_{kj}^l \quad (4)$$

is equivalent to the torsion-free condition (1). Define

$$\mathcal{D}_i e_j^a = \partial_i e_j^a + A_i^{ab} e_j^b - \Gamma_{ij}^l e_l^a. \quad (5)$$

Then (3) is $\mathcal{D}_j e_k^a = 0$, and so $[\mathcal{D}_i, \mathcal{D}_j] e_k^a = 0$. This gives

$$R_{kij}^l e_l^a - F_{ij}^{ab} e_k^b = 0, \quad (6)$$

where R_{kij}^l is the Riemann tensor and

$$F_{ij}^{ab} = \partial_i A_j^{ab} - \partial_j A_i^{ab} + [A_i, A_j]^{ab} \quad (7)$$

(see, e.g., [21]). Using the inverse matrix $\{e^{bk}\}$,

$$e_k^a e^{bk} = \delta^{ab}, \quad (8)$$

we can write

$$F_{ij}^{ab} = R_{kij}^l e_l^a e^{bk}. \quad (9)$$

It may be helpful to link the above with Cartan’s structure equations of general relativity. Cartan’s first equation,

$$de^a + \omega^a_b \wedge e^b = 0, \quad (10)$$

is our equation (1); we have used the symbol A in the place of ω and put all group indices as upper indices. (Since we have SO(3) and not the Lorentz group, there is no need to distinguish between upper and lower indices.) Cartan’s other equation

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b \quad (11)$$

in components is (on using the dreibein to convert the group indices of R^a_{bij} to space indices)

$$R_{kij}^l e_l^a e^{bk} = \partial_i \omega_j^a_b - \partial_j \omega_i^a_b + [\omega_i, \omega_j]^a_b. \quad (12)$$

Equation (12) is (7) and (9) combined (in our notation).

Equation (1) can be written as

$$\epsilon_{ijk} (D_j e_k)^a = 0, \quad (13)$$

where D_j is the gauge-covariant derivative:

$$D_j^{ab} = \delta^{ab} \partial_j + A_j^{ab}. \quad (14)$$

Now write

$$A_i^{ab} = -e^{abc} A_i^c, \quad (15)$$

where A_i^c is the Yang-Mills potential. Then (7) gives the Yang-Mills field strength. A_i^a transforms *inhomogeneously* under an SO(3) gauge transformation, but (13) ensures that the dreibein e_i^a transforms *homogeneously*:

$$e_i^{a'}(x) = O^{ab}(x) e_i^b(x). \quad (16)$$

Here O^{ab} is an SO(3) matrix. The metric $g_{ij}(x)$ as given by (2) is therefore *gauge-invariant*. All gauge invariant objects can be rewritten in terms of g_{ij} . For instance,

$$\text{tr}(F_{i_1 j_1} F_{i_2 j_2} \cdots) = \text{tr}(R_{(i_1 j_1)} g^{-1} R_{(i_2 j_2)} g^{-1} \cdots), \quad (17)$$

where $R_{(ij)}$ is the matrix with the elements $(R_{(ij)})_{kl} = R_{klij}$ and g^{-1} has the elements $(g^{-1})_{km} = g^{km}$. This may be obtained from $F_{ij}^{ab} = R_{klij} g^{km} e_m^a e^{bl}$.

In 3 space dimensions, the Riemann curvature tensor can be completely expressed in terms of the Einstein tensor:

$$R_{kl ij} = -g\epsilon_{klm}\epsilon_{ijn}G^{mn}, \quad (18)$$

where $g = \det g_{ij}$. This allows us to express the Yang-Mills action in 3D

$$S = \frac{1}{2g_{YM}^2} \int d^3x B_i^a B_i^a, \quad (19)$$

(where $B_i^a \equiv (1/2)\epsilon_{ijk}F_{jk}^a$) also in terms of the Einstein tensor, as follows. In (6), we use (18) and also $F_{ij}^{ab} = -\epsilon^{abc}F_{ij}^c$. This gives [1]

$$B_i^a = (\det e) e^{aj} G_{ij}, \quad (20)$$

where $\det e = \det e_i^a = \sqrt{g}$. Then the Yang-Mills action in 3D is

$$S = \frac{1}{2g_{YM}^2} \int d^3x g g^{jk} G_{ij} G_{ik}. \quad (21)$$

This is like R^2 gravity, but *without the diffeomorphism invariance*.

Equation (1) or (13) comprises a set of linear equations for the variables A_i^a :

$$\epsilon_{ijk} (\partial_j e_k^a + \epsilon^{abc} A_j^b e_k^c) = 0. \quad (22)$$

Equation (22) has diffeomorphism covariance provided both A_i^a and e_i^a transform as covariant GL(3) vectors (the Christoffel symbol does not contribute due to antisymmetry of ϵ_{ijk}). Note that ϵ_{ijk} in (22) can be replaced by the Levi-Civita tensor ϵ^{ijk} since the two differ by just a factor of \sqrt{g} .

When the matrix $\{e_i^a\}$ is nonsingular, (22) can be uniquely solved for A_i^a . Indeed, multiplying by e_i^a and summing over a we get

$$\epsilon_{ijk} e_l^a \partial_j e_k^a = a_{il} - \delta_{il} a_{mm}, \quad (23)$$

where $a_{jm} = (\det e) A_j^b e_m^{bm}$. Now set $i = l$ in (23) to get a_{mm} and put it back in (23). Therefore [1, 5]

$$A_l^b = \frac{1}{\det e} \epsilon_{ijk} e_l^a \left(e_i^b \partial_j e_k^a - \frac{1}{2} \delta_{ab} e_i^c \partial_j e_k^c \right). \quad (24)$$

To sum up, (1) or (13) or (22) defines e_i^a . In our context, the dreibein does not arise from transformation between coordinate basis and orthonormal basis. The gauge transformation of A_i^a ensures that e_i^a transforms homogeneously and so g_{ij} as defined by (2) is gauge invariant. Equation (22) is diffeomorphism covariant. But the action is not diffeomorphism invariant, since we are dealing with Yang-Mills theory in flat space and not in curved space.

3. On Existence and Uniqueness of e_i^a for Given A_i^a

When $A_i^a(x) = 0$, (22) implies e_i^a is curl-free for each $a = 1, 2, 3$ and we have a general solution

$$e_i^a(x) = \partial_i \phi^a(x), \quad (25)$$

where $\phi^a(x)$ are arbitrary functions. Thus (22) has a large set of zero modes. In this situation, the curvature F_{ij}^{ab} (or $R_{kl ij}$) vanishes and we have a flat space. From (2), we see that (25) corresponds to $g_{ij}(x) = \partial_i \phi^a \partial_j \phi^a$, that is, a set of curvilinear coordinates $\phi^a(x)$ of the flat space. Now (22) is covariant under diffeomorphisms. If we start with $A_i^a(x) = 0$ and equation (25) and apply the diffeomorphism $x \rightarrow x'(x)$ using the transformation property of covariant vectors, A_i^a stays zero but e_i^a changes to $e_i^{a'}(x) = \partial_i \theta^a(x)$ (where $\theta^a(x') = \phi^a(x)$); that is, we have a new dreibein for the same A_i^a . Thus *we may view the set of zero modes of (22) given by (25) as accidental to the case of $A_i^a = 0$ and as a result of diffeomorphism covariance*.

In [1], it is claimed that all pure gauges, and not just $A_i^a = 0$, give zero modes. But a pure gauge potential $A_i(x) = O^T(x) \partial_i O(x)$ (where $A_i^{ab} = -\epsilon^{abc} A_i^c$ and O is an SO(3) matrix) changes under both diffeomorphism and gauge transformation. If A_i^a were invariant but the dreibein changed under a transformation, we would have had nonunique e_i^a for a given pure gauge A_i^a . This does not appear to be the case. So, while we do not exclude zero modes for nonvanishing pure gauges, such zero modes do not follow from any symmetry (either gauge covariance or diffeomorphism covariance) of (22), unlike the zero modes for vanishing gauge potential.

Now *both vanishing gauge potential and pure gauge A_i^a correspond to $B_i^a = 0$* . The point we make in this section is that, for a *generic gauge potential with $\det B_i^a \neq 0$* , we do not have such large set of nonunique e_i^a . *This issue was earlier addressed in Section 2 of [22]. Here we present a simpler and neater analysis, arriving at the same conclusion.*

First we note that operating on (13) by D_i gives a *consistency condition*, as follows. We have $\epsilon_{ijk} D_i^{ca} D_j^{ab} e_k^b = 0$. Using antisymmetry of ϵ_{ijk} and $[D_i, D_j]^{cb} = F_{ij}^{cb} = -\epsilon^{cba} F_{ij}^a$, we arrive at a condition to be satisfied by any solution to (22):

$$\vec{B}_i(x) \times \vec{e}_i(x) = 0. \quad (26)$$

In the case $\det B_i^a \neq 0$, without any loss of generality, we may expand $\vec{e}_i(x)$ in the basis provided by \vec{B}_i ($i = 1, 2, 3$):

$$e_i^a(x) = B_{ji}^a \alpha_{ji}. \quad (27)$$

Putting (27) in the consistency condition (26) gives us

$$(\det B) \epsilon_{ijk} (B^{-1})_k^a \alpha_{ji} = 0. \quad (28)$$

Multiplying (28) by B_l^a , we get $\epsilon_{ijl} \alpha_{ji} = 0$. Therefore $\alpha_{ij}(x)$ is a symmetric matrix:

$$\alpha_{ij}(x) = \alpha_{ji}(x). \quad (29)$$

In terms of these variables α_{ij} , (22) becomes

$$\epsilon_{ijk} (\partial_j \alpha_{mk} + \beta_{mjl} \alpha_{lk}) = 0 \quad (30)$$

with $i, j, k, l, m = 1, 2, 3$. Here

$$\beta_{mjl} = (B^{-1})_m^a (D_j B_l)^a. \quad (31)$$

Equation (30) gives the following six equations for evolution of α_{ij} in x_3 :

$$\partial_3 \alpha_{11} - \partial_1 \alpha_{13} + \beta_{131} \alpha_{11} - \beta_{111} \alpha_{13} = 0, \quad (32)$$

$$\partial_3 \alpha_{21} - \partial_1 \alpha_{23} + \beta_{231} \alpha_{11} - \beta_{211} \alpha_{13} = 0, \quad (33)$$

$$\partial_3 \alpha_{31} - \partial_1 \alpha_{33} + \beta_{331} \alpha_{11} - \beta_{311} \alpha_{13} = 0, \quad (34)$$

$$\partial_3 \alpha_{12} - \partial_2 \alpha_{13} + \beta_{131} \alpha_{12} - \beta_{121} \alpha_{13} = 0, \quad (35)$$

$$\partial_3 \alpha_{22} - \partial_2 \alpha_{23} + \beta_{231} \alpha_{12} - \beta_{221} \alpha_{13} = 0, \quad (36)$$

$$\partial_3 \alpha_{32} - \partial_2 \alpha_{33} + \beta_{331} \alpha_{12} - \beta_{321} \alpha_{13} = 0. \quad (37)$$

Note that there is no evolution equation for the variable α_{33} . Also, since there are two evolution equations (33) and (35) for the same variable $\alpha_{12} = \alpha_{21}$, we get the consistency requirement

$$\begin{aligned} \partial_3 \alpha_{12} &= \partial_2 \alpha_{13} + \beta_{121} \alpha_{13} - \beta_{131} \alpha_{12} \\ &= \partial_1 \alpha_{23} + \beta_{211} \alpha_{13} - \beta_{231} \alpha_{11}. \end{aligned} \quad (38)$$

This has to be satisfied at all x_3 . The variable α_{33} enters in this consistency equation. The coefficient of α_{33} is $\beta_{123} - \beta_{213}$. If

$$\beta_{123} - \beta_{213} \neq 0 \quad (39)$$

we can solve for α_{33} in favour of the other five variables $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{22}, \alpha_{23}$. Substituting this for α_{33} in the (five) independent equations (32)–(37), we get evolution equations for these five independent variables. The solution is unique with initial choice on $x_3 = x_3^0$ surface. This is ensured by the Cauchy-Kowalevski theorem (see, e.g., [22], which contains further reference, for a statement of this theorem). In the generic case, at least one of $(\beta_{123} - \beta_{213})$, $(\beta_{231} - \beta_{321})$, and $(\beta_{312} - \beta_{132})$ will be nonzero. If $\beta_{231} - \beta_{321} \neq 0$ (or $\beta_{312} - \beta_{132} \neq 0$), we can consider the evolution equation in x_1 (or x_2) instead of x_3 and obtain unique solution with initial choice on $x_1 = x_1^0$ (or $x_2 = x_2^0$) surface.

We have demonstrated that, on fixing the boundary condition, a unique torsion-free dreibein e_i^a exists for a generic Yang-Mills potential A_i^a . So, after taking into account the zero modes discussed in the first two paragraphs of this section, it should be possible to calculate the Jacobian of the transformation. Thus the transformation from A_i^a to e_i^a as given by (1) is indeed possible. However, the Jacobian will not be needed in the calculations of this paper. Only the defining equation of the dreibein will be used.

4. Nonperturbative Monopole Configurations and Conformal Mode of Metric

It is expected that topological degrees of freedom like magnetic monopoles are responsible for nonperturbative properties such as confinement [23]. In the Georgi-Glashow model, confinement is due to the 't Hooft-Polyakov monopole solution [24]. In contrast to this model, there is no scalar (Higgs) field in the SO(3) Yang-Mills theory in three dimensions. But the gauge field part of the 't Hooft-Polyakov

monopole with appropriate boundary conditions (see below) constitutes a finite-action field configuration and contributes to the functional integral. Moreover, it has strong qualitative effects on the Wilson loop.

We now summarize the idea of “centres” of topological field configurations as developed and used in our earlier works [5, 10, 25]. (This framework is in the spirit of, but different from, the Abelian projection procedure of 't Hooft [26].) The topological properties of the 't Hooft-Polyakov monopole can be characterised using only the gauge field and in the interior (in contrast to the usual characterisation by the triplet scalar field at infinity). This is therefore useful in pure gauge theory. This characterisation is done using the eigenvector fields of the gauge invariant quantity $S_{ij} \equiv B_i^a B_j^a$, where $B_i^a \equiv (1/2)\epsilon_{ijk} F_{jk}^a$ is the non-Abelian magnetic field.

For the 't Hooft-Polyakov monopole, $S_{ij} = \alpha(r)\delta_{ij} + \beta(r)x_i x_j$, where α and β are functions of the distance r from the origin. At $r = 0$, $S_{ij} \sim \delta_{ij}$ and so any direction is an eigenvector. Thus the eigenvector x_i with unit winding number is singular at the origin. In general, the points at which the eigenvalues of S_{ij} become triply degenerate or, equivalently, the eigenvector fields of S_{ij} become singular, are called the “centres” of the monopoles and other topological objects in Yang-Mills theory.

Let us then consider the gauge field part of the 't Hooft-Polyakov monopole configuration

$$A_i^a(x) = \epsilon_{aij} x_j \frac{1 - K(r)}{r^2}. \quad (40)$$

Here $K(r) = 1 + O(r^2)$ for $r \rightarrow 0$ and $K(r) \rightarrow 0$ for $r \rightarrow \infty$. (These conditions ensure the finiteness of the action.) Using

$$e_i^a(x) = \delta_i^a e^{-g(r)} \quad (41)$$

in (24), we find that the resulting gauge potential corresponds to (40) with

$$\frac{dg}{dr} = \frac{1 - K(r)}{r}. \quad (42)$$

The configurations (40) and (41) possess spherical symmetry. We next consider a generalization of (41) with $\phi(x)$, an arbitrary scalar function of x_1, x_2 , and x_3 , in the place of $g(r)$:

$$e_i^a(x) = \delta_i^a e^{-\phi(x)}. \quad (43)$$

This corresponds to

$$g_{ij} = \delta_{ij} e^{-2\phi}, \quad (44)$$

a conformally flat metric, and we call ϕ the *conformal mode*. Using (43) in (24) now leads to

$$A_i^a(x) = \epsilon_{aij} \partial_j \phi. \quad (45)$$

This gives

$$B_i^a = \partial_i \partial_a \phi - \delta_{ia} \partial^2 \phi + \partial_i \phi \partial_a \phi. \quad (46)$$

Now consider a point at which the following hold:

- (1) $\partial_i \phi = 0$.
- (2) $\partial_i \partial_a \phi = 0$ for $i \neq a$.
- (3) $\partial^2 \phi / \partial x_1^2 = \partial^2 \phi / \partial x_2^2 = \partial^2 \phi / \partial x_3^2$.

Note that the first condition says that we have a critical point and the next two conditions say that the Hessian matrix is diagonal and its three diagonal elements (eigenvalues) are equal. These are sufficient conditions for ϕ to be an extremum at the point. (ϕ attains a maximum or a minimum depending on whether the eigenvalues are all positive or all negative.)

Now since conditions 2 and 3 are equivalent to $\partial_i \partial_a \phi = (1/3)\delta_{ia}\partial^2\phi$, it follows from (46) that, at the point under consideration, $B_i^a \sim \delta_{ia}$ and so $B_i^a B_j^a \sim \delta_{ij}$. So the matrix S_{ij} is triply degenerate at the point, which, as explained at the beginning of this section, is the criterion for locating the ‘‘centre’’ of a topological configuration in a gauge-invariant description. Thus *at the topological centre, the field ϕ is an extremum, satisfying the three conditions given above.*

For the special, spherical symmetric case given by (40) and (41), using $K(r) = 1 + O(r^2)$ for $r \rightarrow 0$, it can be checked that the function $g(r)$ indeed satisfies the above conditions on $\phi(x)$ at the monopole centre $r = 0$. On the other hand, in the more general form given by (43), the conditions on $\phi(x_1, x_2, x_3)$ stated above show that ϕ needs to be symmetric only up to the second-order terms in the Taylor expansion about the centre. Thus, taking the topological centre to be at $r = 0$, the forms $A + Br^2 + Cx_1^3 + \dots$ and $A + Br^2 + Dx_1^2 x_2 + \dots$ (where A, B, C , and D are constants) are two examples of possible Taylor expansion for the function ϕ satisfying the conditions.

It is interesting to apply our formalism to the Wu-Yang monopole [27]. This is a solution to the Yang-Mills field equations. However, it is a point-like magnetic monopole. The gauge potential for it is obtained on putting $K(r) = 0$ in (40). Then (42) gives $g(r) = \ln r$ (up to an additive constant, which sets the scale). So the conformal mode goes to negative infinity as r goes to zero. Such a singularity, instead of a smooth minimum, in the conformal mode is just what is expected since the Wu-Yang monopole is point-like. The metric in this case is $g_{ij} = \delta_{ij}/r^2$ (up to a multiplicative constant).

5. Ansatz for Gauge Field

We now consider the polar decomposition of the most general 3×3 matrix e_i^a into an orthogonal matrix \mathcal{R} and a symmetric matrix \mathcal{E} :

$$e_i^a(x) = \mathcal{R}_j^a(x) \mathcal{E}_{ji}(x) \quad \forall x. \quad (47)$$

Under a gauge transformation,

$$\mathcal{R}_j^a(x) \longrightarrow O^{ab}(x) \mathcal{R}_j^b(x), \quad (48)$$

while the symmetric matrix \mathcal{E}_{ji} is gauge-invariant and is the symmetric square-root of the metric g_{ij} . The decomposition (47) corresponds to

$$A_i^d(x) = \mathcal{R}_m^d(x) a_{ml}(x) + \omega_l^d(x), \quad (49)$$

where

$$a_{ml}(x) = \frac{1}{\det \mathcal{E}} \epsilon_{ijk} \left(\mathcal{E}_{im} \mathcal{E}_{lp} - \frac{1}{2} \mathcal{E}_{ip} \mathcal{E}_{lm} \right) \partial_j \mathcal{E}_{kp} \quad (50)$$

and ω_l^d is formally a pure gauge:

$$\omega_l^d(x) = -\frac{1}{2} \epsilon^{bcd} \mathcal{R}_m^b(x) \partial_l \mathcal{R}_m^c(x). \quad (51)$$

(To obtain (49), one puts (47) in (24). The terms containing derivative of \mathcal{E} immediately give the first term on the RHS of (49). In the terms containing derivative of \mathcal{R} , we put $\partial_j \mathcal{R}_p^a = -\epsilon^{abc} \omega_j^b \mathcal{R}_p^c$, which is the same as (51). Then using $\epsilon^{abc} \mathcal{R}_n^a \mathcal{R}_p^c = \epsilon_{nqp} \mathcal{R}_q^b$ (since $\det R = 1$) and $\epsilon_{ijk} \mathcal{E}_{in} \mathcal{E}_{kp} = \epsilon_{nrp} (\det \mathcal{E}) (\mathcal{E}^{-1})_{jr}$, we get the ω_l^d term in (49).)

Now the symmetric matrix \mathcal{E} can be further decomposed into a spin-two traceless part and a scalar trace part:

$$\mathcal{E}_{ij} = \tilde{\mathcal{E}}_{ij} + \delta_{ij} e^{-\phi}. \quad (52)$$

(Here $\tilde{\mathcal{E}}_{ij} = \mathcal{E}_{ij} - (1/3)\delta_{ij}\Sigma_k \mathcal{E}_{kk}$ and we define $(1/3)\Sigma_k \mathcal{E}_{kk}$ to be $e^{-\phi}$.) Since $\delta_{ij} e^{-\phi}$ corresponds to the metric (44), we propose that the topological configurations are contained in this part, with ϕ satisfying the three conditions given after (46) at isolated points. At such points, ω_l^d is not strictly a pure gauge. For example, the non-Abelian magnetic field corresponding to ω_l^d has a Dirac string contribution when \mathcal{R} is the singular gauge in which the configuration (40) becomes a Dirac monopole [10].

Thus (49), (50), (51), and (52) comprise our ansatz for the gauge potential, which contains the topological aspects in a natural way. The three gauge degrees of freedom reside in \mathcal{R} and the six gauge-invariant degrees of freedom reside in \mathcal{E} . The conformal mode contained in \mathcal{E} captures the topological configurations.

6. Discussion

In this work, we have used the dreibein and the metric as basic variables for the SO(3) Yang-Mills theory in three Euclidean dimensions. The usefulness of this lies in the fact that the metric is gauge invariant.

By showing that a generic A_i^a corresponds to a unique torsion-free e_i^a on fixing the boundary condition, we have provided justification for the change of variables from A_i^a to e_i^a .

We have applied this formalism to nonperturbative aspects of the theory. The conformal mode of the metric is found to contain the topological aspects of the gauge potential. We have demonstrated that topological centres are located at certain points at which this conformal mode, a scalar field, is an extremum. This criterion is thus an

alternative to that developed by us in [5, 10] (also applied by us in [25]), namely, that the points of triple degeneracy of $B_i^a B_j^a$ are the centres of topological configurations. Both of these criteria are gauge invariant. The Yang-Mills theory expressed in terms of this conformal mode would be a theory of a scalar field, and lattice simulation of this *scalar* theory (as opposed to lattice *gauge* theory) with monopoles located at extrema of the scalar field would serve as a model for confinement.

We have also proposed an ansatz for the gauge potential, which incorporates the topological aspects contained in the conformal mode after separating the gauge variant and the gauge invariant degrees of freedom.

When our formalism is applied to the Hamiltonian formulation of 3 + 1-dimensional Yang-Mills theory, the canonical variables are the metric and its conjugate variable. The physical wave functionals are simply functionals of the metric. However, due to the change of variables, the inner product of the wave functionals will involve the Jacobian of the transformation. For 2 + 1-dimensional $SU(N)$ gauge theory, Nair et al. [28–31] proposed an $N \times N$ complex matrix as a prepotential and, as a consequence of the change of variables, obtained an inner product for physical states which involves the Wess-Zumino-Witten action. They arrived at a trial wave functional which exhibits confinement and gives string tension in striking agreement with lattice gauge theory simulations. Viewing our proposal of using the dreibein as prepotential as a generalization to 3 + 1-dimensions, it will therefore be interesting to calculate the Jacobian of the transformation in our case.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

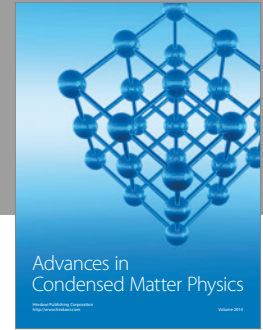
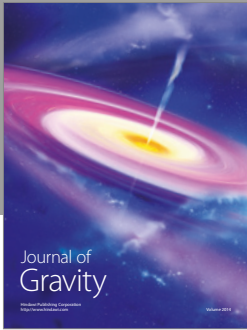
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