# ITERATED MONODROMY GROUPS OF RATIONAL MAPPINGS

A Thesis

by

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ABSTRACT

We give the wreath recursion presentations of iterated monodromy groups of post-critically finite quadratic rational mappings $f_c$ whose ramification portrait are of the form

$$0 \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto \infty \mathbin{\substack{\rightharpoonup \\ \leftharpoondown}} 1$$

To find a pattern of these wreath recursions, we compute the wreath recursions of iterated monodromy groups of capture maps composed with the Basilica polynomial. This computation gives rise to the notion of addresses, which is used to represent wreath recursions. Then we conjecture that each capture map composed with the Basilica polynomial is topologically equivalent to a post-critically finite quadratic rational mapping, and thus we conclude that the iterated monodromy groups of $f_c$ can be represented by addresses.

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# 1. INTRODUCTION

## 1.1 Background and sketch of the method

Iterated monodromy groups of post-critically finite branched coverings can be presented by wreath recursions. Let $\widehat{\mathbb{C}} \cup \{\infty\}$ be the Riemann sphere. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a degree $d$ ($1 < d < \infty$) rational mapping of $\widehat{\mathbb{C}}$ to itself. Denote by $C_f$ the set of all critical points of $f$ and by $P_f := \{f^{\circ n}(c_0) : c_0 \in C_f, n \geq 1\}$ the set of all post-critical points of $f$. The mapping $f$ is called post-critically finite if $P_f$ is finite. The iterated monodromy group of $f$ (denoted IMG($f$)) is defined to be the quotient of the fundamental group $\pi_1(\widehat{\mathbb{C}} \backslash P_f, t)$ by the kernel of its monodromy action of the tree of preimages of the base point $t$ (see [1] for a detailed discussion of the definition of iterated monodromy groups). The wreath recursion of IMG($f$) can be computed using Proposition 2.1 in [2]. The computation requires a choice of generating sets for $\pi_1(\widehat{\mathbb{C}} \backslash P_f, t)$.

There is a rich amount of examples of iterated monodromy groups presented by wreath recursions. One of the most important classes is the iterated monodromy groups of quadratic polynomials (degree 2 branched coverings). These groups are studied in the paper [2]. The wreath recursions for these groups are associated with the *kneading sequences* of the corresponding polynomials, and thus the pattern of the wreath recursions is completely determined by kneading sequences. However, in other degree 2 cases, we do not know a pattern for the wreath recursions, which makes it hard to study the corresponding iterated monodromy groups.

The goal of this paper is to give a method to compute the wreath recursions of iterated monodromy groups of all post-critically finite quadratic rational mappings that are of the form $f_c = \dfrac{z^2 + c}{z^2 - 1}, c \in \mathbb{C}$, with ramification portrait

$$0 \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto \infty \underset{\rightsquigarrow}{\overset{\rightharpoonup}{\rightsquigarrow}} 1$$

Note that $f_c$'s are also degree 2 mappings. Especially, we define a new type of "sequences" called *addresses* to determine the wreath recursions. To do so, we pick a point $a_2$ in a Fatou component on the Basilica Julia set (which is the dynamics of the Basilica polynomial $p_{-1} = z^2 - 1$) and

iterate $p_{-1}$ on this point. The orbit will be quite similar:

$$a_2 \mapsto \cdots \mapsto a_m \mapsto 0 \; \begin{smallmatrix} \curvearrowright \\ \curvearrowleft \end{smallmatrix} \; -1$$

This motivates the usage of *captures* (see [3] for a discussion of captures). The critical points of $p_{-1}$ are $0$ and $\infty$. Let $\sigma_\beta$ be a capture map (define in section 2.5 of [3]) such that $\sigma_\beta(\infty) = a_2$. Then the ramification portrait of $\sigma_\beta \circ p_{-1}$ is

$$\infty \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto 0 \; \begin{smallmatrix} \curvearrowright \\ \curvearrowleft \end{smallmatrix} \; -1$$

Then we can make the following conjecture:

**Conjecture.** The function $\sigma_\beta \circ p_{-1}$ is topologically equivalent to some post-critically finite rational functions $f_c$.

By assuming the conjecture, we can conclude that the wreath recursion of $\mathrm{IMG}(f_c)$ is the same as that of $\mathrm{IMG}(\sigma_\beta \circ p_{-1})$ (Theorem 4.3). Hence $\mathrm{IMG}(f_c)$ will be completely determined by addresses. It remains compute $\mathrm{IMG}(\sigma_\beta \circ p_{-1})$ and give a complete description of addresses.

## 1.2 Outline of this thesis

Chapter 2 introduces a method of choosing a generating set for $\pi_1(\widehat{\mathbb{C}} \backslash P_{\sigma_\beta \circ p_{-1}}, t)$, gives the first observation of wreath recursions, and defines the notion of addresses. Chapter 3 describes the properties of addresses and use them to represent wreath recursion. Chapter 4 is the conclusion.

## 1.3 Notice about the conjecture

It is reasonable to make the above conjecture since $\sigma_\beta \circ p_{-1}$ is also a topological branched covering [3], and thus it makes sense to compute its iterated monodromy group; also, if we know the above ramification portrait is the post-critical orbit of a degree 2 rational mapping, then the only form of this mapping is $\dfrac{z^2 + c}{z^2 - 1}$ (Lemma 4.1).

The proof of this conjecture is beyond the scope of this thesis and will be given in later research. Another similar problem was studied in the paper [4].

### 1.4 Notations

(1) We use letters without brackets to denote homotopy classses of curves. For example, we use $g_\infty$ instead of $[g_\infty]$ to denote a homotopy class of loops going around the point $\infty$. The homotopy product is denoted $\cdot$.

(2) Denote by $\mathcal{B}$ the Basilica Julia set, and $\mathcal{RB}$ the reversed Basilica Julia set. Let $F$ be a Fatou component on $\mathcal{B}$ or $\mathcal{RB}$; the boundary of $F$ is denoted $\partial F$.

(3) For a Fatou component $F$ on the $\mathcal{B}$ or $\mathcal{RB}$, those points on $\partial F$ parameterized by the internal angles $j/2^i$, $i, j \in \mathbb{N}$ are called *joint points*. The point parameterized by $0$ is call the *root point* of this Fatou component. A joint point on $\partial F$ is the root point for the next Fatou component attached to $F$.

## 2. COMPUTING WREATH RECURSIONS

### 2.1  The problem of choosing generating sets

Choose a point $c$ on the parameter plane such that $f_c : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a post-critically finite branched covering with ramification portrait of the form

$$0 \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto \infty \;\substack{\curvearrowright \\ \curvearrowleft}\; 1$$

Critical points of $f_c$ are $0, \infty$. Let $C_{f_c} := \{0, \infty\}$; let $P_{f_c}$ be the set of post-critical points of $f_c$. Note, in particular, that $f_c(\infty) = 1$ and $f_c(1) = \infty$. Hence $1, \infty \in P_{f_c}$, and $\infty$ is in the cycle of period 2, and thus the family of **all** quadratic rational mappings of the form $f_c = \dfrac{z^2 + c}{z^2 - 1}, c \in \mathbb{C}$ is denoted $V_2$ [5]. Let $\mathcal{C}_c = \widehat{\mathbb{C}} \backslash P_{f_c}$. Fixing a base point $t \in \mathcal{C}_c$, the fundamental group $\pi_1(\mathcal{C}_c, t)$ is a free group generated by $|P_{f_c}| - 1$ elements. The generators of $\pi_1(\mathcal{C}_c, t)$ can be taken to be loops going around each point in $P_{f_c}$ in the positive direction. The choice of generating loops of $\pi_1(\mathcal{C}_c, t)$, however, depends not only on the coordinates of each post-critical point, but also on the relative positions of loops and the coordinate of the base point $t$. For instance, let $a, b \in P_{f_c}$ such that $b = f_c(a)$ and suppose $a$ is to the left of $b$ on $\widehat{\mathbb{C}}$, and let $a_1$, $a_2$ be the first and second inverses of $a$ under $f_c$, respectively. Let $\gamma_a$ and $\gamma_b$ be the loops going around $a$ and $b$, respectively. Here $\gamma_a$ might be chosen to the left of $\gamma_b$ or the other way around (See Figure 2.1 below), yet they might result in different wreath recursions: for example, in the former case, the first inverse image of $\gamma_a$ might be a loop going around $a_1$, while in the latter case, the first inverse image of $\gamma_a$ might be a loop going around $a_2$.
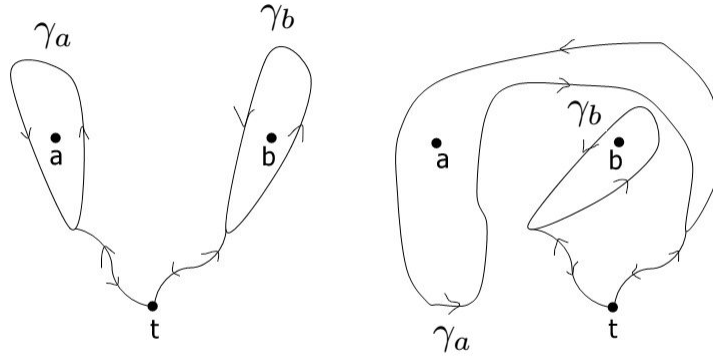
Figure 2.1: Different ways to choose a generating set.

The issue is that we do not know which of these choices is the best, *a priori*, in the sense of dynamical consistency of all IMG($f_c$). Also, if the two inverse images of $\gamma_a$ are simple paths (i.e. not loops), then there is an issue of choosing the connecting paths of the inverse image of $t$ with $t$ to form a new loop. Hence we need a consistent way of choosing generating sets. Ideally, we would like the inverse images of loops to be consistent with the inverse images of points, i.e. the first inverse image of $\gamma_a$ being a loop going around $a_1$ and the second inverse image of $\gamma_a$ being a loop going around $a_2$.

## 2.2 First glance at IMG($f_c$)

Let $f_c$ be a post-critically finite quadratic rational mapping, and consider the forward orbit of $0$ under $f_c$. It eventually goes to $\infty$ which forms a cycle with $1$. Hence $\{f^{\circ n}(0)|n \in \mathbb{N}\} = P_{f_c}$. Let $m$ be the smallest integer such that $f^{\circ m}(0) = \infty$. Fix a base point $t$. It follows that the size of the generating set of $\pi_1(\mathcal{C}_c, t)$ is $m$. Let the generators $g_2$, $g_3$,...,$g_m$, $g_1$ be loops going around $f(0)$, $f^{\circ 2}(0)$,...,$f^{\circ(m-1)}(0)$, $1$, respectively, in counterclockwise (positive) orientation. Also denote by $g_\infty$ the loop going around $\infty$ in positive orientation, which is clockwise orientation around all finite post-critical points. Note that $g_\infty$ is a product of all generators in a certain order and thus it

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is not in the generating set we choose. Nevertheless, we will view $g_\infty$ as a generator to maintain consistency with the extended wreath recursions on Basilica groups to be described below. The (first observation of) wreath recursion of $\text{IMG}(f_c)$ is as follows (using notations from [2]):

$$g_2 = \langle\!\langle *, * \rangle\!\rangle \sigma$$
$$g_{k+1} = \langle\!\langle g_k, 1 \rangle\!\rangle \text{ or } \langle\!\langle 1, g_k \rangle\!\rangle, k = 2, ..., m-1$$
$$g_1 = \langle\!\langle *, * \rangle\!\rangle \sigma \tag{2.1}$$
$$g_\infty = \langle\!\langle g_1, g_m \rangle\!\rangle$$

Where $*$ stands for undetermined loops, and the presentation of $g_{k+1}$ is undetermined.

**Proposition 2.1.** The iterated monodromy group of each post-critically finite $f_c$ has wreath recursion (2.1).

*Proof.* Let P be a point in $\widehat{\mathbb{C}}$. By a standard result of complex analysis, we have that there exist open neighborhoods $U$ and $U'$ of $P$ and $f_c(P)$ respectively, and open neighborhoods $V$ and $V'$ of $0$ in $\mathbb{C}$, and biholomorphisms $g : U \to V$ and $g' : U' \to V'$ sending $0$ and $f_c(0)$ respectively to $0$, such that the map $g' \circ f_c \circ g^{-1} : V \to V'$ is equal to $z \mapsto z^i$, $i = 1$ if $P$ is not a critical point, and $i = 2$ if $P$ is a critical point. Since $0$ and $\infty$ are critical points of $f_c$, and thus $f_c(0)$ and $-1$ are critical values of $f_c$, it follows that the lift of $g_2$ by $f_c$ is mapped biholomorphically to the lift of a loop $\gamma$ going around $0 \in \mathbb{C}$ by $z \mapsto z^2$. Since each point other that $0$ in $\mathbb{C}$ has two preimages under $z \mapsto z^2$, it follows that the lift of $\gamma$ is a simple path. Hence the lift of $g_2$ is also a simple path. Same argument holds for $g_1$. Moreover, all other post-critical points are not critical values, and thus the lifts of each $g_k$, $k = 3, ..., m-1, \infty$ are all loops. By Proposition 2.1 in [2], the wreath recursion can be written as (2.1). $\square$

## 2.3 Extended wreath recursion on the Basilica group

The problem stated in section 2.1 can be solved by computing the extended wreath recursion on the Basilica group and then applying captures on the Basilica Julia set $\mathcal{B}$, which is the Julia set of the quadratic polynomial $p_{-1} = z^2 - 1$. Later we will use the Basilica Julia set and the reversed

Basilica Julia set $\mathcal{RB}$ (which is the Julia set of $f_0 = \dfrac{z^2}{z^2 - 1}$) interchangeably, since they have the same dynamics.

We start by picking an arbitrary finite Fatou component, denoted $F_2$, on $\mathcal{B}$ and apply $p_{-1}$ on $F_2$. Then the image will be another Fatou component. Iterating this process, we obtain an orbit of Fatou components and the image will eventually be $F_0$, which is the Fatou component containing $0$, and form a cycle with $F_{-1}$, the Fatou component containing $-1$. In fact, $p_{-1}$ is a homeomorphism between the closure of each Fatou component, except for $F_0$, on this orbit. In practice, we only need to pick a point $a_2$ inside $F_2$ and iterate $p_{-1}$ on $a_2$. The orbit of this point is in one-to-one correspondence of that of $F_2$, i.e. $p_{-1}^{\circ k}(a_2) \in p_{-1}^{\circ k}(F_2)$. Also let $m$ be the smallest integer such that $p_{-1}^{\circ m}(a_2) \in p_{-1}^{\circ m}(F_2) = F_0$. Note that $m < \infty$ since it takes only finitely many steps for $F_2$ to go to $F_0$ under iteration of $p_{-1}$.

The Basilica group is described in Section 3.3 of [6]. To compute the extended wreath recursion on the Basilica group at $F_2$, we first impose one puncture on each $p_{-1}^{\circ k}(F_2)$ by simply removing the point $p_{-1}^{\circ k}(a_2)$, for $k = 0, 1, ..., m + 1$. Define $P_{F_2} := \{p_{-1}^{\circ k}(a_2) | k = 0, ..., m + 1\}$. Then fix a base point $t = \frac{1 - \sqrt{5}}{2}$. In this way we obtained a partial self-covering $p_{-1} : \widehat{\mathbb{C}} \setminus p_{-1}^{-1}(P_{F_2}) \to \widehat{\mathbb{C}} \setminus P_{F_2}$.

**Definition 1.** The *extended wreath recursion* on the Basilica group at $F_2$ is the wreath recursion of the partial self-covering $p_{-1} : \widehat{\mathbb{C}} \setminus p_{-1}^{-1}(P_{F_2}) \to \widehat{\mathbb{C}} \setminus P_{F_2}$.

**Remark.** The extended wreath recursions on the Basilica group do not define new iterated monodromy groups. Indeed, if we define $\mathrm{IMG}(F_2, t) := \pi_1(\widehat{\mathbb{C}} \setminus P_{F_2}, t) / Ker(\Phi)$, where $Ker(\Phi) = \{\gamma \in \pi_1(\widehat{\mathbb{C}} \setminus P_{F_2}, t) | \forall x \in \bigsqcup_n p_{-1}^{\circ(-n)}(t), \gamma x = x\}$ [6], then all loops going around the imposed puncture points will be in $Ker(\Phi)$, and thus $\mathrm{IMG}(F_2, t)$ is isomorphic to the Basilica group. Nevertheless, for brevity, we still say that the extended wreath recursion on the Basilica group at $F_2$ is the wreath recursion for $\mathrm{IMG}(F_2, t)$, or $\mathrm{IMG}(F_2)$ when $t$ is specified.

We can also use Proposition 2.1 in [2] to compute the extended wreath recursions. The fundamental group $\pi_1(\widehat{\mathbb{C}} \setminus P_{F_2}, t)$ is generated by $m + 1$ elements. Let the generators $g_2, g_3,...,g_0, g_{-1}$ be loops going around $a_2$, $a_3 = p_{-1}(a_2)$,...,$0 = p_{-1}^{\circ m}(a_2)$, $-1 = p_{-1}^{\circ(m+1)}(a_2)$, respectively. It is worth

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noting that technically we require $a_2$ to be a strictly preperiod point of $p_{-1}$ that eventually gets mapped to $0$, so that we are in fact extending the Basilica group by imposing more post-critical points. Yet, in practice, it does not matter which point is removed inside a Fatou component because of the one-one correspondence of $p_{-1}^{\circ k}(a_2)$ and $p_{-1}^{\circ k}(F_2)$. We only need to draw loops close to the boundary of each Fatou components to make sure those loops will go around the (imposed) post-critical points.

**Remark.** We need to show that there is a consistent way of choosing generating sets for all IMG($F_2$). This requires the notion of *addresses*. Therefore, we will give the definition of addresses in this chapter and leave the detailed description to the next chapter.

It is known (see [7]) that the boundary of each Fatou component on $\mathcal{B}$ is homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$ and that two adjacent Fatou components share only one joint point. We now give a consistent parametrization of the boundary of each Fatou component. First, let the homeomorphism $\theta_0 : \mathbb{R}/\mathbb{Z} \to \partial F_0$ be the unique conjugacy with $z^2$ on the unit circle such that $\theta_0(0) = \theta_0(1) = t$. Especially, all joint points are parameterized by angles $j/2^i$, $i, j \in \mathbb{N}$. Then pick an arbitrary joint point $b$ on $\partial F_0$ and consider the attached Fatou component $F_b$. We define another homeomorphism $\theta_b : \mathbb{R}/\mathbb{Z} \to \partial F_b$ such that $\theta_b(0) = \theta_b(1) = b$ and all other points are parameterized in the exact same way as that of $F_0$. Continuing this process, we get a consistent way of parameterizations, and we will identify the boundary of each Fatou component with $\mathbb{R}/\mathbb{Z}$ in this way. Hence every Fatou component can be located by a sequence of angles that is of the form $(\frac{j_1}{2^{i_1}}, ..., \frac{j_l}{2^{i_l}})$.

**Definition 2.** The sequence of angles $(\frac{j_1}{2^{i_1}}, ..., \frac{j_l}{2^{i_l}})$, in which each $j_k, k > 1$ is a positive odd integer less that $2^{i_k}$ while $j_1$ is either $0$ or a positive odd integer less that $2^{i_1}$, is called the *address* associated to a Fatou component.

Especially, the address of $F_0$ is denoted $(\varnothing)$, and the address of $F_{-1}$ is $(0)$. With the notion of addresses, we can easily locate a Fatou component and represent its orbit. We can also locate the two inverses images of a Fatou component by computing the two branches of addresses. Then the most natural way of choosing a generating loop is to draw a simple curve starting at $t$ and go past

the smallest number of Fatou components by crossing the joint points, and then go around a post critical point in counterclockwise orientation and follow the same path back to $t$. Hence each such loop is also associated with the same address as that of $F_2$. We will show in the next chapter that the generating sets obtained in this way is indeed consistent.
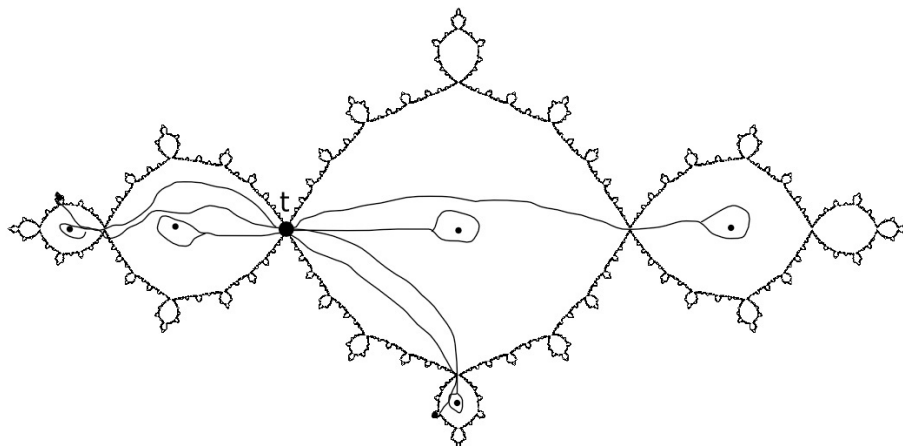


Figure 2.2: An example of generating set: each generating loop is a union of a simple path and a small loop around a puncture. The simple path crosses joint points.

**Proposition 2.2.** The wreath recursions of IMG($F_2$) are of the following form:

$$g_2 = \langle\!\langle 1, 1 \rangle\!\rangle$$

$$g_{k+1} = \langle\!\langle g_k, 1 \rangle\!\rangle \text{ or } \langle\!\langle 1, g_k \rangle\!\rangle, k = 2, ..., m-1$$

$$g_{-1} = \langle\!\langle 1, g_0 \rangle\!\rangle \sigma \tag{2.2}$$

$$g_0 = \langle\!\langle g_{-1}, g_m \rangle\!\rangle$$

Where the presentation of $g_{k+1}$ depends on the address of $F_2$ (to describe in the next chapter).

*Proof.* The argument is exactly the same as that of Proposition 2.1. We merely need to note that the only finite critical value of $p_{-1}$ is $-1 = p_{-1}(0)$. $\qquad\square$

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Comparing (2.1) and (2.2), the difference occurs at $g_2$: in the above wreath recursion, the lifts of $g_2$ are trivial loops, while in (2.1), the lifts of $g_2$ are simple paths. This difference leads to another tool called *captures*.

## 2.4 Captures on the Basilica Julia set

We have seen that it is possible to choose a generating set for $\text{IMG}(f_c)$ such that its wreath recursion is the same, at all generators but $g_2$, as that of $\text{IMG}(F_2)$ for a corresponding $F_2$. In order to make their wreath recursions look completely the same, we apply capture on $F_2$

We interpret the description of captures from Section 2.5 of [3] on the polynomial $p_{-1}$. Recall that an external ray parametrized by an angle $\beta \in \mathbb{R}/\mathbb{Z}$ lands on the root point $b$ of a Fatou component of $\mathcal{B}$ whenever $\beta = \frac{j}{3 \cdot 2^i}$, where $j$ is an integer less than $3 \cdot 2^i$ and does not divide $3 \cdot 2^i$. We also use $\beta$ to denote this external ray, i.e. we view $\beta : [0, 1] \to \widehat{\mathbb{C}}$ as a simple path such that $\beta(0) = \infty$ and $\beta(1) = b$.

Fixing $F_2$, there are two external rays landing on its root point $b_2$. These two rays are parametrized by $\frac{k}{3 \cdot 2^i}$ and $\frac{k+2}{3 \cdot 2^i}$, respectively. Let $\beta = \frac{k}{3 \cdot 2^i}$ and let $a_2 \in F_2$ be the pre-period point that is mapped to $0$ by $p_{-1}$ as before. Connect the points $b_2$ and $a_2$ by a line segment and denote this line segment $L_\beta$. The simple path $L_\beta \cup \beta$ is called the *capture path*. Then define a *path homeomorphism* $\sigma_\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\sigma_\beta(\infty) = a_2$ and $\sigma_\beta = Id$ outside a small neighborhood of $L_\beta \cup \beta$. Suppose the neighborhood of $L_\beta \cup \beta$ is small enough that it does not intersect other Fatou components on the orbit of $F_2$. Then the inverse images of all generators except $g_2$ under $\sigma_\beta \circ p_{-1}$ are the same as those under $p_{-1}$. For $g_2$ we have the following proposition:

**Proposition 2.3.** The lifts of $g_2$ under $\sigma_\beta \circ p_{-1}$ are simple paths.

*Proof.* We need to show that $a_2$ is a critical point of $\sigma_\beta \circ p_{-1}$. Since $\sigma_\beta$ is a homeomorphism, it does have a critical point, and thus $\infty$ is a critical point of $\sigma_\beta \circ p_{-1}$. It follows that $a_2$ is a critical value by the definition of $\sigma_\beta$. $\square$

**Corollary 2.4.** The wreath recursion of the iterated monodromy group of $\sigma_\beta \circ p_{-1}$ is of the form

$$g_2 = \langle\!\langle h, h^{-1} \rangle\!\rangle \sigma$$

$$g_{k+1} = \langle\!\langle g_k, 1 \rangle\!\rangle \text{ or } \langle\!\langle 1, g_k \rangle\!\rangle, k = 2, ..., m-1$$

$$g_{-1} = \langle\!\langle 1, g_0 \rangle\!\rangle \sigma$$   (2.3)

$$g_0 = \langle\!\langle g_{-1}, g_m \rangle\!\rangle$$

Where $h$ is a word depending on the address of $F_2$, and $g_{k+1}$ also depends on the address of $F_2$ (to describe in the next chapter).

Section 3.2 will serve as the proof of this corollary.

# 3. ADDRESSES

The remaining problem from the last chapter is to determine the presentations of $g_k$ for $k \neq -1, 0$ (equivalently, $k \neq 1, \infty$ on the reversed Basilica Julia set $\mathcal{RB}$) in (3). This problem can be solved by computing forward orbits of addresses and the two inverses of each address. Since the generating loops $g_k$ can also be represented by addresses, the wreath recursions will be uniquely determined by inverse addresses.

## 3.1 Forward orbit of a Fatou component represented by forward orbit of addresses

Denote by $D_{-1} := \widehat{\mathbb{C}}$ the dynamic sphere of $p_{-1}$. For an arbitrary Fatou component $F \neq F_0$, the map $p_{-1} : D_{-1} \to D_{-1}$ induces a homeomorphism of $\partial F$ and $p_{-1}(\partial F)$, and thus it induces an angle map $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, since the boundary of each Fatou component can be parametrized by internal angles.

**Definition 3.** For an arbitrarily chosen Fatou component $F$ on $\mathcal{B}$ (equivalently on $\mathcal{RB}$), the *forward orbit of $F$* is a directed graph $(V, E)$ with vertex set $V = \{p_{-1}^{\circ m}(F) | m \in \mathbb{N}\}$ (equivalently $V = \{f_0^{\circ m}(F) | m \in \mathbb{N}\}$), and edge set $E$ defined to be the set of arrows starting at $p_{-1}^{\circ k}(F)$ and ending at $p_{-1}^{\circ(k+1)}(F)$, $k \in \mathbb{N}$.

**Proposition 3.1.** The forward orbit of $F$ can be represented by an orbit of addresses.

*Proof.* First, set $F = F_0$. Then the forward orbit of $F$ is the cycle:

$$F_0 \; \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \; F_{-1}$$

Hence we always have the cycle

$$(\varnothing) \; \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \; (0)$$

Since $0$ is a critical point of $p_{-1}$, the map $p_{-1} : F_0 \to F_{-1}$ induces an angle doubling map of their boundaries $\phi : \partial F_0 \to \partial F_{-1}$ *s.t.* $\alpha \mapsto 2\alpha$. On the other hand, if $F \neq F_0$, then $p_{-1}$ induces an angle preserving map $\phi : \partial F \to \partial p_{-1}(F)$ *s.t.* $\alpha \mapsto \alpha$. It follows that all addresses that are of the

form $(\frac{j}{2^i})$ for $i > 1$ are mapped to $(0, \frac{j'}{2^{i-1}})$, where $j' \equiv j(\mathrm{mod}\ 2^{i-1})$; and $(\frac{1}{2})$ is mapped to $(\varnothing)$. By continuity of $p_{-1}$, all addresses that are of the form $(\frac{j_1}{2^{i_1}}, ..., \frac{j_l}{2^{i_l}})$ for $i_1 > 1$ and will be mapped to $(0, \frac{j'_1}{2^{i_1-1}}, ..., \frac{j_l}{2^{i_l}})$ $(j'_1 \equiv j_1(\mathrm{mod}\ 2^{i_1-1}))$, and then mapped back to $(\frac{j'_1}{2^{i_1-1}}, ..., \frac{j_l}{2^{i_l}})$; while all addresses that are of the form $(\frac{1}{2}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$ will be mapped to $(\frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$. In this way we get an orbit of addresses.  □

The proof of the last proposition allows us to make the following definition:

**Definition 4.** Let $(\frac{j_1}{2^{i_2}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$ be an address. The *forward orbit of this address* is a directed graph satisfying the following rules:

(i) If $\frac{j_1}{2^{i_1}} = 0$, then $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}}) \rightarrow (\frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$

(ii) If $\frac{j_1}{2^{i_1}} = \frac{1}{2}$, then $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}}) \rightarrow (\frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$

(iii) If $\frac{j_1}{2^{i_1}}$ is such that $i_1 > 1$ and $j_1 \neq 0$, then $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}}) \rightarrow (0, \frac{j'_1}{2^{i_1-1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, where $(j'_1 \equiv j_1(\mathrm{mod}\ 2^{i_1-1}))$

(iv) $(\frac{1}{2}) \rightarrow (\varnothing) \overset{\curvearrowright}{\underset{\curvearrowleft}{\phantom{x}}} (0)$

**Remark.** We use an address to represent a Fatou component $F$ as well as each of the points in $F$, especially the root point. Also, by our way of choosing generating set, each generating loop can also be represented by the same address as that of a corresponding Fatou component. We also use a set of addresses to represent a simple curve that crosses only the root points. Hence a generating loop can also be represented by a set of addresses. For instance, if the address of a $g_k$ is $(\frac{1}{2^2}, \frac{1}{2^2})$, then it is also represented by the set $\{(\frac{1}{2^2}), (\frac{1}{2^2}, \frac{1}{2^2})\}$.

**Example 3.2.** Pick the $F_2$ whose address is $(\frac{1}{2^2}, \frac{1}{2^2})$. Then the forward orbit of $F_2$ is represented by:

$$(\tfrac{1}{2^2}, \tfrac{1}{2^2}) \rightarrow (0, \tfrac{1}{2}, \tfrac{1}{2^2}) \rightarrow (\tfrac{1}{2}, \tfrac{1}{2^2}) \rightarrow (\tfrac{1}{2^2}) \rightarrow (0, \tfrac{1}{2}) \rightarrow (\tfrac{1}{2}) \rightarrow (\varnothing) \overset{\curvearrowright}{\underset{\curvearrowleft}{\phantom{x}}} (0)$$

## 3.2 Wreath recursions determined by addresses

First, we represent the lifts of each generating loop $g_k$, for $k \neq 2, 1, \infty$, by branches of the address of the corresponding Fatou component. Divide the dynamic plane $D_{-1}$ into two parts by the union of external rays $\beta_1 = 1/3$ and $\beta_2 = 2/3$, which land on the base point $t = \frac{1-\sqrt{5}}{2}$. Denote respectively by $L$ and $R$ the left part and the right part of $D_{-1}$ divided by $\beta_1 \bigcup \beta_2$ (see Figure 3.1). For an arbitrary Fatou component $F$, we distinguish between two cases: $F$ being on $R$ and $F$ being on $L$.
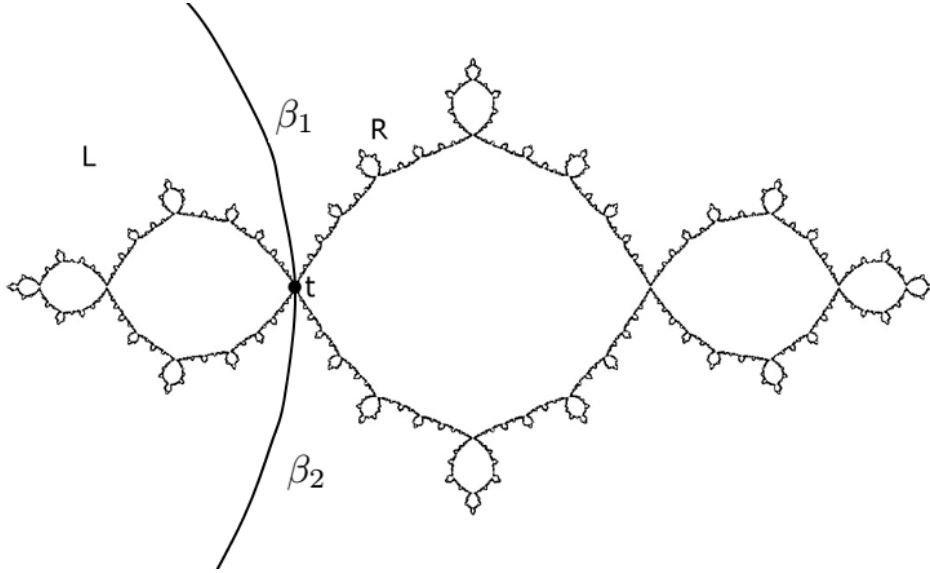


Figure 3.1: Dividing the dynamic plane.

**Definition 5.** (The first and second branches of an address on $R$) Let $F$ be a Fatou component on $R$, then its address is either $(\varnothing)$ or of the form $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, $j_1 \neq 0$. For $(\varnothing)$, the *first branch* is defined to be $(0)$, and the *second branch* is defined to be $(\frac{1}{2})$. For $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, the *first branch* is defined to be $(0, \frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, and the *second branch* is defined to be $(\frac{1}{2}, \frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$.

**Remark.** Intuitively, given a Fatou component on $R$, the first inverse of its address is on $L$ and the second inverse is to the right.

It is a little bit tricky to define the two branches of an address on $L$. Let $\mathbb{H}^+$ denote the upper half plane and $\mathbb{H}^-$ denote the lower half plane. Pick a point $a \in L \cap \mathbb{H}^+$. Then $a$ has two inverse images under $p_{-1}$: one is on $\mathbb{H}^+ \cap R$ and the other is on $\mathbb{H}^- \cap R$. If this $a$ is a root point, then it is associated with an address that is of the form $(0, \frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, where $j_1 < 2^{i_1-1}$. We can set its first branch to be $(\frac{j_1}{2^{i_1+1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, and second branch to be $(\frac{j_1''}{2^{i_1+1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, where $2^{i_1} < j_1'' < 2^{i_1+1}$ and $j_1'' \equiv j_1 \pmod{2^{i_1}}$. However, when $a$ is a root point that is on the $x$-axis, then its branches are ambiguous: the "first" branch could be on $\mathbb{H}^+$ or $\mathbb{H}^-$, and same applies to the "second" branch. In order to rule out this ambiguity, we "require" $a$ to be on $\mathbb{H}^+$. In other words, if we have an imposed post-critical point $p_{-1}^{\circ k}(a_2)$ inside a Fatou component whose address is $(0, \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$, then we first draw a small loop around $p_{-1}^{\circ k}(a_2)$, and connect the base point $t$ with this loop by a path crossing root points and avoiding $\mathbb{H}^-$. Denote this generating loop $g_{k+1}$, and its first inverse image will be inside a Fatou component on $\mathbb{H}^-$; its second inverse image will be on $\mathbb{H}^+$. Therefore, we can make the following definition:

**Definition 6.** (The first and second inverse of an address on $L$) Let $F$ be a Fatou component on $L$, then its address is of the form $(0, \frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$. The *first inverse of this address* is defined to be $(\frac{j_1}{2^{i_1+1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$; the *second inverse of this address* is defined to be $(\frac{j_1''}{2^{i_1+1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$, where $j_1'' \neq j_1$ and $j_1'' \equiv j_1 \pmod{2^{i_1}}$. The two inverses of $(0)$ are equal: they are defined to be $(\varnothing)$

Now we can give a criterion for computing the presentation of $g_k$, for $k \neq 2, 1, 0, \infty$.

**Proposition 3.3.** Let $F_2$ be a Fatou component whose address is $(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}})$. Let $\mathcal{O}$ be the forward orbit of this address. Then each address in $\mathcal{O}$ associated to $g_k$ for $k \neq 2, 1, 0$ (equivalently $k \neq 2, 1, \infty$ on $\mathcal{RB}$) has exactly one branch in $\mathcal{O}$. If the first branch is in $\mathcal{O}$, then $g_k = \langle\langle g_{k-1}, 1 \rangle\rangle$; if the second branch is in $\mathcal{O}$, then $g_k = \langle\langle 1, g_{k-1} \rangle\rangle$. $\qquad \square$

**Example 3.4.** Let $F_2$ be the same as that in Example 3.2. We exhibit the wreath recursion in the following table:

| Address | 1st branch | 2nd branch | Presentation |
|---|---|---|---|
| $(\frac{1}{2^2}, \frac{1}{2^2})$ | $(0, \frac{1}{2^2}, \frac{1}{2^2})$ | $(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^2})$ | To describe below |
| $(0, \frac{1}{2}, \frac{1}{2^2})$ | $(\frac{1}{2^2}, \frac{1}{2^2})$ | $(\frac{3}{2^2}, \frac{1}{2^2})$ | $g_3 = \langle\!\langle g_2, 1 \rangle\!\rangle$ |
| $(\frac{1}{2}, \frac{1}{2^2})$ | $(0, \frac{1}{2}, \frac{1}{2^2})$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2^2})$ | $g_4 = \langle\!\langle g_3, 1 \rangle\!\rangle$ |
| $(\frac{1}{2^2})$ | $(0, \frac{1}{2^2})$ | $(\frac{1}{2}, \frac{1}{2^2})$ | $g_5 = \langle\!\langle 1, g_4 \rangle\!\rangle$ |
| $(0, \frac{1}{2})$ | $(\frac{1}{2^2})$ | $(\frac{3}{2^2})$ | $g_6 = \langle\!\langle g_5, 1 \rangle\!\rangle$ |
| $(\frac{1}{2})$ | $(0, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{2})$ | $g_7 = \langle\!\langle g_6, 1 \rangle\!\rangle$ |
| $(\varnothing)$ | $(0)$ | $(\frac{1}{2})$ | $g_0 = \langle\!\langle g_{-1}, g_7 \rangle\!\rangle$ |
| $(0)$ | $(\varnothing)$ | $(\varnothing)$ | $g_{-1} = \langle\!\langle 1, g_0 \rangle\!\rangle \sigma$ |

Table 3.1: An example of wreath recursion determined by addresses.

It remains to give the recursion for $g_2$. We use $\mathcal{RB}$ to exhibit the lifts of $g_2$ under $\sigma_\beta \circ p_{-1}$. It is worth noting that we are in fact using $\sigma_\beta \circ f_0$ to take the lifts, where $\sigma_\beta$ is the capture mapping that maps $0$ to $a_2$ and fixes all other points outside a small neighborhood of the capture path. Nevertheless, since $\mathcal{RB}$ and $\mathcal{B}$ are the same dynamics, the only difference is the coordinate of each point on the dynamic planes, while the addresses are exactly the same under a modified parametrization by $\mathbb{R}/\mathbb{Z}$. The base point on $\mathcal{RB}$ is chosen to be $t = \frac{1+\sqrt{5}}{2}$, which has itself to be the first inverse image and $-t = -\frac{1+\sqrt{5}}{2}$ to the second inverse image. Denote by $F_\infty$ the Fatou component containing $\infty$, and parameterize its boundary by $\theta : \mathbb{R}/\mathbb{Z} \to \partial F_\infty$ such that $\theta(0) = \theta(1) = t$, and all other points are parameterized counterclockwise. Denote by $(\varnothing)$ the address of $F_\infty$. Then the addresses of all other Fatou components are determined in exactly the same way as those on $\mathcal{B}$.

Denote, respectively, by $g_{21}$ and $g_{22}$ the first and second inverse image of $g_2$. Then $g_{21}$ is a simple path starting at $t$ ending at $-t$ and crossing all the inverse images of the joint points that $g_2$ crosses; $g_{22}$ is also a simple path in the opposite direction that coincides with $g_{21}$ outside $F_0$ and forms a loop with $g_{21}$ on $F_0$. Hence $g_{21}$ and $g_{22}$ are represented by the same set of addresses. The

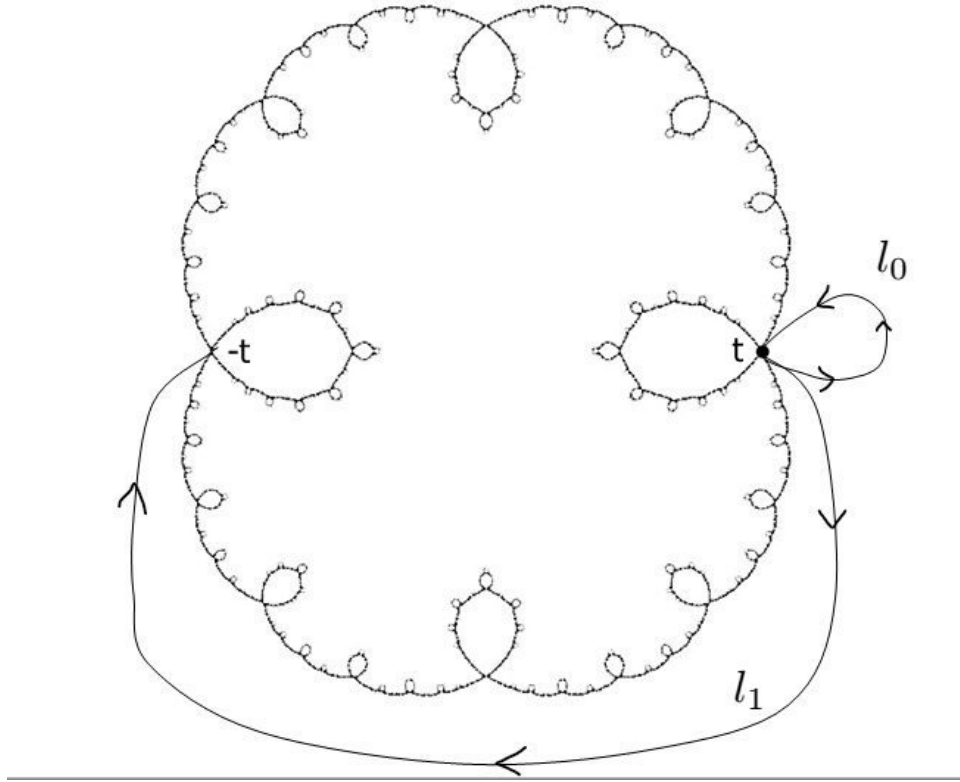two connecting paths $l_0$ and $l_1$ are showed in the following figure:



Figure 3.2: Connecting paths.

The above description results in the following lemma:

**Lemma 3.5.** The presentation of $g_2$ can be written as $g_2 = \langle\!\langle l_1^{-1} \cdot g_{21} \cdot l_0, l_0 \cdot g_{22} \cdot l_1 \rangle\!\rangle \sigma$, where the product notation $\cdot$ is read from right to left. $\qquad\square$

**Proposition 3.6.** Let $\left(\frac{j_1}{2^{i_1}}, \frac{j_2}{2^{i_2}}, ..., \frac{j_l}{2^{i_l}}\right)$ be the address of $g_2$ and $\mathcal{O}$ be the forward orbit of this address. Then $g_2 = \langle\!\langle h, h^{-1} \rangle\!\rangle \sigma$, where $h$ is a composition of generating loops whose addresses are enclosed by the loop $l_1^{-1} \cdot g_{21} \cdot l_0$.

*Proof.* The loops $l_1^{-1} \cdot g_{21} \cdot l_0$ and $l_0 \cdot g_{22} \cdot l_1$ are represented by the same set of addresses as that of $g_{21}$ (as well as $g_{22}$). Hence they go around exactly the same post critical points but in opposite

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orientations. It follows that $h := l_1^{-1} \cdot g_{21} \cdot l_0$ and $h^{-1} = l_0 \cdot g_{22} \cdot l_1$. $\qquad\square$

**Example 3.7.** Let $F_2$ be the same as that in Example 3.2. Then $g_2$ is represented by $\{(\frac{1}{2^2}), (\frac{1}{2^2}, \frac{1}{2^2})\}$; thus $g_{21}$ and $g_{22}$ are represented by $\{(\frac{1}{2^2}, \frac{1}{2^2}), (0, \frac{1}{2^2}, \frac{1}{2^2}), (0, \frac{1}{2^2}), (\frac{1}{2}, \frac{1}{2^2})\}$. Hence $h = g_4 g_7$ and $h^{-1} = g_7^{-1} g_4^{-1}$. All products are read from right to left. Hence $g_2 = \langle\!\langle g_4 g_7, g_7^{-1} g_4^{-1} \rangle\!\rangle \sigma$.
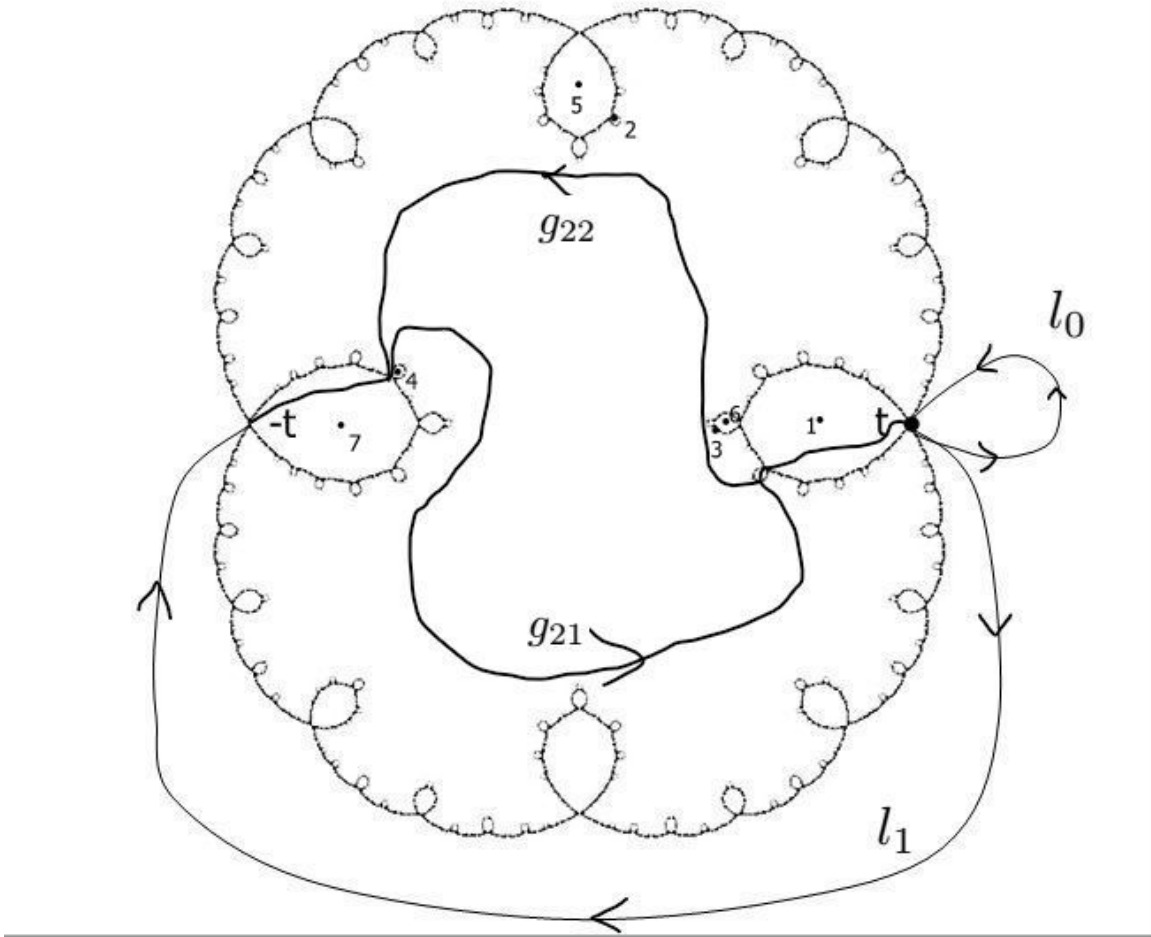


Figure 3.3: Two lifts of $g_2$. Numbers stand for the subscript of each generator.

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# 4. CONJECTURE AND CONCLUSION

**Conjecture.** The function $\sigma_\beta \circ p_{-1}$ is topologically equivalent to some post-critically finite rational functions, i.e. it has no Thurston obstructions (see [8]).

The proof of the above conjecture will be given in later research. We merely assume this conjecture in this paper.

**Lemma 4.1.** Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a degree 2 rational mapping with ramification portrait

$$0 \mapsto a_2 \mapsto \cdots \mapsto a_m \mapsto \infty \overset{\curvearrowright}{\underset{\curvearrowleft}{}} 1$$

where $0$ and $\infty$ are critical points of $f$. Then $f$ is of the form $\dfrac{z^2 + c}{z^2 - 1}$, $c \in \mathbb{C}$.

*Proof.* Write $f = \dfrac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials in $\mathbb{C}[x]$. Since $f$ has degree 2, it follows that either $\deg P(z) = 2$ and $\deg Q(z) \leq 2$, or $\deg P(z) \leq 2$ and $\deg Q(z) = 2$. Moreover, since $f(1) = \infty$, it follows that $Q(z) = z^2 - 1$ or $z - 1$.

(1) If $Q(z) = z^2 - 1$, then

$$f'(z) = \frac{P'(z)(z^2 - 1) - 2zP(z)}{(z^2 - 1)^2}$$

$f'(0) = 0 \implies P'(0) = 0 \implies \deg P(z) = 2$ or $0$. Since $f(\infty) = 1$, $\deg P(z)$ cannot be $0$. It follows that $\deg P(z) = 2$, an thus the coefficient of the highest term of $P(z)$ is $1$.

Write $P(z) = z^2 + bz + c$, $b, c \in \mathbb{C}$. Then

$$\begin{aligned}
f'(z) &= \frac{(2z + b)(z^2 - 1) - (z^2 + bz + c) \cdot 2z}{(z^2 - 1)^2} \\
&= \frac{-bz^2 - (2 + 2c)z - b}{z^4 - 2z^2 + 1}
\end{aligned}$$

Hence it is trivially true that $f'(\infty) = 0$. Since $f'(0) = 0 = -b$, it follows that $b = 0$. Hence $f(z) = \dfrac{z^2 + c}{z^2 - 1}$.

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(2) If $Q(z) = z - 1$, then

$$f'(z) = \frac{P'(z)(z - 1) - P(z)}{(z - 1)^2}$$

$f'(\infty) = 0 \implies \deg P'(z) = 0 \implies \deg P(z) = 1$. This is a contradiction since we assume $f$ has degree 2, but neither $P(z)$ nor $Q(z)$ has degree 2. Hence $Q(z) \neq z - 1$.

Hence the only form of $f$ is $\dfrac{z^2 + c}{z^2 - 1}$. $\qquad\qquad\square$

**Proposition 4.2.** The function $\sigma_\beta \circ p_{-1}$ is topologically equivalent to a post-critically finite $f_c$.

*Proof.* The ramification portrait of $\sigma_\beta \circ f_0$ (which is topologically equivalent to $\sigma_\beta \circ p_{-1}$) is exactly that in the above lemma. Since we assume the conjecture, it follows that $\sigma_\beta \circ f_0$ is of the form $\dfrac{z^2 + c}{z^2 - 1} =: f_c$. $\qquad\qquad\square$

Therefore, we obtained our conclusion:

**Theorem 4.3.** The wreath recursion of IMG($f_c$), in which $f_c$ is post-critically finite, is the same as the wreath recursion of an IMG($\sigma_\beta \circ p_{-1}$). Hence IMG($f_c$) can be represented by the addresses.

*Proof.* This is because two topological mapping are topologically equivalent if and only if their iterated monodromy groups are the same. $\qquad\qquad\square$

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# APPENDIX A

## MATLAB CODE FOR FORWARD ORBIT OF ADDRESSES

```
clc

clear

K=input(prompt);%Input an address

l=length(K);

n=0;

while(length(K) =0)


    if K(1)==0

K(1)=[];

disp(K);

elseif K(1)==1

K(1)=0;

disp(K);

else

K(1)=K(1)-1;

K=[0,K];

disp(K);%Displays each address on this orbit

end

n=n+1;


    end

steps=n-1
```

# APPENDIX B

## MATLAB CODE FOR BRANCHES OF ADDRESSES

```
clc

clear

A=input(prompt); %Input an address

B=A;

C=A;

if isempty(A)

B(1)=0;

C(1)=1/2;

elseif A(1) =0

B=[0,B];

C=[1/2,C];

else

B(1)=[];

B(1)=B(1)2;

C(1)=[];

[a,b]=numden(sym(C(1)));

C(1)=(a+b)/(2*b);

end

disp('The first branch is:');

B %Displays the first branch of the imputed address

disp('The second branch is:');

C %Displays the second branch of the imputed address
```