Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 457163, 9 pages
doi:10.1155/2012/457163

Research Article

Subharmonic Solutions with Prescribed Minimal Periodic for a Class of Second-Order Impulsive Functional Differential Equations

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Received 19 May 2012; Revised 25 June 2012; Accepted 25 June 2012

Academic Editor: Yong Zhou

By using critical point theory and variational methods, we investigate the subharmonic solutions with prescribed minimal period for a class of second-order impulsive functional differential equations. The conditions for the existence of subharmonic solutions are established. In the end, we provide an example to illustrate our main results.

1. Introduction

During the last 40 years, the theory and applications of impulsive differential equations have been developed, see [1–28]. Recently, some researchers studied the minimal period problem or homoclinic solution for some classes of Hamiltonian systems and classical pendulum equations [29–35]. In [30, 31], using the variational methods and decomposition technique, Yu got some sufficient conditions for the existence of periodic solutions with minimal period pT for the following nonautonomous Hamiltonian systems:

$$x''(t) + F'_x(t, x) = 0, \tag{1.1}$$

and a classical forced pendulum equation:

$$x''(t) + A\sin x = f(t), \tag{1.2}$$

respectively. In [35], by using critical point theory and variational methods, Luo et al. considered the existence results of subharmonic solutions with prescribed minimal period for a class of second-order impulsive differential equations:

$$u''(t) + f(t, u(t)) = 0, \quad \text{a.e. } t \in J',$$
$$\Delta u'(t_k) = I_k(u(t_k)), \quad k \in Z_0, \tag{1.3}$$

where $f \in C(R^2, R)$, $Z_0 = Z^+ \cup Z^-$, $J' = R \setminus \{t_k \mid k \in Z_0\}$, $I_k \in C(R, R^+ \cup \{0\})$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u'(t_k^{\pm}) = \lim_{t \to t_k^{\pm}} u'(t)$, $0 < t_1 < \cdots < t_m < T$, $I_{k+m} = I_k$, $T \in R^+$ and $t_k = t_{m+k} - T$ if $k \in Z^+$, while $t_k = t_{m+k+1} - T$ if $k \in Z^-$.

Motivated by [30, 31, 35], in this paper, we consider the existence results of subharmonic solutions with prescribed minimal period for a class of second-order impulsive functional differential equations:

$$u''(t-r) + f(t, u(t), u(t-r), u(t-2r)) = 0, \quad a.e.\ t \in J',$$
$$\Delta u'(t_k) = I_k(u(t_k)), \quad k \in Z_0, \tag{1.4}$$

where $r > 0$, $f \in C(R^4, R)$, $Z_0 = Z^+ \cup Z^-$, $J' = R \setminus \{t_k k \in Z_0\}$, $I_k \in C(R, R^+ \cup \{0\})$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u'(t_k^{\pm}) = \lim_{t \to t_k^{\pm}} u'(t)$, $0 < t_1 < \cdots < t_m < r$, $I_{k+m} = I_k$, $r \in R^+$ and $t_k = t_{m+k} - r$ if $k \in Z^+$, while $t_k = t_{m+k+1} - r$ if $k \in Z^-$.

We make the following assumptions.

(A_1) $f(t, u_1, u_2, u_3) \in C(R^4, R)$ is r-periodic in t for any $u_i \in C([0, pr], R)$, $i = 1, 2, 3$, where p is a positive integer.

(A_2) $F(t, u_1, u_2) \in C(R^3, R)$ is r-periodic in t and continuously differentiable for any $u_i \in C([0, pr], R)$ such that $\limsup_{|u_1|,|u_2| \to +\infty} F(t, u_1, u_2)/(|u_1|^2 + |u_2|^2) \leq 1/2(pr)^2 = \gamma$ and $F'_{u_2}(t, u_1, u_2) + F'_{u_2}(t, u_2, u_3) = f(t, u_1, u_2, u_3)$, where $F'_{u_2}(t, u_1, u_2)$ and $F'_{u_2}(t, u_2, u_3)$ are r-periodic functions in t.

(A_3) There are constants $\alpha > 0$, $\beta > 0$, $d_j \geq 0$, $j = 1, 2, \ldots, m$ such that

$$\begin{aligned} &\left|I_j(u)\right| \leq d_j |u|, \qquad \alpha_2 pr - pr\left(\frac{\omega}{p}\right)^2 - 2mpD > 0, \qquad p^2 < \frac{p_s^2 \omega^2}{\alpha}, \\ &\max\left\{0, \alpha\left(|u_1|^2 + |u_2|^2\right) - \beta\left(|u_1|^4 + |u_2|^4\right)\right\} \leq F(t, u_1, u_2) \\ &- F'_{u_2}(t, 0, 0) u_1 - F'_{u_2}(t, 0, 0) u_2 \leq \alpha\left(|u_1|^2 + |u_2|^2\right), \end{aligned} \tag{1.5}$$

where $D = \max\{d_j, j = 1, 2, \ldots, m\}$.

(A_4) Suppose q is rational. If u is a periodic function with minimal period qr, and $f(t, u_1, u_2, u_3)$ is a periodic function with minimal period qr, then q is necessarily an integer.

From (A_2), we have

$$F'_{u(t-r)}(t, u(t-r), u(t-2r)) + F'_{u(t-r)}(t, u(t), u(t-r)) = f(t, u(t), u(t-r), u(t-2r)). \tag{1.6}$$

Therefore, under the assumptions (A_1)-(A_4), the existence of subharmonic solutions with minimal period for (1.4) has been changed into the existence of subharmonic solutions with minimal period for

$$u''(t-r) + F'_{u(t-r)}(t, u(t-r), u(t-2r)) + F'_{u(t-r)}(t, u(t), u(t-r)) = 0, \quad t \in (t_{k-1}, t_k),$$
$$\Delta u'(t_k) = I_k(u(t_k)), \quad k \in Z_0. \tag{1.4$'$}$$

The outline of the paper is as follows. In Section 2, some preliminaries and basic results are established. In Section 3, by using critical point theory, we give sufficient conditions for the existence of of subharmonic solutions with minimal period for the impulsive systems. In Section 4, we give an example to illustrate the application of our main result

2. Preliminaries and Basic Results

In the following, we introduce some notations and some necessary definitions.

Let $T = pr, p \geq 2$. The norm in $H^1([0,T], R)$ is denoted by $\|\cdot\|_0$. Denote the Sobolov space E by

$$E = \left\{u \in H^1([0,T], R) \mid u \text{ is absolutely continuous}, u(0) = u(T)\right\} \tag{2.1}$$

with the inner product

$$(u, v) = \int_0^T \left[u(t)v(t) + u'(t)v'(t)\right] dt, \quad u, v \in E, \tag{2.2}$$

which induces the norm

$$\|u\| = \|u\|_0 + \left\|u'\right\|_0, \quad u \in E. \tag{2.3}$$

It is easy to verify that E is a reflexive Banach space.

Consider the functional I defined on E by

$$I(u) = \int_0^T \left[\frac{1}{2}\left|u'(t)\right|^2 - F(t, u(t), u(t-r))\right] dt + \sum_{k \in K} \int_0^{u(t_k)} I_k(t) dt, \tag{2.4}$$

where $K = \{k \in Z_0 \setminus t_k \in (0, T]\} = \{1, 2, \ldots, pm\}$.

We should caution that the solutions minimal periods may not be pr. Define $\omega = 2\pi/r$, and p_s as the smallest prime factor of p.

Define $\overline{E} = \{u \in E \mid u(-t) = -u(t)\}$, a subspace of the Sobolev space E. For any $u \in E, u$ has a Fourier series expansion $u(t) = \sum_{n=0}^{\infty}(a_n \cos n\omega t/p + b_n \sin n\omega t/p)$. Moreover, $u \in \overline{E}$ if and only if $u(t) = \sum_{n=0}^{\infty} b_n \sin n\omega t/p$.

We will show that the classic T-solutions of (1.4) or (1.4)$'$ is equivalent to finding the critical points of I.

Similar to the proof [13, 36, 37], we have two lemmas as following.

Lemma 2.1. *Suppose that I_k are continuous. Then, the following statements are equivalent:*

(1) *$u \in E$ is a critical point of I;*

(2) *u is a classical solution of* (1.4) *or* $(1.4)'$.

Lemma 2.2. *If u is a critical point of I on $\overline{E}$, then u is also a critical point of I on X. And the minimal period of u is an integer multiple of r.*

Now we state some results on nonlinear functional analysis and critical point theory. Suppose that X is a Banach space and $\varphi : X \to R$. Say that I is weakly lower semicontinuous if $u_k \rightharpoonup u_0$ means $\liminf_{n\to\infty} I(u_k) \geq I(u_0)$ and I is coercive if $\lim_{\|u\|\to\infty} = +\infty$.

Lemma 2.3 (see [38]). *Let E be a real reflexive Banach space and weak sequentially closed. $\varphi \in C^1(E,R)$ is weakly lower semicontinuous and coercive. Then, φ has a critical point u^* with $\min_{u\in E}\varphi(u) = \varphi(u^*)$.*

Similar to the proof of [35, Lemma 2.3], we have the following lemma.

Lemma 2.4. *Suppose that (A_2)-(A_3) hold. $\overline{E}$ is a weak sequentially closed and φ is coercive and weakly lower semicontinuous on $\overline{E}$.*

3. Main Results

Theorem 3.1. *Suppose that $(A_1)-(A_4)$ hold. If*

$$\left\|F'_{u_*(t)}(t,0,0)\right\|_0 + \left\|F'_{u_*(t-r)}(t,0,0)\right\|_0 \leq \frac{q\omega}{2p}\left(\alpha T - T\frac{\omega^2}{p^2} - 2mpD\right)\sqrt{\frac{2\left(1-\alpha p^2/q^2\omega^2\right)}{3\beta T}}, \tag{3.1}$$

then (1.4) *has at least one classical periodic solution with the minimal period $T = pr$.*

Proof. It follows from Lemmas 2.3 and 2.4 that I has a critical point u_* with $\min \varphi_{u\in E}(u) = \varphi(u^*)$. Next, we show the minimal period of u_* is pr. For the sake of a contradiction, let the minimal period of u_* be pr/q for some integer $q \geq 2$. By Lemma 2.2, we know that q is a factor of p, and so $q \geq p_s$. □

By the Wirtinger inequality and (A_1), we have

$$\begin{aligned}
I(u_*) &= \int_0^T \left[\frac{1}{2}\left|u'_*(t)\right|^2 - F(t,u_*(t),u_*(t-r))\right]dt + \sum_{k\in K}\int_0^{u_*(t_k)} I_k(t)\,dt \\
&\geq \frac{1}{2}\left\|u'_*\right\|_0^2 - \int_0^T \left[F'_{u_*(t)}(t,0,0)u_*(t) + F'_{u_*(t-r)}(t,0,0)u_*(t-r)\right]dt \\
&\quad - \int_0^T \left[F(t,u_*(t),u_*(t-r)) - F'_{u_*(t)}(t,0,0)u_*(t) - F'_{u_*(t-r)}(t,0,0)u_*(t-r)\right]dt
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}\|u_*'\|_0^2 - \left(\left\|F'_{u_*(t)}\ (t,0,0)\right\|_0 + \left\|F'_{u_*(t-r)}(t,0,0)\right\|_0\right)\|u_*\|_0 - \frac{\alpha}{2}\|u_*\|_0^2 \\
&\geq \frac{1}{2}\left(1-\alpha\left(\frac{p}{q\omega}\right)^2\right)\|u_*'\|_0^2 - \frac{p}{q\omega}\left(\left\|F'_{u_*(t)}(t,0,0)\right\|_0 + \left\|F'_{u_*(t-r)}(t,0,0)\right\|_0\right)\|u_*'\|_0.
\end{aligned}
\tag{3.2}
$$

On the other hand, let $\overline{u}(t) = \sqrt{\rho}\sin \omega t/p$. Then, $\overline{u}(t)$ is T-periodic with minimal periodic T. Since $F'_{u(t)}(t,u(t),u(t-r))$ and $F'_{u(t-r)}(t,u(t),u(t-r))$ are r-periodic, we have

$$
\int_0^T F'_{\overline{u}(t)}(t,\overline{u}(t),\overline{u}(t-r))\overline{u}(t)dt = 0, \quad \int_0^T F'_{\overline{u}(t-r)}(t,\overline{u}(t),\overline{u}(t-r))\overline{u}(t-r)dt = 0. \tag{3.3}
$$

By the Wirtinger inequality and (A_3), we also have

$$
\begin{aligned}
I(\overline{u}) &= \int_0^T \left[\frac{1}{2}\left|\overline{u}'(t)\right|^2 - F(t,\overline{u}(t),\overline{u}(t-r))\right]dt + \sum_{k\in K}\int_0^{\overline{u}(t_k)} I_k(t)dt \\
&\leq \frac{\rho T}{4}\left(\frac{\omega}{p}\right)^2 - \int_0^T\left[F'_{\overline{u}(t)}(t,0,0)\overline{u}(t) + F'_{\overline{u}(t-r)}(t,0,0)\overline{u}(t-r)\right]dt \\
&\quad - \int_0^T\left[F(t,\overline{u}(t),\overline{u}(t-r)) - F'_{\overline{u}(t)}(t,0,0)\overline{u}(t) - F'_{\overline{u}(t-r)}(t,0,0)\overline{u}(t-r)\right]dt + \frac{mpD\rho}{2} \\
&\leq \frac{\rho T}{4}\left(\frac{\omega}{p}\right)^2 - \frac{\alpha}{2}\int_0^T|\overline{u}(t)|^2dt + \frac{\beta}{2}\int_0^T|\overline{u}(t)|^4dt + \frac{mpD\rho}{2} \\
&\leq \frac{\rho T}{4}\left(\frac{\omega}{p}\right)^2 - \frac{\alpha\rho T}{4} + \frac{3\beta T\rho^2}{16} + \frac{mpD\rho}{2} \\
&= \frac{3\beta T\rho^2}{16} - \frac{1}{4}\left(\alpha T - T\left(\frac{\omega}{p}\right)^2 - 2mpD\right)\rho.
\end{aligned}
\tag{3.4}
$$

If $I(\overline{u}) < I(u_*)$, then this is clearly in contradiction with the assumption for u_*. Now, we are going to choose some positive number ρ such that

$$
\begin{aligned}
&\frac{3\beta T\rho^2}{16} - 1/4\left(\alpha T - T\left(\frac{\omega}{p}\right)^2 - 2mpD\right)\rho < \frac{1}{2}\left(1-\alpha\left(\frac{p}{q\omega}\right)^2\right)\|u_*\|_0^2. \\
&- \frac{p}{q\omega}\left(\left\|F'_{u_*(t)}\ (t,0,0)\right\|_0 + \left\|F'_{u_*(t-r)}(t,0,0)\right\|_0\right)\|u_*'\|_0.
\end{aligned}
\tag{3.5}
$$

Actually, we can choose $\rho = 4/3\beta T(\alpha T - T(\omega/p)^2 - 2mpD)$. Then, we need to prove

$$\frac{-\left(1/4\left(\alpha T - T(\omega/p)^2 - 2mpD\right)\right)^2}{3\beta T/4} < \frac{-\left(p/q\omega\left(\left\|F'_{u_*(t)}(t,0,0)\right\|_0 + \left\|F'_{u_*(t-r)}(t,0,0)\right\|_0\right)\right)^2}{2\left(1-\alpha(p/q\omega)^2\right)}. \tag{3.6}$$

This is true under the assumption (3.1). Hence, the proof is complete.

4. Example

Suppose

$$F(t,u_1,u_2) = \frac{1}{20}\left(u_1^2 + u_2^2\right) - \frac{1}{20}\sin\frac{2\pi t}{r}\left(u_1^2\arctan u_2^2 + u_2^2\arctan u_1^2 + u_1 + u_2\right). \tag{4.1}$$

Then,

$$\begin{aligned} &F'_{u(t-r)}(t,u(t-r),u(t-2r)) \\ &\quad = \frac{1}{10}u(t-r) - \frac{1}{20}\sin\frac{2\pi t}{r}\left(2u(t-r)\arctan u^2(t-r) + \frac{2u(t-r)u(t-2r)}{1+u^4(t-r)} + 1\right) \\ &F'_{u(t-r)}(t,u(t),u(t-r)) = \frac{1}{10}u(t-r) - \frac{1}{20}\sin\frac{2\pi t}{r}\left(2u(t-r)\arctan u^2(t) + \frac{2u(t)u(t-r)}{1+u^4(t-r)} + 1\right), \\ &F'_{u_1}(t,u_1,u_2)\big|_{u_1=u_2=0}u_1 + F'_{u_2}(t,u_1,u_2)\big|_{u_1=u_2=0}u_2 = -\frac{1}{20}\sin\frac{2\pi t}{r}(u_1+u_2). \end{aligned} \tag{4.2}$$

Let

$$\begin{aligned} &f(t,u(t),u(t-r),u(t-2r)) \\ &\quad = \frac{1}{5}u(t-r) - \frac{1}{20}\sin\frac{2\pi t}{r}\left(2u(t-r)\arctan u^2(t-r) + \frac{2u(t-r)u(t-2r)}{1+u^4(t-r)}\right. \\ &\qquad\qquad \left. + 2u(t-r)\arctan u^2(t) + \frac{2u(t)u(t-r)}{1+u^4(t-r)} + 2\right). \end{aligned} \tag{4.3}$$

Consider the following impulsive system:

$$u''(t-r)+\frac{1}{5}u(t-r)-\frac{1}{20}\sin\frac{2\pi t}{r}\left[2u(t-r)\arctan u^2(t-r)+\frac{2u(t-r)u(t-2r)}{1+u^4(t-r)}\right.$$
$$\left.+2u(t-r)\arctan u^2(t)+\frac{2u(t)u(t-r)}{1+u^4(t-r)}+2\right]=0, \tag{4.4}$$
$$\forall t\in(t_{k-1},t_k),$$
$$\Delta u'(t_k)=I_k(u(t_k))=0.001|u(t_k)|,\quad k\in Z^*,$$

where $t_k=k-1/2$ if $k\in Z^+$, while $t_k=k+1/2$ if $k\in Z^-$.

Proof. Let $r=1, T=1, \gamma=1/20, \alpha=1/20, \beta=1/20, m=1, D=0.001, \omega=2\pi, p_s=2$. It is easy to check all the assumptions of Theorem 3.1 are satisfied. Thus, (4.4) has a periodic solution with the minimal period 30. □

Acknowlegdment

Research was supported by Anhui Provincial Nature Science Foundation (090416237, 1208085MA13), Research Fund for Doctoral Station of Ministry of Education of China (20103401120002, 20113401110001), 211 Project of Anhui University (02303129, 02303303-33030011, 02303902-39020011, KJTD002B), and Foundation of Anhui Education Bureau (KJ2012A019).

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