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Research Article

Global Nonexistence of Solutions for Viscoelastic Wave Equations of Kirchhoff Type with High Energy

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We consider viscoelastic wave equations of the Kirchhoff type $u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + u_t = |u|^{p-1}u$ with Dirichlet boundary conditions, where $\|\cdot\|_p$ denotes the norm in the Lebesgue space L^p. Under some suitable assumptions on g and the initial data, we establish a global nonexistence result for certain solutions with arbitrarily high energy, in the sense that $\lim_{t\to T^{*-}}(\|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2\mathrm{d}s) = \infty$ for some $0 < T^* < +\infty$.

1. Introduction

In this paper we consider the following problem:

$$\begin{gathered}
u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + |u_t|^{m-1}u_t = |u|^{p-1}u, \quad (x,t) \in \Omega \times (0,\infty), \\
u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $p > 1$, $M(s)$ is a nonnegative C^1 function like $M(s) = a + bs^\gamma$ for $s \geq 0$, $a \geq 0$, $b \geq 0$, $a + b > 0$, $\gamma > 0$ and $g(t)$ represents the kernel of memory term.

Problem (1.1) without the viscoelastic term (i.e., $g = 0$) has been extensively studied and many results concerning global existence, decay, and blowup have been established. For example, the following equation:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + g(u_t) = f(u), \quad (x,t) \in \Omega \times (0,\infty), \tag{1.2}$$

has been considered by Matsuyama and Ikehata in [1] for $g(u_t) = \delta|u_t|^{p-1}u_t$ and $f(u) = \mu|u|^{q-1}u$. The authors proved existence of the global solutions by using Faedo-Galerkin method and the decay of energy based on the method of Nakao [2–4]. Later, Ono [5] investigated (1.2) for $M(s) = bs^\gamma$ and $f(u) = |u|^{p-2}u$. When $g(u_t) = -\Delta u_t$, u_t or $|u_t|^\beta u_t$, the author showed that the solutions blow up in finite time with $E(0) \le 0$. For $M(s) = a + bs^\gamma$ and $g(u_t) = u_t$, this model was considered by the same author in [6]. By applying the potential well method he obtained the blow-up properties with positive initial energy $E(0)$. Recently, Zeng et al. [7] studied (1.2) for the case $g(u_t) = u_t$ with the same initial and boundary conditions as that of problem (1.1). By using the concavity argument, they proved that the solutions to (1.2) blow up in finite time with arbitrarily high energy.

In the case of $M \equiv 1$ and in the presence of the viscoelastic term (i.e., $g \neq 0$), the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + |u_t|^{m-1}u_t = |u|^{p-1}u, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.3}$$

was studied by Messaoudi in [8], where the author proved that any weak solution with negative initial energy blows up in finite time if $p > m$ and

$$\int_0^\infty g(s)\mathrm{d}s \le \frac{p-1}{p-1+1/(p+1)}, \tag{1.4}$$

while the solution continues to exist globally for any initial data in the appropriate space if m $\ge p$. This blow-up result was improved by the same author in [9] for positive initial energy under suitable conditions on g, m, and p. More recently, Wang [10] investigated (1.3) and established a blow-up result with arbitrary positive initial energy. In the related work, Cavalcanti et al. [11] studied the following equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + a(x)u_t + |u|^\gamma u = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.5}$$

where $a : \Omega \to R^+$ is a function which may be null on a part of Ω. Under the condition that $a(x) \ge a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometric restrictions and $-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t)$, $t \ge 0$ to guarantee that $\|g\|_{L^1((0,\infty))}$ is small enough, they proved an exponential decay rate.

When $g \neq 0$ and M is not a constant function, problems related to (1.1) have been treated by several authors. Wu and Tsai [12] considered the global existence, asymptotic behavior, and blow-up properties for the following equation:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \Delta u_t = f(u), \quad (x,t) \in \Omega \times (0,\infty), \tag{1.6}$$

with the same initial and boundary conditions as that of problem (1.1). They obtained the blow-up properties of local solution with small positive initial energy by using the direct method of [13]. Global existence and decay properties of the solutions were also obtained there. In [14], Wu then extended the decay result of [12] under a weaker condition on g.

For other papers related to existence, uniform decay and blowup of solutions of nonlinear wave equations, see [15–33] and references therein.

Motivated by the above research, we consider problem (1.1) for $m = 1$ in this paper and establish a global nonexistence result for certain solutions with arbitrarily high energy by using concavity technique. In this way, we can extend the result of [7] to nonzero term g and the result of [10] to nonconstant $M(s)$. Throughout the rest of this paper, we always assume that $m = 1$.

The structure of this paper is as follows. In Section 2, we present some assumptions, notations and the main result. In Section 3, we give the proof of the main result. Some further remarks are stated in Section 4.

2. Preliminaries and Main Result

In this section, we will give some assumptions, notations and state the main result. We first give the following assumptions:

(A1) $g \in C^1([0,\infty))$ is a nonnegative and non-increasing function satisfying

$$1 - \int_0^\infty g(s)\mathrm{d}s = l > 0. \tag{2.1}$$

(A2) The function $e^{t/2}g(t)$ is of positive type in the following sense (see [10]):

$$\int_0^t v(s)\int_0^s e^{(s-z)/2}g(s-z)v(z)\mathrm{d}z\,\mathrm{d}s \geq 0, \quad \forall v \in C^1([0,\infty)),\ \forall t > 0. \tag{2.2}$$

Remark 2.1. Assumption (A2) is needed to prove Lemma 3.1 below.

In order to prove our result, we make the following assumption on M and g:

(A3) there exists a positive constant m_1 such that

$$\frac{p+1}{2}\overline{M}(s) - \left[M(s) + \frac{p+1}{2}\int_0^\infty g(\tau)\mathrm{d}\tau\right]s \geq m_1 s, \quad \forall s \geq 0, \tag{2.3}$$

where $\overline{M}(s) = \int_0^s M(\tau)\mathrm{d}\tau$.

Remark 2.2. It is clear that when $M(s) = a + bs^\gamma$ for $s \geq 0$, $a \geq 0$, $b \geq 0$, $a + b > 0$, $\gamma > 0$ and $p > 1 + 2\gamma$, condition (A3) can be replaced by

$$\int_0^\infty g(\tau)\mathrm{d}\tau < \begin{cases} \dfrac{p-1}{p+1}a, & \text{if } a > 0,\ b \geq 0, \\ \dfrac{(p-1-2\gamma)b}{C_p^\gamma (p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}, & \text{if } a = 0,\ b > 0, \end{cases} \tag{2.4}$$

which is the same as the one in [10, Theorem 1.1] for the case $a = 1$ and $b = 0$, where C_p is the constant from the Poincaré inequality $\|u(t)\|_2^2 \leq C_p\|\nabla u(t)\|_2^2$. Then, the possible choice of the positive constant m_1 in (A3) can be easily obtained (see Section 4.1 for details).

It is necessary to state the local existence theorem for problem (1.1), whose proof follows the arguments in [12, 34].

Theorem 2.3. *Assume that (A1) holds, and $1 < p \leq n/(n-2)$ when $n \geq 3$, $1 < p < \infty$ when $n = 1, 2$. For $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, and $M(\|\nabla u_0\|_2^2) > 0$, problem (1.1) has a unique local solution*

$$u \in C\left([0,T); H_0^1(\Omega) \cap H^2(\Omega)\right), \qquad u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^2\left([0,T); H_0^1(\Omega)\right), \tag{2.5}$$

for the maximum existence time $T > 0$.

The energy functional $E(t)$ and an auxiliary functional $I(u)$ of the solution $u(t)$ of problem (1.1) are defined as follows:

$$\begin{aligned} E(t) &:= E(u(t)) \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\overline{M}\left(\|\nabla u\|_2^2\right) - \frac{1}{2}\int_0^t g(s)\mathrm{d}s\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \end{aligned} \tag{2.6}$$

$$I(u) = M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1}, \tag{2.7}$$

where

$$(g \circ w)(t) = \int_0^t g(t-s)\|w(t,\cdot) - w(s,\cdot)\|_2^2\mathrm{d}s. \tag{2.8}$$

As in [7, 10], we can get

$$\frac{d}{\mathrm{d}t}E(t) = -\|u_t\|_2^2 - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) \leq 0, \quad \forall t \geq 0. \tag{2.9}$$

Then we have

$$E(t) = E(0) - \int_0^t \|u_s\|_2^2\mathrm{d}s + \frac{1}{2}\int_0^t (g' \circ \nabla u)(s)\mathrm{d}s - \frac{1}{2}\int_0^t g(s)\|\nabla u(s)\|_2^2\mathrm{d}s. \tag{2.10}$$

Now we are in a position to state the main result.

Theorem 2.4. *Assume that* $(A1)$ *holds and* $1 < p \leq n/(n-2)$ *when* $n \geq 3$, $1 < p < \infty$ *when* $n = 1, 2$. *Let* u *be a solution of problem* (1.1) *with initial data* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ *and* $M(\|\nabla u_0\|_2^2) > 0$, *and further assume that*

$$E(0) > 0, \tag{2.11}$$

$$I(u_0) < 0, \tag{2.12}$$

$$\int_\Omega u_0 u_1 dx > 0, \tag{2.13}$$

$$\|u_0\|_2^2 > \frac{(p+1)C_p}{m_1} E(0). \tag{2.14}$$

Then the solution of problem (1.1) *blows up in finite time* $0 < T^* < +\infty$, *which means that*

$$\lim_{t \to T^{*-}} \left(\|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2 ds \right) = \infty, \tag{2.15}$$

where C_p *is a constant from the Poincaré inequality and* m_1 *comes from condition* $(A3)$.

Remark 2.5. We note that the set of the initial data which satisfy conditions (2.11)–(2.14) is not empty (see Section 4.2 for details).

3. Proof of the Main Result

In this section we prove our main result, Theorem 2.4, whose proof follows the ideas already used in [7, 10] and relies on the following lemmas.

Lemma 3.1 (see [10, Lemma 2.1]). *Assume that* $g(t)$ *satisfies assumptions* $(A1)$-$(A2)$, *and* $H(t)$ *is a function which is twice continuously differentiable satisfying*

$$\begin{gathered} H''(t) + H'(t) > \int_0^t g(t-s) \int_\Omega \nabla u(s) \nabla u(t) dx\, ds, \\ H(0) > 0, \qquad H'(0) > 0, \end{gathered} \tag{3.1}$$

for every $t \in [0, T)$, *where* $u(t)$ *is the corresponding solution of problem* (1.1) *with* u_0 *and* u_1. *Then the function* $H(t)$ *is strictly increasing on* $[0, T)$.

Lemma 3.2. *Suppose that* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$ *satisfy*

$$\int_\Omega u_0 u_1 dx > 0. \tag{3.2}$$

If the solution $u(t)$ *of problem* (1.1) *exists on* $[0, T)$ *and satisfies*

$$I(u(t)) < 0, \tag{3.3}$$

then $\|u(t)\|_2^2$ *is strictly increasing on* $[0, T)$.

Proof. Since $u(t)$ is the solution of problem (1.1), by a simple computation, we have

$$\begin{aligned}\frac{1}{2}\frac{d^2}{\mathrm{d}t^2}\int_\Omega |u(x,t)|^2\mathrm{d}x &= \int_\Omega \left(|u_t|^2 + uu_{tt}\right)\mathrm{d}x \\ &= \|u_t\|_2^2 - M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1} \\ &\quad + \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\mathrm{d}x\,\mathrm{d}s - \int_\Omega uu_t\mathrm{d}x \\ &> -\int_\Omega uu_t\mathrm{d}x + \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{3.4}$$

where the last inequality is derived by (3.3). Then we get

$$\frac{d^2}{\mathrm{d}t^2}\int_\Omega |u(x,t)|^2\mathrm{d}x + \frac{d}{\mathrm{d}t}\int_\Omega |u(x,t)|^2\mathrm{d}x > \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\mathrm{d}x\,\mathrm{d}s. \tag{3.5}$$

Therefore, by using Lemma 3.1, we finish our proof. □

Lemma 3.3. *If $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ satisfy the assumptions in Theorem 2.4, then the solution $u(t)$ of problem* (1.1) *satisfies*

$$I(u(t)) < 0, \tag{3.6}$$

$$\|u\|_2^2 > \frac{(p+1)C_p}{m_1}E(0), \tag{3.7}$$

for all $t \in [0,T)$.

Proof. We will prove the above lemma by contradiction. First we assume that (3.6) is not true over $[0,T)$, it means that there exists a time t_0 such that

$$t_0 = \min\{t \in (0,T) : I(u(t)) = 0\}. \tag{3.8}$$

Since $I(u(t)) < 0$ on $[0,t_0)$, by Lemma 3.2, we see that $\int_\Omega u^2\mathrm{d}x$ is strictly increasing over $[0,t_0)$, which implies

$$\int_\Omega u^2\mathrm{d}x > \int_\Omega u_0^2\mathrm{d}x > \frac{(p+1)C_p}{m_1}E(0). \tag{3.9}$$

And by the continuity of $\int_\Omega u^2\mathrm{d}x$ on t, we note that

$$\int_\Omega u^2(t_0)\mathrm{d}x \geq \frac{(p+1)C_p}{m_1}E(0). \tag{3.10}$$

On the other hand, by (2.6) and (2.9), we get

$$\overline{M}\left(\|\nabla u(t_0)\|_2^2\right) - \int_0^{t_0} g(s)\mathrm{d}s\|\nabla u(t_0)\|_2^2 + (g \circ \nabla u)(t_0) - \frac{2}{p+1}\|u(t_0)\|_{p+1}^{p+1} \le 2E(0). \tag{3.11}$$

Combining (3.11) with (3.8) yields

$$\begin{aligned}&\frac{p+1}{2}\overline{M}\left(\|\nabla u(t_0)\|_2^2\right) - \frac{p+1}{2}\int_0^{t_0} g(s)\mathrm{d}s\|\nabla u(t_0)\|_2^2 + \frac{p+1}{2}(g \circ \nabla u)(t_0)\\ &\quad - M\left(\|\nabla u(t_0)\|_2^2\right)\|\nabla u(t_0)\|_2^2 \le (p+1)E(0).\end{aligned} \tag{3.12}$$

By (A3), we get

$$\|\nabla u(t_0)\|_2^2 < \frac{p+1}{m_1}E(0). \tag{3.13}$$

By Poincaré's inequality, we have

$$\|u(t_0)\|_2^2 < \frac{(p+1)C_p}{m_1}E(0). \tag{3.14}$$

Obviously, there is a contradiction between (3.10) and (3.14), thus we prove that

$$I(u(t)) < 0, \tag{3.15}$$

for every $t \in (0, T)$. By Lemma 3.2, it follows that $\int_\Omega u^2 \mathrm{d}x$ is strictly increasing on $[0, T)$, which implies that

$$\int_\Omega u^2\,\mathrm{d}x \ge \int_\Omega u_0^2\,\mathrm{d}x > \frac{(p+1)C_p}{m_1}E(0), \tag{3.16}$$

for every $t \in [0, T)$. This completes the proof of Lemma 3.3. □

Proof of Theorem 2.4. We prove our main result by adopting concavity method. We assume by contradiction that the T is sufficiently large. Then we consider the auxiliary function

$$G(t) = \|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2\mathrm{d}s + (T_0 - t)\|u_0\|_2^2 + \beta(t_2 + t)^2, \quad t \in [0, T_0], \tag{3.17}$$

where T_0, t_2, and β are positive constants, which will be chosen later.

A straightforward calculation gives

$$\begin{aligned} G'(t) &= 2\int_{\Omega} uu_t \mathrm{d}x + \|u(t)\|_2^2 - \|u_0\|_2^2 + 2\beta(t_2+t) \\ &= 2\int_{\Omega} uu_t \mathrm{d}x + 2\int_0^t (u(s), u_s(s))\mathrm{d}s + 2\beta(t_2+t), \end{aligned} \tag{3.18}$$

consequently,

$$\begin{aligned} G''(t) &= 2\int_{\Omega} |u_t|^2 \mathrm{d}x + 2\int_{\Omega} uu_{tt} \mathrm{d}x + 2\int_{\Omega} uu_t \mathrm{d}x + 2\beta \\ &= 2\|u_t\|_2^2 - 2M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + 2\int_0^t g(t-s)\int_{\Omega} \nabla u(s)\nabla u(t)\mathrm{d}x\mathrm{d}s \\ &\quad - 2\int_{\Omega} uu_t \mathrm{d}x + 2\int_{\Omega} uu_t \mathrm{d}x + 2\|u\|_{p+1}^{p+1} + 2\beta \\ &= 2\|u_t\|_2^2 - 2M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + 2\|u\|_{p+1}^{p+1} + 2\int_0^t g(t-s)\mathrm{d}s\|\nabla u\|_2^2 \\ &\quad + 2\int_0^t g(t-s)\int_{\Omega} \nabla u(t)(\nabla u(s) - \nabla u(t))\mathrm{d}x\mathrm{d}s + 2\beta. \end{aligned} \tag{3.19}$$

By using Young's inequality, we obtain

$$\int_0^t g(t-s)\int_{\Omega} |\nabla u(t)||\nabla u(s) - \nabla u(t)|\mathrm{d}x\mathrm{d}s \le \int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 + \frac{1}{4}(g \circ \nabla u)(t). \tag{3.20}$$

Substituting (2.6) and (3.20) for the third and the fifth terms of the right hand side of (3.19), respectively, we have

$$\begin{aligned} G''(t) \ge{}& (p+3)\|u_t\|_2^2 + (p+1)\overline{M}\left(\|\nabla u\|_2^2\right) - 2M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 - (p+1)\int_0^t g(s)\mathrm{d}s\|\nabla u\|_2^2 \\ & - 2(p+1)E(t) + \left(p+\frac{1}{2}\right)(g \circ \nabla u)(t) + 2\beta. \end{aligned} \tag{3.21}$$

By (A3), we deduce

$$G''(t) > (p+3)\|u_t\|_2^2 + 2m_1\|\nabla u\|_2^2 - 2(p+1)E(t) + \left(p+\frac{1}{2}\right)(g \circ \nabla u)(t) + 2\beta. \tag{3.22}$$

Noting that (2.10), we obtain that

$$-E(t) \ge -E(0) + \int_0^t \|u_s\|_2^2 \mathrm{d}s. \tag{3.23}$$

Combining (3.22)-(3.23) yields

$$G''(t) > (p+3)\|u_t\|_2^2 + 2m_1\|\nabla u\|_2^2 - 2(p+1)E(0) + \left(p+\frac{1}{2}\right)(g\circ\nabla u)(t) + 2(p+1)\int_0^t \|u_s\|_2^2 \mathrm{d}s + 2\beta. \quad (3.24)$$

By Poincaré's inequality, Lemma 3.2, and (2.14), we see that

$$2m_1\|\nabla u\|_2^2 - 2(p+1)E(0) + \left(p+\frac{1}{2}\right)(g\circ\nabla u)(t) > \frac{2m_1}{C_p}\|u_0\|_2^2 - 2(p+1)E(0) > 0, \quad (3.25)$$

by (3.24)-(3.25), we get

$$G''(t) > (p+3)\|u_t\|_2^2 + \frac{2m_1}{C_p}\|u_0\|_2^2 - 2(p+1)E(0) + 2(p+1)\int_0^t \|u_s\|_2^2 \mathrm{d}s + 2\beta, \quad (3.26)$$

which means that $G''(t) > 0$ for every $t \in [0, T_0]$. Thus, by $G^{'}(0) > 0$ and $G(0) > 0$, we get $G^{'}$ and $G(t)$ are strictly increasing on $[0, T_0]$.

We first choose β small enough satisfying

$$(p+1)\beta < \frac{2m_1}{C_p}\|u_0\|_2^2 - 2(p+1)E(0), \quad (3.27)$$

consequently,

$$G''(t) > (p+3)\|u_t\|_2^2 + 2(p+1)\int_0^t \|u_s\|_2^2 \mathrm{d}s + (p+3)\beta. \quad (3.28)$$

As far as β is fixed, we select t_2 large enough satisfying

$$\frac{p-1}{2}\left(\int_\Omega u_0u_1\mathrm{d}x + \beta t_2\right) > \|u_0\|_2^2. \quad (3.29)$$

From (3.17), (3.18), and (3.29), we now choose T_0 such that $T_0 > (\|u_0\|_2^2 + \beta t_2^2)/(((p-1)/2)(\int_\Omega u_0\, u_1\mathrm{d}x + \beta t_2) - \|u_0\|_2^2) > 0$, which ensures that

$$T_0 > \frac{4}{p-1}\frac{G(0)}{G'(0)}. \quad (3.30)$$

Letting

$$A := \|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2 \mathrm{d}s + \beta(t_2 + t)^2,$$
$$B := \frac{1}{2}G'(t), \tag{3.31}$$
$$C := \|u_t(t)\|_2^2 + \int_0^t \|u_s(s)\|_2^2 \mathrm{d}s + \beta.$$

Since we have assumed that the solution $u(t)$ to problem (1.1) exists for every $t \in [0, T)$, where T is sufficiently large, we have

$$G(t) \geq A,$$
$$G''(t) \geq (p+3)C, \tag{3.32}$$

for every $t \in [0, T_0]$. Then it follows that

$$G''(t)G(t) - \frac{p+3}{4}\left(G'(t)\right)^2 \geq (p+3)\left(AC - B^2\right). \tag{3.33}$$

Furthermore, we have

$$Ar^2 - 2Br + C = \int_\Omega (ru(t) - u_t(t))^2 \mathrm{d}x + \int_0^t \|ru(s) - u_s(s)\|_2^2 \mathrm{d}s + \beta[r(t_2 + t) - 1]^2 \geq 0, \tag{3.34}$$

for every $r \in \mathbb{R}$, which implies that $B^2 - AC \leq 0$. Thus, we obtain

$$G''(t)G(t) - \frac{p+3}{4}\left(G'(t)\right)^2 \geq 0. \tag{3.35}$$

As $(p+3)/4 > 1$, letting $\theta = (p-1)/4$, we have

$$G''(t)G(t) - (1+\theta)\left(G'(t)\right)^2 \geq 0. \tag{3.36}$$

According to concavity technique, there exists a real number T^* such that $T^* \leq G(0)/\theta G'(0) < T_0$ and we have

$$\lim_{t \to T^{*-}} G(t) = \infty, \tag{3.37}$$

that is,

$$\lim_{t \to T^{*-}} \left(\|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2 \,\mathrm{d}s \right) = \infty, \tag{3.38}$$

which contradicts the assumption that the T is sufficiently large.

This completes the proof of Theorem 2.4. □

4. Some Further Remarks

4.1. The Possible Choice of the Positive Constant m_1 in (A3)

When $M(s) = a + bs^{\gamma}$ for $s \geq 0$, $a \geq 0$, $b \geq 0$, $a + b > 0$, $\gamma > 0$ and $p > 1 + 2\gamma$, by straightforward calculation, we obtain

$$\begin{aligned}
&\frac{p+1}{2}\overline{M}(s) - \left[M(s) + \frac{p+1}{2}\int_0^{\infty} g(\tau)\mathrm{d}\tau\right] s \\
&\quad = \frac{p+1}{2}\left(as + \frac{b}{\gamma+1}s^{\gamma+1}\right) - as - bs^{\gamma+1} - \frac{(p+1)s}{2}\int_0^{\infty} g(\tau)\mathrm{d}\tau \\
&\quad = \frac{p-1}{2}as + \frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1} - \frac{(p+1)s}{2}\int_0^{\infty} g(\tau)\mathrm{d}\tau.
\end{aligned} \tag{4.1}$$

If $a > 0$ and $b \geq 0$, it follows from (2.4) that $\int_0^{\infty} g(\tau)\mathrm{d}\tau < ((p-1)/(p+1))\ a$. Thus, we have

$$\begin{aligned}
&\frac{p+1}{2}\overline{M}(s) - \left[M(s) + \frac{p+1}{2}\int_0^{\infty} g(\tau)\mathrm{d}\tau\right] s \\
&\quad > \frac{p-1}{2}as + \frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1} - \frac{(p+1)s}{2}\left[\frac{p-1}{p+1}a - \frac{((p-1)/(p+1))a - \int_0^{\infty} g(\tau)\mathrm{d}\tau}{2}\right] \\
&\quad = \frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1} + \frac{p+1}{4}\left[\frac{p-1}{p+1}a - \int_0^{\infty} g(\tau)\mathrm{d}\tau\right]s \geq \frac{p+1}{4}\left[\frac{p-1}{p+1}a - \int_0^{\infty} g(\tau)\mathrm{d}\tau\right]s,
\end{aligned} \tag{4.2}$$

where we have used a obvious conclusion: $m < n \Rightarrow m < n - (n-m)/2$. Therefore, we can choose $m_1 = ((p+1)/4)[((p-1)/(p+1))\ a - \int_0^{\infty} g(\tau)\mathrm{d}\tau]$ in condition (A3).

If $a = 0$ and $b > 0$, then

$$\begin{aligned}
&\frac{p+1}{2}\overline{M}(s) - \left[M(s) + \frac{p+1}{2}\int_0^{\infty} g(\tau)\mathrm{d}\tau\right] \mathrm{s} \\
&\quad = \frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1} - \frac{(p+1)s}{2}\int_0^{\infty} g(\tau)\mathrm{d}\tau > \frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1} - \frac{(p+1)s}{2} \\
&\qquad \times \left[\frac{(p-1-2\gamma)b}{C_p^{\gamma}(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma} \right.
\end{aligned}$$

$$
\left. - \frac{\left((p-1-2\gamma)b/\left(C_p^{\gamma}\ \ (p+1)(\gamma+1)\right)\right)\|u_0\|_2^{2\gamma} - \int_0^{\infty} g(\tau)\mathrm{d}\tau}{2}\right]
$$

$$
= \frac{(p-1-2\gamma)b}{2(\gamma+1)} s\left(s^{\gamma} - \frac{1}{C_p^{\gamma}}\|u_0\|_2^{2\gamma}\right) + \frac{(p+1)s}{4}\left[\frac{(p-1-2\gamma)b}{C_p^{\gamma}(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma} - \int_0^{\infty} g(\tau)\mathrm{d}\tau\right]. \tag{4.3}
$$

Taking $s = \|\nabla u(t)\|_2^2$, applying Lemma 3.2 and Poincaré's inequality, we can get

$$
\begin{aligned}
&\frac{p+1}{2}\overline{M}\left(\|\nabla u(t)\|_2^2\right) - \left[M\left(\|\nabla u(t)\|_2^2\right) + \frac{p+1}{2}\int_0^{\infty} g(\tau)\mathrm{d}\tau\right]\|\nabla u(t)\|_2^2 \\
&\quad > \frac{(p-1-2\gamma)b}{2(\gamma+1)}\|\nabla u(t)\|_2^2\left(\|\nabla u(t)\|_2^{2\gamma} - \frac{1}{C_p^{\gamma}}\|u_0\|_2^{2\gamma}\right) \\
&\qquad + \frac{p+1}{4}\|\nabla u(t)\|_2^2\left[\frac{(p-1-2\gamma)b}{C_p^{\gamma}(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma} - \int_0^{\infty} g(\tau)\mathrm{d}\tau\right] \\
&\quad \geq \frac{(p-1-2\gamma)b}{2(\gamma+1)}\|\nabla u(t)\|_2^2\left(\|\nabla u(t)\|_2^{2\gamma} - \frac{1}{C_p^{\gamma}}\|u(t)\|_2^{2\gamma}\right) \\
&\qquad + \frac{p+1}{4}\|\nabla u(t)\|_2^2\left[\frac{(p-1-2\gamma)b}{C_p^{\gamma}(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma} - \int_0^{\infty} g(\tau)\mathrm{d}\tau\right] \\
&\quad \geq \frac{(p-1-2\gamma)b}{2(\gamma+1)}\|\nabla u(t)\|_2^2\left(\|\nabla u(t)\|_2^{2\gamma} - \|\nabla u(t)\|_2^{2\gamma}\right) \\
&\qquad + \frac{p+1}{4}\|\nabla u(t)\|_2^2\left[\frac{(p-1-2\gamma)b}{C_p^{\gamma}(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma} - \int_0^{\infty} g(\tau)\mathrm{d}\tau\right] \\
&\quad = \frac{p+1}{4}\left[\frac{(p-1-2\gamma)b}{C_p^{\gamma}(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma} - \int_0^{\infty} g(\tau)\mathrm{d}\tau\right]\|\nabla u(t)\|_2^2.
\end{aligned} \tag{4.4}
$$

So, we can choose $m_1 = ((p+1)/4)\ [(p-1-2\gamma)b/(C_p^{\gamma}\ (p+1)(\gamma+1)\|u_0\ \|_2^{2\gamma}\ ) - \int_0^{\infty} g(\tau)\mathrm{d}\tau]$ in condition (A3).

4.2. The Set of the Initial Data Satisfying Conditions (2.11)–(2.14) *Is Not Empty*

For any real value of the initial energy $E(0) = d > 0$, there exists such initial data which leads to blow up in finite time.

For instance, in the case $M(s) = 1+s^{\gamma}$, then for any $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ with $\int_{\Omega} u_0 u_1 \mathrm{d}x > 0$, we may take some $\lambda, \alpha > 0$, such that (u_0, u_1) satisfies the above conditions (2.11)–(2.14).

Indeed, for $M(s) = 1 + s^{\gamma}$, conditions (2.11)–(2.14) become

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u_0\|_2^{2(\gamma+1)} - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} > 0,$$
$$I(u_0) = \|\nabla u_0\|_2^2 + \|\nabla u_0\|_2^{2(\gamma+1)} - \|u_0\|_{p+1}^{p+1} < 0,$$
$$\int_\Omega u_0 u_1 \mathrm{d}x > 0,$$
$$\|u_0\|_2^2 > \frac{(p+1)C_p}{m_1}E(0). \tag{4.5}$$

Now taking $(v_0, v_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ such that $\int_\Omega v_0 v_1 \mathrm{d}x > 0$, and letting $(u_0, u_1) = (\lambda v_0, \alpha v_1)$ for any scaling parameter $\lambda > 0$ and $\alpha > 0$, then we have

$$E(0) = \frac{1}{2}\alpha^2\|v_1\|_2^2 + \frac{1}{2}\lambda^2\|\nabla v_0\|_2^2 + \frac{1}{2(\gamma+1)}\lambda^{2(\gamma+1)}\|\nabla v_0\|_2^{2(\gamma+1)} - \frac{1}{p+1}\lambda^{p+1}\|v_0\|_{p+1}^{p+1},$$
$$I(u_0) = \lambda^2\|\nabla v_0\|_2^2 + \lambda^{2(\gamma+1)}\|\nabla v_0\|_2^{2(\gamma+1)} - \lambda^{p+1}\|v_0\|_{p+1}^{p+1},$$
$$\|u_0\|_2^2 = \lambda^2\|v_0\|_2^2. \tag{4.6}$$

We suppose that $\lambda^{2(\gamma+1)} < \lambda^{p+1}$(i.e., $p > 1 + 2\gamma$) for $\lambda > 1$, so we can choose sufficiently large λ such that

$$\frac{1}{2}\lambda^2\|\nabla v_0\|_2^2 + \frac{1}{2(\gamma+1)}\lambda^{2(\gamma+1)}\|\nabla v_0\|_2^{2(\gamma+1)} - \frac{1}{p+1}\lambda^{p+1}\|v_0\|_{p+1}^{p+1} < 0,$$
$$I(u_0) = I(\lambda v_0) = \lambda^2\|\nabla v_0\|_2^2 + \lambda^{2(\gamma+1)}\|\nabla v_0\|_2^{2(\gamma+1)} - \lambda^{p+1}\|v_0\|_{p+1}^{p+1} < 0, \tag{4.7}$$
$$\|u_0\|_2^2 = \lambda^2\|v_0\|_2^2 > \frac{(p+1)C_p}{m_1}E(0).$$

And when λ is fixed, we may choose α such that $E(0) = d$.

Similarly, in the case $M(s) = s^{\gamma}$, we can also take initial data (u_0, u_1) satisfying the above conditions (2.11)–(2.14) (see [7, Remark 1.4]).

Thus the set of the initial data which satisfy conditions (2.11)–(2.14) is not empty.

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