Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 217540, 14 pages
http://dx.doi.org/10.1155/2013/217540

Research Article

An Efficient Algorithm for the Reflexive Solution of the Quaternion Matrix Equation $AXB + CX^H D = F$

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Received 3 October 2012; Accepted 4 December 2012

Academic Editor: P. N. Shivakumar

We propose an iterative algorithm for solving the reflexive solution of the quaternion matrix equation $AXB + CX^H D = F$. When the matrix equation is consistent over reflexive matrix X, a reflexive solution can be obtained within finite iteration steps in the absence of roundoff errors. By the proposed iterative algorithm, the least Frobenius norm reflexive solution of the matrix equation can be derived when an appropriate initial iterative matrix is chosen. Furthermore, the optimal approximate reflexive solution to a given reflexive matrix X_0 can be derived by finding the least Frobenius norm reflexive solution of a new corresponding quaternion matrix equation. Finally, two numerical examples are given to illustrate the efficiency of the proposed methods.

1. Introduction

Throughout the paper, the notations $\mathbb{R}^{m\times n}$ and $\mathbb{H}^{m\times n}$ represent the set of all $m \times n$ real matrices and the set of all $m \times n$ matrices over the quaternion algebra $\mathbb{H} = \{a_1 + a_2 i + a_3 j + a_4 k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_1, a_2, a_3, a_4 \in \mathbb{R}\}$. We denote the identity matrix with the appropriate size by I. We denote the conjugate transpose, the transpose, the conjugate, the trace, the column space, the real part, the $mn \times 1$ vector formed by the vertical concatenation of the respective columns of a matrix A by $A^H, A^T, \overline{A}, \mathrm{tr}(A), R(A), \mathrm{Re}(A)$, and $\mathrm{vec}(A)$, respectively. The Frobenius norm of A is denoted by $\|A\|$, that is, $\|A\| = \sqrt{\mathrm{tr}(A^H A)}$. Moreover, $A \otimes B$ and $A \odot B$ stand for the Kronecker matrix product and Hadmard matrix product of the matrices A and B.

Let $Q \in \mathbb{H}^{n\times n}$ be a generalized reflection matrix, that is, $Q^2 = I$ and $Q^H = Q$. A matrix A is called reflexive with respect to the generalized reflection matrix Q, if $A = QAQ$. It is obvious that any matrix is reflexive with respect to I. Let $\mathbb{RH}^{n\times n}(Q)$ denote the set of order n reflexive matrices with respect to Q. The reflexive matrices with respect to a generalized reflection matrix Q have been widely used in engineering and scientific computations [1, 2].

In the field of matrix algebra, quaternion matrix equations have received much attention. Wang et al. [3] gave necessary and sufficient conditions for the existence and the representations of P-symmetric and P-skew-symmetric solutions of quaternion matrix equations $A_a X = C_a$ and $A_b X B_b = C_b$. Yuan and Wang [4] derived the expressions of the least squares η-Hermitian solution with the least norm and the expressions of the least squares anti-η-Hermitian solution with the least norm for the quaternion matrix equation $AXB + CXD = E$. Jiang and Wei [5] derived the explicit solution of the quaternion matrix equation $X - A\widetilde{X}B = C$. Li and Wu [6] gave the expressions of symmetric and skew-antisymmetric solutions of the quaternion matrix equations $A_1 X = C_1$ and $XB_3 = C_3$. Feng and Cheng [7] gave a clear description of the solution set of the quaternion matrix equation $AX - \overline{X}B = 0$.

The iterative method is a very important method to solve matrix equations. Peng [8] constructed a finite iteration method to solve the least squares symmetric solutions of linear matrix equation $AXB = C$. Also Peng [9–11] presented several efficient iteration methods to solve the constrained least squares solutions of linear matrix equations $AXB = C$ and $AXB + CYD = E$, by using Paige's algorithm [12] as the frame method. Duan et al. [13–17] proposed iterative algorithms for the (Hermitian) positive definite solutions of some nonlinear matrix equations. Ding et al. proposed the hierarchical gradient-based iterative algorithms [18] and hierarchical least squares iterative algorithms [19] for solving general (coupled) matrix equations, based on the hierarchical identification principle [20]. Wang et al. [21] proposed an iterative method for the least squares minimum-norm symmetric solution of $AXB = E$. Dehghan and Hajarian constructed finite iterative algorithms to solve several linear matrix equations over (anti)reflexive [22–24], generalized centrosymmetric [25, 26], and generalized bisymmetric [27, 28] matrices. Recently, Wu et al. [29–31] proposed iterative algorithms for solving various complex matrix equations.

However, to the best of our knowledge, there has been little information on iterative methods for finding a solution of a quaternion matrix equation. Due to the noncommutative multiplication of quaternions, the study of quaternion matrix equations is more complex than that of real and complex equations. Motivated by the work mentioned above and keeping the interests and wide applications of quaternion matrices in view (e.g., [32–45]), we, in this paper, consider an iterative algorithm for the following two problems.

Problem 1. For given matrices $A, C \in \mathbb{H}^{m\times n}$, $B, D \in \mathbb{H}^{n\times p}$, $F \in \mathbb{H}^{m\times p}$ and the generalized reflection matrix Q, find $X \in \mathbb{RH}^{n\times n}(Q)$, such that

$$AXB + CX^H D = F. \tag{1}$$

Problem 2. When Problem 1 is consistent, let its solution set be denoted by S_H. For a given reflexive matrix $X_0 \in \mathbb{RH}^{n\times n}(Q)$, find $\check{X} \in \mathbb{RH}^{n\times n}(Q)$, such that

$$\left\|\check{X} - X_0\right\| = \min_{X\in S_H} \left\|X - X_0\right\|. \tag{2}$$

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we introduce an iterative algorithm for solving Problem 1. Then we prove that the given algorithm can be used to obtain a reflexive solution for any initial matrix within finite steps in the absence of roundoff errors. Also we prove that the least Frobenius norm reflexive solution can be obtained by choosing a special kind of initial matrix. In addition, the optimal reflexive solution of Problem 2 by finding the least Frobenius norm reflexive solution of a new matrix equation is given. In Section 4, we give two numerical examples to illustrate our results. In Section 5, we give some conclusions to end this paper.

2. Preliminary

In this section, we provide some results which will play important roles in this paper. First, we give a real inner product for the space $\mathbb{H}^{m\times n}$ over the real field $\mathbb{R}$.

Theorem 3. *In the space $\mathbb{H}^{m\times n}$ over the field $\mathbb{R}$, a real inner product can be defined as*

$$\langle A, B\rangle = \operatorname{Re}\left[\operatorname{tr}\left(B^H A\right)\right] \tag{3}$$

for A, $B \in \mathbb{H}^{m\times n}$. This real inner product space is denoted as $(\mathbb{H}^{m\times n}, \mathbb{R}, \langle\cdot,\cdot\rangle)$.

Proof. (1) For $A \in \mathbb{H}^{m\times n}$, let $A = A_1 + A_2 i + A_3 j + A_4 k$, then

$$\begin{aligned}\langle A, A\rangle &= \operatorname{Re}\left[\operatorname{tr}\left(A^H A\right)\right] \\ &= \operatorname{tr}\left(A_1^T A_1 + A_2^T A_2 + A_3^T A_3 + A_4^T A_4\right).\end{aligned} \tag{4}$$

It is obvious that $\langle A, A\rangle > 0$ and $\langle A, A\rangle = 0 \Leftrightarrow A = 0$.

(2) For $A, B \in \mathbb{H}^{m\times n}$, let $A = A_1 + A_2 i + A_3 j + A_4 k$ and $B = B_1 + B_2 i + B_3 j + B_4 k$, then we have

$$\begin{aligned}\langle A, B\rangle &= \operatorname{Re}\left[\operatorname{tr}\left(B^H A\right)\right] \\ &= \operatorname{tr}\left(B_1^T A_1 + B_2^T A_2 + B_3^T A_3 + B_4^T A_4\right) \\ &= \operatorname{tr}\left(A_1^T B_1 + A_2^T B_2 + A_3^T B_3 + A_4^T B_4\right) \\ &= \operatorname{Re}\left[\operatorname{tr}\left(A^H B\right)\right] = \langle B, A\rangle.\end{aligned} \tag{5}$$

(3) For $A, B, C \in \mathbb{H}^{m\times n}$

$$\begin{aligned}\langle A + B, C\rangle &= \operatorname{Re}\left\{\operatorname{tr}\left[C^H (A + B)\right]\right\} \\ &= \operatorname{Re}\left[\operatorname{tr}\left(C^H A + C^H B\right)\right] \\ &= \operatorname{Re}\left[\operatorname{tr}\left(C^H A\right)\right] + \operatorname{Re}\left[\operatorname{tr}\left(C^H B\right)\right] \\ &= \langle A, C\rangle + \langle B, C\rangle.\end{aligned} \tag{6}$$

(4) For $A, B \in \mathbb{H}^{m\times n}$ and $a \in \mathbb{R}$,

$$\begin{aligned}\langle aA, B\rangle &= \operatorname{Re}\left\{\operatorname{tr}\left[B^H (aA)\right]\right\} = \operatorname{Re}\left[\operatorname{tr}\left(aB^H A\right)\right] \\ &= a \operatorname{Re}\left[\operatorname{tr}\left(B^H A\right)\right] = a\langle A, B\rangle.\end{aligned} \tag{7}$$

All the above arguments reveal that the space $\mathbb{H}^{m\times n}$ over field $\mathbb{R}$ with the inner product defined in (3) is an inner product space. □

Let $\|\cdot\|_\Delta$ represent the matrix norm induced by the inner product $\langle\cdot,\cdot\rangle$. For an arbitrary quaternion matrix $A \in \mathbb{H}^{m\times n}$, it is obvious that the following equalities hold:

$$\|A\|_\Delta = \sqrt{\langle A, A\rangle} = \sqrt{\operatorname{Re}\left[\operatorname{tr}\left(A^H A\right)\right]} = \sqrt{\operatorname{tr}\left(A^H A\right)} = \|A\|, \tag{8}$$

which reveals that the induced matrix norm is exactly the Frobenius norm. For convenience, we still use $\|\cdot\|$ to denote the induced matrix norm.

Let E_{ij} denote the $m \times n$ quaternion matrix whose (i, j) entry is 1, and the other elements are zeros. In inner product space $(\mathbb{H}^{m\times n}, \mathbb{R}, \langle\cdot,\cdot\rangle)$, it is easy to verify that $E_{ij}, E_{ij}i, E_{ij}j, E_{ij}k$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, is an orthonormal basis, which reveals that the dimension of the inner product space $(\mathbb{H}^{m\times n}, \mathbb{R}, \langle\cdot,\cdot\rangle)$ is $4mn$.

Next, we introduce a real representation of a quaternion matrix.

For an arbitrary quaternion matrix $M = M_1 + M_2 i + M_3 j + M_4 k$, a map $\phi(\cdot)$, from $\mathbb{H}^{m\times n}$ to $\mathbb{R}^{4m\times 4n}$, can be defined as

$$\phi(M) = \begin{bmatrix} M_1 & -M_2 & -M_3 & -M_4 \\ M_2 & M_1 & -M_4 & M_3 \\ M_3 & M_4 & M_1 & -M_2 \\ M_4 & -M_3 & M_2 & M_1 \end{bmatrix}. \quad (9)$$

Lemma 4 (see [41]). *Let M and N be two arbitrary quaternion matrices with appropriate size. The map $\phi(\cdot)$ defined by* (9) *satisfies the following properties.*

(a) $M = N \Leftrightarrow \phi(M) = \phi(N)$.

(b) $\phi(M+N) = \phi(M) + \phi(N)$, $\phi(MN) = \phi(M)\phi(N)$, $\phi(kM) = k\phi(M)$, $k \in \mathbb{R}$.

(c) $\phi(M^H) = \phi^T(M)$.

(d) $\phi(M) = T_m^{-1}\phi(M)T_n = R_m^{-1}\phi(M)R_n = S_m^{-1}\phi(M)S_n$, *where*

$$T_t = \begin{bmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{bmatrix}, \qquad R_t = \begin{bmatrix} 0 & -I_{2t} \\ I_{2t} & 0 \end{bmatrix},$$
$$S_t = \begin{bmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{bmatrix}, \quad t = m, n. \quad (10)$$

By (9), it is easy to verify that

$$\|\phi(M)\| = 2\|M\|. \quad (11)$$

Finally, we introduce the commutation matrix.

A commutation matrix $P(m,n)$ is a $mn \times mn$ matrix which has the following explicit form:

$$P(m,n) = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij} \otimes E_{ij}^T = \left[E_{ij}^T\right]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}. \quad (12)$$

Moreover, $P(m,n)$ is a permutation matrix and $P(m,n) = P^T(n,m) = P^{-1}(n,m)$. We have the following lemmas on the commutation matrix.

Lemma 5 (see [46]). *Let A be a $m \times n$ matrix. There is a commutation matrix $P(m,n)$ such that*

$$\operatorname{vec}\left(A^T\right) = P(m,n)\operatorname{vec}(A). \quad (13)$$

Lemma 6 (see [46]). *Let A be a $m\times n$ matrix and B a $p \times q$ matrix. There exist two commutation matrices $P(m,p)$ and $P(n,q)$ such that*

$$B \otimes A = P^T(m,p)(A\otimes B)P(n,q). \quad (14)$$

3. Main Results

3.1. The Solution of Problem 1. In this subsection, we will construct an algorithm for solving Problem 1. Then some lemmas will be given to analyse the properties of the proposed algorithm. Using these lemmas, we prove that the proposed algorithm is convergent.

Algorithm 7 (Iterative algorithm for Problem 1).

(1) Choose an initial matrix $X(1) \in \mathbb{RH}^{n\times n}(Q)$.

(2) Calculate

$$\begin{aligned} R(1) &= F - AX(1)B - CX^H(1)D; \\ P(1) &= \frac{1}{2}\left(A^H R(1) B^H + DR^H(1)C \right. \\ &\quad \left. + QA^H R(1) B^H Q + QDR^H(1)CQ\right); \\ k &:= 1. \end{aligned} \quad (15)$$

(3) If $R(k) = 0$, then stop and $X(k)$ is the solution of Problem 1; else if $R(k) \neq 0$ and $P(k) = 0$, then stop and Problem 1 is not consistent; else $k := k+1$.

(4) Calculate

$$\begin{aligned} X(k) &= X(k-1) + \frac{\|R(k-1)\|^2}{\|P(k-1)\|^2}P(k-1); \\ R(k) &= R(k-1) - \frac{\|R(k-1)\|^2}{\|P(k-1)\|^2} \\ &\quad \times \left(AP(k-1)B + CP^H(k-1)D\right); \\ P(k) &= \frac{1}{2}\left(A^H R(k) B^H + DR^H(k)C \right. \\ &\quad \left. + QA^H R(k) B^H Q + QDR^H(k)CQ\right) \\ &\quad + \frac{\|R(k)\|^2}{\|R(k-1)\|^2}P(k-1). \end{aligned} \quad (16)$$

(5) Go to Step (3).

Lemma 8. *Assume that the sequences $\{R(i)\}$ and $\{P(i)\}$ are generated by Algorithm 7, then $\langle R(i), R(j)\rangle = 0$ and $\langle P(i), P(j)\rangle = 0$ for $i, j = 1, 2, \ldots$, $i \neq j$.*

Proof. Since $\langle R(i), R(j)\rangle = \langle R(j), R(i)\rangle$ and $\langle P(i), P(j)\rangle = \langle P(j), P(i)\rangle$ for $i, j = 1, 2, \ldots$, we only need to prove that $\langle R(i), R(j)\rangle = 0$ and $\langle P(i), P(j)\rangle = 0$ for $1 \le i < j$.

Now we prove this conclusion by induction.

Step 1. We show that

$$\begin{gathered} \langle R(i), R(i+1)\rangle = 0, \\ \langle P(i), P(i+1)\rangle = 0 \quad \text{for } i = 1, 2, \ldots. \end{gathered} \quad (17)$$

We also prove (17) by induction. When $i = 1$, we have

$$\begin{aligned}
\langle R(1), R(2)\rangle
&= \operatorname{Re}\left\{\operatorname{tr}\left[R^H(2) R(1)\right]\right\} \\
&= \operatorname{Re}\left\{\operatorname{tr}\left[\left(R(1) - \frac{\|R(1)\|^2}{\|P(1)\|^2}\left(AP(1)B + CP^H(1)D\right)\right)^H \times R(1)\right]\right\} \\
&= \|R(1)\|^2 - \frac{\|R(1)\|^2}{\|P(1)\|^2} \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(1)\left(A^H R(1) B^H + DR^H(1) C\right)\right]\right\} \\
&= \|R(1)\|^2 - \frac{\|R(1)\|^2}{\|P(1)\|^2} \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(1)\left(\left(A^H R(1) B^H + DR^H(1) C + QA^H R(1) \times B^H Q + QDR^H(1) CQ\right) \times (2)^{-1}\right)\right]\right\} \\
&= \|R(1)\|^2 - \frac{\|R(1)\|^2}{\|P(1)\|^2}\|P(1)\|^2 \\
&= 0.
\end{aligned} \tag{18}$$

Also we can write

$$\begin{aligned}
\langle P(1), P(2)\rangle
&= \operatorname{Re}\left\{\operatorname{tr}\left[P^H(2) P(1)\right]\right\} \\
&= \operatorname{Re}\left\{\operatorname{tr}\left[\left(\frac{1}{2}\left(A^H R(2) B^H + DR^H(2) C + QA^H R(2) B^H Q + QDR^H(2) CQ\right) + \frac{\|R(2)\|^2}{\|R(1)\|^2} P(1)\right)^H P(1)\right]\right\} \\
&= \frac{\|R(2)\|^2}{\|R(1)\|^2}\|P(1)\|^2 + \operatorname{Re}\left\{\operatorname{tr}\left[P^H(1) \times \left(\left(A^H R(2) B^H + DR^H(2) C + QA^H R(2) B^H Q + QDR^H(2) CQ\right) \times (2)^{-1}\right)\right]\right\} \\
&= \frac{\|R(2)\|^2}{\|R(1)\|^2}\|P(1)\|^2 + \operatorname{Re}\left\{\operatorname{tr}\left[P^H(1)\left(A^H R(2) B^H + DR^H(2) C\right)\right]\right\} \\
&= \frac{\|R(2)\|^2}{\|R(1)\|^2}\|P(1)\|^2 + \operatorname{Re}\left\{\operatorname{tr}\left[R^H(2)\left(AP(1)B + CP^H(1)D\right)\right]\right\} \\
&= \frac{\|R(2)\|^2}{\|R(1)\|^2}\|P(1)\|^2 + \frac{\|P(1)\|^2}{\|R(1)\|^2}\operatorname{Re}\left\{\operatorname{tr}\left[R^H(2)(R(1) - R(2))\right]\right\} \\
&= \frac{\|R(2)\|^2}{\|R(1)\|^2}\|P(1)\|^2 - \frac{\|P(1)\|^2}{\|R(1)\|^2}\|R(2)\|^2 \\
&= 0.
\end{aligned} \tag{19}$$

Now assume that conclusion (17) holds for $1 \le i \le s-1$, then

$$\begin{aligned}
\langle R(s), R(s+1)\rangle
&= \operatorname{Re}\left\{\operatorname{tr}\left[R^H(s+1) R(s)\right]\right\} \\
&= \operatorname{Re}\left\{\operatorname{tr}\left[\left(R(s) - \frac{\|R(s)\|^2}{\|P(s)\|^2}\left(AP(s)B + CP^H(s)D\right)\right)^H \times R(s)\right]\right\} \\
&= \|R(s)\|^2 - \frac{\|R(s)\|^2}{\|P(s)\|^2} \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(s)\left(A^H R(s) B^H + DR^H(s) C\right)\right]\right\} \\
&= \|R(s)\|^2 - \frac{\|R(s)\|^2}{\|P(s)\|^2} \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(s) \times \left(\left(A^H R(s) B^H + DR^H(s) C + QA^H R(s) B^H Q + QDR^H(s) CQ\right) \times (2)^{-1}\right)\right]\right\} \\
&= \|R(s)\|^2 - \frac{\|R(s)\|^2}{\|P(s)\|^2} \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(s)\left(P(s) - \frac{\|R(s)\|^2}{\|R(s-1)\|^2} P(s-1)\right)\right]\right\} \\
&= 0.
\end{aligned} \tag{20}$$

And it can also be obtained that

$$
\begin{aligned}
&\langle P(s), P(s+1)\rangle \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[P^{H}(s+1)P(s)\right]\right\} \\
&\quad = \operatorname{Re}\Bigg\{\operatorname{tr}\Bigg[\Bigg(\Big(\big(A^{H}R(s+1)B^{H} + DR^{H}(s+1)C \\
&\qquad\qquad + QA^{H}R(s+1)B^{H}Q \\
&\qquad\qquad + QDR^{H}(s+1)CQ\big)\times(2)^{-1}\Big) \\
&\qquad\qquad + \frac{\|R(s+1)\|^{2}}{\|R(s)\|^{2}}P(s)\Bigg)^{H}P(s)\Bigg]\Bigg\} \\
&\quad = \frac{\|R(s+1)\|^{2}}{\|R(s)\|^{2}}\|P(s)\|^{2} \\
&\qquad + \operatorname{Re}\left\{\operatorname{tr}\left[P^{H}(s)\left(A^{H}R(s+1)B^{H} + DR^{H}(s+1)C\right)\right]\right\} \\
&\quad = \frac{\|R(s+1)\|^{2}}{\|R(s)\|^{2}}\|P(s)\|^{2} \\
&\qquad + \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(s+1)\left(AP(s)B + CP^{H}(s)D\right)\right]\right\} \\
&\quad = \frac{\|R(s+1)\|^{2}}{\|R(s)\|^{2}}\|P(s)\|^{2} \\
&\qquad + \frac{\|P(s)\|^{2}}{\|R(s)\|^{2}}\operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(s+1)(R(s)-R(s+1))\right]\right\} \\
&\quad = 0.
\end{aligned} \tag{21}
$$

Therefore, the conclusion (17) holds for $i = 1, 2, \ldots$.

Step 2. Assume that $\langle R(i), R(i+r)\rangle = 0$ and $\langle P(i), P(i+r)\rangle = 0$ for $i \ge 1$ and $r \ge 1$. We will show that

$$
\langle R(i), R(i+r+1)\rangle = 0, \qquad \langle P(i), P(i+r+1)\rangle = 0. \tag{22}
$$

We prove the conclusion (22) in two substeps.

Substep 2.1. In this substep, we show that

$$
\langle R(1), R(r+2)\rangle = 0, \qquad \langle P(1), P(r+2)\rangle = 0. \tag{23}
$$

It follows from Algorithm 7 that

$$
\begin{aligned}
&\langle R(1), R(r+2)\rangle \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(r+2)R(1)\right]\right\} \\
&\quad = \operatorname{Re}\Bigg\{\operatorname{tr}\Bigg[\Bigg(R(r+1) - \frac{\|R(r+1)\|^{2}}{\|P(r+1)\|^{2}} \\
&\qquad\qquad \times\left(AP(r+1)B + CP^{H}(r+1)D\right)\Bigg)^{H}R(1)\Bigg]\Bigg\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(r+1)R(1)\right]\right\} - \frac{\|R(r+1)\|^{2}}{\|P(r+1)\|^{2}} \\
&\qquad \times \operatorname{Re}\left\{\operatorname{tr}\left[P^{H}(r+1)\left(A^{H}R(1)B^{H} + DR^{H}(1)C\right)\right]\right\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(r+1)R(1)\right]\right\} - \frac{\|R(r+1)\|^{2}}{\|P(r+1)\|^{2}} \\
&\qquad \times \operatorname{Re}\Big\{\operatorname{tr}\Big[P^{H}(r+1)\Big(\big(A^{H}R(1)B^{H} + DR^{H}(1)C \\
&\qquad\qquad + QA^{H}R(1)B^{H}Q \\
&\qquad\qquad + QDR^{H}(1)CQ\big)\times(2)^{-1}\Big)\Big]\Big\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(r+1)R(1)\right]\right\} \\
&\qquad - \frac{\|R(r+1)\|^{2}}{\|P(r+1)\|^{2}}\operatorname{Re}\left\{\operatorname{tr}\left[P^{H}(r+1)P(1)\right]\right\} \\
&\quad = 0.
\end{aligned} \tag{24}
$$

Also we can write

$$
\begin{aligned}
&\langle P(1), P(r+2)\rangle \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[P^{H}(r+2)P(1)\right]\right\} \\
&\quad = \operatorname{Re}\Bigg\{\operatorname{tr}\Bigg[\Bigg(\Big(\big(A^{H}R(r+2)B^{H} + DR^{H}(r+2)C \\
&\qquad\qquad + QA^{H}R(r+2)B^{H}Q + QDR^{H}(r+2)CQ\big) \\
&\qquad\qquad \times(2)^{-1}\Big) + \frac{\|R(r+2)\|^{2}}{\|R(r+1)\|^{2}}P(r+1)\Bigg)^{H}P(1)\Bigg]\Bigg\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[\left(A^{H}R(r+2)B^{H} + DR^{H}(r+2)C\right)^{H}P(1)\right]\right\} \\
&\qquad + \frac{\|R(r+2)\|^{2}}{\|R(r+1)\|^{2}}\operatorname{Re}\left\{\operatorname{tr}\left[P^{H}(r+1)P(1)\right]\right\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(r+2)\left(AP(1)B + CP^{H}(1)D\right)\right]\right\} \\
&\qquad + \frac{\|R(r+2)\|^{2}}{\|R(r+1)\|^{2}}\operatorname{Re}\left\{\operatorname{tr}\left[P^{H}(r+1)P(1)\right]\right\}
\end{aligned}
$$

$$= \frac{\|P(1)\|^2}{\|R(1)\|^2} \operatorname{Re}\left\{\operatorname{tr}\left[R^H(r+2)(R(1)-R(2))\right]\right\}$$
$$+ \frac{\|R(r+2)\|^2}{\|R(r+1)\|^2} \operatorname{Re}\left\{\operatorname{tr}\left[P^H(r+1)P(1)\right]\right\}$$
$$= 0. \tag{25}$$

Substep 2.2. In this substep, we prove the conclusion (22) in Step 2. It follows from Algorithm 7 that

$$\begin{aligned}
&\langle R(i), R(i+r+1)\rangle \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[R^H(i+r+1)R(i)\right]\right\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[\left(R(i+r) - \frac{\|R(i+r)\|^2}{\|P(i+r)\|^2}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\times\left(AP(i+r)B + CP^H(i+r)D\right)\right)^H R(i)\right]\right\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[R^H(i+r)R(i)\right]\right\} - \frac{\|R(i+r)\|^2}{\|P(i+r)\|^2} \\
&\qquad \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(i+r)\left(A^H R(i) B^H + DR^H(i)C\right)\right]\right\} \\
&\quad = -\frac{\|R(i+r)\|^2}{\|P(i+r)\|^2} \\
&\qquad \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(i+r)\left(\left(A^H R(i) B^H + DR^H(i)C\right.\right.\right.\right. \\
&\qquad\qquad + QA^H R(i) B^H Q \\
&\qquad\qquad \left.\left.\left.\left. + QDR^H(i)CQ\right)\times(2)^{-1}\right)\right]\right\} \\
&\quad = -\frac{\|R(i+r)\|^2}{\|P(i+r)\|^2} \\
&\qquad \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(i+r)\left(P(i) - \frac{\|R(i)\|^2}{\|R(i-1)\|^2}P(i-1)\right)\right]\right\} \\
&\quad = \frac{\|R(i+r)\|^2\|R(i)\|^2}{\|P(i+r)\|^2\|R(i-1)\|^2} \operatorname{Re}\left\{\operatorname{tr}\left[P^H(i+r)P(i-1)\right]\right\}, \\
&\langle P(i), P(i+r+1)\rangle \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[P^H(i+r+1)P(i)\right]\right\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[\left(\frac{1}{2}\left(A^H R(i+r+1) B^H + DR^H(i+r+1)C\right.\right.\right.\right. \\
&\qquad\qquad + QA^H R(i+r+1) B^H Q \\
&\qquad\qquad \left. + QDR^H(i+r+1)CQ\right) \\
&\qquad\qquad \left.\left.\left. + \frac{\|R(i+r+1)\|^2}{\|R(i+r)\|^2}P(i+r)\right)^H P(i)\right]\right\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[\left(A^H R(i+r+1) B^H + DR^H(i+r+1)C\right)^H P(i)\right]\right\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[R^H(i+r+1)\left(AP(i)\mathrm{B} + CP^H(i)D\right)\right]\right\} \\
&\quad = \frac{\|P(i)\|^2}{\|R(i)\|^2} \operatorname{Re}\left\{\operatorname{tr}\left[R^H(i+r+1)(R(i)-R(i+1))\right]\right\} \\
&\quad = \frac{\|P(i)\|^2}{\|R(i)\|^2} \operatorname{Re}\left\{\operatorname{tr}\left[R^H(i+r+1)R(i)\right]\right\} \\
&\qquad - \frac{\|P(i)\|^2}{\|R(i)\|^2} \operatorname{Re}\left\{\operatorname{tr}\left[R^H(i+r+1)R(i+1)\right]\right\} \\
&\quad = \frac{\|P(i)\|^2\|R(i+r)\|^2\|R(i)\|^2}{\|R(i)\|^2\|P(i+r)\|^2\|R(i-1)\|^2} \\
&\qquad \times \operatorname{Re}\left\{\operatorname{tr}\left[P^H(i+r)P(i-1)\right]\right\}.
\end{aligned} \tag{26}$$

Repeating the above process (26), we can obtain

$$\begin{aligned}
\langle R(i), R(i+r+1)\rangle &= \cdots = \alpha \operatorname{Re}\left\{\operatorname{tr}\left[P^H(r+2)P(1)\right]\right\}; \\
\langle P(i), P(i+r+1)\rangle &= \cdots = \beta \operatorname{Re}\left\{\operatorname{tr}\left[P^H(r+2)P(1)\right]\right\}.
\end{aligned} \tag{27}$$

Combining these two relations with (24) and (25), it implies that (22) holds. So, by the principle of induction, we know that Lemma 8 holds. □

Lemma 9. *Assume that Problem 1 is consistent, and let $\widetilde{X} \in \mathbb{RH}^{n\times n}(Q)$ be its solution. Then, for any initial matrix $X(1) \in \mathbb{RH}^{n\times n}(Q)$, the sequences $\{R(i)\}$, $\{P(i)\}$, and $\{X(i)\}$ generated by Algorithm 7 satisfy*

$$\left\langle P(i), \widetilde{X} - X(i)\right\rangle = \|R(i)\|^2, \quad i = 1, 2, \ldots. \tag{28}$$

Proof. We also prove this conclusion by induction.

When $i = 1$, it follows from Algorithm 7 that

$$\begin{aligned}
&\left\langle P(1), \widetilde{X} - X(1)\right\rangle \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[\left(\widetilde{X} - X(1)\right)^H P(1)\right]\right\} \\
&\quad = \operatorname{Re}\left\{\operatorname{tr}\left[\left(\widetilde{X} - X(1)\right)^H\right.\right. \\
&\qquad\qquad \times\left(\left(A^H R(1) B^H + DR^H(1)C + QA^H R(1) B^H Q\right.\right. \\
&\qquad\qquad\quad \left.\left.\left.\left. + QDR^H(1)CQ\right)\times(2)^{-1}\right)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{Re}\left\{\operatorname{tr}\left[\left(\widetilde{X}-X(1)\right)^{H}\left(A^{H}R(1)B^{H}+DR^{H}(1)C\right)\right]\right\}\\
&= \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(1)\left(A\left(\widetilde{X}-X(1)\right)B+C\left(\widetilde{X}-X(1)\right)^{H}D\right)\right]\right\}\\
&= \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(1)R(1)\right]\right\}\\
&= \|R(1)\|^{2}.
\end{aligned}
\tag{29}
$$

This implies that (28) holds for $i = 1$.

Now it is assumed that (28) holds for $i = s$, that is

$$
\left\langle P(s), \widetilde{X}-X(s)\right\rangle = \|R(s)\|^{2}. \tag{30}
$$

Then, when $i = s + 1$

$$
\begin{aligned}
&\left\langle P(s+1), \widetilde{X}-X(s+1)\right\rangle\\
&\quad= \operatorname{Re}\left\{\operatorname{tr}\left[\left(\widetilde{X}-X(s+1)\right)^{H}P(s+1)\right]\right\}\\
&\quad= \operatorname{Re}\left\{\operatorname{tr}\left[\left(\widetilde{X}-X(s+1)\right)^{H}\right.\right.\\
&\qquad\quad\times\left(\frac{1}{2}\left(A^{H}R(s+1)B^{H}+DR^{H}(s+1)C\right.\right.\\
&\qquad\qquad\quad+QA^{H}R(s+1)B^{H}Q\\
&\qquad\qquad\quad\left.+QDR^{H}(s+1)CQ\right)\\
&\qquad\qquad\left.\left.\left.+\frac{\|R(s+1)\|^{2}}{\|R(s)\|^{2}}P(s)\right)\right]\right\}\\
&\quad= \operatorname{Re}\left\{\operatorname{tr}\left[\left(\widetilde{X}-X(s+1)\right)^{H}\right.\right.\\
&\qquad\quad\left.\left.\times\left(A^{H}R(s+1)B^{H}+DR^{H}(s+1)C\right)\right]\right\}\\
&\qquad+\frac{\|R(s+1)\|^{2}}{\|R(s)\|^{2}}\operatorname{Re}\left\{\operatorname{tr}\left[\left(\widetilde{X}-X(s+1)\right)^{H}P(s)\right]\right\}\\
&\quad= \operatorname{Re}\left\{\operatorname{tr}\left[R^{H}(s+1)\right.\right.\\
&\qquad\quad\times\left(A\left(\widetilde{X}-X(s+1)\right)B\right.\\
&\qquad\qquad\left.\left.\left.+C\left(\widetilde{X}-X(s+1)\right)^{H}D\right)\right]\right\}\\
&\qquad+\frac{\|R(s+1)\|^{2}}{\|R(s)\|^{2}}\left\{\operatorname{Re}\left\{\operatorname{tr}\left[\left(\widetilde{X}-X(s)\right)^{H}P(s)\right]\right\}\right.\\
&\qquad\qquad\left.-\operatorname{Re}\left\{\operatorname{tr}\left[\left(X(s+1)-X(s)\right)^{H}P(s)\right]\right\}\right\}\\
&\quad= \|R(s+1)\|^{2}+\frac{\|R(s+1)\|^{2}}{\|R(s)\|^{2}}\\
&\qquad\times\left\{\|R(s)\|^{2}-\frac{\|R(s)\|^{2}}{\|P(s)\|^{2}}\operatorname{Re}\left\{\operatorname{tr}\left[P^{H}(s)P(s)\right]\right\}\right\}\\
&\quad= \|R(s+1)\|^{2}.
\end{aligned}
\tag{31}
$$

Therefore, Lemma 9 holds by the principle of induction. □

From the above two lemmas, we have the following conclusions.

Remark 10. If there exists a positive number i such that $R(i) \neq 0$ and $P(i) = 0$, then we can get from Lemma 9 that Problem 1 is not consistent. Hence, the solvability of Problem 1 can be determined by Algorithm 7 automatically in the absence of roundoff errors.

Theorem 11. *Suppose that Problem 1 is consistent. Then for any initial matrix $X(1) \in \mathbb{RH}^{n\times n}(Q)$, a solution of Problem 1 can be obtained within finite iteration steps in the absence of roundoff errors.*

Proof. In Section 2, it is known that the inner product space $(\mathbb{H}^{m\times p}, R, \langle\cdot,\cdot\rangle)$ is $4mp$-dimensional. According to Lemma 9, if $R(i) \neq 0$, $i = 1, 2, \ldots, 4mp$, then we have $P(i) \neq 0$, $i = 1, 2, \ldots, 4mp$. Hence $R(4mp+1)$ and $P(4mp+1)$ can be computed. From Lemma 8, it is not difficult to get

$$
\langle R(i), R(j)\rangle = 0 \quad \text{for } i, j = 1, 2, \ldots, 4mp,\ i \neq j. \tag{32}
$$

Then $R(1), R(2), \ldots, R(4mp)$ is an orthogonal basis of the inner product space $(\mathbb{H}^{m\times p}, R, \langle\cdot,\cdot\rangle)$. In addition, we can get from Lemma 8 that

$$
\langle R(i), R(4mp+1)\rangle = 0 \quad \text{for } i = 1, 2, \ldots, 4mp. \tag{33}
$$

It follows that $R(4mp+1) = 0$, which implies that $X(4mp+1)$ is a solution of Problem 1. □

3.2. The Solution of Problem 2. In this subsection, firstly we introduce some lemmas. Then, we will prove that the least Frobenius norm reflexive solution of (1) can be derived by choosing a suitable initial iterative matrix. Finally, we solve Problem 2 by finding the least Frobenius norm reflexive solution of a new-constructed quaternion matrix equation.

Lemma 12 (see [47]). *Assume that the consistent system of linear equations $My = b$ has a solution $y_0 \in R(M^T)$ then y_0 is the unique least Frobenius norm solution of the system of linear equations.*

Lemma 13. *Problem 1 is consistent if and only if the system of quaternion matrix equations*

$$
\begin{aligned}
AXB + CX^{H}D &= F,\\
AQXQB + CQX^{H}QD &= F
\end{aligned}
\tag{34}
$$

is consistent. Furthermore, if the solution sets of Problem 1 and (34) *are denoted by* S_H *and* S_H^1*, respectively, then, we have* $S_H \subseteq S_H^1$.

Proof. First, we assume that Problem 1 has a solution X. By $AXB + CX^HD = F$ and $QXQ = X$, we can obtain $AXB + CX^HD = F$ and $AQXQB + CQX^HQD = F$, which implies that X is a solution of quaternion matrix equations (34), and $S_H \subseteq S_H^1$.

Conversely, suppose (34) is consistent. Let X be a solution of (34). Set $X_a = (X + QXQ)/2$. It is obvious that $X_a \in \mathbb{RH}^{n\times n}(Q)$. Now we can write

$$\begin{aligned} AX_aB + CX_a^HD &= \frac{1}{2}\left[A(X+QXQ)B + C(X+QXQ)^HD\right] \\ &= \frac{1}{2}\left[AXB + CX^HD + AQXQB + CQX^HQD\right] \\ &= \frac{1}{2}[F+F] \\ &= F. \end{aligned} \quad (35)$$

Hence X_a is a solution of Problem 1. The proof is completed. □

Lemma 14. *The system of quaternion matrix equations* (34) *is consistent if and only if the system of real matrix equations*

$$\begin{aligned} &\phi(A)\left[X_{ij}\right]_{4\times 4}\phi(B) + \phi(C)\left[X_{ij}\right]_{4\times 4}^T\phi(D) = \phi(F), \\ &\phi(A)\phi(Q)\left[X_{ij}\right]_{4\times 4}\phi(Q)\phi(B) \\ &\quad + \phi(C)\phi(Q)\left[X_{ij}\right]_{4\times 4}^T\phi(Q)\phi(D) = \phi(F) \end{aligned} \quad (36)$$

is consistent, where $X_{ij} \in \mathbb{H}^{n\times n}$, $i, j = 1, 2, 3, 4$, *are submatrices of the unknown matrix. Furthermore, if the solution sets of* (34) *and* (36) *are denoted by* S_H^1 *and* S_R^2*, respectively, then, we have* $\phi(S_H^1) \subseteq S_R^2$.

Proof. Suppose that (34) has a solution

$$X = X_1 + X_2 i + X_3 j + X_4 k. \quad (37)$$

Applying (a), (b), and (c) in Lemma 4 to (34) yields

$$\begin{aligned} &\phi(A)\phi(X)\phi(B) + \phi(C)\phi^T(X)\phi(D) = \phi(F), \\ &\phi(A)\phi(Q)\phi(X)\phi(Q)\phi(B) \\ &\quad + \phi(C)\phi(Q)\phi^T(X)\phi(Q)\phi(D) = \phi(F), \end{aligned} \quad (38)$$

which implies that $\phi(X)$ is a solution of (36) and $\phi(S_H^1) \subseteq S_R^2$. Conversely, suppose that (36) has a solution

$$\check{X} = \left[X_{ij}\right]_{4\times 4}. \quad (39)$$

By (d) in Lemma 4, we have that

$$\begin{aligned} &T_m^{-1}\phi(A)T_n\check{X}T_n^{-1}\phi(B)T_p \\ &\quad + T_m^{-1}\phi(C)T_n\check{X}^TT_n^{-1}\phi(D)T_p = T_m^{-1}\phi(F)T_p, \\ &R_m^{-1}\phi(A)R_n\check{X}R_n^{-1}\phi(B)R_p \\ &\quad + R_m^{-1}\phi(C)R_n\check{X}^TR_n^{-1}\phi(D)R_p = R_m^{-1}\phi(F)R_p, \\ &S_m^{-1}\phi(A)S_n\check{X}S_n^{-1}\phi(B)S_p \\ &\quad + S_m^{-1}\phi(C)S_n\check{X}^TS_n^{-1}\phi(D)S_p = S_m^{-1}\phi(F)S_p, \\ &T_m^{-1}\phi(A)\phi(Q)T_n\check{X}T_n^{-1}\phi(Q)\phi(B)T_p \\ &\quad + T_m^{-1}\phi(C)\phi(Q)T_n\check{X}^TT_n^{-1}\phi(Q)\phi(D)T_p = T_m^{-1}\phi(F)T_p, \\ &R_m^{-1}\phi(A)\phi(Q)R_n\check{X}R_n^{-1}\phi(Q)\phi(B)R_p \\ &\quad + R_m^{-1}\phi(C)\phi(Q)R_n\check{X}^TR_n^{-1}\phi(Q)\phi(D)R_p = R_m^{-1}\phi(F)R_p, \\ &S_m^{-1}\phi(A)\phi(Q)S_n\check{X}S_n^{-1}\phi(Q)\phi(B)S_p \\ &\quad + S_m^{-1}\phi(C)\phi(Q)S_n\check{X}^TS_n^{-1}\phi(Q)\phi(D)S_p = S_m^{-1}\phi(F)S_p. \end{aligned} \quad (40)$$

Hence

$$\begin{aligned} &\phi(A)T_n\check{X}T_n^{-1}\phi(B) + \phi(C)\left(T_n\check{X}T_n^{-1}\right)^T\phi(D) = \phi(F), \\ &\phi(A)R_n\check{X}R_n^{-1}\phi(B) + \phi(C)\left(R_n\check{X}R_n^{-1}\right)^T\phi(D) = \phi(F), \\ &\phi(A)S_n\check{X}S_n^{-1}\phi(B) + \phi(C)\left(S_n\check{X}S_n^{-1}\right)^T\phi(D) = \phi(F), \\ &\phi(A)\phi(Q)T_n\check{X}T_n^{-1}\phi(Q)\phi(B) \\ &\quad + \phi(C)\phi(Q)\left(T_n\check{X}T_n^{-1}\right)^T\phi(Q)\phi(D) = \phi(F), \\ &\phi(A)\phi(Q)R_n\check{X}R_n^{-1}\phi(Q)\phi(B) \\ &\quad + \phi(C)\phi(Q)\left(R_n\check{X}R_n^{-1}\right)^T\phi(Q)\phi(D) = \phi(F), \\ &\phi(A)\phi(Q)S_n\check{X}S_n^{-1}\phi(Q)\phi(B) \\ &\quad + \phi(C)\phi(Q)\left(S_n\check{X}S_n^{-1}\right)^T\phi(Q)\phi(D) = \phi(F), \end{aligned} \quad (41)$$

which implies that $T_n\check{X}T_n^{-1}, R_n\check{X}R_n^{-1}$, and $S_n\check{X}S_n^{-1}$ are also solutions of (36). Thus,

$$\frac{1}{4}\left(\check{X} + T_n\check{X}T_n^{-1} + R_n\check{X}R_n^{-1} + S_n\check{X}S_n^{-1}\right) \quad (42)$$

is also a solution of (36), where

$$\begin{aligned}
&\check{X} + T_n \check{X} T_n^{-1} + R_n \check{X} R_n^{-1} + S_n \check{X} S_n^{-1} \\
&\quad = \left[\widetilde{X_{ij}}\right]_{4\times 4}, \quad i, j = 1, 2, 3, 4, \\
&\widetilde{X_{11}} = X_{11} + X_{22} + X_{33} + X_{44}, \\
&\widetilde{X_{12}} = X_{12} - X_{21} + X_{34} - X_{43}, \\
&\widetilde{X_{13}} = X_{13} - X_{24} - X_{31} + X_{42}, \\
&\widetilde{X_{14}} = X_{14} + X_{23} - X_{32} - X_{41}, \\
&\widetilde{X_{21}} = -X_{12} + X_{21} - X_{34} + X_{43}, \\
&\widetilde{X_{22}} = X_{11} + X_{22} + X_{33} + X_{44}, \\
&\widetilde{X_{23}} = X_{14} + X_{23} - X_{32} - X_{41}, \\
&\widetilde{X_{24}} = -X_{13} + X_{24} + X_{31} - X_{42}, \\
&\widetilde{X_{31}} = -X_{13} + X_{24} + X_{31} - X_{42}, \\
&\widetilde{X_{32}} = -X_{14} - X_{23} + X_{32} + X_{41}, \\
&\widetilde{X_{33}} = X_{11} + X_{22} + X_{33} + X_{44}, \\
&\widetilde{X_{34}} = X_{12} - X_{21} + X_{34} - X_{43}, \\
&\widetilde{X_{41}} = -X_{14} - X_{23} + X_{32} + X_{41}, \\
&\widetilde{X_{42}} = X_{13} - X_{24} - X_{31} + X_{42}, \\
&\widetilde{X_{43}} = -X_{12} + X_{21} - X_{34} + X_{43}, \\
&\widetilde{X_{44}} = X_{11} + X_{22} + X_{33} + X_{44}.
\end{aligned} \tag{43}$$

Let

$$\begin{aligned}
\widehat{X} = {} & \frac{1}{4}\left(X_{11} + X_{22} + X_{33} + X_{44}\right) \\
& + \frac{1}{4}\left(-X_{12} + X_{21} - X_{34} + X_{43}\right) i \\
& + \frac{1}{4}\left(-X_{13} + X_{24} + X_{31} - X_{42}\right) j \\
& + \frac{1}{4}\left(-X_{14} - X_{23} + X_{32} + X_{41}\right) k.
\end{aligned} \tag{44}$$

Then it is not difficult to verify that

$$\phi\left(\widehat{X}\right) = \frac{1}{4}\left(\check{X} + T_n \check{X} T_n^{-1} + R_n \check{X} R_n^{-1} + S_n \check{X} S_n^{-1}\right). \tag{45}$$

We have that $\widehat{X}$ is a solution of (34) by (a) in Lemma 4. The proof is completed. □

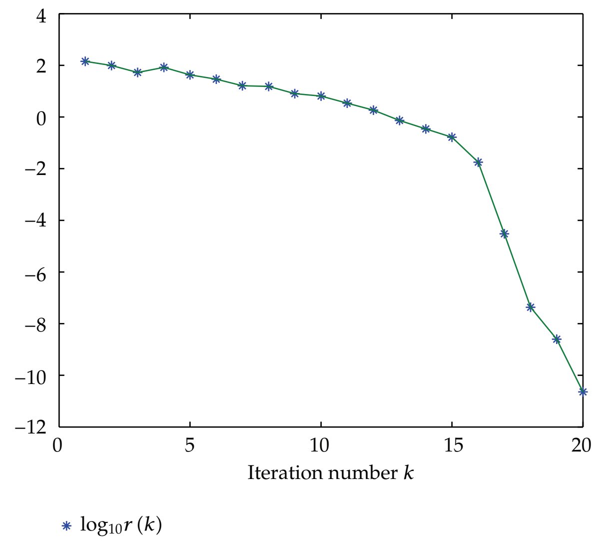

FIGURE 1: The convergence curve for the Frobenius norm of the residuals from Example 18.

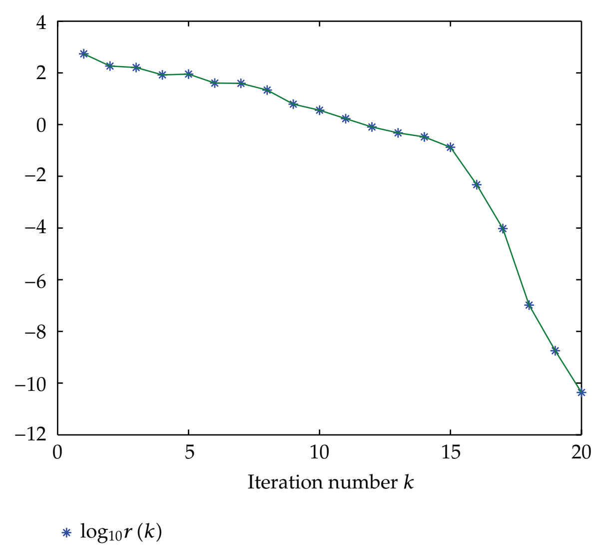

FIGURE 2: The convergence curve for the Frobenius norm of the residuals from Example 19.

Lemma 15. *There exists a permutation matrix P(4n,4n) such that* (36) *is equivalent to*

$$\begin{gathered}
\begin{bmatrix} \phi^T(B) \otimes \phi(A) + \left(\phi^T(D) \otimes \phi(C)\right) P(4n, 4n) \\ \phi^T(B)\phi(Q) \otimes \phi(A)\phi(Q) + \left(\phi^T(D)\phi(Q) \otimes \phi(C)\phi(Q)\right) P(4n, 4n) \end{bmatrix} \\
\times \operatorname{vec}\left(\left[X_{ij}\right]_{4\times 4}\right) = \begin{bmatrix} \operatorname{vec}\left(\phi(F)\right) \\ \operatorname{vec}\left(\phi(F)\right) \end{bmatrix}.
\end{gathered} \tag{46}$$

Lemma 15 is easily proven through Lemma 5. So we omit it here.

Theorem 16. *When Problem 1 is consistent, let its solution set be denoted by S_H. If $\mathring{X} \in S_H$, and $\mathring{X}$ can be expressed as*

$$\mathring{X} = A^H G B^H + D G^H C + Q A^H G B^H Q + Q D G^H C Q, \quad G \in \mathbb{H}^{m\times p}, \tag{47}$$

then, $\mathring{X}$ is the least Frobenius norm solution of Problem 1.

Proof. By (a), (b), and (c) in Lemmas 4, 5, and 6, we have that

$$\begin{aligned}
&\operatorname{vec}\left(\phi\left(\mathring{X}\right)\right) \\
&= \operatorname{vec}\left(\phi^T(A)\phi(G)\phi^T(B) + \phi(D)\phi^T(G)\phi(C)\right. \\
&\qquad + \phi(Q)\phi^T(A)\phi(G)\phi^T(B)\phi(Q) \\
&\qquad \left. + \phi(Q)\phi(D)\phi^T(G)\phi(C)\phi(Q)\right) \\
&= \left[\phi(B)\otimes\phi^T(A) + \left(\phi^T(C)\otimes\phi(D)\right)P(4m,4p),\right. \\
&\qquad \phi(Q)\phi(B)\otimes\phi(Q)\phi^T(A) \\
&\qquad \left. + \left(\phi(Q)\phi^T(C)\otimes\phi(Q)\phi(D)\right)P(4m,4p)\right] \\
&\qquad \times \begin{bmatrix}\operatorname{vec}(\phi(G)) \\ \operatorname{vec}(\phi(G))\end{bmatrix} \\
&= \begin{bmatrix} \phi^T(B)\otimes\phi(A) + (\phi^T(D)\otimes\phi(C))P(4n,4n) \\ \phi^T(B)\phi(Q)\otimes\phi(A)\phi(Q) + (\phi^T(D)\phi(Q)\otimes\phi(C)\phi(Q))P(4n,4n)\end{bmatrix}^T \\
&\qquad \times \begin{bmatrix}\operatorname{vec}(\phi(G)) \\ \operatorname{vec}(\phi(G))\end{bmatrix} \\
&\in R\begin{bmatrix} \phi^T(B)\otimes\phi(A) + \left(\phi^T(D)\otimes\phi(C)\right)P(4n,4n) \\ \phi^T(B)\phi(Q)\otimes\phi(A)\phi(Q) + \left(\phi^T(D)\phi(Q)\otimes\phi(C)\phi(Q)\right)P(4n,4n)\end{bmatrix}^T.
\end{aligned} \tag{48}$$

By Lemma 12, $\phi(\mathring{X})$ is the least Frobenius norm solution of matrix equations (46).

Noting (11), we derive from Lemmas 13, 14, and 15 that $\mathring{X}$ is the least Frobenius norm solution of Problem 1. □

From Algorithm 7, it is obvious that, if we consider

$$X(1) = A^H G B^H + D G^H C + Q A^H G B^H Q + Q D G^H C Q, \quad G \in \mathbb{H}^{m\times p}, \tag{49}$$

then all $X(k)$ generated by Algorithm 7 can be expressed as

$$X(k) = A^H G_k B^H + D G_k^H C + Q A^H G_k B^H Q + Q D G_k^H C Q, \quad G_k \in \mathbb{H}^{m\times p}. \tag{50}$$

Using the above conclusion and considering Theorem 16, we propose the following theorem.

Theorem 17. *Suppose that Problem 1 is consistent. Let the initial iteration matrix be*

$$X(1) = A^H G B^H + D G^H C + Q A^H G B^H Q + Q D G^H C Q, \tag{51}$$

where G is an arbitrary quaternion matrix, or especially, $X(1) = 0$, then the solution X^*, generated by Algorithm 7, is the least Frobenius norm solution of Problem 1.

Now we study Problem 2. When Problem 1 is consistent, the solution set of Problem 1 denoted by S_H is not empty. Then, For a given reflexive matrix $X_0 \in \mathbb{R}\mathbb{H}^{n\times n}(Q)$,

$$\begin{aligned} AXB + CX^H D = F &\Longleftrightarrow A\left(X - X_0\right)B + C\left(X - X_0\right)^H D \\ &= F - AX_0B - CX_0^H D. \end{aligned} \tag{52}$$

Let $\dot{X} = X - X_0$ and $\dot{F} = F - AX_0B - CX_0^H D$, then Problem 2 is equivalent to finding the least Frobenius norm reflexive solution of the quaternion matrix equation

$$A\dot{X}B + C\dot{X}^H D = \dot{F}. \tag{53}$$

By using Algorithm 7, let the initial iteration matrix $\dot{X}(1) = A^H G B^H + D G^H C + Q A^H G B^H Q + Q D G^H C Q$, where G is an arbitrary quaternion matrix in $\mathbb{H}^{m\times p}$, or especially, $\dot{X}(1) = 0$, we can obtain the least Frobenius norm reflexive solution $\dot{X}^*$ of (53). Then we can obtain the solution of Problem 2, which is

$$\check{X} = \dot{X}^* + X_0. \tag{54}$$

4. Examples

In this section, we give two examples to illustrate the efficiency of the theoretical results.

Example 18. Consider the quaternion matrix equation

$$AXB + CX^{H}D = F, \tag{55}$$

with

$$A = \begin{bmatrix} 1+i+j+k & 2+2j+2k & i+3j+3k & 2+i+4j+4k \\ 3+2i+2j+k & 3+3i+j & 1+7i+3j+k & 6-7i+2j-4k \end{bmatrix},$$
$$B = \begin{bmatrix} -2+i+3j+k & 3+3i+4j+k \\ 5-4i-6j-5k & -1-5i+4j+3k \\ -1+2i+j+k & 2+2j+6k \\ 3-2i+j+k & 4+i-2j-4k \end{bmatrix},$$
$$C = \begin{bmatrix} 1+i+j+k & 2+2i+2j & k & 2+2i+2j+k \\ -1+i+3j+3k & -2+3i+4j+2k & 6-2i+6j+4k & 3+i+j+5k \end{bmatrix}, \tag{56}$$
$$D = \begin{bmatrix} -1+4j+2k & 1+2i+3j+3k \\ 7+6i+5j+6k & 3+7i-j+9k \\ 4+i+6j+k & 7+2i+9j+k \\ 1+i+3j-3k & 1+3i+2j+2k \end{bmatrix},$$
$$F = \begin{bmatrix} 1+3i+2j+k & 2-2i+3j+3k \\ -1+4i+2j+k & -2+2i-1j \end{bmatrix}.$$

We apply Algorithm 7 to find the reflexive solution with respect to the generalized reflection matrix.

$$Q = \begin{bmatrix} 0.28 & 0 & 0.96k & 0 \\ 0 & -1 & 0.0 & 0 \\ -0.96k & 0 & -0.28 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{57}$$

For the initial matrix $X(1) = Q$, we obtain a solution, that is

$$X^{*} = X(20)$$
$$= \left[\begin{matrix} 0.27257 - 0.23500i + 0.034789j + 0.054677k & 0.085841 - 0.064349i + 0.10387j - 0.26871k \\ 0.028151 - 0.013246i - 0.091097j + 0.073137k & -0.46775 + 0.048363i - 0.015746j + 0.21642k \\ 0.043349 + 0.079178i + 0.085167j - 0.84221k & 0.35828 - 0.13849i - 0.085799j + 0.11446k \\ 0.13454 - 0.028417i - 0.063010j - 0.10574k & 0.10491 - 0.039417i + 0.18066j - 0.066963k \end{matrix}\right. \tag{58}$$
$$\left.\begin{matrix} -0.043349 + 0.079178i + 0.085167j + 0.84221k & -0.014337 - 0.14868i - 0.011146j - 0.00036397k \\ 0.097516 + 0.12146i - 0.017662j - 0.037534k & -0.064483 + 0.070885i - 0.089331j + 0.074969k \\ -0.21872 + 0.18532i + 0.011399j + 0.029390k & 0.00048529 + 0.014861i - 0.19824j - 0.019117k \\ -0.14099 + 0.084013i - 0.037890j - 0.17938k & -0.63928 - 0.067488i + 0.042030j + 0.10106k \end{matrix}\right],$$

with corresponding residual $\|R(20)\| = 2.2775 \times 10^{-11}$. The convergence curve for the Frobenius norm of the residuals $R(k)$ is given in Figure 1, where $r(k) = \|R(k)\|$.

Example 19. In this example, we choose the matrices A, B, C, D, F, and Q as same as in Example 18. Let

$$X_0 = \begin{bmatrix} 1 & -0.36i - 0.75k & 0 & -0.5625i - 0.48k \\ 0.36i & 1.28 & 0.48j & -0.96j \\ 0 & 1 - 0.48j & 1 & 0.64 - 0.75j \\ 0.48k & 0.96j & 0.64 & 0.72 \end{bmatrix} \tag{59}$$
$$\in \mathbb{RH}^{n\times n}(Q).$$

In order to find the optimal approximation reflexive solution to the given matrix X_0, let $\dot{X} = X - X_0$ and $\dot{F} = F - AX_0B - CX_0^{H}D$. Now we can obtain the least Frobenius norm reflexive solution $\dot{X}^{*}$ of the quaternion matrix equation $A\dot{X}B + C\dot{X}^{H}D = \dot{F}$, by choosing the initial matrix $\dot{X}(1) = 0$, that is

$$\dot{X}^{*} = \dot{X}(20)$$
$$= \left[\begin{array}{cc} -0.47933+0.021111i+0.11703j-0.066934k & -0.52020-0.19377i+0.17420j-0.59403k \\ -0.31132-0.11705i-0.33980j+0.11991k & -0.86390-0.21715i-0.037089j+0.40609k \\ -0.24134-0.025941i-0.31930j-0.26961k & -0.44552+0.13065i+0.14533j+0.39015k \\ -0.11693-0.27043i+0.22718j-0.0000012004k & -0.44790-0.31260i-1.0271j+0.27275k \end{array}\right. \quad (60)$$
$$\left.\begin{array}{cc} 0.24134-0.025941i-0.31930j+0.26961k & 0.089010-0.67960i+0.43044j-0.045772k \\ -0.089933-0.25485i+0.087788j-0.23349k & -0.17004-0.37655i+0.80481j+0.37636k \\ -0.63660+0.16515i-0.13216j+0.073845k & -0.034329+0.32283i+0.50970j-0.066757k \\ 0.00000090029+0.17038i+0.20282j-0.087697k & -0.44735+0.21846i-0.21469j-0.15347k \end{array}\right],$$

with corresponding residual $\|\dot{R}(20)\| = 4.3455 \times 10^{-11}$. The convergence curve for the Frobenius norm of the residuals $\dot{R}(k)$ is given in Figure 2, where $r(k) = \|\dot{R}(k)\|$.

Therefore, the optimal approximation reflexive solution to the given matrix X_0 is

$$\breve{X} = \dot{X}^{*} + X_0$$
$$= \left[\begin{array}{cc} 0.52067+0.021111i+0.11703j-0.066934k & -0.52020-0.55377i+0.17420j-1.3440k \\ -0.31132+0.24295i-0.33980j+0.11991k & 0.41610-0.21715i-0.037089j+0.40609k \\ -0.24134-0.025941i-0.31930j-0.26961k & 0.55448+0.13065i-0.33467j+0.39015k \\ -0.11693-0.27043i+0.22718j+0.48000k & -0.44790-0.31260i-0.067117j+0.27275k \end{array}\right. \quad (61)$$
$$\left.\begin{array}{cc} 0.24134-0.025941i-0.31930j+0.26961k & 0.089010-1.2421i+0.43044j-0.52577k \\ -0.089933-0.25485i+0.56779j-0.23349k & -0.17004-0.37655i-0.15519j+0.37636k \\ 0.36340+0.16515i-0.13216j+0.073845k & 0.60567+0.32283i-0.24030j-0.066757k \\ 0.64000+0.17038i+0.20282j-0.087697k & 0.27265+0.21846i-0.21469j-0.15347k \end{array}\right].$$

The results show that Algorithm 7 is quite efficient.

5. Conclusions

In this paper, an algorithm has been presented for solving the reflexive solution of the quaternion matrix equation $AXB + CX^{H}D = F$. By this algorithm, the solvability of the problem can be determined automatically. Also, when the quaternion matrix equation $AXB + CX^{H}D = F$ is consistent over reflexive matrix X, for any reflexive initial iterative matrix, a reflexive solution can be obtained within finite iteration steps in the absence of roundoff errors. It has been proven that by choosing a suitable initial iterative matrix, we can derive the least Frobenius norm reflexive solution of the quaternion matrix equation $AXB + CX^{H}D = F$ through Algorithm 7. Furthermore, by using Algorithm 7, we solved Problem 2. Finally, two numerical examples were given to show the efficiency of the presented algorithm.

Acknowledgments

This research was supported by Grants from Innovation Program of Shanghai Municipal Education Commission (13ZZ080), the National Natural Science Foundation of China (11171205), the Natural Science Foundation of Shanghai (11ZR1412500), the Discipline Project at the corresponding level of Shanghai (A. 13-0101-12-005), and Shanghai Leading Academic Discipline Project (J50101).

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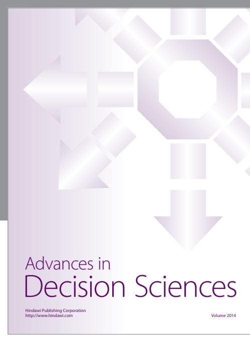
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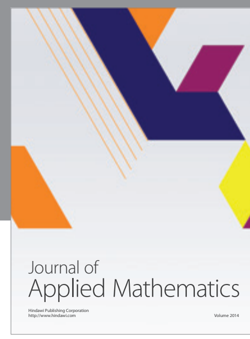
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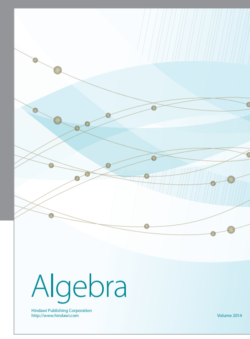
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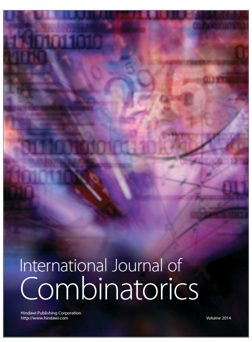
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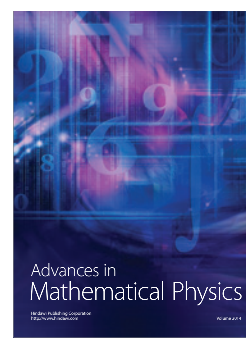
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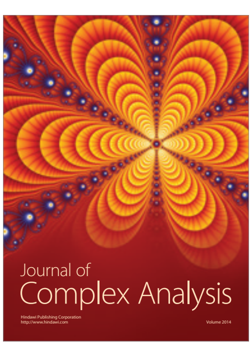
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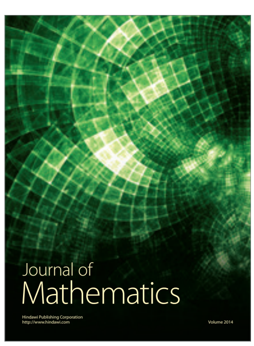
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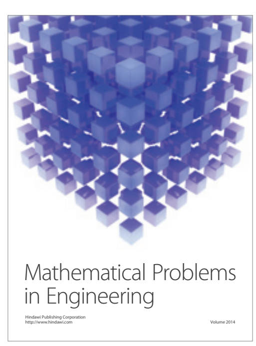
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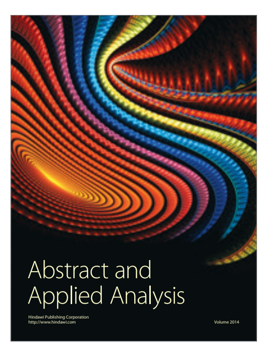
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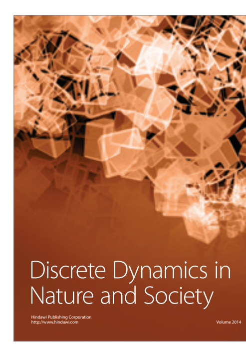
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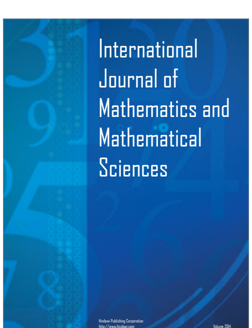
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