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Research Article

Augmented Superfield Approach to Nilpotent Symmetries of the Modified Version of 2D Proca Theory

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We derive the complete set of off-shell nilpotent and absolutely anticommuting Becchi-Rouet-Stora-Tyutin (BRST), anti-BRST, and (anti-)co-BRST symmetry transformations for *all* the fields of the modified version of two $(1 + 1)$-dimensional (2D) Proca theory by exploiting the "augmented" superfield formalism where the (dual-)horizontality conditions and (dual-)gauge invariant restrictions are exploited *together*. We capture the (anti-)BRST and (anti-)co-BRST invariance of the Lagrangian density in the language of superfield approach. We also express the nilpotency and absolute anticommutativity of the (anti-)BRST and (anti-)co-BRST charges within the framework of augmented superfield formalism. This exercise leads to some *novel* observations which have, hitherto, not been pointed out in the literature within the framework of superfield approach to BRST formalism. For the sake of completeness, we also mention, very briefly, a unique bosonic symmetry, the ghost-scale symmetry, and discrete symmetries of the theory and show that the algebra of conserved charges provides a physical realization of the Hodge algebra (satisfied by the de Rham cohomological operators of differential geometry).

1. Introduction

One of the earliest known gauge theories (with $U(1)$ gauge symmetry) is the Abelian 1-form ($A^{(1)} = dx^\mu A_\mu$, $\mu = 0, 1, 2, \ldots, D-1$) Maxwell theory which describes the massless vector boson (A_μ) with $(D-2)$ degrees of freedom in any arbitrary D-dimensions of spacetime. Thus, in the physical *four* dimensions of spacetime, A_μ has two degrees of freedom. Its massive generalization is a Proca theory that describes a vector boson with three degrees of freedom in the physical four $(3 + 1)$-dimensions of spacetime. The central goal of our present investigation is to study the two $(1 + 1)$-dimensional (2D) Stueckelberg-modified [1] version of the Proca theory which also incorporates a pseudoscalar field on physical and mathematical grounds [2, 3]. This model is very *special* because it is endowed with *mass* together with various kinds of *internal* symmetries which originate, primarily, from the gauge symmetry and its "dual" version. The existence of the above symmetries renders the model to become an example for the Hodge theory [2, 3].

Recently, in a set of papers [4–6], we have demonstrated that the $N = 2$ supersymmetric (SUSY) quantum mechanical models also provide a set of physical examples of Hodge theory because of their specific continuous and discrete symmetry transformations which provide the physical realizations of the de Rham cohomological operators and Hodge duality $(*)$ operation of differential geometry [7–12]. However, these SUSY models are *not* gauge theories because they are not endowed with first-class constraints in the terminology of Dirac's prescription for the classification scheme of constraints [13, 14]. One of the characteristic features of these SUSY models is that they have *mass* but do *not* possess gauge symmetries that are primarily generated by the first-class constraints (see, e.g., [14–16]).

We have also provided the physical realizations of the de Rham cohomological operators of the differential geometry in the context of Abelian p-form ($p = 1, 2, 3$) gauge theories in $D = 2p$ dimensions of spacetime within the framework of BRST formalism [17–20]. As a consequence, these theories are also the field theoretic models for the Hodge theory. One of the decisive features of these theories is the observation that they have gauge symmetry (generated by the first-class constraints) but they do not have *mass*. The modified version of 2D Proca theory is, thus, a very *special* field theory which possesses *mass* as well as various kinds of *internal* symmetries and, as has turned out, it *also* presents a field theoretic model for the Hodge theory within the framework of BRST formalism [2, 3].

One of the most intuitive approaches to understand the abstract mathematical properties associated with the (anti-)BRST symmetries is the geometrical superfield formalism (see, e.g., [21–24]), where the celebrated horizontality condition (HC) plays a very important role as far as the derivation of (anti-)BRST symmetry transformations for the gauge fields and associated (anti-)ghost fields, for a given gauge theory, is concerned. In the augmented version [25–28] of the above superfield formalism, the HC blends together with the gauge invariant restrictions (GIRs) in a beautiful fashion enabling us to derive the (anti-)BRST symmetry transformations for the gauge, (anti-)ghost, and *matter* fields of a given interacting gauge theory in a cohesive and consistent manner. The central objective of our present paper is to apply extensively the above augmented version of the geometrical superfield formalism [25–28] to discuss various aspects of the modified version of 2D Proca theory within the framework of BRST formalism.

In our present investigation, we derive the off-shell nilpotent and absolutely anticommuting (anti-)BRST and (anti-)co-BRST symmetry transformations by exploiting the theoretical power of augmented version of superfield formalism. In fact, we exploit the celebrated (dual-)horizontality conditions [(D)HCs] and (dual-)gauge invariant restrictions [(D)GIRs] to obtain the proper (anti-)BRST and (anti-)co-BRST symmetry transformations for *all* the fields of the modified version of 2D Proca theory. We provide the geometrical meaning to the above nilpotent symmetry transformations in the language of translational generators along the Grassmannian directions of the $(2, 2)$-dimensional supermanifold on which our ordinary modified version of 2D Proca theory is generalized.

Some of the key observations of our present investigation are contained in Sections 3.3 and 4.4 where we have expressed the (anti-)BRST and (anti-)co-BRST charges in terms of the superfields (obtained after the applications of (D)HCs and (D)GIRs), Grassmannian partial derivatives, and Grassmannian differentials. The off-shell nilpotency and absolute anticommutativity properties of the (anti-)BRST and (anti-)co-BRST symmetries (and their corresponding generators) emerge very naturally within the framework of our augmented version of superfield formalism. We have also captured the (anti-)BRST and (anti-)co-BRST invariances of the Lagrangian density within the ambit of our augmented version of superfield approach in a very simple and straightforward manner.

The main motivating factors behind our present investigations are as follows. First, it is very important for us to put the basic ideas of our augmented version of superfield formalism on solid footing by applying it to various interesting physical systems which are BRST invariant. Second, it is essential for us to establish the correctness of our earlier results [3] where we have discussed the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations for the 2D modified Proca theory. Finally, the present endeavor is our modest step towards the main goal of applying our basic ideas to find out the 4D massive models for the Hodge theory which might enforce the existence of fields that would turn out to be the candidates for the dark matter [29, 30]. We have already shown the emergence and existence of the latter (as a pseudoscalar field with a negative kinetic term) in our study of the modified version of 2D Proca theory [2, 3].

The contents of our present paper are organized as follows. In Section 2, we recapitulate the bare essentials of the usual Proca theory and discuss the gauge symmetry transformations of the Stueckelberg-modified version of it in any arbitrary D-dimensions of spacetime. Section 3 is devoted to the derivation of off-shell nilpotent (anti-)BRST symmetry transformations within the framework of augmented superfield formalism. In Section 4, we deal with the (anti-)co-BRST symmetry transformations for the 2D Stueckelberg-modified Proca theory by exploiting the augmented version of superfield approach. Section 5 describes, very briefly, a unique bosonic symmetry, the ghost-scale symmetry, and discrete symmetries of our present theory. In Section 6, we present the algebraic structure of all the generators of the above continuous symmetries and establish its connection with the cohomological operators of differential geometry. Finally, we make some concluding remarks in Section 7.

In our Appendices A and B, we perform some explicit computations which have been used in the main body of our present text.

Essential Definitions

(1) On a compact manifold without a boundary, a set of three operators (d, δ, Δ) define the de Rham cohomological operators [7–12] of differential geometry. Here $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) is the exterior derivative and $\delta = \pm * d *$ (with $\delta^2 = 0$) defines the coexterior derivative, where $(*)$ stands for the Hodge duality operation. The Laplacian operator $\Delta = (d+\delta)^2 = \{d, \delta\}$ is defined in terms of d and δ (where $\{d, \delta\} = d\delta + \delta d$).

(2) We have christened the extended version of the usual Bonora-Tonin superfield formalism [21, 22] as the augmented superfield formalism where, in addition to the HC, other physically relevant restrictions (consistent with the HC) are *also* imposed on the superfields defined on the appropriate supermanifold.

2. Preliminaries: Local Gauge Symmetries in the Modified Version of Proca Theory

Let us begin with the Lagrangian density ($\mathscr{L}_0$) of a Proca theory (with a mass parameter m) in any arbitrary D-dimensions of spacetime. This can be expressed in an explicit form as follows:

$$\mathscr{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu. \tag{1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is derived from the 2-form $F^{(2)} = dA^{(1)} = [(dx^\mu \wedge dx^\nu)/2!] F_{\mu\nu}$, where $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) is the exterior derivative and the 1-form $A^{(1)} = dx^\mu A_\mu$ defines the vector boson A_μ. In physical four $(3+1)$-dimensions of spacetime, the bosonic field A_μ has three degrees of freedom and m has the dimension of mass in natural units (where $\hbar = c = 1$). In the massless limit (i.e., $m = 0$), we obtain the 4D Maxwell Lagrangian density from (1) which respects the $U(1)$ gauge invariance under the transformations:

$$A_\mu \longrightarrow A_\mu \mp \frac{1}{m} \partial_\mu \Lambda, \tag{2}$$

where Λ is the local gauge parameter. It is evident that, in the Proca theory, the gauge symmetry transformations (2) are lost because of the presence of mass term. In some sense, a Proca theory is a generalization of Maxwell's theory as the latter is the massless ($m = 0$) limit of the former (where the usual $U(1)$ gauge symmetry invariance is respected).

By exploiting the Stueckelberg formalism, one can restore the gauge symmetry (2) for the original Lagrangian density (1), where the field A_μ is replaced by $A_\mu \mp (1/m)\partial_\mu \phi$. As a consequence, we obtain the following Stueckelberg's Lagrangian density:

$$\mathscr{L}_s = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \mp m A_\mu \partial^\mu \phi, \tag{3}$$

which respects the following local, continuous, and infinitesimal gauge symmetry transformations (δ_g):

$$\begin{aligned} \delta_g A_\mu &= \partial_\mu \Lambda, \\ \delta_g \phi &= \pm m \Lambda, \end{aligned} \tag{4}$$

where ϕ is a real scalar field. The key points, at this stage, are as follows. First, by incorporating the Stueckelberg field ϕ, we have converted the second-class constraints of the original Lagrangian density (1) into the first-class variety in the terminology of Dirac's prescription for the classification scheme [13, 14]. Second, the Lagrangian density (3) describes, in the physical *four* dimensions of spacetime, a theory where the mass and gauge invariance coexist together in a beautiful and meaningful manner.

We close this section with the following remarks. First, the gauge symmetry transformations (4) are valid in any arbitrary dimension of spacetime for the Stueckelberg-modified Lagrangian density ($\mathscr{L}_s$) at the *classical* level. This symmetry, therefore, could be exploited for the (anti-)BRST symmetry transformations at the *quantum* level. Second, the quantity $A_\mu \mp (1/m)\partial_\mu \phi$ is a gauge invariant quantity because $\delta_g [A_\mu \mp (1/m)\partial_\mu \phi] = 0$ (for $\delta_g A_\mu = \partial_\mu \Lambda$ and $\delta_g \phi = \pm m \Lambda$). These observations would play very important roles in our further discussions on the derivation of proper (anti-)BRST symmetries within the framework of augmented version of superfield formalism.

3. Nilpotent (Anti-)BRST Symmetries: Geometrical Superfield Formalism

In this section, we derive the full set of *proper* (anti-)BRST symmetry transformations by exploiting the strength of HC and GIR. Furthermore, we capture the (anti-)BRST invariance of the Lagrangian density and the nilpotency as well as absolute anticommutativity properties of the (anti-)BRST charges within the framework of superfield formalism.

3.1. Derivation of the (Anti-)BRST Symmetries: HC and GIR. According to the prescription, laid down by the superfield approach to BRST formalism [21, 22], we have to generalize the present D-dimensional Stueckelberg-modified theory onto a (D, 2)-dimensional supermanifold which is parameterized by the superspace variable $Z^M = (x^\mu, \theta, \bar\theta)$, where x^μ ($\mu = 0, 1, 2, \ldots, D-1$) are the ordinary D-dimensional spacetime variables and $(\theta, \bar\theta)$ are a pair of Grassmannian variables (with $\theta^2 = \bar\theta^2 = 0$, $\theta\bar\theta + \bar\theta\theta = 0$).

The central role, in the superfield approach [21–24], is played by the HC which requires that the gauge invariant quantity $F_{\mu\nu}$, owing its origin to the exterior derivative, remain independent of the Grassmannian variables when it is generalized onto a (D, 2)-dimensional supermanifold. In other words, the ordinary curvature 2-form $F^{(2)} = dA^{(1)} = (dx^\mu \wedge dx^\nu/2!) F_{\mu\nu}$ must be equal (i.e., $F^{(2)} = \widetilde{\mathscr{F}}^{(2)}$) to the super curvature 2-form ($\widetilde{\mathscr{F}}^{(2)}$):

$$\widetilde{\mathscr{F}}^{(2)} = \tilde d \widetilde{A}^{(1)} \equiv \left(\frac{dZ^M \wedge dZ^N}{2!} \right) \widetilde{\mathscr{F}}_{MN} \left(x, \theta, \bar\theta \right). \tag{5}$$

In the above, the super exterior derivative $\tilde d$ (with $\tilde d^2 = 0$) and super 1-form connection $\widetilde{A}^{(1)}$ are defined on the (D, 2)-dimensional supermanifold as

$$\begin{aligned} \tilde d &= dZ^M \partial_M \equiv dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar\theta \partial_{\bar\theta}, \\ \widetilde{A}^{(1)} &= dZ^M A_M \\ &\equiv dx^\mu \mathscr{B}_\mu \left(x, \theta, \bar\theta \right) + d\theta \bar F \left(x, \theta, \bar\theta \right) \\ &\quad + d\bar\theta F \left(x, \theta, \bar\theta \right). \end{aligned} \tag{6}$$

We have taken $\partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar\theta})$ as the superspace derivative on the (D, 2)-dimensional supermanifold. Physically, the equality $dA^{(1)} = \tilde d \widetilde{A}^{(1)}$ of the HC implies that the gauge invariant electric and magnetic fields of the ordinary theory should *not* be affected by the presence of the Grassmannian variables θ and $\bar\theta$ of the supermanifold on which the ordinary theory has been generalized within the ambit of superfield formalism.

The superfields $\mathscr{B}_\mu(x,\theta,\bar\theta)$, $F(x,\theta,\bar\theta)$, and $\bar F(x,\theta,\bar\theta)$ of (6) are the generalizations of the gauge field (A_μ), ghost field (C), and anti-ghost field ($\bar C$), respectively, of the ordinary D-dimensional BRST invariant theory because the above superfields can be expanded along the Grassmannian directions of the (D, 2)-dimensional supermanifold as (see, e.g., [21])

$$\begin{aligned}\mathscr{B}_\mu\left(x,\theta,\bar\theta\right) &= A_\mu(x) + \theta R_\mu^{(1)}(x) + \bar\theta R_\mu^{(2)}(x) \\ &\quad + i\theta\bar\theta S_\mu(x), \\ F\left(x,\theta,\bar\theta\right) &= C(x) + i\theta B_1(x) + i\bar\theta B_2(x) + i\theta\bar\theta s(x), \\ \bar F\left(x,\theta,\bar\theta\right) &= \bar C(x) + i\theta B_3(x) + \bar\theta B_4(x) + i\theta\bar\theta\bar s(x),\end{aligned} \tag{7}$$

where $(A_\mu, C, \bar C)$ are the basic fields of any arbitrary D-dimensional (anti-)BRST invariant Abelian theory (an Abelian BRST invariant theory, in any arbitrary dimension of spacetime, contains the gauge-fixing and FP-ghost terms in addition to the kinetic term for A_μ) and rest of the fields, on the r.h.s. of (7), are the secondary fields which can be expressed in terms of the basic and auxiliary fields of the ordinary D-dimensional theory by exploiting HC. It is clear that $(R_\mu^{(1)}, R_\mu^{(2)}, s, \bar s)$ and $(S_\mu, B_1, B_2, B_3, B_4)$ are the fermionic and bosonic secondary fields, respectively, on the r.h.s. of (7).

One can expand the expression $\tilde d\tilde A^{(1)}$ of (5) in the following explicit form using (6). This expansion, in its full blaze of glory, is as follows:

$$\begin{aligned}\tilde d\tilde A^{(1)} &= \left(\frac{dx^\mu\wedge dx^\nu}{2!}\right)\left(\partial_\mu\mathscr{B}_\nu - \partial_\nu\mathscr{B}_\mu\right) \\ &\quad + \left(dx^\mu\wedge d\theta\right)\left(\partial_\mu\bar F - \partial_\theta\mathscr{B}_\mu\right) \\ &\quad + \left(dx^\mu\wedge d\bar\theta\right)\left(\partial_\mu F - \partial_{\bar\theta}\mathscr{B}_\mu\right) \\ &\quad - \left(d\theta\wedge d\theta\right)\left(\partial_\theta\bar F\right) - \left(d\bar\theta\wedge d\bar\theta\right)\left(\partial_{\bar\theta}F\right) \\ &\quad - \left(d\theta\wedge d\bar\theta\right)\left(\partial_\theta F + \partial_{\bar\theta}\bar F\right).\end{aligned} \tag{8}$$

In a similar fashion, one can also expand the r.h.s. of (5) as follows:

$$\begin{aligned}&\left(\frac{dx^\mu\wedge dx^\nu}{2!}\right)\tilde{\mathscr{F}}_{\mu\nu} + \left(dx^\mu\wedge d\theta\right)\tilde{\mathscr{F}}_{\mu\theta} \\ &\quad + \left(dx^\mu\wedge d\bar\theta\right)\tilde{\mathscr{F}}_{\mu\bar\theta} + \left(\frac{d\theta\wedge d\theta}{2!}\right)\tilde{\mathscr{F}}_{\theta\theta} \\ &\quad + \left(\frac{d\bar\theta\wedge d\bar\theta}{2!}\right)\tilde{\mathscr{F}}_{\bar\theta\bar\theta} + \left(d\theta\wedge d\bar\theta\right)\tilde{\mathscr{F}}_{\theta\bar\theta}.\end{aligned} \tag{9}$$

The HC requires that $F^{(2)} = [dx^\mu\wedge dx^\nu/2!]F_{\mu\nu}$ be equal to $\tilde{\mathscr{F}}^{(2)} = [dZ^M\wedge dZ^N/2!]\mathscr{F}_{MN}$. This implies that $\tilde{\mathscr{F}}_{\mu\theta} = \tilde{\mathscr{F}}_{\mu\bar\theta} = \tilde{\mathscr{F}}_{\theta\theta} = \tilde{\mathscr{F}}_{\theta\bar\theta} = \tilde{\mathscr{F}}_{\bar\theta\bar\theta} = 0$.

Written in an explicit form, we have the following relationships (from the comparison between (8) and (9)) due to the celebrated HC; namely,

$$\begin{aligned}\tilde{\mathscr{F}}_{\mu\theta} &= \partial_\mu\bar F - \partial_\theta\mathscr{B}_\mu, \\ \tilde{\mathscr{F}}_{\mu\bar\theta} &= \partial_\mu F - \partial_{\bar\theta}\mathscr{B}_\mu, \\ \frac{1}{2!}\tilde{\mathscr{F}}_{\theta\theta} &= -\partial_\theta\bar F, \\ \frac{1}{2!}\tilde{\mathscr{F}}_{\bar\theta\bar\theta} &= -\partial_{\bar\theta}F, \\ \tilde{\mathscr{F}}_{\theta\bar\theta} &= -\left(\partial_\theta F + \partial_{\bar\theta}\bar F\right), \\ \tilde{\mathscr{F}}_{\mu\nu} &\equiv \left(\partial_\mu\mathscr{B}_\nu - \partial_\nu\mathscr{B}_\mu\right) \Longrightarrow \\ F_{\mu\nu} &\equiv \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right).\end{aligned} \tag{10}$$

The substitution of the expansions (7) into the above equation yields the following relationships amongst the secondary fields and the basic as well as auxiliary fields of the ordinary 2D theory (our method of derivation is somewhat different from the original work of Bonora-Tonin superfield formalism [21, 22] even though our present work is motivated by the latter works (i.e., [21, 22])); namely,

$$\begin{aligned}R_\mu^{(2)} &= \partial_\mu C, \\ R_\mu^{(1)} &= \partial_\mu\bar C, \\ S_\mu &= \partial_\mu B_4 \equiv -\partial_\mu B_1, \\ B_2 &= B_3 = 0, \\ s &= \bar s = 0, \\ B_4 + B_1 &= 0.\end{aligned} \tag{11}$$

The last entry in the above is nothing but the celebrated Curci-Ferrari condition [31] which turns out to be trivial in the case of Abelian 1-form modified Proca gauge theory. Taking the help of (11), we have the following expansions (if we choose $B_4 = B$, $B_1 = -B$); namely,

$$\begin{aligned}\mathscr{B}_\mu^{(h)}\left(x,\theta,\bar\theta\right) &= A_\mu(x) + \theta\left(\partial_\mu\bar C\right) + \bar\theta\left(\partial_\mu C\right) \\ &\quad + \theta\bar\theta\left(i\partial_\mu B\right) \\ &\equiv A_\mu(x) + \theta\left(s_{ab}A_\mu\right) + \bar\theta\left(s_b A_\mu\right) \\ &\quad + \theta\bar\theta\left(s_b s_{ab}A_\mu\right), \\ F^{(h)}\left(x,\theta,\bar\theta\right) &= C(x) + \theta(-iB) + \bar\theta(0) + \theta\bar\theta(0) \\ &\equiv C + \theta\left(s_{ab}C\right) + \bar\theta\left(s_b C\right) \\ &\quad + \theta\bar\theta\left(s_b s_{ab}C\right),\end{aligned}$$

$$\overline{F}^{(h)}\left(x,\theta,\overline{\theta}\right) = \overline{C}(x) + \theta(0) + \overline{\theta}(iB) + \theta\overline{\theta}(0) \equiv \overline{C} + \theta\left(s_{ab}\overline{C}\right) + \overline{\theta}\left(s_b\overline{C}\right) + \theta\overline{\theta}\left(s_b s_{ab}\overline{C}\right), \quad (12)$$

which yields the following off-shell nilpotent (anti-)BRST symmetries for the gauge (A_μ) and Faddeev-Popov (FP) (anti-)ghost fields $(\overline{C})C$ of the theory:

$$\begin{aligned} s_b A_\mu &= \partial_\mu C, \\ s_b C &= 0, \\ s_b \overline{C} &= iB, \\ s_b B &= 0, \\ s_{ab} A_\mu &= \partial_\mu \overline{C}, \\ s_{ab}\overline{C} &= 0, \\ s_{ab} C &= -iB, \\ s_{ab} B &= 0. \end{aligned} \quad (13)$$

A few noteworthy points, at this stage, are as follows. First, the superscript (h) on the superfields in (12) denotes the expansion of the superfields after the application of HC. Second, the transformations $s_{(a)b}B = 0$ on the Nakanishi-Lautrup auxiliary fields B have been derived from the nilpotency condition. Third, it can be checked that the last entry of (10) is satisfied: $\widetilde{\mathscr{F}}^{(h)}_{\mu\nu} = \partial_\mu \mathscr{B}^{(h)}_\nu - \partial_\nu \mathscr{B}^{(h)}_\mu \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$, due to expansions in (12). Finally, we have the following mappings (see, e.g., [25–28] for details):

$$\begin{aligned} s_b &\longleftrightarrow \lim_{\theta=0}\frac{\partial}{\partial\overline{\theta}}, \\ s_{ab} &\longleftrightarrow \lim_{\overline{\theta}=0}\frac{\partial}{\partial\theta}, \\ s_b s_{ab} &\longleftrightarrow \frac{\partial}{\partial\overline{\theta}}\frac{\partial}{\partial\theta}. \end{aligned} \quad (14)$$

Thus, we note that the Grassmannian translation generators $(\partial_\theta, \partial_{\overline{\theta}})$ are connected with the off-shell nilpotent ($s^2_{(a)b} = 0$) and absolutely anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) fermionic (anti-)BRST symmetry transformations $s_{(a)b}$. These properties have their origin in the properties $\partial^2_\theta = 0$, $\partial^2_{\overline{\theta}} = 0$, $\partial_\theta\partial_{\overline{\theta}} + \partial_{\overline{\theta}}\partial_\theta = 0$ of the Grassmannian translation generators $(\partial_\theta, \partial_{\overline{\theta}})$ when the above relations are taken in their operator form.

Truly speaking, the exact relationship between the (anti-)BRST symmetry transformations $s_{(a)b}$ and Grassmannian translational generators $(\partial_\theta, \partial_{\overline{\theta}})$ is $s_b M(x) = [(\partial/\partial\overline{\theta})\widetilde{M}^{(h)}(x,\theta,\overline{\theta})]|_{\theta=0}$ for the BRST transformations and $s_{ab} M(x) = [(\partial/\partial\theta)\widetilde{M}^{(h)}(x,\theta,\overline{\theta})]|_{\overline{\theta}=0}$ for the anti-BRST symmetry transformations, where $M(x)$ is the D-dimensional ordinary field and $\widetilde{M}^{(h)}(x,\theta,\overline{\theta})$ is the corresponding superfield (obtained after the application of the HC). However, we will continue with the mapping (14) but will keep in mind that the precise connection between the (anti-)BRST transformations $s_{(a)b}$ and $(\partial_\theta, \partial_{\overline{\theta}})$ is $s_b \leftrightarrow \partial_{\overline{\theta}}$ and $s_{ab} \leftrightarrow \partial_\theta$.

Now we exploit the strength of the augmented version of superfield formalism [25–28] to derive the (anti-)BRST symmetry transformations for the real scalar field ϕ. To this end in mind, we recall that the quantity $(A_\mu \mp (1/m)\partial_\mu\phi)$ is a *gauge invariant* quantity (cf. Section 2). Thus, this physical quantity should remain unaffected by the presence of the Grassmannian variables $(\theta, \overline{\theta})$ when it is generalized onto a $(D, 2)$-dimensional supermanifold. In other words, in the language of differential geometry, the following is true:

$$\tilde{d}\widetilde{A}^{(1)}_{(h)}\left(x,\theta,\overline{\theta}\right) \mp \frac{1}{m}\tilde{d}\Phi\left(x,\theta,\overline{\theta}\right) = dA^{(1)}(x) \mp \frac{1}{m} d\phi(x). \quad (15)$$

Here $\widetilde{A}^{(1)}_{(h)} = dx^\mu \mathscr{B}^{(h)}_\mu + d\theta \overline{F}^{(h)} + d\overline{\theta} F^{(h)}$ [cf. (12)] and the zero-form superfield $\Phi(x,\theta,\overline{\theta})$ has the following superexpansion along the Grassmannian directions $(\theta,\overline{\theta})$ of the $(D, 2)$-dimensional supermanifold; namely,

$$\Phi\left(x,\theta,\overline{\theta}\right) = \phi(x) + \theta\overline{f}(x) + \overline{\theta} f(x) + i\theta\overline{\theta} b(x). \quad (16)$$

In the above, it is evident that the pair of secondary fields $(f(x), \overline{f}(x))$ is fermionic and $(\phi(x), b(x))$ are bosonic in nature. In the limit $(\theta, \overline{\theta}) = 0$, we retrieve back our real scalar Stueckelberg field $\phi(x)$ of the original D-dimensional ordinary theory.

The gauge invariant restriction (GIR) in (15) leads to the following relationships:

$$\begin{aligned} \overline{f} &= \pm m\overline{C}, \\ f &= \pm mC, \\ b &= \pm mB. \end{aligned} \quad (17)$$

The substitution of (17) into (16) yields

$$\Phi^{(g)}\left(x,\theta,\overline{\theta}\right) = \phi(x) + \theta\left(\pm m\overline{C}\right) + \overline{\theta}(\pm mC) + \theta\overline{\theta}(\pm imB), \equiv \phi(x) + \theta\left(s_{ab}\phi\right) + \overline{\theta}\left(s_b\phi\right) + \theta\overline{\theta}\left(s_b s_{ab}\phi\right), \quad (18)$$

where the superscript (g) on the superfield $\Phi(x,\theta,\overline{\theta})$ corresponds to the superexpansion of this superfield after the application of GIR. It is clear, from the above equation, that we have the following:

$$\begin{aligned} s_b\phi &= \pm mC, \\ s_{ab}\phi &= \pm m\overline{C}, \\ s_b s_{ab}\phi &= \pm imB. \end{aligned} \quad (19)$$

We note that, ultimately, it is the combination of HC and GIR which leads to the derivation of full set of correct off-shell nilpotent ($s^2_{(a)b} = 0$) and absolutely anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-)BRST transformations for *all* the fields of the D-dimensional modified version of Proca theory.

3.2. Lagrangian Densities: (Anti-)BRST Invariance. Exploiting the full set of (anti-)BRST symmetry transformations, we can derive the (anti-)BRST invariant Lagrangian densities (which incorporate the gauge-fixing and Faddeev-Popov ghost terms) by exploiting the standard techniques of BRST approach; namely,

$$\begin{aligned}\mathscr{L}_B &= \mathscr{L}_s + s_b s_{ab}\left[\frac{i}{2}A_\mu A^\mu - \frac{i}{2}\phi^2 + \frac{1}{2}C\overline{C}\right] \\ &= \mathscr{L}_s + s_b\left[i\left(A_\mu\partial^\mu\overline{C} \mp m\phi\overline{C} - \frac{1}{2}B\overline{C}\right)\right] \\ &= \mathscr{L}_s + s_{ab}\left[-i\left(A_\mu\partial^\mu C \mp m\phi C - \frac{1}{2}BC\right)\right].\end{aligned} \tag{20}$$

In explicit form, the total (anti-)BRST invariant Lagrangian densities (containing two signatures) look as the following form:

$$\begin{aligned}\mathscr{L}_B = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \mp mA_\mu\partial^\mu\phi \\ &+ B\left(\partial\cdot A \pm m\phi\right) + \frac{1}{2}B^2 - i\partial_\mu\overline{C}\partial^\mu C + im^2\overline{C}C.\end{aligned} \tag{21}$$

Using the full set of (anti-)BRST symmetries (13) and (19), we can check that the above Lagrangian densities transform to the total spacetime derivatives as

$$\begin{aligned}s_b\mathscr{L}_B &= \partial_\mu\left[B\partial^\mu C\right], \\ s_{ab}\mathscr{L}_B &= \partial_\mu\left[B\partial^\mu\overline{C}\right].\end{aligned} \tag{22}$$

As a consequence, the action integral $S = \int d^{D-1}x\mathscr{L}_B$ remains invariant for the physically well-defined fields of the theory. The above infinitesimal and continuous symmetry transformations, according to Noether's theorem, lead to the following expressions for the (anti-)BRST charges $Q_{(a)b}$:

$$\begin{aligned}Q_{ab} &= \int d^{D-1}x\left[B\dot{\overline{C}} - \dot{B}\overline{C}\right], \\ Q_b &= \int d^{D-1}x\left[B\dot{C} - \dot{B}C\right],\end{aligned} \tag{23}$$

which are found to be conserved ($\dot{Q}_{(a)b} = 0$) and nilpotent ($Q^2_{(a)b} = 0$) of order two. These charges are the generators of transformations listed in (13) and (19) and they are derived from the following Noether conserved currents:

$$\begin{aligned}J^\mu_b &= -F^{\mu\nu}\left(\partial_\nu C\right) + B\left(\partial^\mu C\right) + mC\left(\partial^\mu\phi - mA^\mu\right), \\ J^\mu_{ab} &= -F^{\mu\nu}\left(\partial_\nu\overline{C}\right) + B\left(\partial^\mu\overline{C}\right) + m\overline{C}\left(\partial^\mu\phi - mA^\mu\right).\end{aligned} \tag{24}$$

In the proof of the conservation laws ($\partial_\mu J^\mu_{(a)b} = 0$), we have to use the following Euler-Lagrange (E-L) equations of motion:

$$\begin{aligned}&\left(\Box + m^2\right)C = 0, \\ &\left(\Box + m^2\right)A_\mu - \partial_\mu\left(\partial\cdot A \pm m\phi + B\right) = 0, \\ &\left(\Box + m^2\right)\overline{C} = 0, \\ &\Box\phi - m\left(\partial\cdot A + B\right) = 0, \\ &B = -\left(\partial\cdot A \pm m\phi\right),\end{aligned} \tag{25}$$

which emerge from the Lagrangian densities (21).

The (anti-)BRST invariance of the Lagrangian density (21) can be also captured in the language of superfield formalism. To this end in mind, first of all, we note that the Stueckelberg Lagrangian density $\mathscr{L}_s$ [cf. (3)] can be written as

$$\begin{aligned}\mathscr{L}_s &\longrightarrow \widetilde{\mathscr{L}}_s \\ &= -\frac{1}{4}\widetilde{\mathscr{F}}^{(h)}_{\mu\nu}\widetilde{\mathscr{F}}^{\mu\nu(h)} + \frac{m^2}{2}\mathscr{B}^{(h)}_\mu\mathscr{B}^{\mu(h)} \\ &\quad + \frac{1}{2}\partial_\mu\Phi^{(g)}\partial^\mu\Phi^{(g)} \mp m\mathscr{B}^{(h)}_\mu\partial^\mu\Phi^{(g)},\end{aligned} \tag{26}$$

within the framework of superfield formalism, where the superfields are obtained after the applications of HC and GIR [cf. (12) and (18)]. It is straightforward to check that the following is true; namely,

$$\begin{aligned}&\lim_{\theta=0}\frac{\partial}{\partial\bar{\theta}}\widetilde{\mathscr{L}}_s = 0, \\ &\lim_{\bar{\theta}=0}\frac{\partial}{\partial\theta}\widetilde{\mathscr{L}}_s = 0, \\ &\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\widetilde{\mathscr{L}}_s = 0.\end{aligned} \tag{27}$$

In view of the mappings (14), it is evident that the Stueckelberg Lagrangian densities are (anti-)BRST invariant (i.e., $s_b\mathscr{L}_s = 0$, $s_{ab}\mathscr{L}_s = 0$, and $s_b s_{ab}\mathscr{L}_s = 0$).

Exploiting the techniques of superfield formalism, the full (anti-)BRST invariant Lagrangian densities (21) (which incorporate the gauge-fixing and Faddev-Popov ghost terms) can be expressed in *three* different ways (modulo a total spacetime derivative); namely,

$$\begin{aligned}\widetilde{\mathscr{L}}_B &= \widetilde{\mathscr{L}}_s + \frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\left[\frac{i}{2}\mathscr{B}^{(h)}_\mu\mathscr{B}^{\mu(h)} - \frac{i}{2}\left(\Phi^{(g)}\Phi^{(g)}\right)\right. \\ &\quad\left. + \frac{1}{2}\left(F^{(h)}\overline{F}^{(h)}\right)\right], \\ &\equiv \widetilde{\mathscr{L}}_s + \lim_{\theta=0}\frac{\partial}{\partial\bar{\theta}}\left[i\left\{\mathscr{B}^{(h)}_\mu\partial^\mu\overline{F}^{(h)} \mp m\left(\Phi^{(g)}\overline{F}^{(h)}\right)\right.\right. \\ &\quad\left.\left. - \frac{1}{2}\left(B(x)\overline{F}^{(h)}\right)\right\}\right],\end{aligned}$$

$$\equiv \widetilde{\mathscr{L}}_s + \lim_{\bar{\theta}=0} \frac{\partial}{\partial\theta} \left[-i \left\{ \mathscr{B}^{(h)}_\mu \partial^\mu F^{(h)} \mp m \left(\Phi^{(g)} F^{(h)} \right) - \frac{1}{2} \left(B(x) F^{(h)} \right) \right\} \right]. \quad (28)$$

Taking into account the nilpotency ($\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$) and anticommutativity ($\partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0$) property of the generator (∂_θ, $\partial_{\bar{\theta}}$), it is straightforward to prove that

$$\begin{aligned}
\lim_{\bar{\theta}=0} \frac{\partial}{\partial\theta} \widetilde{\mathscr{L}}_B = 0 &\longleftrightarrow s_{ab} \mathscr{L}_B = 0, \\
\lim_{\theta=0} \frac{\partial}{\partial\bar{\theta}} \widetilde{\mathscr{L}}_B = 0 &\longleftrightarrow s_b \mathscr{L}_B = 0, \\
\frac{\partial}{\partial\bar{\theta}} \frac{\partial}{\partial\theta} \widetilde{\mathscr{L}}_B = 0 &\longleftrightarrow s_b s_{ab} \mathscr{L}_B = 0,
\end{aligned} \quad (29)$$

where the mappings (14) and results from (27) have been taken into consideration. We would like to lay emphasis on the fact that there is *no* contradiction amongst (20), (22), (28), and (29). This is due to the observation that, in reality, we have $(1/2) s_b s_{ab} [i A_\mu A^\mu - i\phi^2 + C\bar{C}] = -\partial_\mu (A^\mu B) + B(\partial \cdot A \pm m\phi) + (B^2/2) - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C$. However, we have thrown away the total spacetime derivative term from the Lagrangian density (21). If we keep this term in (21), then we have $s_{(a)b} \mathscr{L}_B = 0$ instead of the expressions in (22). Thus, there is no inconsistency anywhere.

3.3. Conserved Charges: Superfield Approach. We can also express the (anti-)BRST charges in terms of superfields (obtained after the application of HC and GIR), the Grassmannian partial derivatives ($\partial_\theta, \partial_{\bar{\theta}}$), and differentials ($d\theta, d\bar{\theta}$). For instance, we note that the following expression for the BRST charge is true; namely,

$$\begin{aligned}
Q_b &= \frac{\partial}{\partial\bar{\theta}} \frac{\partial}{\partial\theta} \left(\int d^{D-1}x \left[i F^{(h)} \mathscr{B}^{(h)}_0 \right] \right) \\
&= \int d^{D-1}x \int d\bar{\theta} \int d\theta \left[i F^{(h)} \mathscr{B}^{(h)}_0 \right],
\end{aligned} \quad (30)$$

in the language of superfields (after the application of HC). It is clear, from the mappings (14), that the above expression implies

$$Q_b = \int d^{D-1}x \left[s_b s_{ab} \left(i C A_0 \right) \right], \quad (31)$$

in the ordinary D-dimensions of spacetime where the (anti-)BRST transformations $s_{(a)b}$ and ordinary fields are defined. The proof of the nilpotency of the BRST charge becomes quite simple now due to the nilpotency ($s_b^2 = 0$) of s_b and that of the translation generator $\partial_{\bar{\theta}}$ (because $\partial_{\bar{\theta}}^2 = 0$). In exactly similar fashion, we can express the anti-BRST charge Q_{ab}, within the framework of superfield formalism, as

$$\begin{aligned}
Q_{ab} &= \frac{\partial}{\partial\theta} \frac{\partial}{\partial\bar{\theta}} \left(\int d^{D-1}x \left[-i \bar{F}^{(h)} \mathscr{B}^{(h)}_0 \right] \right) \\
&= \int d^{D-1}x \int d\theta \int d\bar{\theta} \left[-i \bar{F}^{(h)} \mathscr{B}^{(h)}_0 \right].
\end{aligned} \quad (32)$$

Once again, the proof of nilpotency of the anti-BRST charge Q_{ab} becomes pretty simple because of the fact that, in the ordinary D-dimensional spacetime, expression (32) can be written in the following form by exploiting the mappings (14); namely,

$$Q_{ab} = \int d^{D-1}x \left[s_{ab} s_b \left(-i \bar{C} A_0 \right) \right], \quad (33)$$

where $s_{ab}^2 = 0$ implies that $Q_{ab}^2 = 0$ (due to $s_{ab} Q_{ab} = i\{Q_{ab}, Q_{ab}\} = 0$). Within the framework of superfield formalism, the nilpotency of Q_{ab} is encoded in the nilpotency of ∂_θ (because $\partial_\theta^2 = 0$). In other words, we can explicitly verify the nilpotency of the conserved (anti-)BRST charges in terms of the translational generators along the Grassmannian directions $(\theta, \bar{\theta})$ of the $(2, 2)$-dimensional supermanifold as $\lim_{\theta=0} (\partial/\partial\bar{\theta}) Q_b = 0$, $\lim_{\bar{\theta}=0} (\partial/\partial\theta) Q_{ab} = 0$ because $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$.

There are other alternative forms of the conserved and nilpotent (anti-)BRST charges, within the framework of the superfield formalism, that are *also* interesting. For instance, it can be checked that the anti-BRST charge can be expressed as:

$$\begin{aligned}
Q_{ab} &= \int d^{D-1}x \int d\theta \left[B(x) \mathscr{B}^{(h)}_0 \left(x, \theta, \bar{\theta} \right) + i \bar{F}^{(h)} \left(x, \theta, \bar{\theta} \right) \dot{F}^{(h)} \left(x, \theta, \bar{\theta} \right) \right] \\
&= \lim_{\bar{\theta}=0} \frac{\partial}{\partial\theta} \int d^{D-1}x \left[B(x) \mathscr{B}^{(h)}_0 \left(x, \theta, \bar{\theta} \right) + i \bar{F}^{(h)} \left(x, \theta, \bar{\theta} \right) \dot{F}^{(h)} \left(x, \theta, \bar{\theta} \right) \right] \\
&\equiv \int d^{D-1}x \left[s_{ab} \left(B A_0 + i \bar{C} \dot{C} \right) \right].
\end{aligned} \quad (34)$$

In exactly similar fashion, we can express the conserved BRST charge as

$$\begin{aligned}
Q_b &= \int d^{D-1}x \int d\bar{\theta} \left[B(x) \mathscr{B}^{(h)}_0 \left(x, \theta, \bar{\theta} \right) - i F^{(h)} \left(x, \theta, \bar{\theta} \right) \dot{\bar{F}}^{(h)} \left(x, \theta, \bar{\theta} \right) \right], \\
&= \lim_{\theta=0} \frac{\partial}{\partial\bar{\theta}} \int d^{D-1}x \left[B(x) \mathscr{B}^{(h)}_0 \left(x, \theta, \bar{\theta} \right) - i F^{(h)} \left(x, \theta, \bar{\theta} \right) \dot{\bar{F}}^{(h)} \left(x, \theta, \bar{\theta} \right) \right] \\
&\equiv \int d^{D-1}x \left[s_b \left(B A_0 - i C \dot{\bar{C}} \right) \right].
\end{aligned} \quad (35)$$

The nilpotency ($Q^2_{(a)b} = 0$) of the (anti-)BRST charges $Q_{(a)b}$ is encoded in the observation that the following are true; namely,

$$\lim_{\bar\theta=0} \frac{\partial}{\partial\theta} Q_{ab} = 0, \qquad \lim_{\theta=0} \frac{\partial}{\partial\bar\theta} Q_b = 0, \tag{36}$$

where the nilpotency ($\partial^2_\theta = \partial^2_{\bar\theta} = 0$) of ∂_θ and $\partial_{\bar\theta}$ plays an important role.

We close this subsection with the remark that the following observations in the context of expressions for the (anti-)BRST charges:

$$Q_{ab} = \int d^{D-1}x \left[s_b \left(-i\bar C \dot{\bar C} \right) \right], \qquad Q_b = \int d^{D-1}x \left[s_{ab} \left(iC\dot C \right) \right], \tag{37}$$

lead to the proof of absolute anticommutativity of the (anti-)BRST charges because it can be readily checked that the following are true; namely,

$$s_b Q_{ab} = i\{Q_{ab}, Q_b\} = \int d^{D-1}x \left[s_b^2 \left(-i\bar C \dot{\bar C} \right) \right] = 0, \quad \left(s_b^2 = 0 \right),$$
$$s_{ab} Q_b = i\{Q_b, Q_{ab}\} = \int d^{D-1}x \left[s_{ab}^2 \left(iC\dot C \right) \right] = 0, \quad \left(s_{ab}^2 = 0 \right), \tag{38}$$

because of the nilpotency of (anti-)BRST transformations $s_{(a)b}$. These observations can also be captured in the language of the superfield formalism; namely,

$$Q_{ab} = \lim_{\theta=0} \frac{\partial}{\partial\bar\theta} \int d^{D-1}x \left[-i\bar F^{(h)} \dot{\bar F}^{(h)} \right], \qquad Q_b = \lim_{\bar\theta=0} \frac{\partial}{\partial\theta} \int d^{D-1}x \left[iF^{(h)} \dot F^{(h)} \right]. \tag{39}$$

The above expressions imply the following:

$$\lim_{\theta=0} \frac{\partial}{\partial\bar\theta} Q_{ab} = 0, \qquad \lim_{\bar\theta=0} \frac{\partial}{\partial\theta} Q_b = 0. \tag{40}$$

A close look at (38), (39), and (40) shows that the nilpotency and anticommutativity property are interrelated. In other words, the properties $\partial^2_\theta = \partial^2_{\bar\theta} = 0$ and $\partial_\theta\partial_{\bar\theta} + \partial_{\bar\theta}\partial_\theta = 0$ are interconnected. For instance, in the latter relation when we set $\partial_\theta = \partial_{\bar\theta}$, we obtain the former relation $\partial^2_\theta = \partial^2_{\bar\theta} = 0$ which actually provides the connection between the properties of anticommutativity and nilpotency associated with the (anti-)BRST symmetry transformations ($s_{(a)b}$).

We wish to make a final remark that it is the strength of the superfield approach to BRST formalism that we have obtained various expressions for the (anti-)BRST charges in the language of (anti-)BRST symmetry transformations. Some of the results are completely *novel* as, to the best of our knowledge, these expressions have not been pointed out in the literature. In fact, these new expressions are responsible, with the help of mapping in (14), to establish the nilpotency and absolute anticommutativity of the (anti-)BRST symmetries (and corresponding charges) within the framework of superfield formalism. For instance, the relationships, given in (38), demonstrate that the nilpotency property and absolute anticommutativity property (of $s_{(a)b}$ and $Q_{(a)b}$) are intertwined together.

4. (Anti-)co-BRST Symmetries: Superfield Approach

In this section, first of all, we discuss the dual-gauge transformations for the gauge-fixed Lagrangian densities and show that a particular kind of restriction must be imposed on the dual-gauge parameter if we wish to maintain the dual-gauge symmetry in the theory. Then, we derive the off-shell nilpotent and absolutely anticommuting (anti-)co-BRST symmetry transformations by exploiting the strength of dual-HC (DHC) and dual-GIR (DGIR). After this, we prove the (anti-)co-BRST invariance of the Lagrangian densities within the framework of superfield formalism. Finally, we capture the (anti-)co-BRST invariance of the conserved charges, their nilpotency, and absolute anticommutativity within the ambit of the augmented version of superfield approach to BRST formalism.

4.1. Dual-Gauge Transformations for the Gauge-Fixed Lagrangian Densities in Two-Dimensions of Spacetime. Analogous to the infinitesimal, continuous, and local gauge symmetry transformations (4), we wish to discuss, in this subsection, the dual-gauge transformations which would be, finally, generalized to the (anti-)co-BRST symmetry transformations (in the two (1 + 1)-dimensions of spacetime, a particular part of the Lagrangian density [i.e., $-(1/4)F^{\mu\nu}F_{\mu\nu}$] becomes $[(1/2)E^2]$ because there is only one nonvanishing component of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which is $F_{01} = -\varepsilon^{\mu\nu}\partial_\mu A_\nu = E$; this is a pseudoscalar field because it changes sign under the operation of parity transformation and it has only one existing component). In their most general form, the two (1 + 1)-dimensional (2D) gauge-fixed Lagrangian densities for the modified Proca theory (without the fermionic (anti-)ghost fields) are as follows (for the 2D theory, we adopt the convention and notations such that the *background* flat Minkowskian spacetime manifold is endowed with a metric $\eta_{\mu\nu}$ with signatures (+1, −1) so that $P \cdot Q = \eta_{\mu\nu}P^\mu Q^\nu = P_0 Q_0 - P_1 Q_1$ is the dot product between two nonnull 2D vectors P_μ and Q_μ; we also choose the antisymmetric Levi-Civita tensor $\varepsilon_{\mu\nu}$ such that $\varepsilon_{01} = +1 = \varepsilon^{10}$, $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = -2!$, $\varepsilon_{\mu\nu}\varepsilon^{\nu\lambda} = \delta^\lambda_\mu$, etc.) (see, e.g., [3]):

$$\mathscr{L}_{(b_1)} = \mathscr{B}\left(E - m\tilde\phi \right) - \frac{1}{2}\mathscr{B}^2 + mE\tilde\phi - \frac{1}{2}\partial_\mu\tilde\phi\partial^\mu\tilde\phi + \frac{1}{2}m^2 A_\mu A^\mu - mA_\mu\partial^\mu\phi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

$$+ B\left(\partial \cdot A + m\phi\right) + \frac{1}{2}B^2,$$

$$\mathscr{L}_{(b_2)} = \overline{\mathscr{B}}\left(E + m\widetilde{\phi}\right) - \frac{1}{2}\overline{\mathscr{B}}^2 - mE\widetilde{\phi} - \frac{1}{2}\partial_\mu\widetilde{\phi}\partial^\mu\widetilde{\phi}$$
$$+ \frac{1}{2}m^2 A_\mu A^\mu + mA_\mu\partial^\mu\phi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$
$$+ \overline{B}\left(\partial \cdot A - m\phi\right) + \frac{1}{2}\overline{B}^2, \tag{41}$$

where $(B, \overline{B}, \mathscr{B}, \overline{\mathscr{B}})$ are the Nakanishi-Lautrup type auxiliary fields and $\widetilde{\phi}$ is a pseudoscalar field that has been incorporated in the theory on mathematical as well as physical grounds [2, 3]. It will be noted that the (pseudo-)scalar fields $(\widetilde{\phi})\phi$ have been added/subtracted in a symmetrical fashion to the kinetic and gauge-fixing terms, respectively, so that we would have appropriate discrete symmetry transformations in the theory [cf. (89) below].

Let us discuss the dual-gauge transformations $\delta_{dg}^{(1,2)}$:

$$\delta_{dg}^{(1,2)} A_\mu = -\varepsilon_{\mu\nu}\partial^\nu\Sigma,$$
$$\delta_{dg}^{(1,2)}\left(\partial \cdot A \pm m\phi\right) = 0,$$
$$\delta_{dg}^{(1,2)}\phi = 0,$$
$$\delta_{dg}^{(1,2)}\widetilde{\phi} = \mp m\Sigma, \tag{42}$$
$$\delta_{dg}^{(1,2)} E = \Box\Sigma,$$
$$\delta_{dg}^{(1,2)}\left[B, \overline{B}, \mathscr{B}, \overline{\mathscr{B}}\right] = 0,$$

where $\Box = \partial_0^2 - \partial_1^2$ is the d'Alembertian operator, $\Sigma(x)$ is the local and infinitesimal dual-gauge parameter, and the superscripts $(1,2)$ denote the dual-gauge transformations for the Lagrangian densities $\mathscr{L}_{(b_1)}$ and $\mathscr{L}_{(b_2)}$, respectively. We note that the Lagrangian densities $\mathscr{L}_{(b_1,b_2)}$ transform, under the above dual-gauge transformations $\delta_{dg}^{(1,2)}$, as follows:

$$\delta_{dg}^{(1)}\mathscr{L}_{(b_1)} = \partial_\mu\left[m\varepsilon^{\mu\nu}\left(mA_\nu\Sigma + \phi\partial_\nu\Sigma\right) + m\widetilde{\phi}\partial^\mu\Sigma\right]$$
$$+ \mathscr{B}\left(\Box + m^2\right)\Sigma,$$
$$\delta_{dg}^{(2)}\mathscr{L}_{(b_2)} = \partial_\mu\left[m\varepsilon^{\mu\nu}\left(mA_\nu\Sigma - \phi\partial_\nu\Sigma\right) - m\widetilde{\phi}\partial^\mu\Sigma\right] \tag{43}$$
$$+ \overline{\mathscr{B}}\left(\Box + m^2\right)\Sigma.$$

Thus, it is clear that, to maintain the dual-gauge symmetries in the 2D gauge-fixed theory, we have to impose the condition $(\Box + m^2)\Sigma(x) = 0$, from outside, on the dual-gauge parameter $\Sigma(x)$. We note that the operation of coexterior derivative $\delta = -*d*$ on the connection 1-form $(A^{(1)} = dx^\mu A_\mu)$ yields $(\partial \cdot A)$ which is a zero form. We can add/subtract a scalar field ϕ to it as is the case with the gauge-fixing terms $(\partial \cdot A \pm m\phi)$ that have been incorporated in $\mathscr{L}_{(b_1,b_2)}$. This scalar field ϕ is nothing but the Stueckelberg field.

A few noteworthy points, at this juncture, are as follows. First, we point out that the nomenclature of the dual-gauge symmetry is appropriate because we have $\delta_{dg}^{(1,2)}(\partial \cdot A \pm m\phi) = 0$. In other words, the total gauge-fixing terms $(\partial \cdot A \pm m\phi)$, owing their *fundamental* origin to the dual-exterior derivative, remain invariant. Second, the dual-gauge parameter has to be restricted by $(\Box + m^2)\Sigma = 0$ to maintain the dual-gauge symmetry in the theory. One can take care of this restriction by introducing the (anti-)ghost fields $(\overline{C})C$ within the framework of BRST formalism as we will see in our Section 4.3. Third, the dual-gauge symmetry transformations exist only in the specific two $(1 + 1)$-dimensions of spacetime for the Abelian 1-form gauge theory, whereas the local gauge and (anti-)BRST symmetries exist in any arbitrary dimension of spacetime. Fourth, the *perfect* analogue of the gauge symmetry [cf. (4)] does *not* exist for the dual-gauge symmetry (because we have to impose the restriction $(\Box + m^2)\Sigma = 0$ from outside). Finally, in the forthcoming sections, we will see that one can have *perfect* (anti-)dual-BRST (or (anti-)co-BRST) symmetries in the theory, where Σ will be replaced by the (anti-)ghost fields $(\overline{C})C$ (without any ad hoc restrictions from outside).

We claim that there would *not* be any restrictions on anything (from outside) when we will discuss the full (anti-)BRST and (anti-)co-BRST invariant Lagrangian densities of our present theory.

4.2. Nilpotent (Anti-)co-BRST Symmetry Transformations: Geometrical Superfield Technique. As prescribed by the superfield formalism, first of all, we generalize the 2D theory onto the $(2,2)$-dimensional supermanifold and promote the ordinary coexterior derivative $\delta = -*d*$ onto the same supermanifold, as

$$\delta = - * d * \Longrightarrow$$
$$\widetilde{\delta} = - \star \widetilde{d} \star, \tag{44}$$

where the $(\star)$ operator is the Hodge duality operation, defined on the $(2,2)$-dimensional supermanifold. The working rule for the operation of $(\star)$ has been worked out in our earlier paper [32] and we exploit these results in the following dual-HC (DHC):

$$\widetilde{\delta}\widetilde{A}^{(1)} = \delta A^{(1)},$$
$$\delta A^{(1)} = \left(\partial \cdot A\right), \tag{45}$$

where the l.h.s. is $(- \star \widetilde{d} \star \widetilde{A}^{(1)})$ and r.h.s. is obviously equal to the Lorentz condition for the gauge-fixing $(\partial \cdot A)$. The definition of $\widetilde{d}$ and $\widetilde{A}^{(1)}$ is quoted in (6) and the superexpansions of the superfields are listed in (7).

The explicit expression for the computation of the l.h.s. of the DHC, in (45), is as follows (we have performed explicit computation of $(- \star \widetilde{d} \star \widetilde{A}^{(1)})$ in our Appendix A and derived clearly (46) which plays an important role in our further discussions) (see, e.g., [32] for details):

$$\partial \cdot \mathscr{B} - \left(\partial_\theta \overline{F} + \partial_{\overline{\theta}} F\right) - S^{\theta\theta}\left(\partial_\theta F\right) - S^{\overline{\theta}\overline{\theta}}\left(\partial_{\overline{\theta}} \overline{F}\right), \tag{46}$$

where $(S^{\theta\theta}, S^{\bar{\theta}\bar{\theta}})$ coefficients, in the above, have turned up while taking the Hodge duality $(\star)$ operation on the following super 4-forms (defined on the $(2,2)$-dimensional supermanifold), while the computations of $d \star A^{(1)}$ are performed; namely,

$$\star \left(dx^\mu \wedge dx^\nu \wedge d\theta \wedge d\theta\right) = \varepsilon^{\mu\nu} S^{\theta\theta}, \\ \star \left(dx^\mu \wedge dx^\nu \wedge d\bar{\theta} \wedge d\bar{\theta}\right) = \varepsilon^{\mu\nu} S^{\bar{\theta}\bar{\theta}}. \tag{47}$$

It is to be noted that $(\tilde{d} \star \tilde{A}^{(1)})$ is a super 4-form on the $(2,2)$-dimensional supermanifold and when we perform another $(\star)$ operation on it, the differentials of (47) appear. In the above, $S^{\theta\theta}$ and $S^{\bar{\theta}\bar{\theta}}$ are symmetric in θ and $\bar{\theta}$ and all the other coefficients of the l.h.s. of (45) have been worked out in our earlier paper [32]. On the comparison of the l.h.s. and r.h.s. of (45), we obtain

$$\partial \cdot R^{(1)} = 0, \\ \partial \cdot R^{(2)} = 0, \\ \partial \cdot S = 0, \\ s = \bar{s} = 0, \\ B_1 = B_4 = 0, \\ B_2 + B_3 = 0. \tag{48}$$

It is clear that, unlike the HC where all the secondary fields of expansions (7) are exactly and uniquely determined, in the case of DHC, the secondary fields are not uniquely determined and there can be various (non-)local choices for the solution of (48) (see, e.g., [33] for details). Thus, we have the complete freedom to make the choices. Finally, we select (by exploiting the *augmented* version of superfield formalism, we have derived these *exact* expressions in our Appendix B; thus, results of (49) are mathematically precise and exact) the following *local* expressions for the solution of (48); namely,

$$R_\mu^{(1)} = -\varepsilon_{\mu\nu}\partial^\nu C, \\ R_\mu^{(2)} = -\varepsilon_{\mu\nu}\partial^\nu \bar{C}, \\ S_\mu = \varepsilon_{\mu\nu}\partial^\nu \mathscr{B}, \\ B_3 = -\mathscr{B}, \\ B_2 = \mathscr{B}, \tag{49}$$

which, unambiguously, satisfy $\partial \cdot R^{(1)} = \partial \cdot R^{(2)} = \partial \cdot S = 0$ and $B_2 + B_3 = 0$. From now on, we will focus only on the Lagrangian density $\mathscr{L}_{(b_1)}$ of (41) and its generalization to the (anti-)co-BRST invariant Lagrangian density (57) (see below). However, it is straightforward to make the local choices for the Lagrangian density $\mathscr{L}_{(b_2)}$, too. For instance, we can choose $R_\mu^{(1)} = -\varepsilon_{\mu\nu}\partial^\nu C$, $R_\mu^{(2)} = -\varepsilon_{\mu\nu}\partial^\nu \bar{C}$, $S_\mu = \varepsilon_{\mu\nu}\partial^\nu \overline{\mathscr{B}}$, $B_3 = -\overline{\mathscr{B}}$, and $B_2 = \overline{\mathscr{B}}$ for the (anti-)co-BRST invariant version of $\mathscr{L}_{(b_2)}$.

Ultimately, we obtain the following expansions for the superfields along the Grassmannian $(\theta, \bar{\theta})$-directions of the $(2,2)$-dimensional supermanifold after the application of DHC; namely,

$$\begin{aligned} \mathscr{B}_\mu^{(dh)}\left(x,\theta,\bar{\theta}\right) &= A_\mu + \theta\left(-\varepsilon_{\mu\nu}\partial^\nu C\right) + \bar{\theta}\left(-\varepsilon_{\mu\nu}\partial^\nu \bar{C}\right) \\ &\quad + \theta\bar{\theta}\left(i\varepsilon_{\mu\nu}\partial^\nu \mathscr{B}\right) \\ &\equiv A_\mu + \theta\left(s_{ad}A_\mu\right) + \bar{\theta}\left(s_d A_\mu\right) \\ &\quad + \theta\bar{\theta}\left(s_d s_{ad} A_\mu\right), \\ F^{(dh)}\left(x,\theta,\bar{\theta}\right) &= C + \theta(0) + \bar{\theta}(-i\mathscr{B}) + \theta\bar{\theta}(0) \\ &\equiv C + \theta\left(s_{ad}C\right) + \bar{\theta}\left(s_d C\right) \\ &\quad + \theta\bar{\theta}\left(s_d s_{ad} C\right), \\ \bar{F}^{(dh)}\left(x,\theta,\bar{\theta}\right) &= \bar{C} + \theta(i\mathscr{B}) + \bar{\theta}(0) + \theta\bar{\theta}(0) \\ &\equiv \bar{C} + \theta\left(s_{ad}\bar{C}\right) + \bar{\theta}\left(s_d \bar{C}\right) \\ &\quad + \theta\bar{\theta}\left(s_d s_{ad} \bar{C}\right), \end{aligned} \tag{50}$$

where the superscript (dh) denotes the expansions of the superfields after the application of DHC. A close look at the above expansions demonstrates that we have already obtained the (anti-)co-BRST symmetry transformations for the gauge field (A_μ) and corresponding (anti-)ghost fields $(\bar{C})C$. Physically, the DHC states that the dual-gauge invariant quantity [i.e., $\delta_{dg}^{(1,2)}(\partial \cdot A = 0)$], which is nothing but the Lorentz condition $(\partial \cdot A)$ for the gauge-fixing, does *not* depend on the Grassmannian variables θ and $\bar{\theta}$ in any form.

To obtain the (anti-)co-BRST symmetry transformations for the $\tilde{\phi}$ field, we exploit the strength of augmented superfield formalism where we demand that all the dual-gauge (or (anti-)co-BRST) invariant quantities should remain independent of the Grassmannian variables θ and $\bar{\theta}$. In this context, we observe that $\delta_{dg}^{(1)}[A_\mu(x) - (1/m)\varepsilon_{\mu\nu}\partial^\nu \tilde{\phi}(x)] = 0$ under the dual-gauge transformations (42). Thus, we demand the following dual-GIR on the superfields of the $(2,2)$-dimensional supermanifold; namely,

$$\mathscr{B}_\mu^{(dh)}\left(x,\theta,\bar{\theta}\right) - \frac{1}{m}\varepsilon_{\mu\nu}\partial^\nu \tilde{\Phi}\left(x,\theta,\bar{\theta}\right) \\ = A_\mu(x) - \frac{1}{m}\varepsilon_{\mu\nu}\partial^\nu \tilde{\phi}(x). \tag{51}$$

We note that the DGIR combines DHC and the dual-gauge invariance *together* in a fruitful fashion. Taking the help from (50) and using the following expansion for the superfield $\tilde{\Phi}(x,\theta,\bar{\theta})$ along the Grassmannian $(\theta,\bar{\theta})$-directions of the $(2,2)$-dimensional supermanifold:

$$\tilde{\Phi}\left(x,\theta,\bar{\theta}\right) = \tilde{\phi}(x) + \theta f_1(x) + \bar{\theta} f_2(x) + i\theta\bar{\theta} b_1(x), \tag{52}$$

we obtain the following results:

$$f_1(x) = (-mC),\quad f_2(x) = (-m\overline{C}),\quad b_1(x) = (m\mathscr{B}). \tag{53}$$

It is obvious, from the above, that $f_1(x)$ and $f_2(x)$ are fermionic in nature and $b_1(x)$ is bosonic. Plugging in the above values into (52), we deduce the following:

$$\begin{aligned}\widetilde{\Phi}^{(dg)}(x,\theta,\bar{\theta}) &= \tilde{\phi}(x) + \theta(-mC) + \bar{\theta}(-m\overline{C}) \\ &\quad + \theta\bar{\theta}(im\mathscr{B}) \\ &\equiv \tilde{\phi} + \theta(s_{ad}\tilde{\phi}) + \bar{\theta}(s_d\tilde{\phi}) \\ &\quad + \theta\bar{\theta}(s_d s_{ad}\tilde{\phi}),\end{aligned} \tag{54}$$

where the superscript (dg) denotes the superexpansion after the application of dual-GIR (DGIR) on the superfields of the $(2,2)$-dimensional supermanifold.

A careful observation of (50) and (54) leads to the derivation of the following fermionic (anti-)co-BRST symmetry transformations for the whole theory; namely,

$$\begin{aligned} s_{ad}A_\mu &= -\varepsilon_{\mu\nu}\partial^\nu C, \\ s_{ad}C &= 0, \\ s_{ad}\overline{C} &= i\mathscr{B}, \\ s_{ad}\mathscr{B} &= 0, \\ s_{ad}(\partial\cdot A) &= 0, \\ s_{ad}\phi &= 0, \\ s_{ad}\tilde{\phi} &= -mC, \\ s_{ad}E &= \Box C, \\ s_d A_\mu &= -\varepsilon_{\mu\nu}\partial^\nu \overline{C}, \\ s_d\overline{C} &= 0, \\ s_d C &= -i\mathscr{B}, \\ s_d\mathscr{B} &= 0, \\ s_d(\partial\cdot A) &= 0, \\ s_d\phi &= 0, \\ s_d\tilde{\phi} &= -m\overline{C}, \\ s_d E &= \Box\overline{C}, \end{aligned} \tag{55}$$

which would be the symmetry transformations for the appropriately modified [cf. (57) below] form of the Lagrangian density (21). A careful observation at the transformations (55) demonstrates that the (anti-)co-BRST symmetry transformations are off-shell nilpotent as we do *not* use any equation of motion in the proof of $s_{(a)d}^2 = 0$. Further, these transformations are absolutely anticommuting in nature because it can be checked that $s_d s_{ad} + s_{ad} s_d = 0$. Finally, we have christened the transformations (55) as (anti-)dual-BRST (or (anti-)co-BRST) transformations because the total gauge-fixing term $(\partial\cdot A + m\phi)$, originating basically from the coexterior derivative, remains invariant under the nilpotent ($s_{(a)d}^2 = 0$) and absolutely anticommuting ($s_d s_{ad} + s_{ad} s_d = 0$) transformations $s_{(a)d}$.

It is clear that the 2D nilpotent (anti-)co-BRST symmetry transformations (55) are derived from the superexpansions (50) and (54) which are present on the $(2,2)$-dimensional supermanifold. Hence, there should be some connection between the 2D nilpotent (anti-)co-BRST symmetries and the superfield formalism on $(2,2)$-dimensional superspace. A careful observation at the superexpansions in (50) and (54) leads to the following relationships:

$$\begin{aligned} &\lim_{\bar{\theta}=0}\frac{\partial}{\partial\theta}\widetilde{N}^{(dh,dg)}(x,\theta,\bar{\theta}) = s_{ad}N(x), \quad \partial_\theta \longleftrightarrow s_{ad}, \\ &\lim_{\theta=0}\frac{\partial}{\partial\bar{\theta}}\widetilde{N}^{(dh,dg)}(x,\theta,\bar{\theta}) = s_d N(x), \quad \partial_{\bar{\theta}} \longleftrightarrow s_d, \\ &\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\widetilde{N}^{(dh,dg)}(x,\theta,\bar{\theta}) = s_d s_{ad} N(x), \\ &\qquad\qquad \partial_{\bar{\theta}}\partial_\theta \longleftrightarrow s_d s_{ad}, \end{aligned} \tag{56}$$

where $\widetilde{N}^{(dh,dg)}(x,\theta,\bar{\theta})$ is the generic superfield obtained after the application of DHC and DGIR on the $(2,2)$-dimensional supermanifold and $N(x)$ is the ordinary 2D field of our present (anti-)co-BRST invariant theory. It is evident that the transformations (55) would be automatically off-shell nilpotent and absolutely anticommuting because these are identified with the translational operators $(\partial_\theta, \partial_{\bar{\theta}})$, along the Grassmannian directions $(\theta,\bar{\theta})$ of the $(2,2)$-dimensional supermanifold, which satisfy $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$, $\partial_\theta\partial_{\bar{\theta}} + \partial_{\bar{\theta}}\partial_\theta = 0$ due to their inherent properties. We end this subsection with the remark that $\Phi(x,\theta,\bar{\theta}) = \phi(x)$ because $\phi(x)$ is a dual-gauge invariant quantity (i.e., $s_{(a)d}\phi = 0$).

4.3. Lagrangian Densities: (Anti-)co-BRST Invariance. The (anti-)co-BRST invariant version of the 2D Lagrangian density $\mathscr{L}_{(b_1)}$ of (41) is the one that incorporates the FP-ghost terms; namely,

$$\begin{aligned}\mathscr{L}_{\mathscr{B}} &= \mathscr{B}(E - m\tilde{\phi}) - \frac{1}{2}\mathscr{B}^2 + mE\tilde{\phi} - \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} \\ &\quad + \frac{1}{2}m^2 A_\mu A^\mu - mA_\mu\partial^\mu\phi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}B^2 \\ &\quad + B(\partial\cdot A + m\phi) - i\partial_\mu\overline{C}\partial^\mu C + im^2\overline{C}C.\end{aligned} \tag{57}$$

We note that the gauge-fixing and Faddeev-Popov ghost terms of the above Lagrangian density are same as that of the (anti-)BRST invariant Lagrangian density (21). Under the off-shell nilpotent and absolutely anticommuting (anti-)co-BRST symmetry transformations $s_{(a)d}$ [cf. (55)], the above

Lagrangian density transforms to the total spacetime derivatives as illustrated in the following:

$$\begin{aligned} s_d \mathscr{L}_{\mathscr{B}} &= \partial_\mu \left[\mathscr{B}\partial^\mu \overline{C} + m\varepsilon^{\mu\nu}\left(mA_\nu \overline{C} + \phi\partial_\nu \overline{C}\right) + m\widetilde{\phi}\partial^\mu \overline{C}\right], \\ s_{ad} \mathscr{L}_{\mathscr{B}} &= \partial_\mu \left[\mathscr{B}\partial^\mu C + m\varepsilon^{\mu\nu}\left(mA_\nu C + \phi\partial_\nu C\right) + m\widetilde{\phi}\partial^\mu C\right]. \end{aligned} \tag{58}$$

Hence, the action integral $S = \int dx \mathscr{L}_{\mathscr{B}}$ of our theory remains invariant.

We note that the gauge-fixing and FP-ghost terms of the Lagrangian densities (21) have been derived by exploiting the off-shell nilpotent (anti-)BRST symmetry transformations [cf. (20)]. In exactly similar fashion, it is interesting to observe that

$$\begin{aligned} s_d s_{ad}\left[\frac{i}{2}A_\mu A^\mu + \frac{i}{2}\widetilde{\phi}^2 + \frac{1}{2}C\overline{C}\right] &= \mathscr{B}\left(E - m\widetilde{\phi}\right) - \frac{1}{2}\mathscr{B}^2 - i\partial_\mu \overline{C}\partial^\mu C + im^2\overline{C}C, \\ s_d\left[(-i)\left\{\varepsilon^{\mu\nu}A_\mu\partial_\nu C + m\widetilde{\phi}C + \frac{1}{2}\mathscr{B}C\right\}\right] &= \mathscr{B}\left(E - m\widetilde{\phi}\right) - \frac{1}{2}\mathscr{B}^2 - i\partial_\mu \overline{C}\partial^\mu C + im^2\overline{C}C, \\ s_{ad}\left[(+i)\left\{\varepsilon^{\mu\nu}A_\mu\partial_\nu \overline{C} + m\widetilde{\phi}\overline{C} + \frac{1}{2}\mathscr{B}\overline{C}\right\}\right] &= \mathscr{B}\left(E - m\widetilde{\phi}\right) - \frac{1}{2}\mathscr{B}^2 - i\partial_\mu \overline{C}\partial^\mu C + im^2\overline{C}C. \end{aligned} \tag{59}$$

The above expressions show that there are *three* different ways (modulo a total spacetime derivative term) to write the kinetic term plus the FP-ghost terms in the language of the off-shell nilpotent ($s_{a(d)}^2 = 0$) (anti-)co-BRST symmetry transformations $s_{a(d)}$ which are also absolutely anticommute (i.e., $s_d s_{ad} + s_{ad} s_d = 0$) with each other in their operator form.

We can express the above three relations in the language of superfield formalism because we observe that the following expressions:

$$\begin{aligned} &\frac{\partial}{\partial\overline{\theta}}\frac{\partial}{\partial\theta}\left[\frac{i}{2}\mathscr{B}_\mu^{(dh)}\mathscr{B}^{\mu(dh)} + \frac{i}{2}\widetilde{\Phi}^{(dg)}\widetilde{\Phi}^{(dg)} + \frac{1}{2}F^{(dh)}\overline{F}^{(dh)}\right] \\ &\lim_{\theta=0}\frac{\partial}{\partial\overline{\theta}}\left[(-i)\left\{\varepsilon^{\mu\nu}\mathscr{B}_\mu^{(dh)}\partial_\nu F^{(dh)} + m\widetilde{\Phi}^{(dg)}F^{(dh)} + \frac{1}{2}\mathscr{B}(x)F^{(dh)}\right\}\right] \\ &\lim_{\overline{\theta}=0}\frac{\partial}{\partial\theta}\left[(+i)\left\{\varepsilon^{\mu\nu}\mathscr{B}_\mu^{(dh)}\partial_\nu \overline{F}^{(dh)} + m\widetilde{\Phi}^{(dg)}\overline{F}^{(dh)} + \frac{1}{2}\mathscr{B}(x)\overline{F}^{(dh)}\right\}\right], \end{aligned} \tag{60}$$

also lead to the derivation of the sum of a part of kinetic term and FP-ghost terms. Ultimately, this exercise implies that the sum of kinetic and FP-ghost terms,

$$\mathscr{B}\left(E - m\widetilde{\phi}\right) - \frac{1}{2}\mathscr{B}^2 - i\partial_\mu \overline{C}\partial^\mu C + im^2\overline{C}C, \tag{61}$$

is *always* (anti-)dual-BRST invariant quantity (modulo a total spacetime derivative) because this is trivially true when we take into account the nilpotency and absolute anticommutativity of the (anti-)co-BRST symmetry transformations. In other words, we conclude that $s_{(a)d}[\mathscr{B}(E - m\widetilde{\phi}) - (1/2)\mathscr{B}^2 - i\partial_\mu \overline{C}\partial^\mu C + im^2\overline{C}C] = 0$ modulo a total spacetime derivative. Thus, a part of Lagrangian density (57) is invariant under $s_{(a)d}$.

We have already seen that a part of the kinetic term and the total of FP-ghost terms can be expressed in terms of the superfields obtained after the application of DHC and DGIR [cf. (60)]. Furthermore, the kinetic term $(1/2)\partial_\mu\phi\partial^\mu\phi$ for the ϕ field and the total gauge-fixing term $[B(\partial\cdot A + m\phi) + 1/2(B^2)]$ would remain intact within the framework of superfield formalism as they are the dual-gauge (or (anti-)co-BRST) invariant quantities. We note that $B(\partial \cdot A + m\phi) \rightarrow B(\partial_\mu \mathscr{B}^{\mu(dh)} + m\phi)$ in the superfield formalism and it is trivial to check that $B(\partial \cdot \mathscr{B}^{(dh)}) = B(\partial \cdot A)$ [cf. (50)] so that $B(\partial\cdot A + m\phi) \rightarrow B(\partial\cdot A + m\phi)$ without any change whatsoever when we generalize it onto the $(2,2)$-dimensional supermanifold. The rest of the terms can be generalized onto the $(2,2)$-dimensional supermanifold as

$$\begin{aligned} &- m\varepsilon^{\mu\nu}\partial_\mu\mathscr{B}_\nu^{(dh)}\widetilde{\Phi}^{(dg)} - \frac{1}{2}\partial_\mu\widetilde{\Phi}^{(dg)}\partial^\mu\widetilde{\Phi}^{(dg)} \\ &+ \frac{1}{2}m^2\mathscr{B}_\mu^{(dh)}\mathscr{B}^{\mu(dh)} - m\mathscr{B}_\mu^{(dh)}\partial^\mu\phi(x), \end{aligned} \tag{62}$$

where the symbols have already been explained earlier and they are nothing but the superexpansions after the application of the DHC and DGIR [cf. (50) and (54)].

It is interesting to note that the *last* term of (62) is *always* (anti-)co-BRST invariant quantity because we observe the following:

$$\begin{aligned} - m\mathscr{B}_\mu^{(dh)}\partial^\mu\phi = -m\Big[A_\mu &+ \theta\left(-\varepsilon_{\mu\nu}\partial^\nu C\right) \\ &+ \overline{\theta}\left(-\varepsilon_{\mu\nu}\partial^\nu \overline{C}\right) + \theta\overline{\theta}\left(i\varepsilon_{\mu\nu}\partial^\nu\mathscr{B}\right)\Big]\partial^\mu\phi, \end{aligned} \tag{63}$$

where we have taken the expansions of $\mathscr{B}_\mu^{(dh)}(x,\theta,\overline{\theta})$ from (50). Taking the help of the mappings (56), we note the following:

$$\begin{aligned} \lim_{\theta=0}\frac{\partial}{\partial\overline{\theta}}\left[-m\mathscr{B}_\mu^{(dh)}\partial^\mu\phi\right] &= \partial_\mu\left[m\varepsilon^{\mu\nu}\phi\partial_\nu\overline{C}\right] \\ &\equiv s_d\left[-mA_\mu\partial^\mu\phi\right], \\ \lim_{\overline{\theta}=0}\frac{\partial}{\partial\theta}\left[-m\mathscr{B}_\mu^{(dh)}\partial^\mu\phi\right] &= \partial_\mu\left[m\varepsilon^{\mu\nu}\phi\partial_\nu C\right] \\ &\equiv s_{ad}\left[-mA_\mu\partial^\mu\phi\right], \end{aligned}$$

$$\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\left[-m\mathscr{B}_\mu^{(dh)}\partial^\mu\phi\right] = \partial_\mu\left[-im\varepsilon^{\mu\nu}\phi\partial_\nu\mathscr{B}\right]$$
$$\equiv s_d s_{ad}\left[-mA_\mu\partial^\mu\phi\right], \tag{64}$$

which demonstrates that $s_{(a)d}[-mA_\mu\partial^\mu\phi]$ is always a total spacetime derivative. The rest of the terms in (62) are also (anti-)co-BRST invariant quantity because we check that

$$\lim_{\theta=0}\frac{\partial}{\partial\bar{\theta}}\left[-m\varepsilon^{\mu\nu}\partial_\mu\mathscr{B}_\nu^{(dh)}\widetilde{\Phi}^{(dg)} - \frac{1}{2}\partial_\mu\widetilde{\Phi}^{(dg)}\partial^\mu\widetilde{\Phi}^{(dg)}\right.$$
$$\left.+\frac{1}{2}m^2\mathscr{B}_\mu^{(dh)}\mathscr{B}^{\mu(dh)}\right] = \partial_\mu\left[m\tilde{\phi}\partial^\mu\overline{C} + m^2\varepsilon^{\mu\nu}A_\nu\overline{C}\right]$$
$$\equiv s_d\left[mE\tilde{\phi} - \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}m^2A_\mu A^\mu\right],$$
$$\lim_{\bar{\theta}=0}\frac{\partial}{\partial\theta}\left[-m\varepsilon^{\mu\nu}\partial_\mu\mathscr{B}_\nu^{(dh)}\widetilde{\Phi}^{(dg)} - \frac{1}{2}\partial_\mu\widetilde{\Phi}^{(dg)}\partial^\mu\widetilde{\Phi}^{(dg)}\right.$$
$$\left.+\frac{1}{2}m^2\mathscr{B}_\mu^{(dh)}\mathscr{B}^{\mu(dh)}\right] = \partial_\mu\left[m\tilde{\phi}\partial^\mu C + m^2\varepsilon^{\mu\nu}A_\nu C\right] \tag{65}$$
$$\equiv s_{ad}\left[mE\tilde{\phi} - \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}m^2A_\mu A^\mu\right],$$
$$\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\left[-m\varepsilon^{\mu\nu}\partial_\mu\mathscr{B}_\nu^{(dh)}\widetilde{\Phi}^{(dg)} - \frac{1}{2}\partial_\mu\widetilde{\Phi}^{(dg)}\partial^\mu\widetilde{\Phi}^{(dg)}\right.$$
$$\left.+\frac{1}{2}m^2\mathscr{B}_\mu^{(dh)}\mathscr{B}^{\mu(dh)}\right] = \partial_\mu\left[m^2\left(C\partial^\mu\overline{C} - \overline{C}\partial^\mu C\right)\right.$$
$$\left.- im\left(\tilde{\phi}\partial^\mu\mathscr{B} + m\varepsilon^{\mu\nu}A_\nu\mathscr{B}\right)\right] \equiv s_d s_{ad}\left[mE\tilde{\phi}\right.$$
$$\left.- \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}m^2A_\mu A^\mu\right].$$

Ultimately, we conclude that $s_{(a)d}[mE\tilde{\phi} - (1/2)\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + (1/2)m^2A_\mu A^\mu]$ is *always* a total spacetime derivative. As a consequence, this specific part of the Lagrangian density (i.e., $mE\tilde{\phi} - (1/2)\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + (1/2)m^2A_\mu A^\mu$) is (anti-)co-BRST invariant quantity.

Ultimately, we have the total expression for the 2D Lagrangian density (57) in the superfield formalism, on the $(2,2)$-dimensional supermanifold, as

$$\mathscr{L}_{\mathscr{B}} \longrightarrow \widetilde{\mathscr{L}}_{\mathscr{B}} = \frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\left[\frac{i}{2}\mathscr{B}_\mu^{(dh)}\mathscr{B}^{\mu(dh)}\right.$$
$$\left.+\frac{i}{2}\widetilde{\Phi}^{(dg)}\widetilde{\Phi}^{(dg)} + \frac{1}{2}F^{(dh)}\overline{F}^{(dh)}\right] + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}$$
$$\cdot B^2(x) - m\varepsilon^{\mu\nu}\partial_\mu\mathscr{B}_\nu^{(dh)}\widetilde{\Phi}^{(dg)} - \frac{1}{2}\partial_\mu\widetilde{\Phi}^{(dg)}\partial^\mu\widetilde{\Phi}^{(dg)} \tag{66}$$
$$+\frac{1}{2}m^2\mathscr{B}_\mu^{(dh)}\mathscr{B}^{\mu(dh)} - m\mathscr{B}_\mu^{(dh)}\partial^\mu\phi + B(x)$$
$$\cdot\left(\partial_\mu\mathscr{B}^{\mu(dh)} + m\phi(x)\right),$$

where all the symbols have been explained earlier. The (anti-)co-BRST invariance of the Lagrangian density, within the framework of superfield formalism, is

$$\lim_{\theta=0}\frac{\partial}{\partial\bar{\theta}}\widetilde{\mathscr{L}}_{\mathscr{B}} = \partial_\mu\left[\mathscr{B}\partial^\mu\overline{C} + m\varepsilon^{\mu\nu}\phi\partial_\nu\overline{C} + m\tilde{\phi}\partial^\mu\overline{C}\right.$$
$$\left.+ m^2\varepsilon^{\mu\nu}A_\nu\overline{C}\right] \equiv s_d\mathscr{L}_{\mathscr{B}},$$
$$\lim_{\bar{\theta}=0}\frac{\partial}{\partial\theta}\widetilde{\mathscr{L}}_{\mathscr{B}} = \partial_\mu\left[\mathscr{B}\partial^\mu C + m\varepsilon^{\mu\nu}\phi\partial_\nu C + m\tilde{\phi}\partial^\mu C\right.$$
$$\left.+ m^2\varepsilon^{\mu\nu}A_\nu C\right] \equiv s_{ad}\mathscr{L}_{\mathscr{B}}, \tag{67}$$
$$\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\widetilde{\mathscr{L}}_{\mathscr{B}} = -\partial_\mu\left[i\mathscr{B}\partial^\mu\mathscr{B} + m^2\partial^\mu\left(\overline{C}C\right)\right.$$
$$\left.+ im\left\{\tilde{\phi}\partial^\mu\mathscr{B} + \varepsilon^{\mu\nu}\left(mA_\nu\mathscr{B} + \phi\partial_\nu\mathscr{B}\right)\right\}\right]$$
$$\equiv s_d s_{ad}\mathscr{L}_{\mathscr{B}}.$$

Due to the above observations, it is clear that the action integral would remain invariant under the nilpotent (anti-)co-BRST symmetry transformations.

Finally, we would like to state that we have accomplished our goal of capturing the (anti-)co-BRST invariance of the action integral within the framework of superfield formalism where we have used the superfields that have been obtained after the application of DHC and DGIR. We further note that the expressions in (58) and (67) match very nicely. The appearance of the terms like $\mathscr{B}\partial^\mu\overline{C}$, $\mathscr{B}\partial^\mu C$, and $i\mathscr{B}\partial^\mu\mathscr{B}$ in the parenthesis of above equation is due to the same kind of arguments as offered at the end of (29) in the context of (anti-)BRST symmetries and corresponding invariance of the action integral under these continuous and nilpotent symmetry transformations.

4.4. Nilpotency and Anticommutativity of the Conserved (Anti-)co-BRST Charges: Superfield Formulation. Exploiting the standard technique of the Noether theorem and using the appropriate equations of motion, we obtain the following expressions for the conserved ($\dot{Q}_{(a)d} = 0$) and off-shell nilpotent ($\dot{Q}_{(a)d}^2 = 0$) (anti-)co-BRST (or (anti-)dual-BRST) charges:

$$Q_{ad} = \int dx\left[\mathscr{B}\dot{C} - \dot{\mathscr{B}}C\right] \equiv \int dx J^0_{(ad)},$$
$$Q_d = \int dx\left[\mathscr{B}\dot{\overline{C}} - \dot{\mathscr{B}}\overline{C}\right] \equiv \int dx J^0_{(d)}, \tag{68}$$

which have been derived from the Lagrangian density (57) that has led to the following conserved (i.e., $\partial_\mu J^\mu_{(a)d} = 0$) Noether currents:

$$J^\mu_{ad} = \mathscr{B}\partial^\mu C - \varepsilon^{\mu\nu}B\partial_\nu C + mC\partial^\mu\tilde{\phi} - m\varepsilon^{\mu\nu}\phi\partial_\nu C,$$
$$J^\mu_d = \mathscr{B}\partial^\mu\overline{C} - \varepsilon^{\mu\nu}B\partial_\nu\overline{C} + m\overline{C}\partial^\mu\tilde{\phi} - m\varepsilon^{\mu\nu}\phi\partial_\nu\overline{C}. \tag{69}$$

The conservation law (i.e., $\partial_\mu J^\mu_{(a)d} = 0$) can be proven by exploiting the following equations of motion emerging from the Lagrangian density (57); namely,

$$\begin{aligned}
&B = -(\partial \cdot A + m\phi),\\
&\Box\tilde{\phi} - m(\mathscr{B} - E) = 0,\\
&(\Box + m^2)\bar{C} = 0,\\
&\mathscr{B} = E - m\tilde{\phi},\\
&\Box\phi - m(\partial \cdot A + B) = 0,\\
&(\Box + m^2) C = 0,\\
&\epsilon^{\mu\nu}\partial_\mu(\mathscr{B} + m\tilde{\phi}) - \partial^\nu B + m^2 A^\nu - m\partial^\nu\phi = 0.
\end{aligned} \tag{70}$$

It is straightforward to check that the (anti-)co-BRST charges can be expressed in terms of the (anti-)co-BRST symmetry transformations as

$$\begin{aligned}
Q_{ad} &= \int dx\,[s_{ad}s_d(iA_1C)],\\
Q_d &= \int dx\,[s_d s_{ad}(-iA_1\bar{C})].
\end{aligned} \tag{71}$$

Exploiting the mapping (56), it can be seen that the above expressions could be recast in the language of the superfields, obtained after the application of DHC and DGIR, as

$$\begin{aligned}
Q_{ad} &= \frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}\int dx\,[i\mathscr{B}_1^{(dh)}F^{(dh)}]\\
&\equiv \int dx \int d\theta \int d\bar{\theta}\,[i\mathscr{B}_1^{(dh)}F^{(dh)}],\\
Q_d &= \frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\int dx\,[-i\mathscr{B}_1^{(dh)}\bar{F}^{(dh)}]\\
&\equiv \int dx \int d\theta \int d\bar{\theta}\,[-i\mathscr{B}_1^{(dh)}\bar{F}^{(dh)}].
\end{aligned} \tag{72}$$

From the above expressions, too, one can prove the off-shell nilpotency ($Q^2_{(a)d} = 0$) of the charges $Q_{(a)d}$ by observing that the following are true; namely,

$$\begin{aligned}
&\lim_{\bar{\theta}=0}\frac{\partial}{\partial\theta}Q_{ad} = 0 \Longleftrightarrow\\
&\qquad s_{ad}Q_{ad} = 0 \equiv i\{Q_{ad}, Q_{ad}\},\\
&\lim_{\theta=0}\frac{\partial}{\partial\bar{\theta}}Q_d = 0 \Longleftrightarrow\\
&\qquad s_d Q_d = 0 \equiv i\{Q_d, Q_d\}.
\end{aligned} \tag{73}$$

The above observation of the nilpotency of $Q_{(a)d}$ is intimately connected with the nilpotency $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$ of translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ along the Grassmannian directions of the supermanifold.

The nilpotency of $Q_{(a)d}$ can *also* be proven by the following expressions of $Q_{(a)d}$ in terms of the (anti-)co-BRST symmetry transformations $s_{(a)d}$; namely,

$$\begin{aligned}
Q_{ad} &= \int dx s_{ad}\left[\mathscr{B}(x)A_1(x) + i\dot{\bar{C}}C\right],\\
Q_d &= \int dx s_d\left[\mathscr{B}(x)A_1(x) + i\bar{C}\dot{C}\right].
\end{aligned} \tag{74}$$

Thus, it is clear that the following will be true; namely,

$$\begin{aligned}
s_d Q_d &= i\{Q_d, Q_d\} = \int dx s_d^2\left[\mathscr{B}(x)A_1(x) + i\dot{\bar{C}}C\right]\\
&= 0, \quad (s_d^2 = 0),\\
s_{ad}Q_{ad} &= i\{Q_{ad}, Q_{ad}\}\\
&= \int dx s_{ad}^2\left[\mathscr{B}(x)A_1(x) + i\bar{C}\dot{C}\right] = 0,\\
&\qquad (s_{ad}^2 = 0),
\end{aligned} \tag{75}$$

due to the nilpotency of $s_{(a)d}$ (i.e., $s^2_{(a)d} = 0 \Leftrightarrow Q^2_{(a)d} = 0$). In the language of superfield formalism, expressions (74) can be written as

$$\begin{aligned}
Q_d &= \lim_{\theta=0}\frac{\partial}{\partial\bar{\theta}}\int dx\left[\mathscr{B}(x)\mathscr{B}_1^{(dh)}(x,\theta,\bar{\theta})\right.\\
&\quad\left. + i\bar{F}^{(dh)}(x,\theta,\bar{\theta})\dot{F}^{(dh)}(x,\theta,\bar{\theta})\right],\\
Q_{ad} &= \lim_{\bar{\theta}=0}\frac{\partial}{\partial\theta}\int dx\left[\mathscr{B}(x)\mathscr{B}_1^{(dh)}(x,\theta,\bar{\theta})\right.\\
&\quad\left. + i\dot{\bar{F}}^{(dh)}(x,\theta,\bar{\theta})F^{(dh)}(x,\theta,\bar{\theta})\right],
\end{aligned} \tag{76}$$

which demonstrate trivially the following:

$$\begin{aligned}
&\lim_{\theta=0}\frac{\partial}{\partial\bar{\theta}}Q_d = 0 \Longleftrightarrow\\
&\qquad s_d Q_d = 0,\\
&\lim_{\bar{\theta}=0}\frac{\partial}{\partial\theta}Q_{ad} = 0 \Longleftrightarrow\\
&\qquad s_{ad}Q_{ad} = 0,
\end{aligned} \tag{77}$$

where the nilpotency of ∂_θ and $\partial_{\bar{\theta}}$ (i.e., $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$) plays a decisive role.

To prove the absolute anticommutativity of $Q_{(a)d}$, we note the following interesting expressions for the conserved (anti-)co-BRST charges:

$$\begin{aligned}
Q_d &= \int dx\left[s_{ad}\left(-i\bar{C}\dot{\bar{C}}\right)\right],\\
Q_{ad} &= \int dx\left[s_d\left(iC\dot{C}\right)\right].
\end{aligned} \tag{78}$$

The above expressions automatically imply the following beautiful relationships:

$$s_{ad}Q_d = i\{Q_d, Q_{ad}\} = \int dx\left[s_{ad}^2\left(-i\bar{C}\dot{\bar{C}}\right)\right] = 0, \quad \left(s_{ad}^2 = 0\right),$$
$$s_d Q_{ad} = i\{Q_{ad}, Q_d\} = \int dx\left[s_d^2\left(iC\dot{C}\right)\right] = 0, \quad \left(s_d^2 = 0\right). \tag{79}$$

Thus, we point out a very interesting observation that the absolute anticommutativity property of the (anti-)co-BRST charges is deeply and clearly connected with the nilpotency of the (anti-)co-BRST symmetry transformations (i.e., $s_{(a)d}^2 = 0$). These expressions (78) could be also written in terms of superfields, translational generators $(\partial_\theta, \partial_{\bar\theta})$, and differentials $(d\theta, d\bar\theta)$ defined on the $(2, 2)$-dimensional supermanifold, as

$$Q_d = \lim_{\bar\theta=0}\frac{\partial}{\partial\theta}\int dx\left[-i\bar{F}^{(dh)}\dot{\bar{F}}^{(dh)}\right] \equiv \int dx\int d\theta\left[-i\bar{F}^{(dh)}\dot{\bar{F}}^{(dh)}\right],$$
$$Q_{ad} = \lim_{\theta=0}\frac{\partial}{\partial\bar\theta}\int dx\left[iF^{(dh)}\dot{F}^{(dh)}\right] \equiv \int dx\int d\bar\theta\left[iF^{(dh)}\dot{F}^{(dh)}\right]. \tag{80}$$

The above expressions capture the anticommutativity property of the (anti-)co-BRST charges in the language of superfield formalism, as

$$\lim_{\bar\theta=0}\frac{\partial}{\partial\theta}Q_{ad} = 0,$$
$$\lim_{\theta=0}\frac{\partial}{\partial\bar\theta}Q_{ad} = 0, \tag{81}$$

where the properties $\partial_\theta^2 = 0, \partial_{\bar\theta}^2 = 0$ play important roles when we use the expressions for $Q_{(a)d}$ from (80). The anticommutativity property is hidden in (81) in view of the mapping (56) which implies that (81) can be written as $s_dQ_{ad} = i\{Q_{ad}, Q_d\} = 0$ and $s_{ad}Q_d = i\{Q_d, Q_{ad}\} = 0$ primarily due to $\partial_\theta^2 = 0, \partial_{\bar\theta}^2 = 0$.

We end this subsection with the remark that the nilpotency and absolute anticommutativity properties of the (anti-)co-BRST symmetry transformations (and their corresponding conserved charges) are related to the properties $\partial_\theta^2 = 0$, $\partial_{\bar\theta}^2 = 0$ and $\partial_\theta\partial_{\bar\theta} + \partial_{\bar\theta}\partial_\theta = 0$. These relations are, in turn, interconnected with each other because the limiting case of the latter (i.e., $\partial_\theta\partial_{\bar\theta} + \partial_{\bar\theta}\partial_\theta = 0$) leads to the derivation of the former ($\partial_\theta^2 = 0, \partial_{\bar\theta}^2 = 0$) when we set $\partial_\theta = \partial_{\bar\theta}$ in the latter relationship of anticommutativity between ∂_θ and $\partial_{\bar\theta}$.

5. On a Unique Bosonic Symmetry, the Ghost-Scale Symmetry, and the Discrete Symmetries

From the four nilpotent ($s_{(a)b}^2 = s_{(a)d}^2 = 0$) symmetries of the theory, we can construct a unique bosonic symmetry $s_\omega = \{s_b, s_d\} \equiv -\{s_{ab}, s_{ad}\}$, under which, the relevant fields of our present theory (described by the Lagrangian density (57)) transform as

$$s_\omega A_\mu = \varepsilon_{\mu\nu}\partial^\nu B + \partial_\mu \mathscr{B},$$
$$s_\omega \tilde\phi = mB,$$
$$s_\omega \phi = m\mathscr{B},$$
$$s_\omega(\partial\cdot A) = \Box\mathscr{B}, \tag{82}$$
$$s_\omega E = -\Box B,$$
$$s_\omega\left(B, \mathscr{B}, C, \bar{C}\right) = 0,$$

modulo an overall factor of $(-i)$. We note that $\{s_d, s_{ad}\} = 0$, $\{s_d, s_{ab}\} = 0$, $\{s_b, s_{ad}\} = 0$, and $\{s_b, s_{ab}\} = 0$. We point out that the fundamental symmetries of the theory are $s_{(a)b}$ and $s_{(a)d}$ which have been derived from the augmented superfield formalism. The bosonic symmetry transformation s_ω is derived from the above *basic* off-shell nilpotent ($s_{(a)b}^2 = s_{(a)d}^2 = 0$) symmetries $s_{(a)b}$ and $s_{(a)d}$. One of the decisive features of the above bosonic symmetry is the observation that the ghost part of the Lagrangian density remains invariant.

Under the above transformations (82), the Lagrangian density (57) transforms as

$$s_\omega\mathscr{L}_B = \partial_\mu\left[B\partial^\mu\mathscr{B} - \mathscr{B}\partial^\mu B - m\tilde\phi\partial^\mu B - m\varepsilon^{\mu\nu}\left(\phi\partial_\nu B + mA_\nu B\right)\right]. \tag{83}$$

As a consequence, the action integral $S = \int dx\mathscr{L}_{\mathscr{B}}$ remains invariant. The above symmetry transformation, according to Noether's theorem, leads to the derivation of the following conserved charge (as the analogue of the Laplacian operator):

$$Q_\omega = \int dx J_\omega^0 = \int dx\left[B\dot{\mathscr{B}} - \dot{B}\mathscr{B}\right], \tag{84}$$

which emerges from the Noether conserved ($\partial_\mu J_\omega^\mu = 0$) current

$$J_\omega^\mu = \varepsilon^{\mu\nu}\left(B\partial_\nu B - \mathscr{B}\partial_\nu\mathscr{B} - m\tilde\phi\partial_\nu\mathscr{B} + m\phi\partial_\nu B + m^2A_\nu B\right) + m\mathscr{B}\partial^\mu\phi - mB\partial^\mu\tilde\phi - m^2A^\mu\mathscr{B}. \tag{85}$$

The conserved charge (84) is the generator of the continuous and infinitesimal bosonic symmetry transformations (82) which can be checked by using the standard formula between the continuous symmetry and its generator.

Our theory, described by the Lagrangian density (57), is endowed with the following ghost-scale symmetry transformations (with a *global* (i.e., spacetime independent) scale parameter Ω); namely,

$$\begin{aligned} C &\longrightarrow e^{(+1)\cdot\Omega} C, \\ \bar{C} &\longrightarrow e^{(-1)\cdot\Omega} \bar{C}, \\ \Psi &\longrightarrow e^{(0)\cdot\Omega} \Psi, \quad \left(\Psi = A_\mu, \phi, \tilde{\phi}, B, \mathscr{B}\right), \end{aligned} \tag{86}$$

where the numerals in the exponentials denote the ghost numbers of the fields. The infinitesimal version of the above scale transformations (s_g) is

$$\begin{aligned} s_g C &= +C, \\ s_g \bar{C} &= -\bar{C}, \\ s_g \Psi &= 0, \quad \left(\Psi = A_\mu, \phi, \tilde{\phi}, B, \mathscr{B}\right), \end{aligned} \tag{87}$$

modulo a factor of Ω that can be set equal to *one* for the sake of brevity. The above transformations are generated by the following ghost charge Q_g [2, 3]:

$$Q_g = i \int dx \left[\bar{C}\dot{C} - \dot{\bar{C}}C\right] \equiv \int dx J_g^0. \tag{88}$$

This charge has been derived from the conserved current $J_g^\mu = i(\bar{C}\partial^\mu C - \partial^\mu \bar{C} C)$. The conservation law $\partial_\mu J_g^\mu = 0$ can be proven by using the Euler-Lagrange equations of motion $(\Box + m^2)\bar{C} = 0$ and $(\Box + m^2)C = 0$ which emerge from (57).

In addition to the above continuous symmetries, we have a set of suitable discrete symmetries in the theory. These symmetries are as follows:

$$\begin{aligned} A_\mu &\longrightarrow \pm i\varepsilon_{\mu\nu} A^\nu, \\ E &\longrightarrow \mp i (\partial \cdot A), \\ (\partial \cdot A) &\longrightarrow \mp i E, \\ B &\longrightarrow \pm i \mathscr{B}, \\ \mathscr{B} &\longrightarrow \pm i B, \\ C &\longrightarrow \mp i \bar{C}, \\ \bar{C} &\longrightarrow \mp i C. \end{aligned} \tag{89}$$

It is straightforward to check that the Lagrangian density (57) remains invariant under the above discrete symmetry transformations. Further, it can be readily verified that the following is true; namely,

$$\begin{aligned} *Q_b &= +Q_d, \\ *Q_d &= +Q_b, \\ *Q_\omega &= +Q_\omega, \\ *Q_{ab} &= +Q_{ad}, \\ *Q_{ad} &= +Q_{ab}, \\ *Q_g &= -Q_g, \\ *\left(*Q_r\right) &= +Q_r, \quad (r = b, ab, d, ad, \omega), \end{aligned} \tag{90}$$

where the operator $(*)$ is nothing but the operation of the above discrete symmetry transformations on the conserved charges of the theory. We note that two successive operations of the discrete symmetry transformations leave the conserved charges intact. On the other hand, a single operation of the discrete symmetry transformations interchanges each of the pairs (Q_b, Q_d) and (Q_{ab}, Q_{ad}) such that $(Q_b \leftrightarrow Q_d, Q_{ab} \leftrightarrow Q_{ad})$ and the ghost charge transforms as $Q_g \rightarrow -Q_g$.

6. Algebraic Structures and Cohomological Aspects

It can be checked that the six conserved (i.e., $\dot{Q}_r = 0$) charges (i.e., $Q_r, r = b, ab, d, ad, \omega, g$) of the theory obey the following extended BRST algebra:

$$\begin{aligned} Q_{(a)b}^2 &= 0, \\ Q_{(a)d}^2 &= 0, \\ \{Q_b, Q_{ab}\} &= 0, \\ \{Q_b, Q_{ad}\} &= 0, \\ \{Q_d, Q_{ad}\} &= 0, \\ \{Q_d, Q_{ab}\} &= 0, \\ i\left[Q_g, Q_d\right] &= -Q_d, \\ i\left[Q_g, Q_b\right] &= +Q_b, \\ i\left[Q_g, Q_{ab}\right] &= -Q_{ab}, \\ i\left[Q_g, Q_{ad}\right] &= +Q_{ad}, \\ \{Q_b, Q_d\} &= Q_\omega \equiv -\{Q_{ad}, Q_{ab}\}, \\ \left[Q_\omega, Q_r\right] &= 0, \quad (r = b, ab, d, ad, g, \omega). \end{aligned} \tag{91}$$

The above algebra is exactly like the algebra satisfied by the de Rham cohomological operators of differential geometry [7–12]; namely,

$$\begin{aligned} d^2 &= 0, \\ \delta^2 &= 0, \\ \{d, \delta\} &= \Delta \equiv (d + \delta)^2, \\ [\Delta, d] &= 0, \\ [\Delta, \delta] &= 0, \end{aligned} \tag{92}$$

where $\delta = - * d *$ (with $\delta^2 = 0$) and $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) are the (co-)exterior derivatives and $\Delta = \{d, \delta\}$ is the Laplacian operator of differential geometry. In the above, the symbol $(*)$ stands for the Hodge duality operation on a given spacetime manifold. For the *even* dimensional manifold, the relation $\delta = -*d*$ is always true.

There is a simpler way to check the sanctity of the extended BRST algebra listed in (91) where we use the well-known relationship between the continuous symmetry transformations and their generators. For instance, the above algebra can be obtained from the following transformations on the conserved charges; namely,

$$\begin{aligned}
s_r Q_r &= i\{Q_r, Q_r\} = 0 \Longrightarrow \\
Q_r^2 &= 0, \\
&\qquad (r = b, ab, d, ad), \\
s_r Q_{ar} &= i\{Q_{ar}, Q_r\} = 0 \Longrightarrow \\
\{Q_{ar}, Q_r\} &= 0, \\
&\qquad (r = b, d), \\
s_\omega Q_r &= -i[Q_r, Q_\omega] = 0 \Longrightarrow \\
[Q_\omega, Q_r] &= 0, \\
&\qquad (r = b, ab, d, ad, g, \omega), \\
s_g Q_r &= -i[Q_r, Q_g] = +Q_r \Longrightarrow \\
i[Q_g, Q_r] &= +Q_r, \\
&\qquad (r = b, ad), \\
s_b Q_{ad} &= +i\{Q_{ad}, Q_b\} = 0 \Longrightarrow \\
\{Q_b, Q_{ad}\} &= 0, \\
s_d Q_{ab} &= +i\{Q_{ab}, Q_d\} = 0 \Longrightarrow \\
\{Q_d, Q_{ab}\} &= 0, \\
s_g Q_r &= -i[Q_r, Q_g] = -Q_r \Longrightarrow \\
i[Q_g, Q_r] &= -Q_r, \\
&\qquad (r = d, ab),
\end{aligned} \tag{93}$$

where the l.h.s. can be calculated in a straightforward manner by exploiting the expressions for the *six* conserved charges and the corresponding continuous symmetry transformations that have been mentioned in the main body of our text.

A comparison between (91) and (92) demonstrates that Q_ω and Δ are the Casimir operators (the algebras (91) and (92) are *not* the Lie algebras; hence, the charge Q_ω and operator Δ are *not* the Casimir operators in the sense of such objects in the case of Lie algebra) for the above algebras in the sense that they absolutely commute with the rest of the operators. A close look at these algebras leads to the following clear-cut two-to-one mappings:

$$\begin{aligned}
(Q_b, Q_{ad}) &\longrightarrow d, \\
(Q_d, Q_{ab}) &\longrightarrow \delta, \\
\{Q_b, Q_d\} = -\{Q_{ab}, Q_{ad}\} &\longrightarrow \Delta,
\end{aligned} \tag{94}$$

between the conserved charges and the cohomological operators. Furthermore, we note that we have the following beautiful relationship [2, 3]:

$$s_{(a)d}\Psi = -*s_{(a)b}*\Psi, \quad \left(\Psi = A_\mu, \phi, \tilde{\phi}, C, \bar{C}, B, \mathscr{B}\right), \tag{95}$$

which provides the physical realization of the relationship (i.e., $\delta = -*d*$) between the (co-)exterior derivatives $(\delta)d$ defined on an *even* dimensional spacetime manifold. In (95), we observe that it is the interplay between the continuous symmetries (i.e., $s_{(a)b}, s_{(a)d}$) and the discrete symmetries (89) that provide the analogue of relationship $\delta = -*d*$. In fact, the latter (i.e., (89)) leads to the physical realization of the Hodge duality $(*)$ operation of the differential geometry. Thus, we note that the $(*)$ in (95) is nothing but the discrete symmetry transformations (89). The minus sign on the r.h.s of (95) is governed by two successive operations of the discrete symmetry transformations (89) on the generic field Ψ; namely, $*(*\Psi) = -\Psi$ (see, e.g., [34] for details).

One of the distinguishing features of the cohomological operators (d, δ, Δ) is the observation that when they operate on a differential form of a specific degree, the consequences turn out to be completely different. For instance, when the (co-)exterior derivatives operate on a form (f_n) of degree n, they (lower) raise the degree of the form by one (i.e., $\delta f_n \sim f_{n-1}$, $df_n \sim f_{n+1}$). On the contrary, when Δ acts on a form of degree n, it does *not* change the degree at all (i.e., $\Delta f_n \sim f_n$). We have to capture these properties in the language of the symmetry properties and conserved charges of our present modified version of 2D Proca theory so that we could establish precise analogy.

The above algebraic features could be also captured in the language of conserved charges. To this end in mind, let us define a state $|\psi\rangle_n$ in the quantum Hilbert space of states as

$$iQ_g|\psi\rangle_n = n|\psi\rangle_n, \tag{96}$$

where the eigenvalue n is the ghost number because Q_g is the ghost charge [cf. (88)]. Due to the algebra (91), respected by the various charges, it can be readily checked that the following relationships are true; namely,

$$\begin{aligned}
iQ_g Q_b|\psi\rangle_n &= (n+1)Q_b|\psi\rangle_n, \\
iQ_g Q_{ad}|\psi\rangle_n &= (n+1)Q_{ad}|\psi\rangle_n, \\
iQ_g Q_d|\psi\rangle_n &= (n-1)Q_d|\psi\rangle_n, \\
iQ_g Q_{ab}|\psi\rangle_n &= (n-1)Q_{ab}|\psi\rangle_n, \\
iQ_g Q_\omega|\psi\rangle_n &= nQ_\omega|\psi\rangle_n.
\end{aligned} \tag{97}$$

Thus, we note that the ghost numbers for the states $Q_b|\psi\rangle_n$, $Q_d|\psi\rangle_n$ and $Q_\omega|\psi\rangle_n$ are $(n+1)$, $(n-1)$ and n, respectively. In exactly similar fashion, the states $Q_{ad}|\psi\rangle_n$, $Q_{ab}|\psi\rangle_n$ and $Q_\omega|\psi\rangle_n$ also carry the ghost numbers $(n+1)$, $(n-1)$ and n, respectively. These properties are exactly like the consequences that ensue from the operations of the cohomological operators (d, δ, Δ) on a differential form of degree n defined on a given manifold.

We conclude that, if the degree of a form is identified with the ghost number, then the operation of (d, δ, Δ) on this given form is exactly like the operations of the set (Q_b, Q_d, Q_ω) and/or $(Q_{ad}, Q_{ab}, Q_\omega)$ on the state with ghost number equal to the degree of the form. Thus, the mappings (94) are *correct* as far as the algebraic structures of (91) and (92) are concerned and we have two-to-one mapping from the conserved charges of the theory to the de Rham cohomological operators (d, δ, Δ) of differential geometry. A careful look at (90) and (91) leads to the conclusion that the algebra (91) remains invariant under any number of operations of discrete (duality) symmetry transformations (89). This establishes that our present 2D theory is a *perfect* model for the Hodge theory where the continuous symmetry transformations (and corresponding generators) provide the physical realizations of the cohomological operators. On the other hand, it is the discrete symmetry transformations of the theory that are the physical analogue of the Hodge duality $(*)$ operation of differential geometry. Finally, we observe that the ghost number of a specific state in the quantum Hilbert space provides the physical analogue of the degree of a form of differential geometry as far as its cohomological aspects are concerned.

7. Conclusions

In our present endeavor, we have applied the augmented version of superfield formalism to derive the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations for the modified version of 2D Proca theory. We have exploited the theoretical strength of horizontality condition (HC) and gauge invariant restriction (GIR) to derive the (anti-)BRST symmetries for *all* the fields of our present 2D theory. In addition, we have made use of the dual-HC (DHC) and dual-GIR (DGIR) to obtain the complete set of (anti-)co-BRST symmetry transformations for *all* the fields of our present theory. The local gauge symmetry transformations [cf. (4)] are the perfect “classical” version of the (anti-)BRST symmetries which exist in any arbitrary dimension of spacetime. However, there is no such *perfect* “classical” analogue (see, e.g., Section 4.1) for the (anti-)co-BRST symmetries of our present theory. The latter symmetries exist only in specific dimensions of spacetime and they are always “quantum” in nature. For instance, for the Abelian 1-form gauge theory, these “quantum” symmetries exist only in two dimensions of spacetime.

In Sections 3.3 and 4.4, we have expressed the (anti-)BRST and (anti-)co-BRST charges in various forms due to our knowledge of the augmented superfield approach to BRST formalism. In these subsections, we have been able to provide the meaning of their nilpotency and absolute anticommutativity in the language of superfield formalism. We have been *also* able to establish connections between the properties of nilpotency and absolute anticommutativity. In fact, it is the strength of the augmented superfield formalism that we have expressed the (anti-)BRST and (anti-)co-BRST charges in a completely *novel* fashions (which have, hitherto, not been pointed out in literature). Thus, there are completely novel results in Sections 3.3 and 4.4 as far as our present investigation on the superfield approach to BRST formalism is concerned.

In addition to the above results, there are applications of DHC and DGIR in deducing the full set of (anti-)co-BRST symmetry transformations for *all* the fields of our present theory. These derivations are *also* novel results. In particular, the application of DGIR, in the derivation of the (anti-)co-BRST symmetry transformations for the pseudoscalar field $(\tilde{\phi})$, is a completely new result which has *not* been discussed in the literature. The symmetries of the theory enforce the pseudoscalar field to have a negative kinetic term. Since this field is massive [i.e., $(\Box + m^2)\tilde{\phi} = 0$], it is a very good candidate for the dark matter [29, 30]. We lay emphasis on the fact that the Stueckelberg scalar field (ϕ) has always a *positive* kinetic term and, hence, it is an *ordinary* matter (due to $(\Box + m^2)\,\phi = 0$).

In our investigation, we have provided physical realizations of the de Rham cohomological operators in the language of the continuous symmetry transformations (and their corresponding charges). Further, we have shown that a set of discrete symmetry transformations provide the physical analogue of the Hodge duality $(*)$ operation of differential geometry. Ultimately, we have shown that, at the algebraic level, the set of six conserved charges of our theory obey exactly the same algebra as that of the de Rham cohomological operators of differential geometry. This algebra remains invariant [cf. (90)] under the discrete symmetry transformations (89) which are the analogue of Hodge theory $(*)$ operation. The degree of a form finds its physical analogue as the ghost number of a state (in the quantum Hilbert space of states). Thus, our present 2D modified version of Proca theory turns out to be a *perfect* model for the Hodge theory. The unique feature of our present theory is the coexistence of *mass* and various kinds of *internal* symmetries *together* in a physically and mathematically meaningful manner.

It would be nice future endeavor to study the above kind of possibilities in the cases of 3D and 4D massive gauge theories [35, 36] where the gauge invariance and mass would coexist together. In other words, we would like to study whether Stueckelberg’s type of technique would be able to modify the above theories in such a way that they could also become *massive* field theoretic models for the Hodge theory. We speculate that such kind of situation will exist and these models will provide candidates for the *dark* matter in more physical 3D and 4D of spacetime (analogous to the massive pseudoscalar $\tilde{\phi}$ of our present 2D theory). Our speculation is based on the fact that we have already shown that the 4D free Abelian 2-form gauge theory is a model for the Hodge theory where a *massless* pseudoscalar field does exist with a *negative* kinetic term (see, e.g., [18, 19] for details). We are currently intensively involved with such kind of problems and we will be able to report about our progress in our future publications.

Appendices

A. On the Verification of (46)

Here we compute (46) step by step which is nothing but the expression for $(-\star\tilde{d}\star\tilde{A}^{(1)})$. Taking the expression for $\tilde{A}^{(1)}$ (from (6)) and applying a single $(\star)$ on it, we obtain the following on a (2, 2)-dimensional supermanifold (see, e.g., [32] for details):

$$\begin{aligned}\star\,\tilde{A}^{(1)} &= \varepsilon^{\mu\nu}\left(dx_\nu\wedge d\theta\wedge d\bar{\theta}\right)B_\mu\left(x,\theta,\bar{\theta}\right)\\&+\frac{1}{2!}\varepsilon_{\mu\nu}\left(dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\right)\bar{F}\left(x,\theta,\bar{\theta}\right)\\&+\frac{1}{2!}\varepsilon_{\mu\nu}\left(dx^\mu\wedge dx^\nu\wedge d\theta\right)F\left(x,\theta,\bar{\theta}\right),\end{aligned}\quad\text{(A.1)}$$

which is nothing but a super 3-form on the above supermanifold. In the above computation, we have used the following relationship on the given (2, 2)-dimensional supermanifold (see, e.g., [32] for details):

$$\begin{aligned}\star\left(dx^\mu\right) &= \varepsilon^{\mu\nu}\left(dx_\nu\wedge d\theta\wedge d\bar{\theta}\right),\\ \star\left(d\theta\right) &= \frac{1}{2!}\varepsilon_{\mu\nu}\left(dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\right),\\ \star\left(d\bar{\theta}\right) &= \frac{1}{2!}\varepsilon_{\mu\nu}\left(dx^\mu\wedge dx^\nu\wedge d\theta\right).\end{aligned}\quad\text{(A.2)}$$

Now we obtain a super 4-form on the (2, 2)-dimensional supermanifold by applying a $\tilde{d}$ on (A.1). This is given by the following expression:

$$\begin{aligned}&\tilde{d}\star\tilde{A}^{(1)}\\&= \varepsilon^{\mu\nu}\left(dx_\lambda\wedge dx_\nu\wedge d\theta\wedge d\bar{\theta}\right)\partial^\lambda B_\mu\left(x,\theta,\bar{\theta}\right)\\&+\frac{1}{2!}\varepsilon_{\mu\nu}\left(dx^\lambda\wedge dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\right)\partial_\lambda\bar{F}\left(x,\theta,\bar{\theta}\right)\\&+\frac{1}{2!}\varepsilon_{\mu\nu}\left(dx^\lambda\wedge dx^\mu\wedge dx^\nu\wedge d\theta\right)\partial_\lambda F\left(x,\theta,\bar{\theta}\right)\\&+\varepsilon^{\mu\nu}\left(d\theta\wedge dx_\nu\wedge d\theta\wedge d\bar{\theta}\right)\partial_\theta B_\mu\left(x,\theta,\bar{\theta}\right)\\&-\frac{1}{2!}\varepsilon_{\mu\nu}\left(d\theta\wedge dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\right)\partial_\theta\bar{F}\left(x,\theta,\bar{\theta}\right)\\&-\frac{1}{2!}\varepsilon_{\mu\nu}\left(d\theta\wedge dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\right)\partial_\theta F\left(x,\theta,\bar{\theta}\right)\\&+\varepsilon^{\mu\nu}\left(d\bar{\theta}\wedge dx_\nu\wedge d\theta\wedge d\bar{\theta}\right)\partial_{\bar{\theta}}B_\mu\left(x,\theta,\bar{\theta}\right)\\&-\frac{1}{2!}\varepsilon_{\mu\nu}\left(d\bar{\theta}\wedge dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\right)\partial_{\bar{\theta}}\bar{F}\left(x,\theta,\bar{\theta}\right)\\&-\frac{1}{2!}\varepsilon_{\mu\nu}\left(d\bar{\theta}\wedge dx^\mu\wedge dx^\nu\wedge d\theta\right)\partial_{\bar{\theta}}F\left(x,\theta,\bar{\theta}\right).\end{aligned}\quad\text{(A.3)}$$

It is clear, from the above, that the second and third terms would be zero because there are wedge-products which contain *three* spacetime differentials (which is *not* allowed on a (2, 2)-dimensional supermanifold). Furthermore, fourth and seventh terms would be zero because the wedge product with *three* Grassmannian differentials is *not* allowed on a (2, 2)-dimensional supermanifold. Hence, we have the existing super 4-form as

$$\begin{aligned}&\tilde{d}\star\tilde{A}^{(1)}\\&= \varepsilon^{\mu\nu}\left(dx_\lambda\wedge dx_\nu\wedge d\theta\wedge d\bar{\theta}\right)\partial^\lambda B_\mu\left(x,\theta,\bar{\theta}\right)\\&-\frac{1}{2!}\varepsilon_{\mu\nu}\left(d\theta\wedge dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\right)\partial_\theta\bar{F}\left(x,\theta,\bar{\theta}\right)\\&-\frac{1}{2!}\varepsilon_{\mu\nu}\left(dx^\mu\wedge dx^\nu\wedge d\theta\wedge d\bar{\theta}\right)\partial_\theta F\left(x,\theta,\bar{\theta}\right)\\&-\frac{1}{2!}\varepsilon_{\mu\nu}\left(d\bar{\theta}\wedge dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\right)\partial_{\bar{\theta}}\bar{F}\left(x,\theta,\bar{\theta}\right)\\&-\frac{1}{2!}\varepsilon_{\mu\nu}\left(d\bar{\theta}\wedge dx^\mu\wedge dx^\nu\wedge d\theta\right)\partial_{\bar{\theta}}F\left(x,\theta,\bar{\theta}\right),\end{aligned}\quad\text{(A.4)}$$

where we have taken into account the following rules:

$$\begin{aligned}\left(dx^\mu\wedge d\theta\right) &= -\left(d\theta\wedge dx^\mu\right),\\ \left(d\theta\wedge d\bar{\theta}\right) &= \left(d\bar{\theta}\wedge d\theta\right).\end{aligned}\quad\text{(A.5)}$$

Taking a $[-(\star)]$ on (A.4), we obtain

$$\begin{aligned}-\star\,\tilde{d}\star\tilde{A}^{(1)} &= -\varepsilon^{\mu\nu}\varepsilon_{\lambda\nu}\partial^\lambda B_\mu-\partial_\theta\bar{\mathscr{F}}-S^{\bar{\theta}\bar{\theta}}\partial_{\bar{\theta}}\bar{\mathscr{F}}\\&-S^{\theta\theta}\partial_\theta\mathscr{F}-\partial_{\bar{\theta}}\mathscr{F},\end{aligned}\quad\text{(A.6)}$$

where we have used the following inputs (see, e.g., [32]):

$$\begin{aligned}\star\left(dx_\lambda\wedge dx_\nu\wedge d\theta\wedge d\bar{\theta}\right) &= \varepsilon_{\lambda\nu},\\ \star\left(dx^\mu\wedge dx^\nu\wedge d\theta\wedge d\bar{\theta}\right) &= \varepsilon^{\mu\nu},\\ \star\left(dx^\mu\wedge dx^\nu\wedge d\theta\wedge d\theta\right) &= \varepsilon^{\mu\nu}S^{\theta\theta},\\ \star\left(dx^\mu\wedge dx^\nu\wedge d\bar{\theta}\wedge d\bar{\theta}\right) &= \varepsilon^{\mu\nu}S^{\bar{\theta}\bar{\theta}}.\end{aligned}\quad\text{(A.7)}$$

Thus, we finally obtain the explicit expression for $(-\star\tilde{d}\star\tilde{A}^{(1)})$ as follows:

$$\begin{aligned}-\star\,\tilde{d}\star\tilde{A}^{(1)} &= (\partial\cdot\mathscr{B})-\left(\partial_\theta\bar{F}+\partial_{\bar{\theta}}\mathscr{F}\right)-S^{\theta\theta}\left(\partial_\theta F\right)\\&-S^{\bar{\theta}\bar{\theta}}\left(\partial_{\bar{\theta}}\bar{F}\right),\end{aligned}\quad\text{(A.8)}$$

which has been mentioned in (46).

B. On the Verification of (49)

By exploiting the basic ideas behind the augmented superfield formulation, we demonstrate here that the choices made in (49) are *exact*. Towards this goal in mind, we note that the following (anti-)co-BRST invariant quantity:

$$s_{(a)d}\left[\varepsilon^{\mu\nu}\left(\partial_\mu\mathscr{B}\right)A_\nu-i\partial_\mu\bar{C}\partial^\mu C\right]=0,\quad\text{(B.1)}$$

should remain independent of the "soul" coordinates θ and $\bar{\theta}$ when it is generalized onto the $(1,1)$-dimensional (anti-)chiral super-submanifolds. This is physically allowed and it can be readily utilized within the framework of the augmented superfield formalism. In other words, the following equality:

$$\begin{aligned}&\varepsilon^{\mu\nu}\left(\partial_\mu \mathscr{B}(x)\right)\mathscr{B}_\nu\left(x,\theta,\bar{\theta}\right)\\&\quad - i\partial_\mu \bar{F}^{(dh)}\left(x,\theta,\bar{\theta}\right)\partial^\mu F^{(dh)}\left(x,\theta,\bar{\theta}\right)\\&= \varepsilon^{\mu\nu}\left(\partial_\mu \mathscr{B}(x)\right)A_\nu(x) - i\partial_\mu \bar{C}(x)\partial^\mu C(x),\end{aligned} \tag{B.2}$$

should hold good as far as the (super)fields of our present theory are concerned. Plugging in the superexpansions from (7) and (50) for $\mathscr{B}_\mu(x,\theta,\bar{\theta})$, $F^{(dh)}(x,\theta,\bar{\theta})$, and $\bar{F}^{(dh)}(x,\theta,\bar{\theta})$, we obtain the following relationships:

$$\begin{aligned}&\varepsilon^{\mu\nu}\left(\partial_\mu \mathscr{B}(x)\right)\bar{R}_\nu(x) + \partial_\mu C(x)\partial^\mu \mathscr{B}(x) = 0,\\&\varepsilon^{\mu\nu}\left(\partial_\mu \mathscr{B}(x)\right)R_\nu(x) + \partial_\mu \bar{C}(x)\partial^\mu \mathscr{B}(x) = 0,\\&\varepsilon^{\mu\nu}\left(\partial_\mu \mathscr{B}(x)\right)S_\nu(x) + \partial_\mu \mathscr{B}(x)\partial^\mu \mathscr{B}(x) = 0,\end{aligned} \tag{B.3}$$

which are obtained when we set equal to zero the coefficients of θ and $\bar{\theta}$ and $\theta\bar{\theta}$ in the equality (B.2). From (B.3), it is clear that we obtain

$$\begin{aligned}\bar{R}_\mu &= -\varepsilon_{\mu\nu}\partial^\nu C,\\R_\mu &= -\varepsilon_{\mu\nu}\partial^\nu \bar{C},\\S_\mu &= \varepsilon_{\mu\nu}\partial^\nu \mathscr{B}.\end{aligned} \tag{B.4}$$

The substitution of these values into (7) leads to the derivation of the expansions $\mathscr{B}_\mu^{(dh)}(x,\theta,\bar{\theta})$. It is worth pointing out that the expansions for $F^{(dh)}(x,\theta,\bar{\theta})$ and $\bar{F}^{(dh)}(x,\theta,\bar{\theta})$ have been obtained due to the dual-HC (given in (45)). This demonstrates that, for our 2D theory, the choices made in (49) can be computed *exactly* in a precise manner.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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