FLOW OF STRESS POWER LAW FLUIDS IN CYLINDERS OF NON CIRCULAR CROSS SECTIONS

A Dissertation

by

JUAN PABLO GOMEZ CONSTANTE

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Chair of Committee,	Kumbakonam R. Rajagopal
Co-Chair of Committee,	Igor Zelenko
Committee Members,	Alan Freed
	Raytcho Lazarov
Head of Department,	Andreas A. Polycarpou

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ABSTRACT

Inspired in the work of Poiseuille, and taking advantage of the much more realistic model for describing the mechanical properties of blood in large vessels derived by K. R. Rajagopal, this dissertation explains the behavior of blood on non circular cross sections, given an Implicit Constitutive Relation.

First, by assuming that the axial direction is dominant in this phenomenom, an axial flow problem is solved. Given the non linear and implicit nature of the problem, a non linear system of coupled partial differential equations will be solved. Using the Minty-Browder Theorem and the theory for linear elliptic operators, some elementary results about existense and uniqueness of solutions can be stated.

Next, assuming a lower order velocity field in the other two directions, a secondary flow problem in the cross sectional area is stated. Taking into account just the terms of the same order and the solution of the previous step, the problem becomes a linear coupled system of partial differential equations. Using the theorem of existence of a streamfunction in a two dimensional problem, we can actually prove that there are no secondary flows for this model.

Finally, a numerical approximation directly based on the Lions-Mercier Splitting Algorithm is given. Some generalizations of the problem are proposed as future work.

DEDICATION

To my parents and my brothers who always trusted in me. To my wife Lia, for her unconditional love and sacrifice in order to help me achieve one of my biggest dreams. To my children Juan Sebastian and Sofia Victoria, whose mere existence is what gives my life sense and strength to keep on.

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All the work conducted for the dissertation was completed by the student independently.

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NOMENCLATURE

Ω	Open, bounded and connected subset of $\mathbb{R}^2$
$\mathcal{D}(\overline{\Omega})$	Space of smooth functions with compact support in Ω
$supp(f)$	Support of the function f
$\overline{A}^X$	Closure of the subset $A \subset X$ in the topology of the space X
$\|u\|_H$	Norm of the element $u \in H$ in the space H
$\langle f, u \rangle_{H',H}$	Duality in between the dual element $f \in H'$ and the element $u \in H$ in the space H
$(u, v)_H$	Inner product in between the elements $u \in H$ and $v \in H$ in the space H
$L^2(\Omega)$	Space of square-Lebesgue-integrable functions
$H^1(\Omega)$	Space of weakly differentiable functions of $L^2(\Omega)$
$H^2(\Omega)$	Space of twice weakly differentiable functions of $L^2(\Omega)$

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1. INTRODUCTION

1.1 History

In the first half of the XIX century, a young, brilliant French physician named Jean Leonard Marie Poiseuille experimentally demostrated, amongst other things, that there was a direct relationship in between the pressure and the volumetric flow in an artery with blood as the working fluid. Around the same time, a German Engineer called Gotthilf Heinrich Ludwig Hagen published a paper in which he demostrated a similar but less accurate relationship for the flow of water on circular pipes.

Figure 1.1: Jean Leonard Marie Poiseuille (1797-1869). From a photographic portrait that appeared with the article by Brillouin (1930); oil-painted enhancement by SPS.

Independently of Poiseuille's and Hagen's work, Sir George Gabriel Stokes solved the problem of a fluid flowing on a circular pipe and found a very similar result. Although, Stokes did not published his results as he was not sure about the no slip condition on the boundary. Years later, he will accept the non slip boundary condition for viscous fluids as a result of his studies on the drag around solid objects. Since both Hagen and Poiseuille worked independently over the same problem and found similar results, the scientific comunity called this particular physical problem as the Hagen-Poiseuille Flow.[1] .

Since them, as the mathematical tools available became more and more advanced, general versions of this problem have been solved (compressible flow, porous media, non circular cross section, time dependent pressure gradient, etc.). Nevertheless, most of these new problems rely on the same physical hypothesis: the Constitutive Relation in between the symmetric part of the velocity gradient and the traceless part of the Cauchy Stress tensor can be solved in terms of the last one, i.e., the Constitutive Relation is Explicit.

The present work deals with the case of what happens when the Constitutive Relation is implicit, and the effects (if any) of the geometry of the cross section of the pipe.

1.2 General definitions

Before we start with the statement of the problem, it is neccesary to state some basic definitions that will be used throughout the body of this research.

1.2.1 Implicit Constitutive Relation

In the classical theory of Fluid (Solid) Mechanics, the Constitutive Relation that relates the Cauchy stress tensor and the gradients of velocity (position) respectively is given as function of the later:

$$\mathbb{T} = \mathbb{F}(\mathbb{D}) \tag{1.1}$$

Of course, this approach has the advantage that the divergence of the stress can be directly replaced into the Conservation of Linear Momentum to get a system of differential equations in terms of the velocity only. Obviously, in this case the boundary conditions are only necessary for the velocity, since the problem does not depend on the Cauchy Stress tensor anymore.

The main advantage of the classical approach is the fact that it reduces the number of unknowns of the problem, and therefore the number of boundary conditions. On the other hand, it cannot represent accurately the physical behaviour of some types of fluids, even some classical ones as Bingham Fluids. Moreover, philosophycally it is also inconsistent with the causality of the problem, since the cause (Cauchy stress) is written in terms of the effect (motion).

Rajagopal[1], in a historical change of view of Mechanics, developed a series of theories called "Implicit Constitutive Theories". The term "implicit" comes from the facts that on his models, the kinematic invariants (velocity and deformation gradient dependent) are given as a tensorial function of the stress and that, in general, the Constitutive Relation cannot be inverted to get an explicit Constitutive Relation:

$$\mathbb{F}(\mathbb{T}, \mathbb{D}) = \mathbb{O} \tag{1.2}$$

In a subsequent paper[2], Rajagopal presented a class of fluids called stress power-law fluids which can fairly represent the behaviour of blood flowing on relatively large vessels. This class of incompressible fluids is represented by the tensorial equation:

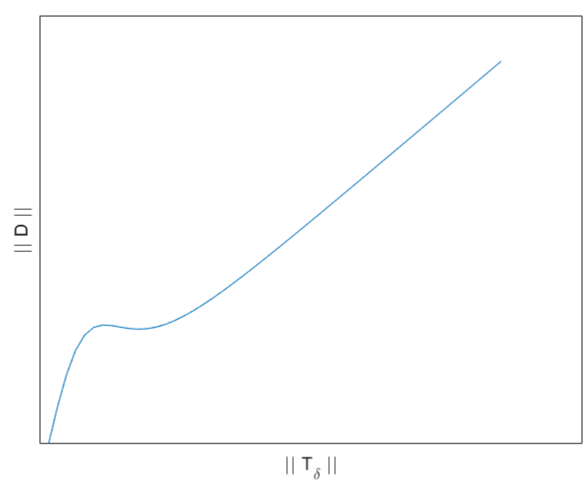


Figure 1.2: Norm of the symmetric gradient of velocity in terms of the deviatoric Cauchy stress tensor.

[1] K. R. Rajagopal: On implicit constitutive theories. Appl Math. 48 (2003), 279-319.

$$\mathbb{D} = \left[\alpha(1+\beta\left\|\mathbb{T}_\delta\right\|^2)^{-n} + \gamma\right]\mathbb{T}_\delta \tag{1.3}$$

where $\mathbb{T}_\delta = \mathbb{T} - (\frac{1}{3}\text{tr }\mathbb{T})\mathbb{I}$, α, β, γ are positive constants, and $n \geq \frac{1}{2}$. Rajagopal proved that when $\frac{\gamma}{\alpha} < d_n$ with $d_n = 2(\frac{2n-1}{2n+2})^{n+1}$ the norm of (1.1) is non monotonic, therefore the tensorial function not only is not invertible but it is not even a function of the symmetric part of the velocity gradient either.

1.2.2 Secondary Flow

In Fluid Mechanics, a secondary flow is defined as a lower order flow assumed to be perpendicular to the main direction of the stream, superimposed to a simplified primary flow that flows parallel to the main direction. Most of the times, the analytic primary flow solution can be found. This assumption is made in order to simplify the resolution of the differential equations, because we tranform the non linearity of the conservation of linear momentum into a linear recursion in terms of the previous step.

Moreover, the non Newtonian behavior of the model expressed in the fact that the normal stress components are different gives rise to interesting non linear effects such as "rod climbing" and "die swell". In fact, Fosdick and Serrin [3] proved that for a steady rectilinear flow in a pipe with bounded and connected cross sectional area and with material functions ϕ and μ satisfying appropiate analytic and monotonic conditions and such that they are not porportional in between them for small shear rates, the only possible cross section of the pipe must be either circular or in between two concentric circles. Therefore, we can reasonably expect to have a secondary flow in our problem since it is defined in a general bounded, simply connected cross sectional pipe.

The secondary flow assumption will be extended for the Cauchy stress tensor too. This generalization is very natural and simple to obtain, since it follows from the directions of both the primary and the secondary velocity fields.

1.3 Problem Statement

As it was previously stated, for the sake of simplicity, we will focus on the study of the behaviour of blood flowing inside an elliptical cross sectional pipe. Since the general problem is excedenly hard to solve, we will start with a somewhat physically meaningful simplification of the general phenomenom.

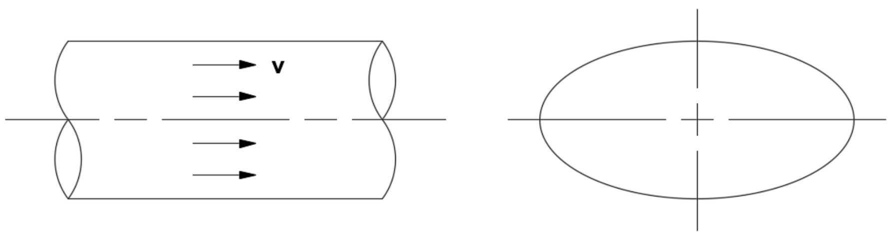


Figure 1.3: Elliptic pipe.

1.3.1 Hypothesis and assumptions

To start, we will assume that the fluid will be flowing on a steady state ($\frac{\partial \mathbf{v}}{\partial t} = \mathbf{0}$), that the velocity field is fully developed (which eliminates the dependence on the axial direction) and that velocity and stress can be expressed in terms of a first order Taylor expansion:

$$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 \tag{1.4}$$

where $\mathbf{v}_0$ is the primary axial flow, $\mathbf{v}_1$ is the secondary flow (in the plane perpendicular to the primary flow) and:

$$\mathbb{T} = \mathbb{T}_0 + \epsilon \mathbb{T}_1 \tag{1.5}$$

where $\mathbb{T}_0$ is the stress tensor associated with the primary axial flow and $\mathbb{T}_1$ is the stress tensor associated to the secondary flow. Next, recall the conservation of linear momentum:

$$\rho \frac{d\mathbf{v}}{dt} = \text{div } \mathbb{T} + \rho \mathbf{b} \tag{1.6}$$

and the incompressible version of the conservation of mass:

$$\text{div } \mathbf{v} = 0 \tag{1.7}$$

For simplicity in the analysis, we will also assume no body forces. All the previous assumptions are made to transform the general tridimensional problem into a two dimensional one in which every cross sectional area have the same vector fields (velocity) and tensor fields (stress).

1.3.2 Nondimensional analysis

In order to make an easier and more meaningful physical analysis of the problem, we define the dimensionless variables:

$$\begin{array}{ccc} \mathbf{x}^* = \frac{1}{L}\mathbf{x} & t^* = \frac{V}{L}t & \mathbf{v}^* = \frac{1}{V}\mathbf{v} \\ \mathbb{D}^* = \frac{L}{V}\mathbb{D} & \mathbf{T}^* = \frac{\alpha L}{V}\mathbf{T} & \mathbb{T}^*_\delta = \frac{\alpha L}{V}(\mathbb{T}^*)_\delta \end{array} \tag{1.8}$$

where L is a characteristic length, V is the characteristic maximum speed and α is the characteristic inverse of the dynamic viscosity. These characteristic properties depend on both the geometry of the cross section and in the actual velocity profile obtained from the calculations.

Now, neglecting the gravitational and other source of body forces, equations (1.3) (1.6) and (1.7) become:

$$\mathbb{D}^* = \left[(1 + R_2 \left\|\mathbb{T}^*_\delta\right\|^2)^{-n} + R_3\right] \mathbb{T}^*_\delta \tag{1.9}$$

$$\frac{d\mathbf{v}^*}{dt^*} = \frac{1}{R_1}\text{div } \mathbb{T}^* \tag{1.10}$$

and

$$\text{div } \mathbf{v}^* = 0 \tag{1.11}$$

where $R_1 = \alpha\rho VL$, $R_2 = \frac{\beta V^2}{\alpha^2 L^2}$, and $R_3 = \frac{\gamma}{\alpha}$. For the sake of simplicity, from now on we will drop the asterisk notation and work only with the parameters R_1, R_2 and R_3.

The advantage of the non dimensional equations comes into the physical meaning of the coefficients R_1, R_2 and R_3: Reynolds number, shear thinning/thickening by Newtonian shear stress ratio, and Non linear shear by Newtonian shear ratio respectively.

1.3.3 Primary and Secondary flow equations

By replacing equations (1.4), (1.5) into equations (1.10) and (1.11) and taking in account the assumptions previously discussed it yields:

$$[\text{grad } (\mathbf{v}_0 + \epsilon \mathbf{v}_1)](\mathbf{v}_0 + \epsilon \mathbf{v}_1) = \frac{1}{R_1} \text{div } (\mathbb{T}_0 + \epsilon \mathbb{T}_1) \tag{1.12}$$

$$\text{div } (\mathbf{v}_0 + \epsilon \mathbf{v}_1) = 0 \tag{1.13}$$

now, if we replace (1.4) and (1.5) into (1.9) and use the linearity of trace we get:

$$\mathbb{D}_0 + \epsilon \mathbb{D}_1 = \left[(1 + R_2 \left\| \mathbb{T}_{0\delta} + \epsilon \mathbb{T}_{1\delta} \right\|^2)^{-n} + R_3 \right] (\mathbb{T}_{0\delta} + \epsilon \mathbb{T}_{1\delta}) \tag{1.14}$$

next, since the axial flow $\mathbf{v}_0$ has only one component across the axial direction, and that component depends only on the other independent variables, then $(\text{grad } \mathbf{v}_0)\mathbf{v}_0 = \mathbf{0}$. Similarly, it is easy to see that $(\text{grad } \mathbf{v}_1)\mathbf{v}_0 = \mathbf{0}$. Moreover, because of the nature of the secondary flow the traction on both the primary and the secondary Cauchy Stress tensor are always perpendicular in between them, which then leads to the useful relation:

$$\left\| \mathbb{T}_{0\delta} + \epsilon \mathbb{T}_{1\delta} \right\|^2 = \left\| \mathbb{T}_{0\delta} \right\|^2 + \epsilon^2 \left\| \mathbb{T}_{1\delta} \right\|^2 \approx \left\| \mathbb{T}_{0\delta} \right\|^2 \tag{1.15}$$

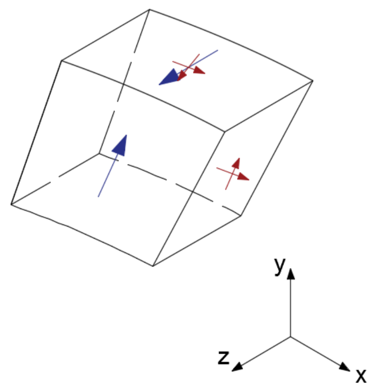


Figure 1.4: Primary shear traction (blue) and secondary shear and normal traction (red). The axial direction is given by z.

This small property will make a huge difference in terms of solvability of the problem as we will see next.

Re arranging terms on equations (1.12), (1.13) and replacing equation (1.14) along with the definition of trace of a tensor leads to solving the problem:

$$\epsilon(\text{grad }\mathbf{v}_0)\mathbf{v}_1+\epsilon^2(\text{grad }\mathbf{v}_1)\mathbf{v}_1 = \frac{1}{R_1}\left[\text{div }\mathbb{T}_{0\delta} + \text{grad }(\frac{1}{3}\text{tr }\mathbb{T}_0)\mathbb{I}\right]+\epsilon\frac{1}{R_1}\left[\text{div }\mathbb{T}_{1\delta} + \text{grad }(\frac{1}{3}\text{tr }\mathbb{T}_1)\mathbb{I}\right] \tag{1.16}$$

$$\mathbb{D}_0 + \epsilon\mathbb{D}_1 = \left[(1 + R_2 \left\|\mathbb{T}_{0\delta}\right\|^2)^{-n} + R_3\right] \mathbb{T}_{0\delta} + \epsilon \left[(1 + R_2 \left\|\mathbb{T}_{0\delta}\right\|^2)^{-n} + R_3\right] \mathbb{T}_{1\delta} \tag{1.17}$$

$$\text{div }\mathbf{v}_0 + \epsilon \text{ div }\mathbf{v}_1 = 0 \tag{1.18}$$

Finally, equating the terms of the polynomial in the variable ϵ on each one of the three previous equations, and neglecting the only second order term ((grad $\mathbf{v}_1)\mathbf{v}_1$) that appears, the problem of finding a secondary flow becomes two coupled problems of systems of partial differential equations given by:

$$\begin{cases} -\text{grad}\left(\frac{1}{3}tr\mathbb{T}_0\right) = \text{div } \mathbb{T}_{0\delta} \\ \mathbb{D}_0 = \left[(1 + R_2 \left\|\mathbb{T}_{0\delta}\right\|^2)^{-n} + R_3\right] \mathbb{T}_{0\delta} \\ \text{div } \mathbf{v}_0 = 0 \end{cases} \tag{1.19}$$

$$\begin{cases} -\text{grad}\left(\frac{1}{3}tr\mathbb{T}_1\right) = -(\text{grad } \mathbf{v}_0)\mathbf{v}_1 + \text{div } \mathbb{T}_{1\delta} \\ \mathbb{D}_1 = \left[(1 + R_2 \left\|\mathbb{T}_{0\delta}\right\|^2)^{-n} + R_3\right] \mathbb{T}_{1\delta} \\ \text{div } \mathbf{v}_1 = 0 \end{cases} \tag{1.20}$$

Notice that the system of equations (1.20) is linear in both $\mathbf{v}_1$ and $\mathbb{T}_{1\delta}$, thus it can be reduced to a classical Navier Stokes problem, with one of its terms depending on $\mathbb{T}_{0\delta}$ known from the previous step. Also, it is clear that the system of Partial Differential Equations (PDE) for the secondary flow can only be solved once that the solution for the primary flow is obtained.

Now that we have already set the appropiate system of PDE for both problems, we will develop all the theoretical scheme necessary to solve them.

2. PRIMARY AXIAL FLOW

From the previous Chapter, it was proven that finding the secondary flow problem reduces to solving two coupled systems of PDE, one after the other. To do so, we have first to calculate and determine how the conservation equations and the Constitutive Equations are represented in the Cartesian coordinate system for the main axial flow.

2.1 Mathematical statement of the problem

Since we assumed a fully developed fluid, we can focus our analysis on what happens on a fixed cross sectional are of the pipe, thus (given $a > 0, b > 0$) let $\Omega = \left\{(x, y) \in \mathbb{R} : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\} \subset \mathbb{R}^2$ be the domain, then the associated system of PDE is given by:

$$\begin{cases} \frac{\partial}{\partial x}(\frac{1}{3}\text{tr } \mathbb{T}_0) = 0 & \text{in } \Omega \\ \frac{\partial}{\partial y}(\frac{1}{3}\text{tr } \mathbb{T}_0) = 0 & \text{in } \Omega \\ \frac{\partial}{\partial z}(\frac{1}{3}\text{tr } \mathbb{T}_0) = \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} & \text{in } \Omega \\ \frac{\partial v_z}{\partial x} = 2G(T_{xz}, T_{yz})T_{xz} & \text{in } \Omega \\ \frac{\partial v_z}{\partial y} = 2G(T_{xz}, T_{yz})T_{yz} & \text{in } \Omega \\ v_z = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\mathbf{v}_0 = (0, 0, v_z)$ and T_{xz}, T_{yz} are the only non null elements of the stress tensor $\mathbb{T}_{0\delta}$. It is easy to see that $(\frac{1}{3}\text{tr } \mathbb{T})$ is a function of z only, and since the fluid is fully developed the system becomes:

$$\begin{cases} \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} = \lambda & \text{in } \Omega \\ \frac{\partial v_z}{\partial x} = 2G(T_{xz}, T_{yz})T_{xz} & \text{in } \Omega \\ \frac{\partial v_z}{\partial y} = 2G(T_{xz}, T_{yz})T_{yz} & \text{in } \Omega \\ v_z = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

with $G(T_{xz}, T_{yz}) = [1 + 2R_2(T_{xz}^2 + T_{yz}^2)]^{-n} + R_3$

2.2 Variational formulation

It is clear that seeking for classical solutions of this system of differential equations is a futile endeavor. Instead, we will find what is called a weak solution of the problem defined by the variational formulation of the problem.

Before we start, notice that if we take a smooth function v with compact support in Ω and a smooth two dimensional vector field $\mathbf{T}$, we have the following:

$$\int_\Omega \text{div } (v\mathbf{T}) \text{ dx} = \int_{\partial\Omega} v\mathbf{T} \cdot \widehat{\mathbf{n}} \text{ dS} = \int_\Omega \text{grad } v \cdot \mathbf{T} \text{ dx} + \int_\Omega v\text{div } \mathbf{T} \text{ dx} \tag{2.3}$$

this clearly leads to the interesting relation:

$$\int_\Omega -\text{grad } v \cdot \mathbf{T} \text{ dx} = \int_\Omega v \text{ div } \mathbf{T} \text{ dx} \tag{2.4}$$

the previous equality holds for any v and $\mathbf{T}$ since they are arbitrary. This relation suggests that both the gradient and divergence are dual operators in between them on some appropiate spaces.

Next, let us define the vector $\mathbf{T} = (T_{xz}, T_{yz})$, then the problem (2.2) can be written as:

$$\begin{cases} \text{div } \mathbf{T} = \lambda & \text{in } \Omega \\ \text{grad } v_z = 2G(|\mathbf{T}|^2)\mathbf{T} & \text{in } \Omega \\ v_z = 0 & \text{in } \partial\Omega \end{cases} \tag{2.5}$$

Now, let w be a smooth function with compact support in Ω and $\mathbf{S}$ be a smooth two dimensional vector field. Multiplying the first and second equations of (2.5) by w and $\mathbf{S}$ respectively, and integrating the result over Ω it yields:

$$\begin{cases} \int_\Omega \text{div } \mathbf{T}w \text{ dx} = \int_\Omega \lambda w \text{ dx} \\ \int_\Omega \text{grad } v_z \cdot \mathbf{S} \text{ dx} = \int_\Omega 2G(|\mathbf{T}|^2)\mathbf{T} \cdot \mathbf{S} \text{ dx} \end{cases} \tag{2.6}$$

Using the property (2.4) we end up with:

$$\begin{cases} -\int_\Omega \mathbf{T}\cdot \text{ grad } w \text{ dx} = \int_\Omega \lambda w \text{ dx} \\ \int_\Omega \text{ grad } v_z \cdot \mathbf{S} \text{ dx} = \int_\Omega 2G(|\mathbf{T}|^2)\mathbf{T}\cdot\mathbf{S} \text{ dx} \end{cases} \tag{2.7}$$

wich is true for any w and $\mathbf{S}$ as previously defined. This is what is called the variational formulation of the problem. Finally, by invoquing some classical results about density (Appendix A) we know that (2.7) holds for $v \in H_0^1(\Omega)$ and $\mathbf{T} \in (L^2(\Omega))^2$.

2.3 Abstract approach

As it was mentioned before, the core analytical approach to solve this problem comes from ideas originally developed to solve the Navier Stokes equations [4], but combined with some other results on the Theory of Monotone Operators.

Let X and M be two Hilbert spaces with norms $\|\cdot\|_M$ and $\|\cdot\|_X$ respectively. and define two continuous forms: $a(\cdot;\cdot,\cdot) : X \times X \times X \to \mathbb{R}$ and $b(\cdot,\cdot) : X \times M \to \mathbb{R}$ such that the first one is bilinear in the second and third entries, and the second form is bilinear.

Then consider the problem: Given $f \in M'$ find a pair $(\mathbf{T}, v) \in X \times M$ such that for any $\mathbf{S} \in X$ and $w \in M$ we have:

$$\begin{aligned} a(\mathbf{T};\mathbf{T},\mathbf{S}) + b(\mathbf{S}, v) = 0 \\ b(\mathbf{T}, w) = \langle f, w\rangle_{M',M} \end{aligned} \tag{2.8}$$

By Riesz Representation Theorem we know that associated with both forms, we have two linear operators $A_{(\mathbf{R})} \in \mathcal{L}(X, X')$ and $B \in \mathcal{L}(X, M')$ such that they satisfy:

$$\begin{aligned} \langle A_{(\mathbf{R})}\mathbf{S}, \mathbf{T}\rangle_{X',X} = a(\mathbf{T};\mathbf{T},\mathbf{S}) \\ \langle B\mathbf{S}, v\rangle_{M',M} = b(\mathbf{T}, w) \end{aligned} \tag{2.9}$$

Therefore, the equations on (2.8) become:

$$\begin{aligned} \langle A_{(\mathbf{T})}\mathbf{T}, \mathbf{S}\rangle_{X',X} + \langle B^* v, \mathbf{S}\rangle_{X',X} = 0 \\ \langle B\mathbf{T}, v\rangle_{M',M} = \langle f, w\rangle_{M',M} \end{aligned} \tag{2.10}$$

which is valid for any $\mathbf{T} \in X$ and any $w \in M$. Thus, our problem is equivalent to:

Given $f \in M'$, find a pair $(\mathbf{T}, v) \in X \times M$ such that:

$$\begin{aligned} \left\langle A_{(\mathbf{T})}\mathbf{T}, \mathbf{S} \right\rangle_{X',X} + \left\langle B^* v, \mathbf{S} \right\rangle_{X',X} = 0 \qquad & \forall \mathbf{S} \in X \\ \left\langle B\mathbf{T}, w \right\rangle_{M',M} = \left\langle f, w \right\rangle_{M',M} \qquad & \forall w \in M \end{aligned} \tag{2.11}$$

or equivalently:

$$\begin{aligned} A_{(\mathbf{T})}\mathbf{T} + B^* v = 0 \qquad & \text{in } X' \\ B\mathbf{T} = f \qquad & \text{in } M' \end{aligned} \tag{2.12}$$

It is clear that the operator B in the abstract form acts as the divergence operator in our original problem. Therefore, the existence and uniqueness of the solution to the problem will depend on the properties of both operators $A_{(\mathbf{T})}$ and B.

Remark 1. *In the classical approach our system is of the form:*

$$\begin{aligned} \mathbf{T} + b(v) B^* v = 0 \qquad & \textit{in } X' \\ B\mathbf{T} = f \qquad & \textit{in } M' \end{aligned} \tag{2.13}$$

where $b(v)$ is scalar field. Therefore, it is trivial to see that the previous system reduces to solving:

$$-B(b(v)B^* v) = f \textit{ in } M' \tag{2.14}$$

which is not only independent of the value of $\mathbf{T}$, but also in the general divergence form of an elliptic second order PDE.

2.4 Existence and uniqueness of solutions

Clearly, the functional $A_{(\mathbf{T})}\mathbf{T}$ from the previous section given by:

$$A_{(\mathbf{T})}\mathbf{T} = 2G(|\mathbf{T}|)\mathbf{T} = 2\left\{ \left(1 + 2R_2 |\mathbf{T}|^2\right)^{-n} + R_3 \right\} \mathbf{T} \tag{2.15}$$

is not linear, but it still has some useful properties that can help in our task of finding a solution. In fact, it is easy to see that:

$$\left\| \left\{ \left(1 + 2R_2 \left|\mathbf{T}\right|^2\right)^{-n} + R_3 \right\} \mathbf{T} \right\|_{L^2(\Omega)} \leq (1 + R_3) \left\|\mathbf{T}\right\|_{L^2(\Omega)} \tag{2.16}$$

i.e., $A_{(\mathbf{T})}\mathbf{T} \in (L^2(\Omega))^2$, and also

$$R_3 \left\|\mathbf{T}\right\|^2_{L^2(\Omega)} \leq \int_\Omega \left\{ \left(1 + 2R_2 \left|\mathbf{T}\right|^2\right)^{-n} + R_3 \right\} \mathbf{T} \cdot \mathbf{T} \,\mathrm{dx} \tag{2.17}$$

Now, take $\mathbf{T}, \mathbf{S} \in (L^2(\Omega))^2 \bigcap (C^1(\Omega))^2$ and let $t > 0$ and calculate the following:

$$\int_\Omega [G(|\mathbf{T} + t\mathbf{S}|^2)(\mathbf{T} + t\mathbf{S}) - G(|\mathbf{T}|^2)\mathbf{T}] \cdot [(\mathbf{T} + t\mathbf{S}) - \mathbf{T}] dx \tag{2.18}$$

after arranging terms and dividing by $\frac{1}{t^2}$ it yields:

$$\int_\Omega \frac{1}{t}[G(|\mathbf{T} + t\mathbf{S}|^2) - G(|\mathbf{T}|^2)]\mathbf{T} \cdot \mathbf{S} dx + \int_\Omega G(|\mathbf{T} + t\mathbf{S}|^2)\mathbf{S} \cdot \mathbf{S} dx \tag{2.19}$$

by letting $t \to 0$ we get:

$$\int_\Omega (\text{grad } G(|\mathbf{T}|^2) \cdot \mathbf{T})(\mathbf{T} \cdot \mathbf{S}) dx + \int_\Omega G(|\mathbf{T}|^2)\mathbf{S} \cdot \mathbf{S} dx \tag{2.20}$$

finally, by calculating the actual value of the differential of G, replacing it back into equation (2.20) and by Theorem of Monotonicity of Differentiable Mappings (Appendix A), we have that the monotonicity of the operator holds if, for any $\mathbf{T}, \mathbf{S} \in (L^2(\Omega))^2$:

$$\int_\Omega |\mathbf{S}|^2 \left[-4R_2 n(1 + 2R_2 \left|\mathbf{T}\right|^2)^{-n-1} \left|\mathbf{T}\right|^2 + G(|\mathbf{T}|^2)\right] dx \geq 0 \tag{2.21}$$

the critical point when $|\mathbf{T}| = 0$ gives the final condition that the operator is monotonic if:

$$R_3 \geq \left(\frac{2n - 1}{2n + 2}\right)^{n+1} \tag{2.22}$$

It turns out this is the same condition for monotonicity found by Le Roux and Rajagopal [1] Obviously, for negative values of n, the monotonicity is uniform, i.e., it is independent of any of the other parameters.

Finally, since the operator $G(|\mathbf{T}|)\mathbf{T}$ is continuous, bounded, monotone, and coercive, the Browder-Minty Theorem (Appendix B) implies that for any $\mathbf{S} \in (L^2(\Omega))^2$ there exists a unique $\mathbf{T} \in (L^2(\Omega))^2$ such that:

$$G(|\mathbf{T}|)\mathbf{T} = \mathbf{S} \tag{2.23}$$

as long as (2.22) holds.

To summarize all the previous ideas, let us prove the next.

Proposition 1. *Let $\lambda \in \mathbb{R}$, $R_2 > 0, R_3 > 0$ and n be such that $R_3 \geq (\frac{2n-1}{2n+2})^{n+1}$. Then, there exists a unique pair $(\mathbf{T}, v) \in L^2(\Omega)^2 \times H_0^1(\Omega)$ such that:*

$$\begin{cases} -\int_\Omega \mathbf{T} \cdot \textit{grad}\ w\ dx = \int_\Omega \lambda w\ dx \\ \int_\Omega \textit{grad}\ v_z \cdot \mathbf{S}\ dx = \int_\Omega 2G(|\mathbf{T}|^2)\mathbf{T} \cdot \mathbf{S}\ dx \end{cases} \tag{2.24}$$

holds for any $(\mathbf{S}, w) \in L^2(\Omega)^2 \times H_0^1(\Omega)$

Proof. Associated to λ we define the function $\lambda = -\text{div}\ \mathbf{F}_\lambda$, then the problem becomes:

$$\begin{cases} \int_\Omega (\mathbf{T} - \mathbf{F}_\lambda) \cdot \text{grad}\ w\ \text{dx} = 0 \\ \int_\Omega \text{grad}\ v_z \cdot \mathbf{S}\ \text{dx} = \int_\Omega 2G(|\mathbf{T}|^2)\mathbf{T} \cdot \mathbf{S}\ \text{dx} \end{cases} \tag{2.25}$$

which holds for any $(\mathbf{S}, w) \in L^2(\Omega)^2 \times H_0^1(\Omega)$

In particular, for $\mathbf{T} \in L^2(\Omega)^2$ the second equation on (2.25) becomes:

$$\int_\Omega \text{grad}\ v_z \cdot \mathbf{T}\ \text{dx} = \int_\Omega 2G(|\mathbf{T}|^2)\mathbf{T} \cdot \mathbf{T}\ \text{dx} \geq R_3 \|\mathbf{T}\|^2_{(L^2(\Omega))^2} \tag{2.26}$$

[1] Shear Flows of a New Class of Power-Law Fluids, Applications of Mathematics, 2013

then, by the Cauchy-Schwartz and Poincare inequality we finally have that:

$$\|\mathbf{T}\|_{(L^2(\Omega))^2} \leq \frac{C_p}{2R_3} \|v_z\|_{H_0^1(\Omega)} \tag{2.27}$$

also, by the upper boundedness of the non linear term and by the second equation of (2.5) we have that:

$$\|v_z\|_{H_0^1(\Omega)} \leq \frac{2(1+R_3)}{C_p} \|\mathbf{T}\|_{(L^2(\Omega))^2} \tag{2.28}$$

next notice that in the first equation on (2.25), in particular we have that

$$\int_\Omega \mathbf{T} \cdot \text{grad } v_z \text{ dx} = \int_\Omega \mathbf{F}_\lambda \cdot \text{grad } v_z \text{ dx} \tag{2.29}$$

thus, we can finally see that the stress has a bound given by:

$$\|\mathbf{T}\|_{(L^2(\Omega))^2} \leq \frac{(1+R_3)}{R_3} \|\mathbf{F}_\lambda\|_{(L^2(\Omega))^2} \tag{2.30}$$

and since the velocity is bounded by the stress, then so is the velocity. Therefore, since $L^2(\Omega)^2$ and $H_0^1(\Omega)$ are reflexive spaces, we can solve the finite dimensional projected problem.

Finally, let us assume that we have two solutions $(\mathbf{T}_1, v_1)$ and $(\mathbf{T}_2, v_2)$ such that they satisfy (2.5), then substracting one from another we have that:

$$\begin{cases} \text{div } (\mathbf{T}_1 - \mathbf{T}_2) = 0 & \text{in } \Omega \\ \text{grad } (v_1 - v_2) = 2(G(|\mathbf{T}_1|^2)\mathbf{T}_1 - G(|\mathbf{T}_2|^2)\mathbf{T}_2) & \text{in } \Omega \end{cases} \tag{2.31}$$

then, by calculating the inner product of the second equation with $\mathbf{T}_1 - \mathbf{T}_2$ it yields:

$$(\text{grad } (v_1 - v_2), \mathbf{T}_1 - \mathbf{T}_2)_{(L^2(\Omega))^2} = 2((G(|\mathbf{T}_1|^2)\mathbf{T}_1 - G(|\mathbf{T}_2|^2)\mathbf{T}_2), \mathbf{T}_1 - \mathbf{T}_2)_{(L^2(\Omega))^2} \tag{2.32}$$

then notice that the left hand side has the property:

$$(\text{grad }(v_1 - v_2), \mathbf{T}_1 - \mathbf{T}_2)_{(L^2(\Omega))^2} = -(v_1 - v_2, \text{div }(\mathbf{T}_1 - \mathbf{T}_2))_{(L^2(\Omega))^2} = 0 \tag{2.33}$$

this implies by the monoticity of the non linear operator that $\mathbf{T}_1 = \mathbf{T}_2$ which then implies that $v_1 = v_2$ □

Remark 2. *Notice that the bounds for both velocity and stress do not depend on anything else than the size of the domain, the Poincare constant C_p and the non dimensional parameter R_3. Moreover, from (2.30) we see that the smaller R_3 is, the bigger right hand side bound we have. This means that for very small values of R_3 the problem becomes more and more unstable, which is exactly what the monotonicity condition for R_3 from the Browder-Mintty Theorem stated.*

3.1 Mathematical statement of the problem

For the secondary flow analysis, recall the system of equations of the secondary flow:

$$\begin{cases} -\frac{1}{R_1}\text{grad }(\frac{1}{3}tr\mathbb{T}_1) = -(\text{grad }\mathbf{v}_0)\mathbf{v}_1 + \frac{1}{R_1}\text{div }\mathbb{T}_{1\delta} \\ \mathbb{D}_1 = \left[(1 + R_2 \left\|\mathbb{T}_{0\delta}\right\|^2)^{-n} + R_3\right] \mathbb{T}_{1\delta} \\ \text{div }\mathbf{v}_1 = 0 \end{cases} \tag{3.1}$$

at this time, the terms $\mathbf{v}_0$ and $\mathbb{T}_{0\delta}$ are already known, so this problem is linear on $\mathbf{v}_1$ and $\mathbf{T}_{1\delta}$. Moreover, it can be written only in terms of velocity and the trace of τ as:

$$\begin{cases} -\frac{1}{R_1}\text{grad }(\frac{1}{3}tr\mathbb{T}_1) = -(\text{grad }\mathbf{v}_0)\mathbf{v}_1 + \frac{1}{R_1}\text{div }[\frac{1}{2G(\mathbb{T}_0)}(\text{grad }\mathbf{v}_1 + \text{grad }\mathbf{v}_1^T)] \\ \text{div }\mathbf{v}_1 = 0 \end{cases} \tag{3.2}$$

where $\mathbf{v}_1 = (v_x, v_y, 0)$ and the term $(\frac{1}{3}tr\mathbb{T}_1)$ are unknown. Notice that only the gradient on the left and the first term on the right hand side of the first equation have components on the z direction.

3.2 Existence and uniqueness

The two dimensional nature of the problem and the fact that the divergence of the velocity is zero implies the existence of a streamfunction ψ such that:

$$\begin{cases} v_x = \frac{\partial\psi}{\partial y} \\ v_y = -\frac{\partial\psi}{\partial x} \end{cases} \tag{3.3}$$

Next after calculating the curl of the conservation of linear momentum (3.2) and replacing the streamfunction (3.3) it yields:

$$
\begin{cases} \triangle^2\psi + G\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{G}\right)\right]\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right) + 4G\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{G}\right)\frac{\partial^2\psi}{\partial x\partial y} = 0 \\ \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0 \\ \psi = \psi_0 \end{cases} \tag{3.4}
$$

where the symbol Δ^2 stands for the bilaplacian operator. The boundary conditions come naturally from the definition of ψ and also from the fact that the boundary itself is also a streamline.

Notice that this is a linear PDE for ψ with the function G given from the previous step of the axial flow. In particular, if the exponent $n = 0$, the function G becomes constant and the equation reduces to a biharmonic equation, whose solution is a flat surface at a distance ψ_0 from the origin, i.e., there are no streamlines being formed.

Without any loose of generality on the boundary conditions, we can rewrite the problem as follows:

$$
\begin{cases} \frac{\partial^2}{\partial x^2}\left[\frac{1}{G}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)\right] - \frac{\partial^2}{\partial y^2}\left[\frac{1}{G}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)\right] + 4\frac{\partial^2}{\partial x\partial y}\left[\frac{1}{G}\frac{\partial^2\psi}{\partial x\partial y}\right] = 0 \\ \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0 \\ \psi = 0 \end{cases} \tag{3.5}
$$

The previous equations suggests that the solution to our problem would be on the space $H_0^2(\Omega)$. Next, let $\phi \in \mathcal{D}(\overline{\Omega})$ be given. After multuplying and integrating over the domain, we get the variational form given by:

$$
\int_\Omega \frac{1}{G}\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\phi}{\partial x^2} + 4\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\phi}{\partial y^2}\right] = 0 \tag{3.6}
$$

After rearranging terms, we define the symmetric bilinear form:

$$
B(\psi,\phi) = \int_\Omega \frac{1}{G}\left[\Delta\psi\Delta\phi - 2\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - 2\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\phi}{\partial x^2} + 4\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\phi}{\partial x\partial y}\right] dx \tag{3.7}
$$

from the boundedness of G (upper and lower) we can get the estimates:

$$\begin{cases} |B(\psi, \phi)| \leq \frac{1}{R_3} \left\| D^2\psi \right\|_{L^2(\Omega)} \left\| D^2\phi \right\|_{L^2(\Omega)} \\ B(\psi, \psi) \geq \frac{1}{1+R_3} \left\| \Delta\psi \right\|^2_{L^2(\Omega)} \end{cases} \tag{3.8}$$

notice that the derivatives of ψ are functions of $H^1_0(\Omega)$, therefore we can apply the Poincare Inequality to them, sum them up and get:

$$\left\| D\psi \right\|_{L^2(\Omega)} \leq c_p \left\| D^2\psi \right\|_{L^2(\Omega)} \tag{3.9}$$

also, we have the interesting property:

$$\int_\Omega \frac{\partial^2\psi}{\partial x \partial y} \frac{\partial^2\psi}{\partial x \partial y} dx = -\int_\Omega \frac{\partial\psi}{\partial x} \frac{\partial^3\psi}{\partial x \partial y^2} dx = \int_\Omega \frac{\partial^2\psi}{\partial x^2} \frac{\partial^2\psi}{\partial y^2} dx \tag{3.10}$$

then, it is easy to see that the two previous equations imply the important relation:

$$\left\| \psi \right\|_{L^2(\Omega)} \leq c_p^2 \left\| \Delta\psi \right\|_{L^2(\Omega)} \tag{3.11}$$

that holds for any $\psi \in \mathcal{D}(\overline{\Omega})$, and by density in $H^2_0(\Omega)$

In other words, the Laplacian operator is continuous and bounded in $H^2_0(\Omega)$. Now, since the Laplacian is also symmetric, and positive definite we can directly claim (without the help of Lax-Milgram Theorem) that the bilinear form is also an inner product. Finally, we claim that the problem has solution and that it is unique for any given right hand side element in $H^{-2}(\Omega)$.

Finally, since the boundary condition for ψ is constant all over it, and since (3.5) involves derivatives only, we have finally proved that there are no secondary flows being formed for this particular Implicit Constitutive Model in our approach.

4. NUMERICAL ANALYSIS

4.1 Numerical Method

Recall the variational formulation for the primary flow:

$$\begin{cases} -\int_\Omega \mathbf{T} \cdot \text{grad } w \text{ dx} = \int_\Omega \lambda w \text{ dx} \\ \int_\Omega \text{ grad } v_z \cdot \mathbf{S} \text{ dx} = \int_\Omega 2G(|\mathbf{T}|^2)\mathbf{T} \cdot \mathbf{S} \text{ dx} \end{cases} \tag{4.1}$$

Then, let us take a Delaunay triangulation of the domain, and set V_h to be the defined by:

$$V_h = \left\{ w_h \in C^0(\Omega) : w_h \in \mathcal{P}_1(\tau_h), \tau_h \in \mathcal{T}_h \right\} \tag{4.2}$$

where $\mathcal{P}_1(\tau_h)$ is the space of first degree polynomials on the triangle τ_h of the triangulation $\mathcal{T}_h$. For the stress we can take L_h to be:

$$L_h = \left\{ \mathbf{T}_h \in L^2(\Omega) : \mathbf{T}_h = T_x \widehat{e}_x + T_y \widehat{e}_y \right\} \tag{4.3}$$

As we expected, the elements of the L_h space are functions that are constant on each triangle. Now let's take a triangle τ_h of the mesh with vertices i,j and k and define the i^{th} nodal basis element of V_h as:

$$\phi_i(x) = \begin{cases} 1 & \text{if } x = P_i \\ 0 & \text{if } x = P_j \text{ or } x = P_k \\ \text{linear} & \text{over } \tau_h \end{cases} \tag{4.4}$$

in particular, for a triangle with vertices P_i, P_j and P_k the i^{th} element $\phi_i(x, y)$ is given by

$$\phi_i(x, y) = \frac{a_i}{\Delta} + \frac{b_i}{\Delta} x + \frac{c_i}{\Delta} y \tag{4.5}$$

where the constants a_i, b_i and c_i are calculated in terms of coordinates of the vertices, and Δ is twice the size of the triangle.

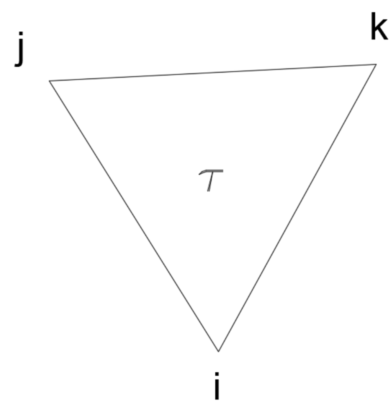


Figure 4.1: triangular element

Next, we define $\mathbf{T}_h = (T_x, T_y)$, $\mathbf{S}_h = (S_x, S_y)$, $v_h = \Phi_\tau^T \mathbf{V}$ and $w_h = \Phi_\tau^T \mathbf{W}$ where $\Phi_\tau = (\phi_i, \phi_j, \phi_k)^T$, $\mathbf{V} = (V_i, V_j, V_k)$ and $\mathbf{W} = (W_i, W_j, W_k)$. This leads to:

$$\begin{aligned} \int_{\tau_h} \text{grad } \Phi_\tau^T \mathbf{V} \cdot \mathbf{S}_h \text{ dx} &= \int_{\tau_h} (\nabla\phi_i V_i + \nabla\phi_j V_j + \nabla\phi_k V_k) \cdot (S_x \widehat{e}_x + S_y \widehat{e}_y) \text{ dx} \\ &= \begin{pmatrix} S_x & , & S_y \end{pmatrix} \frac{1}{2} \begin{pmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix} \begin{pmatrix} V_i \\ V_j \\ V_k \end{pmatrix} \end{aligned} \tag{4.6}$$

$$\begin{aligned} \int_{\tau_h} \mathbf{T}_h \cdot \text{grad } \Phi_\tau^T \mathbf{W} \text{ dx} &= \int_{\tau_h} (T_x \widehat{e}_x + T_y \widehat{e}_y) \cdot (\nabla\phi_i W_i + \nabla\phi_j W_j + \nabla\phi_k W_k) \text{ dx} \\ &= \begin{pmatrix} W_i & , & W_j & , & W_k \end{pmatrix} \frac{1}{2} \begin{pmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{pmatrix} \begin{pmatrix} T_x \\ T_y \end{pmatrix} \end{aligned} \tag{4.7}$$

$$
\begin{aligned}
\int_{\tau_h} \mathbf{T}_h \cdot \mathbf{S}_h \,\mathrm{dx} &= \int_{\tau_h} (T_x \widehat{e}_x + T_y \widehat{e}_y) \cdot (S_x \widehat{e}_x + S_y \widehat{e}_y) \,\mathrm{dx} \\
&= \begin{pmatrix} S_x & , & S_y \end{pmatrix} \frac{\Delta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_x \\ T_y \end{pmatrix}
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
\int_{\tau_h} G(\mathbf{T}_h)\mathbf{T}_h \cdot \mathbf{S}_h \,\mathrm{dx} &= G(T_x, T_y) \int_{\tau_h} (T_x \widehat{e}_x + T_y \widehat{e}_y) \cdot (S_x \widehat{e}_x + S_y \widehat{e}_y) \,\mathrm{dx} \\
&= \begin{pmatrix} S_x & , & S_y \end{pmatrix} \frac{\Delta}{2} G(T_x, T_y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_x \\ T_y \end{pmatrix}
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
\int_{\tau_h} \Phi_\tau^T \mathbf{F} \Phi_\tau^T \mathbf{W} \,\mathrm{dx} &= \int_{\tau_h} (\phi_i F_i + \phi_j F_j + \phi_k F_k)(\phi_i W_i + \phi_j W_j + \phi_k W_k) \,\mathrm{dx} \\
&= \begin{pmatrix} W_i & , & W_j & , & W_k \end{pmatrix} \int_{\tau_h} \begin{pmatrix} \phi_i\phi_i & \phi_i\phi_j & \phi_i\phi_k \\ \phi_j\phi_i & \phi_j\phi_j & \phi_j\phi_k \\ \phi_k\phi_i & \phi_k\phi_j & \phi_k\phi_k \end{pmatrix} \,\mathrm{dx} \begin{pmatrix} F_i \\ F_j \\ F_k \end{pmatrix} = \mathbf{W}^T A_\tau^0 \mathbf{F}
\end{aligned} \tag{4.10}
$$

where A_τ^0 is known as the mass matrix of the element. Notice that G is constant over each triangle since our elements are constant too.

4.1.1 Splitting Algorithm

Unfortunately, the non linear nature of the problem makes any direct approach insufficient, which is why we have to use a very clever trick based in the paper by Mercier and Lions [5], that exploits the fact that some non linear monotonic operators can be splitted into two non linear monotonic operators. In our case, since one of the two operators is linear, we can use it as a initializating value, in the algorithm that is explained as follows:

- **step 0:** set $n = 0$ in the non linear operator so it becomes a linear operator, and solve the associated linear problem:

$$
\begin{cases}
\int_\Omega (1 + R_3)\mathbf{T}^0 \cdot \mathbf{S} \,\mathrm{dx} - \int_\Omega \frac{1}{2} \,\mathrm{grad}\, v^0 \cdot \mathbf{S} \,\mathrm{dx} = 0 \\
- \int_\Omega \mathbf{T}^0 \cdot \mathrm{grad}\, w \,\mathrm{dx} = \int_\Omega \lambda w \,\mathrm{dx}
\end{cases} \tag{4.11}
$$

- **step 1:** using the previous k step and a fake time derivative, calculate the linear problem for the step $k+\frac{1}{2}$ given by:

$$\begin{cases} \int_\Omega \left[\frac{1}{h}(\mathbf{T}^{k+\frac{1}{2}} - \mathbf{T}^k) + R_3\mathbf{T}^{k+\frac{1}{2}} - \nabla v^{k+\frac{1}{2}}\right]\cdot \mathbf{S}\ \mathrm{dx} = -\int_\Omega (1+2R_2\left|\mathbf{T}^k\right|^2)^{-n}\mathbf{T}^k\cdot\mathbf{S}\ \mathrm{dx} \\ -\int_\Omega \mathbf{T}^{k+\frac{1}{2}}\cdot\nabla w\ \mathrm{dx} = \int_\Omega \lambda w\ \mathrm{dx} \end{cases} \tag{4.12}$$

- **step 2:** using the previous $k+\frac{1}{2}$ step, calculate the nonlinear $k+1$ term given by:

$$\int_\Omega \left[\frac{1}{h}(\mathbf{T}^{k+1} - \mathbf{T}^{k+\frac{1}{2}}) + (1+2R_2\left|\mathbf{T}^{k+1}\right|^2)^{-n}\mathbf{T}^{k+1}\right]\cdot\mathbf{S}\ \mathrm{dx} = \int_\Omega (\nabla v^{k+\frac{1}{2}} - R_3\mathbf{T}^{k+\frac{1}{2}})\cdot\mathbf{S}\ \mathrm{dx} \tag{4.13}$$

  now feed the first step with the result of the second step recursively until the solution converges to a steady state.

Finally, notice that the finite dimensional scheme for step 0 has the structure:

$$\begin{pmatrix} (1+R_3)\mathbf{D} & -\frac{1}{2}\mathbf{B}^T \\ \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{T}_x} \\ \overrightarrow{\mathbf{T}_y} \end{pmatrix} \\ \overrightarrow{\mathbf{V}} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \overrightarrow{0} \\ \overrightarrow{0} \end{pmatrix} \\ A^0\overrightarrow{\mathbf{\Lambda}} \end{pmatrix} \tag{4.14}$$

and the scheme for step 1 has the structure:

$$\begin{pmatrix} (\frac{1}{h}+R_3)\mathbf{D} & -\frac{1}{2}\mathbf{B}^T \\ \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{T}_x} \\ \overrightarrow{\mathbf{T}_y} \end{pmatrix} \\ \overrightarrow{\mathbf{V}} \end{pmatrix} = \begin{pmatrix} \mathbf{D}(\overrightarrow{\mathbf{T}^0}) \begin{pmatrix} \overrightarrow{\mathbf{T}^0_x} \\ \overrightarrow{\mathbf{T}^0_y} \end{pmatrix} \\ A^0\overrightarrow{\mathbf{\Lambda}} \end{pmatrix} \tag{4.15}$$

where $\mathbf{D} \in \mathbb{R}^{2M\times 2M}$ is a diagonal matrix, $\mathbf{B} \in \mathbb{R}^{N\times 2M}$ is a rectangular matrix and $\mathbf{0} \in \mathbb{R}^{N\times N}$ is a null matrix.

Remark 3. *Notice that by the choice of the elements on the stress, the second step (4.13) is in reality a nonlinear algebraic system of two equations, that can be solved triangle by triangle by applying, for example, the classical Newton's Method.*

4.2 Numerical Results

For the experimental results, a Matlab® algorithm was developed using a Delaunay triangular mesh obtained with Abaqus®. The code had to be written since most simulation programs do not allow to compute Implicit Constitutive Relations.

4.2.1 Effects on the minor to major axis ratio for the elliptic cross section

As it can be seen from the results (fig. 4.2), if all the parameters of the model are kept fixed, the minor to major axis ratio has a minor effect on the axial velocity. It escentially just decreases when the ratio decreases, which is consistent with the decreased cross sectional area. Overall, the surface that represents the axial velocity remains very similar to a paraboloid.

From the plots of the norm of the shear stress (fig. 4.3), it can be seen that the maximum stress occurs all over the boundary for a circular cross section, and it smoothly changes to being just two single points on the ends of the minor axes when the ratio in between them changes.

4.2.2 Effects on the geometry of the domain

As it was stated before, the numerical model also works on different domains (fig. 4.4), as long as they are bounded and simply connected. In this case, the maximum stress will appear on the boundary point which is the closest to the center of the domain, and the velocity profile would remain to be a paraboloid-like surface.

4.2.3 Numerical experiments

To compare how well the numercal approximation works, it is important to compare it against a particular given analitycal solution. It turns out that when the cross section of the pipe is circular, the problem becomes trivial, and the velocity is then given analitically by the equation:

$$v_z = \frac{1}{(-n+1)R_2\lambda}\left\{\left[1+\frac{R_2\lambda^2}{2}(x^2+y^2)\right]^{-n+1} - \left(1+\frac{R_2\lambda^2}{2}\right)^{-n+1}\right\} + \frac{R_3\lambda}{2}(x^2+y^2-1) \tag{4.16}$$

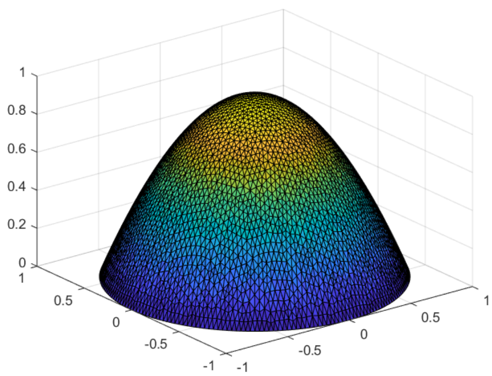
From the formulation of the problem, we see that the most critical power exponent such that the model is implicit is when $n = \frac{1}{2}$, and also by the monotonicity of the nonlinear term, the critical value for the model to be monotonic is when $R_3 \geq 2e^{-\frac{3}{2}}$ therefore we pick $R_3 = \frac{1}{2}$. A similar analysis tell us that a meaningful value for the last parameter will be $R_2 = 1$.

Table 4.1: Experimental error for different pressure gradients

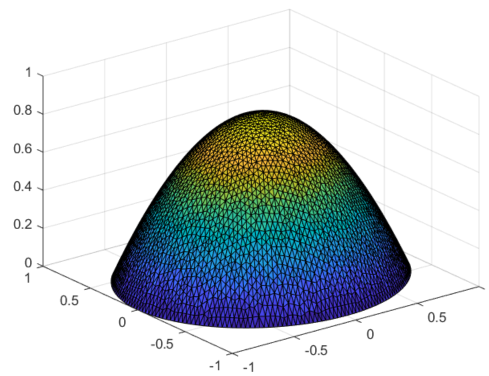
pressure gradient	0.1	0.25	0.5	1
coarse	0.0089	0.0084	0.0079	0.0065
refined	8.6981×10^{-4}	7.8671×10^{-4}	7.1506×10^{-4}	5.7597×10^{-4}

From the experimental errors, it is easy to see that the smaller the pressure gradient, the greater the error, which is consistent with the theory since the nonlinearity is dominant for small values of the pressure gradient. For large enough pressure gradients the error becomes smaller, since the model tends to a Newtonian fluid.

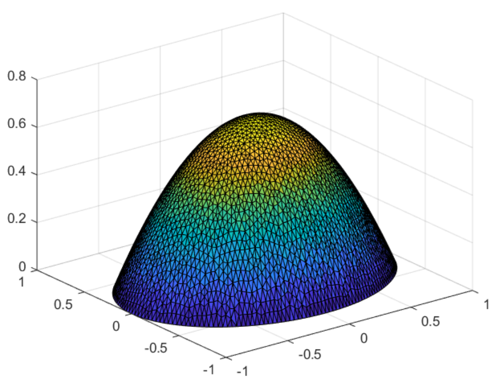
Remark 4. *Notice that the stress field cannot be given in terms of an analytical solution since the problem itself is implicit. Therefore, there is no point on comparing the numerical results for stress agaisnt another numerical approximation.*



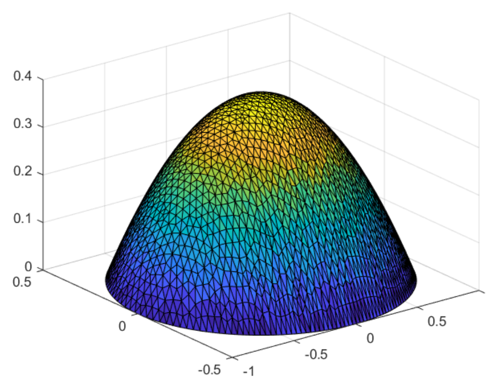
$\frac{b}{a} = 1$



$\frac{b}{a} = 0.9$

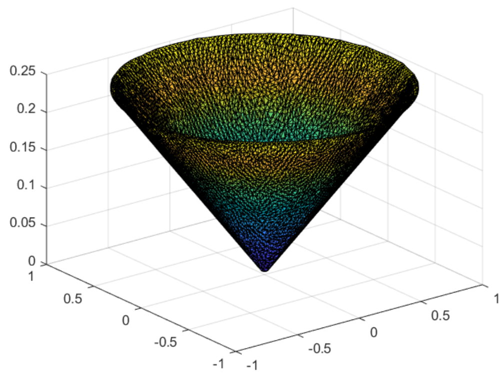


$\frac{b}{a} = 0.75$

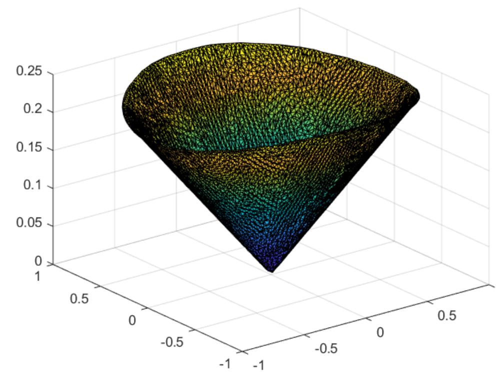


$\frac{b}{a} = 0.5$

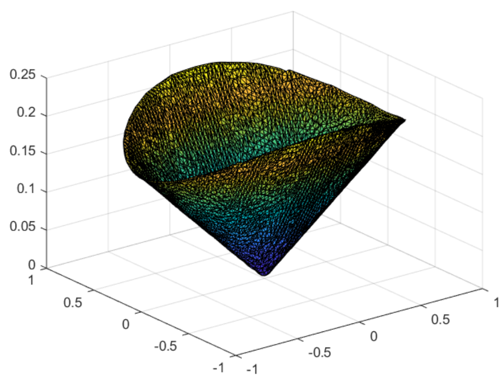
Figure 4.2: Effect of the axes ratio $\frac{b}{a}$ on the axial velocity for given values on the model's parameters



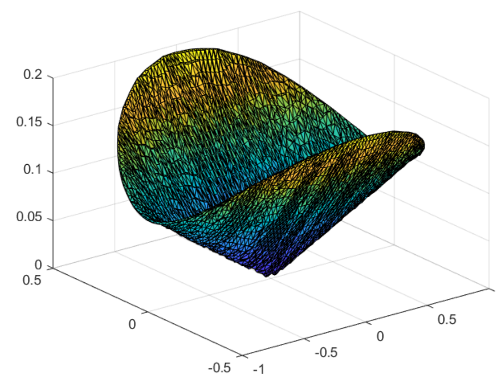
$\frac{b}{a} = 1$



$\frac{b}{a} = 0.9$

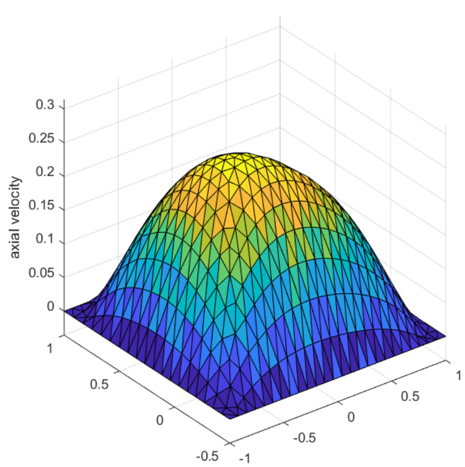


$\frac{b}{a} = 0.75$

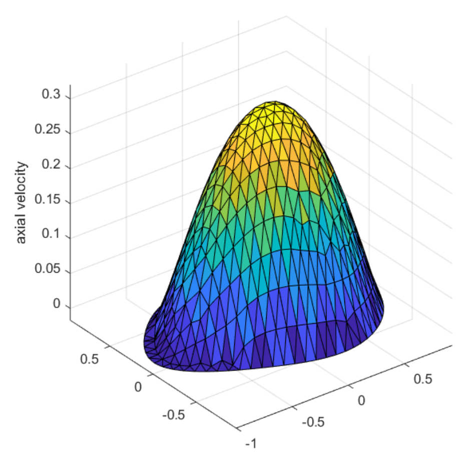


$\frac{b}{a} = 0.5$

Figure 4.3: Effect of the axes ratio $\frac{b}{a}$ on the shear stress for given values on the model's parameters



L-shaped domain



Irregular domain

Figure 4.4: Axial velocity for different geometries

5. CONCLUSIONS

- Solving the secondary flow approximated problem (first order perturbation of velocity and stress) is equivalent to solve the axial flow problem only.
- The result does not depend on the geometry of the boundary, as long as the domain is bounded and simply connected.
- Notice also, that the problem does not depend on the Reynolds number R_1, which is a hughe problem when it comes to stability analysis.
- For the elliptical case, the non linearities are barely noticeable since the profile remain closer to the paraboloid profile of the classical solution.
- For general shape domains, the the axial velocity remains as a paraboloid-like surface where the maximum velocity occurs around some inner point of the domain, whereas the maximum stress occurs on the closest point to the maximum velocity in the boundary.
- The algorithm behaves quite well for small values of n and λ, but it starts to become slower and slower when these parameters increase.

5.1 Challenges

Recall the abstract formulation of the problem (equation 2.12) from the axial flow:

$$\begin{aligned} A_{(\mathbf{T})}\mathbf{T} + B^{*}v = 0 \qquad & \text{in } X' \\ B\mathbf{T} = f \qquad & \text{in } M' \end{aligned} \tag{5.1}$$

In our perturbation approach , and because of the space of solutions chosen, the second equation depends linearly on only one of the unknowns. Without having any perturbation on the velocity or the Cauchy stress tensor, the problem becomes:

$$A_{(\mathbf{T})}\mathbf{T} + B^{*}v = 0 \qquad \text{in } X'$$
$$C(v)v + B\mathbf{T} = f \qquad \text{in } M' \tag{5.2}$$

where $C(v)v$ is also a nonlinear operator. Obviously, this problem must be solved simultaneously for velocity and stress. Also, the ellipticity condition, in general, is no longer valid. Nevertheless, we can still apply the Minty-Browder Theorem (Appendix A) for the nonlinear term $A_{(\mathbf{T})}\mathbf{T}$ and try to apply something similar to $C(v)v$. Also, notice that the fully developed flow assumption can still be applied.

5.2 Further Study

The next natural step from here is to analyze the transient axial flow problem. The only change that we will have is that now we will assume a pulsatile flow, which brings an extra term on the Conservation of Linear Momentum, wich transforms the elliptic nature of this equation into a parabolic PDE. Notice that in this case, the Reynolds number R_1 will play a role on the model.

Since our main interest is the study of the flow of human blood in artificial pipes, and given that this pipes are flexible, the more interesting problem to solve is the flow of the blood under the same Stress Power-Law Constitutive Relation but for a curved pipe. There are currently available some studies on the classical approach [6] that can be generalized for the type of fluids we are interested in. The main challenge of this problem is the fact that the curvature of the pipe plays a very important rol and cannot be neglected nor taken as a small parameter. An analytical approach using the concepts of differential manifolds with curvilinear coordinate systems should be considered for the differential operators involved in the problem.

REFERENCES

[1] S. P. Sutera and R. Skalak, "The history of poiseuille's law," *Annu. Rev. Fluid Mech.*, vol. 25, pp. 1–19, 1993.

[2] C. L. Roux and K. R. Rajagopal, "Shear flows of a new class of power law fluids," *Applications of Mathematics*, vol. 58, no. 2, pp. 153–177, 2011.

[3] R. L. Fosdick and J. Serrin, "Rectilinear steady flow of simple fluids," *Proccedings of the Royal Society*, no. 332, pp. 311–333, 1973.

[4] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier Stokes Equations*. Springer Series on Computational Mathematics, 1986.

[5] P. L. Lions and B. Mercier, "Splitting algorithms for the sum of two nonlinear operators," *SIAM J. Numer. Anal.*, no. 16, 1979.

[6] J. H. Siggers and S. L. Waters, "Steady flows in pipes with finite curvature," *Physics of Fluids*, no. 17, 2005.

APPENDIX A

SOME CLASSICAL RESULTS ON FUNCTIONAL ANALYSIS

A.1 Sobolev Spaces and L^2 Spaces

Definition 1. *Let $\Omega \subset \mathbf{R}^n$ be a non empty, open, bounded set. We define the space $H^1(\Omega)$ to be:*

$$H^1(\Omega) = \left\{ w \in L^2(\Omega) : \frac{\partial w}{\partial x_i} \in L^2(\Omega), i = 1, ..., n \right\} \tag{A.1}$$

with inner product defined by:

$$(u, v)_{H^1(\Omega)} = \int_\Omega \left(uv + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) \, dx \tag{A.2}$$

Likewise, we define $(H^2(\Omega))^n$ to be:

$$H^2(\Omega) = \left\{ w \in L^2(\Omega) : \frac{\partial w}{\partial x_i} \in L^2(\Omega), \frac{\partial^2 w}{\partial x_i \partial x_j} \in L^2(\Omega) i, j = 1, ..., n \right\} \tag{A.3}$$

with inner product defined by:

$$(\mathbf{S}, \mathbf{T})_{H^2(\Omega)} = \int_\Omega \left(uv + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \, dx \tag{A.4}$$

where differentiation is meant to be in the sense of a Lebesgue measure.

Finally, we define $(L^2(\Omega))^n$ to be:

$$(L^2(\Omega))^n = \underbrace{L^2(\Omega) \times ... \times L^2(\Omega)}_{n\text{-times}} \tag{A.5}$$

with inner product defined by:

$$(\mathbf{S}, \mathbf{T})_{L^2(\Omega)} = \int_\Omega \mathbf{S} \cdot \mathbf{T} \, dx \tag{A.6}$$

Now, we invoque the next classical result from the theory of Sobolev spaces in order to find an appropiate space in which our velocity would be in.

Definition 2. *Under the same hypothesis of the previous definition, we define the closed subspace* $H_0^1(\Omega)$ *to be given by*

$$H_0^1(\Omega) = \overline{\mathcal{D}(\overline{\Omega})}^{H^1(\Omega)} \tag{A.7}$$

$$H_0^2(\Omega) = \overline{\mathcal{D}(\overline{\Omega})}^{H^2(\Omega)} \tag{A.8}$$

where

$$\mathcal{D}(\overline{\Omega}) = \left\{ v \in \mathcal{C}^\infty(\Omega) : \overline{supp(v)}^{\mathbb{R}^2} \subset \Omega \right\} \tag{A.9}$$

A.2 Some results about existence and uniqueness of solutions to Partial Differential Equations

A.2.1 Linear operators

Theorem 1. *(Lax-Milgram) Let* H *be a Hilbert space and let* $b : H \times H \mapsto \mathbb{R}$ *be a bilinear form such that:*

$$\begin{aligned} |b(u,v)| \leq c \left\| u \right\|_H \left\| v \right\|_H \quad &, \forall u, v \in H \\ b(u,u) \geq \alpha \left\| u \right\|_H^2 \quad &, \forall u \in H \end{aligned} \tag{A.10}$$

for given positive constants c *and* α*. Then, given* $f \in H'$*, there exists a unique* $u \in H$ *such that:*

$$b(u,v) = \langle f, v \rangle_{H',H} \tag{A.11}$$

for any $v \in H$*.*

If the bilinear form is continuous but not elliptic, we can still guarantee the existence and uniqueness of the solution of the problem by the next:

Theorem 2. *(Banach-Necas-Babuska) Let X be a Banach space and M be a reflexive Banach space. Let $b : X \times Y \mapsto \mathbb{R}$ a bilinear form and $f \in M'$ be such that:*

1. *there exists a constant $\alpha > 0$ such that:*

$$\inf_{v \in M} \sup_{\mathbf{T} \in X} \frac{b(\mathbf{T}, v)}{\|\mathbf{T}\|_X \|v\|_M} \geq \alpha \tag{A.12}$$

2. *for all $v \in M$ and for all $\mathbf{T} \in X$:*

$$b(\mathbf{T}, v) = 0 \Rightarrow v = 0 \tag{A.13}$$

then, for $f \in M'$ there exists a unique $\mathbf{T}$ such that for all $v \in M$ we have:

$$b(\mathbf{T}, v) = \langle f, v \rangle_{M', M} \tag{A.14}$$

Moreover, for all $f \in M'$ the following estimate holds:

$$\|\mathbf{T}\|_X \leq \frac{1}{\alpha} \|f\|_{M'} \tag{A.15}$$

A.2.2 Non Linear operators

When the operator is non linear, there is still a way to prove existence and uniqueness o solutions via the next:

Theorem 3 (Browder-Minty)**.** *Let X be a reflexive Banach space, and let $F : X \mapsto X'$ be continuous, bounded, and such that:*

$$\frac{\langle F(u), u \rangle_{X', X}}{\|u\|_X} \to \infty \textit{ as } \|u\|_X \to \infty \tag{A.16}$$

$$\langle F(u) - F(v), u - v \rangle_{X', X} \geq 0 \; \forall u, v \in X \tag{A.17}$$

then, for every $g \in X'$, there exists an element $u \in X$ such that:

$$F(u) = g \tag{A.18}$$

In order to prove the monotonicity of the functional, sometimes it is useful to use the next result:

Theorem 4 (Monotonicity of differentiable mappings)**.** *A differentiable mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is monotone if and only if for each $x \in \mathbb{R}^n$ the Jacobian matrix $DF(x)$ is positive-semidefinite. It is strictly monotone if the matrix is positive-definite (but this condition is only sufficient, not necessary).*

APPENDIX B

NUMERICAL SCHEME

B.1 Algorithm

```
function [error_v] = implicit(ellipse,n,R2,R3,lambda)
h = 100;     %time step
node = table2array(ellipse);
TRI = delaunay(node(:,2),node(:,3));
N = length(node(:,1));
N0 = length(node(:,1))-sum(node(:,4));
M = length(TRI(:,1));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% STEP 0 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
D = sparse(2*M,2*M);
B = sparse(N,2*M);
A0 = sparse(N,N);

for i=1:M
    P1 = [node(TRI(i,1),2);node(TRI(i,1),3)];
    P2 = [node(TRI(i,2),2);node(TRI(i,2),3)];
    P3 = [node(TRI(i,3),2);node(TRI(i,3),3)];
    [At0,b,c] = matricesfinal(P1,P2,P3);
    delta = det([[P1,P2,P3];[1 1 1]]);

    D(i,i) = delta/2;
    D(i+M,i+M) = delta/2;
    B(TRI(i,1),i) = B(TRI(i,1),i) + b(1)/2*node(TRI(i,1),4);
    B(TRI(i,2),i) = B(TRI(i,2),i) + b(2)/2*node(TRI(i,2),4);
    B(TRI(i,3),i) = B(TRI(i,3),i) + b(3)/2*node(TRI(i,3),4);
    B(TRI(i,1),i+M) = B(TRI(i,1),i+M) + c(1)/2*node(TRI(i,1),4);
    B(TRI(i,2),i+M) = B(TRI(i,2),i+M) + c(2)/2*node(TRI(i,2),4);
```

```
    B(TRI(i,3),i+M) = B(TRI(i,3),i+M) + c(3)/2*node(TRI(i,3),4);
    A0(TRI(i,1),TRI(i,1)) = A0(TRI(i,1),TRI(i,1)) + At0(1,1)*node(TRI(i,1),4);
    A0(TRI(i,2),TRI(i,2)) = A0(TRI(i,2),TRI(i,2)) + At0(2,2)*node(TRI(i,2),4);
    A0(TRI(i,3),TRI(i,3)) = A0(TRI(i,3),TRI(i,3)) + At0(3,3)*node(TRI(i,3),4);
    A0(TRI(i,1),TRI(i,2)) = A0(TRI(i,1),TRI(i,2)) + At0(1,2)*node(TRI(i,1),4);
    A0(TRI(i,1),TRI(i,3)) = A0(TRI(i,1),TRI(i,3)) + At0(1,3)*node(TRI(i,1),4);
    A0(TRI(i,2),TRI(i,3)) = A0(TRI(i,2),TRI(i,3)) + At0(2,3)*node(TRI(i,2),4);
    A0(TRI(i,2),TRI(i,1)) = A0(TRI(i,2),TRI(i,1)) + At0(2,1)*node(TRI(i,2),4);
    A0(TRI(i,3),TRI(i,1)) = A0(TRI(i,3),TRI(i,1)) + At0(3,1)*node(TRI(i,3),4);
    A0(TRI(i,3),TRI(i,2)) = A0(TRI(i,3),TRI(i,2)) + At0(3,2)*node(TRI(i,3),4);
end

A = [(1+R3)*D , -.5*B' ; B , sparse(N,N)];
A( ~any(A,2), : ) = [];     %delete null rows
A( :, ~any(A,1) ) = [];     %delete null columns
l = lambda*ones(N-N0,1);
f = [sparse(2*M,1);A0(N0+1:N,N0+1:N)*l];
x0 = A\f;
T0 = x0(1:2*M);
v0 = [zeros(N0,1) ; x0(2*M+1:2*M+N-N0)];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% STEP 1 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
D1 = sparse(2*M,2*M);
A1 = [(1/h+R3)*D , -.5*B' ; B sparse(N,N)];
A1( ~any(A1,2), : ) = [];     %delete null rows
A1( :, ~any(A1,1) ) = [];     %delete null columns

T1 = ones(2*M,1);
T2 = zeros(2*M,1);

while(norm(T1-T2)>1e-3)
    for i=1:M
        P1 = [node(TRI(i,1),2);node(TRI(i,1),3)];
```

```
        P2 = [node(TRI(i,2),2);node(TRI(i,2),3)];
        P3 = [node(TRI(i,3),2);node(TRI(i,3),3)];
        delta = det([[P1,P2,P3];[1 1 1]]);

        norm2 = T0(i)^2+T0(i+M)^2;
        norm2 = 1/h - (1+2*R2*norm2)^(-n);
        D1(i,i) = delta*norm2/2;
        D1(i+M,i+M) = delta*norm2/2;
    end
    f1 = [D1*T0 ; A0(N0+1:N,N0+1:N)*l];
    x1 = A1\f1;
    T1 = x1(1:2*M);
    v1 = [zeros(N0,1) ; x1(2*M+1:2*M+N-N0)];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% STEP 2 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    for i=1:M
        P1 = [node(TRI(i,1),2);node(TRI(i,1),3)];
        P2 = [node(TRI(i,2),2);node(TRI(i,2),3)];
        P3 = [node(TRI(i,3),2);node(TRI(i,3),3)];
        delta = det([[P1,P2,P3];[1 1 1]]);
        [~,b,c] = matricesfinal(P1,P2,P3);
        Tt_1 = [T1(i);T1(i+M)];
        vt_1 = [v1(TRI(i,1));v1(TRI(i,2));v1(TRI(i,3))];

        Tt_2 = Newton(n,R2,R3,Tt_1,vt_1,b,c,delta,h);
        T2(i) = Tt_2(1);
        T2(i+M) = Tt_2(2);
    end
    T0 = T2;
end
v_th = -1/((-n+1)*R2*lambda)*((1+(R2*lambda^2/2)*(node(:,2).^2+node(:,3).^2)).^(-n+1)
-(1+R2*lambda^2/2)^(-n+1)) - (R3*lambda/2)*(node(:,2).^2+node(:,3).^2-1);
```

```
tri1 = triangulation(TRI,node(:,2),node(:,3),v1);
tri = triangulation(TRI,node(:,2),node(:,3),v_th);
subplot(2,1,1)
trisurf(tri1);
title('experimental velocity');
subplot(2,1,2)
trisurf(tri);
title('theoretical velocity');
error_v = norm(v1-v_th)/norm(v_th);
end
```