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Entanglement detection and parameter estimation of quantum channels

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We derive a general criterion to detect entangled states in multipartite systems based on the symmetric logarithmic derivative quantum Fisher information. This criterion is a direct consequence of the fact that separable states do not improve the accuracy upon estimating the one-parameter family of quantum channels. Our result is a generalization of the previously known criterion for the one-parameter unitary channel to any one-parameter quantum channel. Several variants of the proposed criterion are also given, and then the general structure is revealed behind this sort of entanglement criteria based on quantum Fisher information. We discuss several examples to illustrate our criterion. In the last part, we briefly show how the proposed criterion can be extended to a more general setting that is applicable for a certain class of open quantum systems, and we discuss how to detect entangled states even in the presence of decoherence.

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I. INTRODUCTION

The main objective of this paper is to address a problem of the general relationship between entanglement in multipartite systems and quantum Fisher information from the channel-parameter estimation perspective. This problem has received great interest in the field of so-called quantum metrology, that is, quantum-mechanically enhanced precision measurements [1–4]. Our work is largely motivated by the seminal work of Pezzé and Smerzi and the experimental verification of their criterion [5–7]. In Ref. [5], they observed that the symmetric logarithmic derivative (SLD) quantum Fisher information for any separable state cannot be greater than the total number of qubits when qubit states undergo a global rotation along some axis. Their result was further generalized to detect k-producible states and to derive other criteria by taking averages with respect to rotation axes [8–11] (see also updated references cited in review articles [2–4]).

In a recent paper [7], a beautiful experimental result was reported showing that global rotation of atomic spin states was used to detect non-Gaussian entangled states. We also note that Pezzé *et al.* [12] recently attempted to generalize the above types of criteria using the concept of information distance and demonstrated the usefulness of such an approach. However, almost all previous results focused only on unitary channels, which belong to a special class of general quantum channels. The main achievement of this paper is to generalize all previous criteria to the most general setting, i.e., any quantum channel, and to reveal the general structure behind these sorts of entanglement criteria based on quantum Fisher information.

The second motivation of our work is to examine whether entanglement has a benefit for estimating parameters for nonunitary channels, i.e., general quantum channels. To answer this question, we shall use the standard language of the quantum parameter estimation theory developed by Helstrom, Yuen and Lax, Holevo, and others [13–17]. A formal channel-parameter estimation problem in quantum systems was initiated by Fujiwara and his collaborators [18,19], in which they utilize basic tools developed in the quantum parameter estimation theory mentioned above.

There exist at least four known no-go theorems regarding the observation of a quantum metrologically enhanced measurement upon estimating the one-parameter family of quantum channels. Ji *et al.* [20] showed the rather remarkable result that no programmable channels can be estimated with quantum metrological enhancement. Here whether a given channel is programmable or not is defined by Ref. [21]. Fujiwara and Imai [22] provided another no-go theorem stating that quantum metrological enhancement cannot occur for any full-rank channels changing smoothly with the parameter. Their result is very general and implies that almost no realistic quantum channels exhibit such quantum metrological enhancement. Matsumoto gave a simple criterion where no classically simulated channel can be estimated with quantum metrological enhancement [23]. Two results in Refs. [22,23] are well summarized in Ref. [24], where the authors applied these two criteria for physically important quantum channels. Last, Hayashi [25] gave a very powerful argument; no quantum metrological enhancement occurs when a given channel admits a finite amount of the right logarithmic derivative (RLD) quantum Fisher information.

These no-go theorems state that there are quantum channels in which we cannot utilize quantum entanglement to go beyond the standard quantum limit [26]. This poses a question of whether or not nonunitary channels which satisfy the above no-go criteria can be used to detect entanglement. In this paper, we answer this question first by showing that no separable states bring any benefit to estimating one-parameter channels. This is then translated into a simple yet general criterion: If the amount of SLD quantum Fisher information of the output state for a given family of a one-parameter quantum channel is above a certain threshold, then the input state must be entangled. We emphasize that the proposed criterion is not meant to detect *any* entangled states, as typically done by a quantum witness, but rather, it can detect *useful* entanglement upon estimating a given quantum channel. This important point and difference from the usual quantum witness approach were already discussed in the literature [6,12]. We then examine

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several examples to demonstrate the obtained criterion, such as the unitary channel, dephasing channel, depolarizing channel, and transpose channel. We show that detection of entanglement is possible for a certain range of parameters of these channels in at least the two-qubit case.

This paper is organized as follows. In Sec. II we summarize notations and discuss the relationship between classical information quantities and quantum Fisher information. In Sec. III we prove the main result of this paper and then give several variants of the proposed criterion. In Sec. IV we apply our criterion to the bipartite case and compare it to results for different quantum channels. In Sec. V we extend our criterion from the identically and independently distributed (i.i.d.) setting to a more general setting in order to apply it to open quantum systems. In the last section, we summarize our results.

II. CHANNEL-PARAMETER ESTIMATION IN QUANTUM SYSTEMS

We provide the basic terminology and notations used in this paper. We then summarize the basic result known for one-parameter channel estimation problems in quantum systems.

A. Preliminaries

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and $\mathcal{S}(\mathcal{H})$ be the set of all density operators on $\mathcal{H}$, which are semidefinite positive. Let Γ_θ be a trace-preserving and completely positive (TP-CP) map (also called a quantum channel, a quantum map, etc.) from $\mathcal{S}(\mathcal{H})$ to itself that is parametrized by a single parameter θ:

$$\Gamma_\theta : \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{S}(\mathcal{H}). \tag{1}$$

Assume that the parameter θ takes values in an open subset of real numbers, $\Theta \subset \mathbb{R}$; then the output state $\rho_\theta = \Gamma_\theta(\rho)$ for a given input state $\rho \in \mathcal{S}(\mathcal{H})$ can be regarded as a quantum-statistical model parametrized by this parameter $\theta \in \Theta$:

$$\mathcal{M} = \{\rho_\theta = \Gamma_\theta(\rho)\,|\,\theta \in \Theta \subset \mathbb{R}\}. \tag{2}$$

Depending on the channel and the given input state, the rank of the output states $\Gamma_\theta(\rho)$ may vary with respect to the parameter θ in general. For mathematical convenience, we further assume that the rank of the quantum statistical model $\mathcal{M}$ does not change for all values $\theta \in \Theta$, at least for each fixed input quantum state.

The SLD operator about $\rho_\theta \in \mathcal{M}$ is defined by a Hermitian operator L_θ satisfying

$$\frac{d}{d\theta}\rho_\theta = \frac{1}{2}(\rho_\theta L_\theta + L_\theta \rho_\theta). \tag{3}$$

The SLD quantum Fisher information about ρ_θ is defined as

$$g_\theta[\rho_\theta] := \mathrm{tr}\big(\rho_\theta L_\theta^2\big). \tag{4}$$

By definition, it also holds that $g_\theta[\rho_\theta] = \mathrm{tr}(L_\theta \frac{d}{d\theta}\rho_\theta)$. For full-rank states, the solution to operator equation (3) is unique; that is, the SLD operator is uniquely defined. For low-rank states such as pure states, on the other hand, the SLD operator is not uniquely determined from the above equation. In this case, one has to consider equivalent classes to define a proper inner product first and then to define the SLD operator resulting in the unique SLD quantum Fisher information [27].

There are several important properties of the SLD quantum Fisher information. First, it is non-negative, i.e., $g_\theta[\rho_\theta] \geqslant 0$.

Second, quantum Fisher information is additive; that is, for any product state $\rho_\theta = \rho_\theta^1 \otimes \rho_\theta^2 \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$,

$$g_\theta\big[\rho_\theta^1 \otimes \rho_\theta^2\big] = g_\theta\big[\rho_\theta^1\big] + g_\theta\big[\rho_\theta^2\big]. \tag{5}$$

Third, it cannot increase when a CP-TP Γ is applied to the state, that is, the following inequality holds:

$$g_\theta[\Gamma(\rho_\theta)] \leqslant g_\theta[\rho_\theta]. \tag{6}$$

This property is usually referred to as the monotonicity of SLD quantum Fisher information [28]. As a special case of quantum channels, let us consider a general measurement, described by a positive operator-valued measure (POVM), $\Pi = \{\Pi_x\,|\,x \in \mathcal{X}\}$. The map from a state ρ to a probability distribution $p_\theta(x) = \mathrm{tr}(\rho_\theta E_x)$ is regarded as the quantum to classical channel since the state $\rho_\Pi = \mathrm{diag}(p_\theta(x_1), p_\theta(x_2), \dots)$ describes the probability distribution for measurement outcomes. The SLD quantum Fisher information about this (classical) state is equal to the (classical) Fisher information $G_\theta^c[p_\theta]$ defined by

$$G_\theta^c[p_\theta] := \sum_{x\in\mathcal{X}} p_\theta(x)\left[\frac{d}{d\theta}\ln p_\theta(x)\right]^2 = \sum_{x\in\mathcal{X}} \frac{\left[\frac{d}{d\theta}p_\theta(x)\right]^2}{p_\theta(x)}. \tag{7}$$

Since the probability distribution for measurement outcomes is determined by a given POVM Π, we also write it as $G_\theta^c[\Pi, \rho_\theta]$. Thus, the following inequality for any POVM holds:

$$g_\theta[\rho_\theta] \geqslant G_\theta^c[\Pi, \rho_\theta]. \tag{8}$$

We call this property the quantum to classical (q-c) monotonicity of the SLD quantum Fisher information. Importantly, the necessary and sufficient condition is explicitly known for a measurement to attain the equality in Eq. (8) [29–31].

Last, quantum Fisher information is convex with respect to quantum states. Let $\rho_\theta^j \in \mathcal{S}(\mathcal{H})$ ($j = 1,2$) be two families of states with the same parameter set Θ. The convex property states

$$g_\theta\big[\lambda\rho_\theta^1 + (1-\lambda)\rho_\theta^2\big] \leqslant \lambda g_\theta\big[\rho_\theta^1\big] + (1-\lambda) g_\theta\big[\rho_\theta^2\big] \tag{9}$$

for any $\lambda \in [0,1]$. This convexity can be proven in many ways. The simplest is given in Ref. [18] using the monotonicity of the SLD quantum Fisher information. It seems that the equality condition for the above convex inequality is, in general, complicated. Since this condition is important, we examine it for a simple unitary model, which is given at the end of Sec. III B.

The main objective of channel-parameter estimation in quantum systems is to find the ultimate precision bound and an optimal strategy upon estimating the value of parameters describing a given channel. Here we stress that there is no unique way to define the optimality, and one has to analyze a given problem according to a suitable figure of merit. A strategy upon estimating the value of the given quantum channel consists of three elements: an input state, a measurement, and an estimator. One way to get the optimal strategy is as follows. For a fixed input state ρ, we optimize over all possible quantum measurements described by a POVM

Π and an estimator $\hat{\theta}$ which is classical data processing. The set $(\Pi,\hat{\theta})$ is called a quantum estimator, or simply an estimator in this paper. With this optimal estimator, we optimize over all possible input states available. A triplet $(\rho,\Pi,\hat{\theta})$ is called an estimation strategy for the quantum channel. For a one-parameter problem, this procedure gives at least the asymptotically optimal one [32].

When one regards the mean-square error (MSE), defined by

$$E_\theta[\Pi,\hat{\theta}] := \sum_{x\in\mathcal{X}}[\hat{\theta}(x)-\theta]^2\mathrm{tr}[\Gamma_\theta(\rho)\Pi_x], \tag{10}$$

as a figure of merit for the channel estimation, one can derive lower bounds for the MSE depending upon resources and estimation schemes under consideration. Let $\mathcal{H}^{\otimes N}$ and $\mathcal{S}(\mathcal{H}^{\otimes N})$ be the N tensor product of the Hilbert space and the totality of positive density operators on it, respectively. Consider an Nth i.i.d. extension of the given channel, and denote it as

$$\Gamma_\theta^N := \underbrace{\Gamma_\theta\otimes\Gamma_\theta\otimes\cdots\otimes\Gamma_\theta}_{N} : \mathcal{S}(\mathcal{H}^{\otimes N}) \longrightarrow \mathcal{S}(\mathcal{H}^{\otimes N}).$$

When one only uses the Nth i.i.d. extension of input states $\rho^{\otimes N}$ to estimate the channel, the problem is to find an optimal input state maximizing the SLD quantum Fisher information for the channel Γ_θ. Let ρ^* be one such optimal input state and g_θ^* be the maximum of the SLD quantum Fisher information:

$$g_\theta^*(\Gamma_\theta) := \max_{\rho\in\mathcal{S}(\mathcal{H})}\{g_\theta[\Gamma_\theta(\rho)]\}. \tag{11}$$

Importantly, the convexity property of the SLD Fisher information guarantees that the optimal input state attaining $g_\theta^*(\Gamma_\theta)$ can be a pure state [18].

The additivity of SLD quantum Fisher information becomes $g_\theta[\Gamma_\theta^N(\rho^{\otimes N})] = Ng_\theta[\Gamma_\theta(\rho)]$. For any locally unbiased estimators $(\Pi,\hat{\theta})$, the MSE is equal to the variance of estimating the value of parameter and the SLD Cramér-Rao (CR) bound is given by

$$\mathrm{Var}_\theta[\Pi,\hat{\theta}] \geqslant \frac{1}{N}[g_\theta^*(\Gamma_\theta)]^{-1}. \tag{12}$$

In general, this bound is attained adaptively in the N infinite limit unless the channel possesses a special symmetry. See the discussion given in Ref. [33] and an experimental demonstration of the adaptive estimation [34]. Alternatively, one can use the two-step method proposed in Refs. [35,36].

When one estimates the Nth i.i.d. extension of the channel Γ_θ^N, one can also use other resources such as entangled states $\rho\in\mathcal{S}(\mathcal{H}^{\otimes N})$ for input states or ancillary states. In this case, the variance for estimation can be further lowered. This enhancement effect, known as quantum metrology, is important for quantum information processing protocols and has been investigated actively [1–3].

B. Experimental detection of SLD quantum Fisher information

In this section, we discuss a general strategy of how to detect the amount of quantum Fisher information about the output state of a given family of quantum channels in experiment. We assume that the parameter for the quantum channel can be tuned at will and there are identical resources to repeat the same experiment sufficiently many times. A prominent step was already reported in Ref. [7]. In this paper we shall present a more general framework to supplement their result.

First of all, if one knows an input state ρ completely, then one can directly evaluate the SLD quantum Fisher information about the state $\Gamma_\theta[\rho]$ by substituting an estimated value of θ into formula (4). If one attempts to evaluate the SLD quantum Fisher information without knowing input states, one has to follow a different strategy as follows. For a given one-parameter family of quantum channels Γ_θ, let us fix an unknown input state ρ and consider a fixed measurement Π on the output state $\rho_\theta = \Gamma_\theta(\rho)$. Then, the family of probability distributions for the measurement outcomes is regarded as a classical statistical model:

$$\mathcal{M}(\Pi,\Gamma_\theta,\rho) = \{p_\theta[\Pi]\,|\,\theta\in\Theta\},$$
$$p_\theta[\Pi] = \{p_\theta(x) = \mathrm{tr}[\Gamma_\theta(\rho)\Pi_x]\,|\,x\in\mathcal{X}\}. \tag{13}$$

By performing a sufficiently large repetition of the same measurement for a fixed value of the parameter θ, we can obtain experimental data according to the classical probability distribution $p_\theta[\Pi]$. These data then give a density estimation of the distribution $p_\theta[\Pi]$. We next change the channel parameter θ and redo the same step as before. After sufficiently many observations with respect to the changes in θ, say M different choices, we can obtain the set of classical probability distributions $\{p_\theta\,|\,\theta\in\{\theta_1,\theta_2,\ldots,\theta_M\}\}$. If we choose the parameter set $\{\theta_1,\theta_2,\ldots,\theta_M\}$ $(\theta_{k+1}>\theta_k)$ such that the differences $\Delta_k = \theta_{k+1}-\theta_k$ are sufficiently small, then we can directly calculate the classical Fisher information $G_\theta^c[p_\theta]$ approximately from the definition (7).

Alternatively, one can estimate other information quantities first and then calculate the classical Fisher information as follows. In classical information theory, the most general information quantity is the f divergence [37]. This family of information quantity is a measure of "distance" between two probability distributions. The formal definition of f divergence for two probability distributions p,q on $\mathcal{X}$ is

$$D_f(p||q) := \sum_{x\in\mathcal{X}} p(x) f\left(\frac{q(x)}{p(x)}\right), \tag{14}$$

where $f:\mathbb{R}^+\to\mathbb{R}^+$ is a monotonically decreasing and convex function and $f(1)=0$ is a standard convention. Familiar examples are $f(t) = -\log_2(t)$ (the relative entropy), $f(t) = 1-\sqrt{t}$ (the Hellinger distance), and $f(t)=t^\alpha$ (the relative Rényi entropy). One important property of the f divergence is the following relation to Fisher information:

$$G_\theta^c[p_\theta] = 2\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}D_f(p_\theta||p_{\theta+\epsilon}). \tag{15}$$

From experimental data obtained after many repetitions, we can construct a curve for f divergence $D_f(p_\theta||p_{\theta'})$ for various different values of θ,θ'. It is easy to see that formula (15) provides an approximated value for the classical Fisher information.

We next show that this experimentally obtained Fisher information can attain the SLD quantum Fisher information by the optimal measurement. As noted in the previous section, q-c monotonicity of the SLD quantum Fisher information implies

$$g_\theta[\Gamma_\theta(\rho)] \geqslant G_\theta^c[\Pi], \tag{16}$$

where the equality holds if and only if Π is optimal and is given by the projection measurement about the SLD operator L_θ [29–31]. By choosing the optimal measurement, the classical Fisher information obtained with the above-described method yields the approximated value of the quantum Fisher information. However, this requires exact knowledge of the input state, and hence, one can only get a rough estimate of the lower bound for the SLD quantum Fisher information in general.

III. RESULT

A. Separability criterion

The main result of this paper is the following theorem.

Theorem 1. For a given channel Γ_θ parametrized by a single parameter θ, let Γ_θ^N be the Nth i.i.d. extension of Γ_θ and $g_\theta^*(\Gamma_\theta)$ be the largest value of SLD quantum Fisher information, which is given by Eq. (11). For each value θ, if a density operator ρ on $\mathcal{S}(\mathcal{H}^{\otimes N})$ is separable, then the SLD quantum Fisher information $g_\theta[\Gamma_\theta^N(\rho)]$ is smaller than or equal to the value $Ng_\theta^*(\Gamma_\theta)$.

Several remarks are in order. First, taking the contraposition of this theorem, it is equivalent to state that if the value of SLD quantum Fisher information for the output states $\Gamma_\theta^N(\rho)$ is larger than $Ng_\theta^*(\Gamma_\theta)$, then the input state ρ on $\mathcal{S}(\mathcal{H}^{\otimes N})$ is entangled.

Second, the special case of this separability criterion was shown by Pezzé and Smerzi [5], where the channel is given by a rotation along a given axis on N qubits in the context of quantum metrology. In this case, $g_\theta^* = 1$ holds for all values of θ due to the symmetry of this unitary transformation. This special case will be examined in the next section. Thus, our contribution is to first prove their criterion in a general setting and to provide a more general criterion.

Third, since the MSE for estimation of the value θ is bounded by the inverse of SLD quantum Fisher information, Theorem 1 states that separable states are not efficient following the usage of the Nth extension of a given channel. But, of course, this theorem does not indicate if all entangled states are more efficient than separable ones or not.

Fourth, it is straightforward to see this theorem can be extended to more general channels parameterized by several parameters. In this case, the SLD quantum Fisher information becomes a matrix, and the corresponding inequality is given by a matrix inequality. It is also not difficult to see from the proof that SLD quantum Fisher information can be replaced by other quantum Fisher information. See Sec. III C for more details.

Last, the most important point is that the parameter θ is arbitrary in Theorem 1. Since we can vary it as an arbitrary value, we can then consider the union of all possible parameter regions of entangled states. Let $r_{\text{ent}}(\theta)$ be the entangled region of the states derived from the inequality $g_\theta[\Gamma_\theta^N(\rho)] > Ng_\theta^*(\Gamma_\theta)$ for a fixed value θ, i.e.,

$$r_{\text{ent}}(\theta) = \{\rho \in \mathcal{S}(\mathcal{H}^{\otimes N}) \mid g_\theta[\Gamma_\theta^N(\rho)] > Ng_\theta^*(\Gamma_\theta)\}; \tag{17}$$

then, the union

$$R_{\text{ent}} := \bigcup_{\theta\in\Theta} r_{\text{ent}}(\theta) \tag{18}$$

provides the most powerful criterion. Since the subset of states $R_{\text{ent}} \subset \mathcal{S}(\mathcal{H}^{\otimes N})$ is solely determined by the given quantum channel Γ_θ, we denote it as $R_{\text{ent}}[\Gamma_\theta]$. With these notations, our contribution is to derive the criterion

$$\rho \in R_{\text{ent}}[\Gamma_\theta] \;\Rightarrow\; \rho \text{ is entangled.} \tag{19}$$

This point will be illustrated by several examples in Sec. IV.

The proof of Theorem 1 is straightforward and is given as follows.

Proof. Consider an arbitrary separable state on $\mathcal{S}(\mathcal{H}^{\otimes N})$ of the form

$$\rho_{\text{sep}} = \sum_j p_j \rho_j^{(1)} \otimes \rho_j^{(2)} \otimes \cdots \otimes \rho_j^{(N)}, \tag{20}$$

where $\sum_j p_j = 1 \,\forall\, p_j \geqslant 0$ and $\rho_j^{(k)}$ are states on the kth Hilbert space. Then, the following sequence of inequalities holds:

$$g_\theta[\Gamma_\theta^N(\rho_{\text{sep}})] = g_\theta\left[\Gamma_\theta^N\left(\sum_j p_j\rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)}\right)\right] \tag{21}$$

$$= g_\theta\left[\sum_j p_j \Gamma_\theta^N\left(\rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)}\right)\right] \tag{22}$$

$$\leqslant \sum_j p_j g_\theta\left[\Gamma_\theta^N\left(\rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)}\right)\right] \tag{23}$$

$$= \sum_j p_j g_\theta\left[\Gamma_\theta\left(\rho_j^{(1)}\right) \otimes \cdots \otimes \Gamma_\theta\left(\rho_j^{(N)}\right)\right] \tag{24}$$

$$= \sum_j p_j \sum_{k=1}^N g_\theta\left[\Gamma_\theta\left(\rho_j^{(k)}\right)\right] \tag{25}$$

$$\leqslant \sum_j p_j \sum_{k=1}^N g_\theta^*[\Gamma_\theta] \tag{26}$$

$$= Ng_\theta^*[\Gamma_\theta]. \tag{27}$$

Here equality (22) follows from the linearity of quantum channels, and inequality (23) follows from the convexity of SLD quantum Fisher information with respect to the states. The i.i.d. assumption and additivity of the SLD Fisher information give expression (25). Inequality (26) is due to the definition of $g_\theta^*[\Gamma_\theta]$, Eq. (11), and this completes the proof. ■

B. Shift-parameter model

As noted in the remarks following Theorem 1, the theorem is simplified when the channel is given by a unitary transformation of the form

$$\Gamma_\theta(\rho) = e^{i\theta A}\rho\, e^{-i\theta A}, \tag{28}$$

where a Hermitian operator A on $\mathcal{H}$ is called a generator of the unitary transformation. The parameter region is any 2π interval of real numbers, e.g., $\Theta = [0, 2\pi)$. The quantum-statistical model for the output states is given by

$$\mathcal{M}_A = \{\rho_\theta = e^{i\theta A}\rho_0 e^{-i\theta A}\,|\,\theta \in \mathbb{R}\}. \tag{29}$$

Here ρ_0 is called a reference state. This model was referred to as a shift-parameter model and a displacement model in Refs. [13,15] and is also known as a unitary model in the physics community.

The following lemma is fundamental for the shift-parameter model.

Lemma 1. For a shift-parameter model, the SLD quantum Fisher information is independent of the parameter θ and is bounded from above as

$$g_\theta = g_{\theta=0} \leqslant 4\Delta_{\rho_0}A, \tag{30}$$

where $\Delta_\rho A := \mathrm{tr}(\rho A^2) - [\mathrm{tr}(\rho A)]^2$ is the square of the variance of operator A with respect to state ρ.

This lemma can be proven in different ways; here we sketch the most transparent one due to Holevo [15].

Proof. For a given state ρ, let $\mathcal{D}_\rho$ be a superoperator acting on Hermitian operators X on $\mathcal{H}$, which is formally defined by the solution to the following operator equation:

$$\rho\,\mathcal{D}_\rho(X) + \mathcal{D}_\rho(X)\,\rho = \frac{1}{i}[\rho\,,X]. \tag{31}$$

It follows from the definition that the SLD operator is expressed as

$$L_\theta = 2\mathcal{D}_{\rho_\theta}(A) = e^{-i\theta A}L_0\,e^{i\theta A}, \tag{32}$$

$$L_0 = 2\mathcal{D}_{\rho_0}(A). \tag{33}$$

This relation proves the first equality in Eq. (30).

We define a symmetric inner product for linear operators X,Y on $\mathcal{H}$ by

$$\langle X,Y\rangle_\rho := \tfrac{1}{2}\mathrm{tr}[\rho(YX^\dagger + X^\dagger Y)]; \tag{34}$$

then the SLD quantum Fisher information for the shift-parameter model (29) is written as

$$g_0 = \langle L_0,L_0\rangle_{\rho_0} = 4\langle \mathcal{D}_{\rho_0}(A),\mathcal{D}_{\rho_0}(A)\rangle_{\rho_0}. \tag{35}$$

Next, we note that the relation

$$\langle X,X\rangle_\rho - \langle \mathcal{D}_\rho(X),\mathcal{D}_\rho(X)\rangle_\rho = \big\langle X,\big(1+\mathcal{D}_\rho^2\big)(X)\big\rangle_\rho \geqslant 0 \tag{36}$$

holds for any Hermitian operators X and a state ρ on $\mathcal{H}$ since the superoperator $1+\mathcal{D}_\rho^2$ is positive with respect to the inner product. Writing the variance as $\Delta_{\rho_0}A = \langle A-\bar{A},A-\bar{A}\rangle_{\rho_0}$ with $\bar{A} = \mathrm{tr}(\rho A)$ and using the relation $\mathcal{D}_{\rho_0}(A) = \mathcal{D}_{\rho_0}(A-\bar{A})$, we prove the inequality in Eq. (30). ■

We note that the equality condition for the inequality in Eq. (30) is equivalent to the condition [15]

$$\big(1+\mathcal{D}_\rho^2\big)(A-\bar{A}) = 0 \Leftrightarrow \rho_0 A\rho_0 = \bar{A}\rho_0^2. \tag{37}$$

It is clear that this is satisfied if ρ_0 is a pure state, and one might expect that the converse also holds. This is true if the dimension of the Hilbert space is 2, i.e., qubit. However, we note that the condition that ρ_0 is pure is just a sufficient condition in general. The sufficiency is immediate if we use the second condition in Eq. (37). A simple counterexample of mixed states satisfying the upper bound is given by a rank-2 state in $\dim\mathcal{H} = 3$ as follows:

$$\rho_0 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 & c^* \\ 0 & a & d^* \\ c & d & b \end{pmatrix}, \tag{38}$$

where $\lambda \in (0,1)$ and a,b and c,d are real and complex numbers, respectively.

The variance of all possible states $\rho \in \mathcal{S}(\mathcal{H})$ is maximized when we take an equally weighted superposition of the eigenstates whose eigenvalues are maximum and minimum [38]. With this observation, Theorem 1 and Lemma 1 can be combined to give the following corollary.

Corollary 1. For a given shift-parameter model, let $U_A^N = \bigotimes_{k=1}^N e^{i\theta A}$ be a global unitary for $\mathcal{S}(\mathcal{H}^{\otimes N})$. If a density operator ρ on $\mathcal{S}(\mathcal{H}^{\otimes N})$ is separable, then the SLD quantum Fisher information $g_\theta[U_A^N\,\rho\,(U_A^N)^\dagger]$ is smaller than or equal to the value $N(a_{\max} - a_{\min})^2$, where $a_{\max}$ ($a_{\min}$) is the maximum (minimum) of the eigenvalues of A.

Since we have an achievable bound for unitary channels, one can consider various extensions of the criterion. An immediate one is to consider a set of rotations around several axes and to take the average SLD Fisher information. Another one is to consider k-producible states rather than a completely separable state of N qubits. These extensions seem to work quite well, as reported in Refs. [9–11]. In the next section, we also examine how these variants can be derived from our general theorem.

Before closing this section, we shall analyze the equality condition for the convexity of SLD quantum Fisher information, i.e., the equality condition in inequality (9), in the case of this simple unitary model. Consider mixed qubit states generated by the following unitary:

$$\rho_\theta^j = e^{-i\theta\boldsymbol{n}\cdot\boldsymbol{\sigma}/2}\rho_0^j\,e^{i\theta\boldsymbol{n}\cdot\boldsymbol{\sigma}/2} \quad (j=1,2), \tag{39}$$

where $\boldsymbol{n}$ is a given unit vector and θ is the rotation angle and their convex mixture

$$\rho_\theta^\lambda = \lambda\rho_\theta^1 + (1-\lambda)\rho_\theta^2. \tag{40}$$

Let $g_\theta^j = g_\theta[\rho_\theta^j]$ and $g_\theta^\lambda = g_\theta[\rho_\theta^\lambda]$ be the SLD quantum Fisher information about states ρ_θ^j and ρ_θ^λ, respectively. A straightforward calculation shows that

$$g_\theta^j = g_{\theta=0}^j = |\boldsymbol{n}\times\boldsymbol{s}_j|^2 \tag{41}$$

when expressed in terms of the Bloch vector of the state, $\rho_0^j = (I + \boldsymbol{s}_j\cdot\boldsymbol{\sigma})/2$. Let us define the difference $\Delta g_\theta^\lambda = \lambda g_\theta^1 + (1-\lambda)g_\theta^2 - g_\theta^\lambda$; then it reads

$$\Delta g_\theta^\lambda = \lambda(1-\lambda)|\boldsymbol{n}\times(\boldsymbol{s}_1 - \boldsymbol{s}_2)|^2. \tag{42}$$

Therefore, the equality in convexity inequality (9) holds if and only if the difference of the two Bloch vectors $\boldsymbol{s}_1 - \boldsymbol{s}_2$ is parallel to the rotation direction $\boldsymbol{n}$. This is equivalent to satisfying the condition $\boldsymbol{s}_2 = \boldsymbol{s}_1 - 2(\boldsymbol{n}\cdot\boldsymbol{s}_1)\boldsymbol{n}$.

C. Variants of the proposed criterion

So far we have presented a simple yet very general criterion upon detecting entangled states from parameter estimation of quantum channels. In this section, we wish to provide several variants of the proposed criterion and to convince readers that our criterion is indeed the most general one, including all previous known criteria as variants. In the following we provide three possible variations of the criterion and then analyze the general structure behind this sort of entanglement criterion based on quantum Fisher information. Proofs for these variants are omitted because they follow almost the same format as that of Theorem 1.

1. Average criterion

The first variant is to consider several families of quantum channels:

$$\Gamma_\theta = \{\Gamma_\theta^{(j)} \,|\, \theta \in \Theta\} \quad (i \in I_J), \tag{43}$$

where the discrete index $j \in I_J = \{1,2,\dots,J\}$ distinguishes different families of quantum channels. An extension to the continuous-index-set case is straightforward and is omitted below. Let us define the average SLD quantum Fisher information about a given state $\rho \in \mathcal{S}(\mathcal{H}^{\otimes N})$ with respect to the index set I_J by

$$\bar{g}_\theta[\Gamma_\theta(\rho)] := \sum_{j\in I_J} g_\theta\big[\Gamma_\theta^{(j)}(\rho)\big] \tag{44}$$

and the maximal SLD quantum Fisher information for the single-input state by

$$\bar{g}_\theta^*(\Gamma_\theta) := \max_{\rho\in\mathcal{S}(\mathcal{H})} \left\{\sum_{j\in I_J} g_\theta\big[\Gamma_\theta^{(j)}(\rho)\big]\right\}. \tag{45}$$

Then, the average criterion states the following.

Lemma 2. If state $\rho \in \mathcal{S}(\mathcal{H}^{\otimes N})$ is separable, the following inequality always holds:

$$\bar{g}_\theta[\Gamma_\theta(\rho)] \leqslant N\bar{g}_\theta^*(\Gamma_\theta). \tag{46}$$

Importantly, this variant provides a stronger criterion than the original one after taking the average with respect to the index set. This is because the following relation holds in general:

$$\max_{\rho\in\mathcal{S}(\mathcal{H})} \left\{\sum_{j\in I_J} g_\theta\big[\Gamma_\theta^{(j)}(\rho)\big]\right\} \leqslant \sum_{j\in I_J} \max_{\rho\in\mathcal{S}(\mathcal{H})} \big\{g_\theta\big[\Gamma_\theta^{(j)}(\rho)\big]\big\}. \tag{47}$$

In other words, $\bar{g}_\theta^*(\Gamma_\theta) \leqslant g_\theta^*(\Gamma_\theta)$.

In Refs. [9–11], the authors pointed out that taking the average with respect to different rotational axes sharpens the entanglement criterion for the family of qubit unitary channels. In this case the enhancement factor is 2/3, i.e., $\bar{g}_\theta^*(\Gamma_\theta) = 2g_\theta^*(\Gamma_\theta)/3$, and the mathematical reason is clearly explained here.

2. Approximation criterion

To apply the entanglement criterion studied in this paper, it is necessary to compute SLD quantum Fisher information for composite systems as well as the maximum value for a single-input state. Although such a computation is straightforward numerically, the analytical result cannot be expected except in very special cases. Here we point out that an approximated value enables us to detect entangled states.

Case 1 (bounds for the single-input state). Suppose we have found a bound for the SLD quantum Fisher information about single-input states by some function f_θ as

$$g_\theta[\Gamma_\theta(\rho)] \leqslant f_\theta, \tag{48}$$

which holds for each parameter value θ. Then, the variant criterion states that if a state ρ is separable, then the following inequality holds:

$$g_\theta[\Gamma_\theta(\rho)] \leqslant N f_\theta. \tag{49}$$

Equivalently, if $g_\theta[\Gamma_\theta(\rho)] > N f_\theta$, then ρ is entangled.

Case 2 (bounds for the composite state). In many cases, computation of the SLD quantum Fisher information become harder as the number of composite systems increases. Suppose we have an inequality for the SLD quantum Fisher information about the output state $\Gamma_\theta(\rho)$ as

$$g_\theta[\Gamma_\theta(\rho)] \geqslant F_\theta(\rho); \tag{50}$$

then, the second variant of the approximation criterion states the following. For a given composite state ρ, if the inequality

$$F_\theta(\rho) > N g_\theta^*(\Gamma_\theta) \tag{51}$$

holds, then state ρ is entangled.

An important example of the function of states $F_\theta(\rho)$ is the inverse of the MSE. This is due to the SLD CR bound:

$$E_\theta[\Pi,\hat{\theta}] \geqslant \{g_\theta[\Gamma_\theta(\rho)]\}^{-1}. \tag{52}$$

This result enables us to conclude the simple fact that if the inverse value of the MSE after estimating a given channel exceeds a certain threshold $N g_\theta^*(\Gamma_\theta)$, then the composite state must be entangled.

3. Nonidentical channel criterion

In the last variant of criteria, we consider nonidentical channels. For simplicity, let us consider two different parametric families of quantum channels:

$$\{\Gamma_\theta^{(j)} \,|\, \theta \in \Theta\} \quad (j = 1,2), \tag{53}$$

acting on the same state space $\mathcal{S}(\mathcal{H})$. We now divide the N-composite system into the first k system and the remaining $N-k$ system, corresponding to the splitting of the Hilbert space: $\mathcal{H}^{\otimes N} = \mathcal{H}^{\otimes k} \otimes \mathcal{H}^{\otimes N-k}$. Let us apply the first channel $\Gamma_\theta^{(1)}$ identically and independently to the k-partite states and the second channel $\Gamma_\theta^{(2)}$ to the rest, that is, consider a quantum channel of the form $\Gamma_\theta := (\Gamma_\theta^{(1)})^{\otimes k} \otimes (\Gamma_\theta^{(2)})^{\otimes N-k}$. Then, the variant criterion states the following.

Lemma 3. If a given state $\rho \in \mathcal{S}(\mathcal{H}^{\otimes N})$ is separable, then the SLD quantum Fisher information about $\Gamma_\theta(\rho)$ satisfies the inequality

$$g_\theta[\Gamma_\theta(\rho)] \leqslant k g_\theta^*\big(\Gamma_\theta^{(1)}\big) + (N-k) g_\theta^*\big(\Gamma_\theta^{(2)}\big), \tag{54}$$

where $g_\theta^*(\Gamma_\theta^{(j)}) = \max\{g_\theta[\Gamma_\theta^{(j)}(\rho)]\}$ is the maximum value of the SLD quantum Fisher information for the jth channel.

We note that an extension to more than two channels is straightforward.

4. General structure

In this section, we analyze the general structure behind the proposed criterion based on quantum Fisher information. The key ingredients for our discussion are as follows. The existence of the "information quantity" $I(\rho)$, which is a functional of states, satisfies three axioms: (1) positivity, i.e., $I(\rho) \geqslant 0$ for all states $\rho \in \mathcal{S}(\mathcal{H})$, (2) additivity about product states, i.e., $I(\rho \otimes \sigma) = I(\rho) + I(\sigma)$, and (3) convexity about the states, i.e., $I[\lambda\rho + (1-\lambda)\sigma] \leqslant \lambda I(\rho) + (1-\lambda)I(\sigma)$ for $\lambda \in [0,1]$. An additional element is boundedness: $\sup_{\rho\in\mathcal{S}(\mathcal{H})}\{I(\rho)\}<\infty$. With these assumptions, we have the following general theorem.

Theorem 2. If a state $\rho \in \mathcal{S}(\mathcal{H}^{\otimes N})$ is (completely) separable, then the information quantity is always bounded as

$$I(\rho) \leqslant N I^*, \tag{55}$$

where $I^* = \sup_{\rho\in\mathcal{S}(\mathcal{H})}\{I(\rho)\}$ is the maximum information for a single-input state.

The proof of this theorem is along the same lines as that of Theorem 1.

One of the main achievements of information theory is to establish nontrivial equality between the information quantity defined via calculus and the operationally defined quantity. The general form (55) emphasizes that this class of entanglement criteria is not meant to detect *any entangled states* but to pick up *useful entanglement* for a given information processing task. To put it differently, if a given composite state ρ passes through the above criterion, then entanglement presented in state ρ provides an advantage over a state without entanglement. In the case of quantum Fisher information studied in this paper, the proposed criterion can detect useful entanglement upon estimating the value of a parameter for a given quantum channel. Additional analysis of the above general criterion is required before any conclusive statement can be made about whether it is useful or not. That is beyond the scope of this paper and will be performed in due course.

IV. EXAMPLES

In this section we analyze several examples to illustrate the proposed criterion to detect entanglement, in particular, criterion (19). To get analytical results, we simplify the setting to the two-qubit case, that is, $N = 2$ and $\dim\mathcal{H} = 2$. The input states analyzed in this section are the Bell-diagonal states defined by

$$\rho_{BD}(c_1,c_2,c_3) := \frac{1}{4}\left(I + \sum_{j=1}^{3} c_j\sigma_j \otimes \sigma_j\right). \tag{56}$$

Here σ_j are the usual Pauli spin operators, and the coefficients are restricted from the positivity condition as

$$\begin{aligned} 1 - c_1 - c_2 - c_3 &\geqslant 0,\\ 1 - c_1 + c_2 + c_3 &\geqslant 0,\\ 1 + c_1 - c_2 + c_3 &\geqslant 0,\\ 1 + c_1 + c_2 - c_3 &\geqslant 0. \end{aligned} \tag{57}$$

This state space is geometrically represented by a tetrahedron in the (c_1,c_2,c_3) coordinate system. All separable states for the two-qubit case are given by the well-known positive partial transpose (PPT) criterion [39,40]. For the Bell-diagonal states, it is given by a simple inequality:

$$\rho_{DB} \text{ is separable } \Leftrightarrow |c_1| + |c_2| + |c_3| \leqslant 1, \tag{58}$$

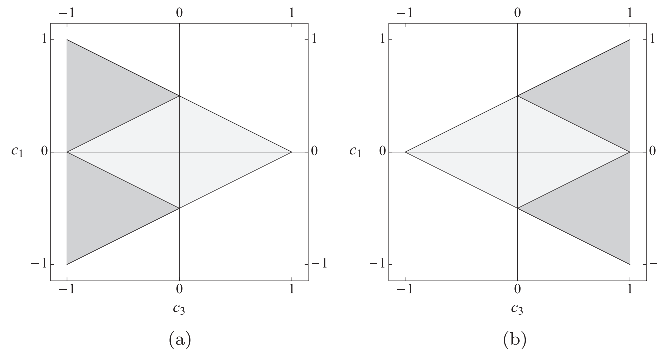

FIG. 1. Two-dimensional cross sections of state space for Bell-diagonal states (56): c_1 (vertical axis) vs c_3 (horizontal axis) for (a) case (a), $c_1 = c_2$, and (b) case (b), $c_1 = -c_2$. In Fig. 1(a), the largest triangle shows the state space, and two smaller triangles indicate entangled regions. The same convention holds for (b).

which is geometrically represented by an octahedron located inside the tetrahedron. In the following, we mainly analyze two-dimensional cross sections of the Bell-diagonal states to draw two-dimensional plots rather than three-dimensional ones. The first is to set $c_1 = c_2$ [case (a)], and the other is to set $c_1 = -c_2$ [case (b)]. For case (a), the state space is given by a triangle in Fig. 1(a), and the entangled regions are shown by the gray areas. Similarly, the matter for case (b) is shown in Fig. 1(b).

Further simplification arises if we restrict the following two subfamilies for case (a):

$$\begin{aligned} \rho_\lambda^+ &= \rho_{BD}(\lambda,\lambda,-\lambda),\\ \rho_\lambda^- &= \rho_{BD}(-\lambda,-\lambda,-\lambda). \end{aligned} \tag{59}$$

States $\rho_\lambda^\pm$ are also written as

$$\rho_\lambda^\pm = \lambda|\psi_\pm\rangle\langle\psi_\pm| + \tfrac{1}{4}(1-\lambda)I, \tag{60}$$

where $|\psi_\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ are the Bell states. ρ_λ^- is known as the Werner state, and ρ_λ^+ is locally equivalent to it. For λ in $\Lambda := (-1/3,1)$, both states $\rho_\lambda^\pm$ are strictly positive. Further, $\rho_\lambda^\pm$ are entangled if and only if $\lambda \in \Lambda_{\mathrm{ent}} := (1/3,1]$. The difference between $\rho_\lambda^\pm$ is that ρ_λ^- is a rotationally invariant state (spin-singlet state), whereas ρ_λ^+ is only invariant around the z axis.

Our concern is to find a set of entangled states which can be detected by a given quantum channel. This quantity is represented by Eq. (17) or Eq. (18).

A. Unitary channels

1. Rotation around the z axis

We first consider a rotation around the z axis on a single-qubit system as

$$U_z(\theta) = e^{i\theta\sigma_z/2}, \tag{61}$$

with $\theta \in \Theta = [0,2\pi)$. The maximum variance of the generator $\sigma_z/2$ is 1. In this case, Theorem 1 reduces to the Pezzé-Smerzi criterion that compares the value of SLD quantum Fisher information about the rotated state $U_z^N(\theta)\rho[U_z^N(\theta)]^\dagger$ with the total number of qubit systems, i.e., N. Here $U_z^N(\theta) =$

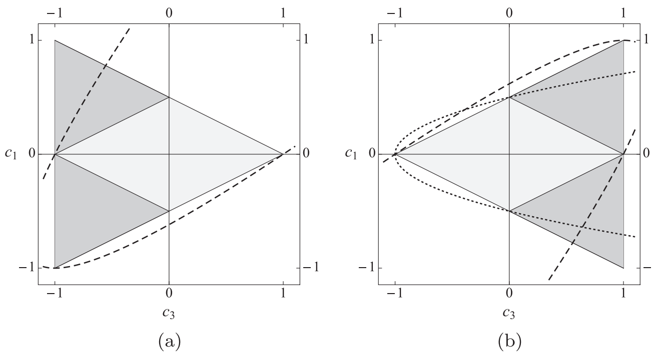

FIG. 2. Entangled regions detected by unitary channels for Bell-diagonal states (56): (a) Case (a), $c_1 = c_2$, and (b) case (b), $c_1 = -c_2$. State space and entangled regions follow the same convention as in Fig. 1. The dotted curve corresponds to the boundary due to a rotation around the z axis, and dashed curves are those due to a rotation around the x axis.

$\bigotimes_{k=1}^N e^{i\theta\sigma_z/2} = e^{i\theta J_z}$, with J_z being the z component of the total angular momentum operator. The SLD quantum Fisher information for the two-qubit Bell-diagonal states is

$$g_\theta^z := g_\theta\big[U_z^2(\theta)\rho_{BD}\big(U_z^2\big)^\dagger\big] = 2\frac{(c_1-c_2)^2}{1+c_3}, \tag{62}$$

and hence, the Pezzé-Smerzi criterion states that ρ_{BD} is entangled if

$$2\frac{(c_1-c_2)^2}{1+c_3} > 2. \tag{63}$$

For case (a), this condition cannot be satisfied since the numerator is always zero. Thus, the above criterion does not detect entangled states at all. Case (b), on the other hand, reads $4c_1^2 > 1 + c_3$, and the entangled region detected by this unitary is shown in Fig. 2(b). In this figure, the dotted curve represents the parabola $4c_1^2 = 1 + c_3$, and the entangled region is outside of this curve. Due to the symmetry $c_1 \leftrightarrow -c_1$, the result is symmetric with respect to reflection about the c_3 axis. This is in contrast to the case of the unitary channel generated by a rotation around the x axis.

As noted before, the SLD quantum Fisher information for states $\rho_\lambda^\pm$ is zero due to the fact that J_z commutes with $\rho_\lambda^\pm$. Thus, one cannot get any useful information about $\rho_\lambda^\pm$ by applying any global rotation around the z axis.

2. Rotation around the x axis

We next consider a rotation around the x axis as

$$U_x(\theta) = e^{i\theta\sigma_x/2}. \tag{64}$$

The SLD quantum Fisher information for the two-qubit Bell-diagonal states is

$$g_\theta^x := g_\theta\big[U_x^2(\theta)\rho_{BD}\big(U_x^2\big)^\dagger\big] = 2\frac{(c_2-c_3)^2}{1+c_1}, \tag{65}$$

and thus, the Pezzé-Smerzi criterion is that ρ_{BD} is entangled if

$$2\frac{(c_2-c_3)^2}{1+c_1} > 2. \tag{66}$$

For case (a), this is expressed as $(c_1 - c_3)^2 > 1 + c_1$, and the detected entangled region is plotted in Fig. 2(a). Case (b) reads $(c_1 + c_3)^2 > 1 + c_1$ and is shown in Fig. 2(b). In these figures, the dashed curves represent parabolas $(c_1 \pm c_3)^2 = 1 + c_1$, and the entangled regions are outside of these curves. Comparing two unitary channels, we see that U_x can detect entangled states only within the regions $c_1 > 0$ and $c_1 < 0$ for cases (a) and (b), respectively. The unitary channel U_z, on the other hand, can detect entangled states only for case (b) within the region with the reflection symmetry $c_1 \leftrightarrow -c_1$.

Upon considering the one-parameter case, the SLD quantum Fisher information about ρ_λ^+ is calculated as

$$g_\theta\big[U_x^2\,\rho\,\big(U_x^2\big)^\dagger\big] = \frac{8\lambda^2}{1+\lambda}. \tag{67}$$

Since the maximum SLD quantum Fisher information for the single system is 1, as discussed in Sec. III B, the Pezzé-Smerzi criterion states that state ρ_λ^+ is entangled if the following inequality holds:

$$\frac{8\lambda^2}{1+\lambda} > 2 \;\Leftrightarrow\; 4\lambda^2 - \lambda - 1 > 0. \tag{68}$$

Solving this inequality leads to the sufficient condition for the entangled region:

$$R_{\rm ent} = \left(\frac{\sqrt{17}+1}{8},1\right). \tag{69}$$

The numerical value $(\sqrt{17}+1)/8 \simeq 0.64$ is larger than the true boundary $1/3$, as it should be.

This example shows that entanglement in state ρ_λ^- cannot be detected by a rotation around any axis. For state ρ_λ^+, on the other hand, a rotation around the x axis can detect entanglement.

B. Dephasing channel

We consider the following dephasing channel:

$$\Gamma_\theta^{Dph}(\rho) := \frac{1+\theta}{2}\rho + \frac{1-\theta}{2}\sigma_z\rho\sigma_z, \tag{70}$$

where the channel parameter θ represents the strength of dephasing: $\theta \in \Theta = (0,1)$, e.g., no error $\Leftrightarrow \theta = 1$ and complete dephasing $\Leftrightarrow \theta = 0$. The maximum value of the SLD quantum Fisher information for a single-input state is given by

$$g_\theta^*\big[\Gamma_\theta^{Dph}\big] = \frac{1}{1-\theta^2}. \tag{71}$$

The SLD quantum Fisher information is calculated for the Bell-diagonal state (56) as

$$\begin{aligned} &g_\theta\big[\Gamma_\theta^{Dph}(\rho_{BD})\big] \\ &= \theta^2\Bigg[\frac{(c_1+c_2)^2}{1-c_3-\theta^2(c_1+c_2)} + \frac{(c_1+c_2)^2}{1-c_3+\theta^2(c_1+c_2)} \\ &\quad + \frac{(c_1-c_2)^2}{1-c_3-\theta^2(c_1-c_2)} + \frac{(c_1-c_2)^2}{1-c_3+\theta^2(c_1-c_2)}\Bigg]. \end{aligned} \tag{72}$$

Thus, the proposed entangled criterion is $g_\theta[\Gamma_\theta^{Dph}(\rho_{BD})] > 2g_\theta^*[\Gamma_\theta^{Dph}] = 2/(1-\theta^2)$.

To visualize the above result, we consider case (a) ($c_1 = c_2$) first. This reduces the expression $g_\theta[\Gamma_\theta^{Dph}(\rho_{BD})] > 2/(1-\theta^2)$ as

$$\theta^2\left[\frac{4c_1^2}{1-c_3-2\theta^2 c_1}+\frac{4c_1^2}{1-c_3+2\theta^2 c_1}\right] > \frac{2}{1-\theta^2} \Leftrightarrow 4\theta^4 c_1^2 + 4\theta^2(1-\theta^2)c_1^2(1-c_3) - (1-c_3)^2 > 0$$
$$\Rightarrow\ c_3 > 1 - 2\theta^2\left[(1-\theta^2)c_1^2 + |c_1|\sqrt{1-(1-\theta^2)^2 c_1^2}\right],$$

within the parameter regions for (c_1,c_3) allowed by positivity of the Bell-diagonal state. Therefore, varying the parameter θ yields criterion (19) as follows:

$$c_3 > \min_{\theta\in[0,1]}\left\{1-2\theta^2\left[(1-\theta^2)c_1^2+|c_1|\sqrt{1-(1-\theta^2)^2c_1^2}\right]\right\} = 1 - 2\max_{\theta\in[0,1]}\left\{\theta^2\left[(1-\theta^2)c_1^2+|c_1|\sqrt{1-(1-\theta^2)^2c_1^2}\right]\right\}$$
$$= 1-2|c_1|, \tag{73}$$

where the maximization is a straightforward exercise. We note that condition (73) does not detect entangled states since it is equivalent to the boundary of positivity of the state.

We next consider case (b) ($c_1 = -c_2$). The result is exactly the same as that for case (a) due to the symmetry $c_2 \leftrightarrow -c_2$ in the setting. Hence, no entangled state is detected. To understand this better, we finally consider states $\rho_\lambda^\pm$. In this case, the above criterion before carrying out the minimization procedure is

$$4\theta^2(1-\theta^2)\lambda^3 - (1-4\theta^2)\lambda^2 - 2\lambda - 1 > 0 \tag{74}$$

for both $\rho_\lambda^\pm$. The left-hand side of this inequality is always negative as a function of $\lambda \in (-1/3,1)$ and $\theta \in (0,1)$.

C. Depolarizing channel

The depolarizing channel for a two-dimensional quantum system is defined by

$$\Gamma_\theta(\rho) := \theta\rho + \frac{1-\theta}{2}\mathrm{tr}(\rho)I. \tag{75}$$

Here the channel parameter θ represents the probability of errors taking values in $\Theta = (0,1)$, e.g., no error $\Leftrightarrow \theta = 1$ [41].

Optimal parameter estimation strategies for this channel were studied based on various figures of merit. See, for example, Refs. [18,42,43]. It was shown that this channel is programmable and hence θ cannot be estimated with quantum metrological enhancement [20]. The maximum value of the SLD quantum Fisher information for a single-input state is given by an arbitrary pure state as

$$g_\theta^*[\Gamma_\theta^{DP}] = \frac{1}{1-\theta^2}. \tag{76}$$

The SLD quantum Fisher information for the Bell-diagonal state (56) is easily computed as

$$g_\theta\left[\Gamma_\theta^{DP}(\rho_{BD})\right]$$
$$= \theta^2\left[\frac{(c_1+c_2+c_3)^2}{1-\theta^2(c_1+c_2+c_3)} + \frac{(-c_1+c_2+c_3)^2}{1+\theta^2(-c_1+c_2+c_3)} + \frac{(c_1-c_2+c_3)^2}{1+\theta^2(c_1-c_2+c_3)} + \frac{(c_1+c_2-c_3)^2}{1+\theta^2(c_1+c_2-c_3)}\right]. \tag{77}$$

A difference from the dephasing channel is that the channel parameter θ also appears as a multiplication factor for c_3. From this we expect that the SLD quantum Fisher information changes depending on c_3 by varying θ. The proposed entanglement criterion is expressed as $g_\theta[\Gamma_\theta^{DP}(\rho_{BD})] > 2g_\theta^*[\Gamma_\theta^{DP}]$, which can be easily checked numerically. To get analytical insight into this entanglement criterion, we consider two cases [cases (a) and (b)] as before. First, consider case (a) ($c_1 = c_2$). From expression (77), the entanglement criterion gets simplified as

$$c_1^2 > \frac{1-\theta^2 c_3}{4\theta^2}\,\frac{1-\theta^2(2-\theta^2)c_3^2}{1+\theta^2(2-\theta^2)c_3} =: f(\theta,c_3). \tag{78}$$

We next consider an optimization for the right-hand side of Eq. (78) by varying the channel parameter θ. With thorough analysis we can show that

$$c_1^2 > \min_{\theta\in(0,1)} f(\theta,c_3)$$
$$= \begin{cases} f^*(c_3) & \text{if } c_3 < -\frac{1+\sqrt{17}}{6},\\ \frac{1}{2}(1-c_3) & \text{otherwise,}\end{cases}$$

where $f^*(c_3)$ is a very complicated function of c_3 whose explicit expression is omitted in this paper. It is easy to see that the condition $c_1^2 > (1-c_3)/2$ cannot detect entangled states at all. Therefore, entangled states can be detected only for the region $-1 < c_3 < -(1+\sqrt{17})/6$. Upon analyzing the above optimization, we observe that the channel parameter θ needs to be in the set $\theta \in (1/\sqrt{3},1)$ in order to detect entangled states. Otherwise, inequality (78) only provides the region that is outside the state space. Physically speaking, the channel cannot be too noisy to detect entangled states. Figure 3(a) shows the optimal curves $c_1 = \pm\sqrt{f^*(c_3)}$ detecting entanglement located in two corners of the triangle-shaped state space. (Entangled regions are outside of these curves.) In Fig. 3(b), we also plot several curves $c_1 = \sqrt{f(\theta,c_3)}$ for four different values of θ, $\theta = 0.3, 0.5, 0.7, 0.9$ (from the dotted curve to the dashed ones). Case (b) is omitted due to the symmetry $c_3 \leftrightarrow -c_3$.

We finally consider the case of one-parameter subfamilies $\rho_\lambda^\pm$. The SLD quantum Fisher information is the same for the two input states $\rho_\lambda^\pm$ and is calculated as

$$g_\theta\left[\Gamma_\theta^2(\rho_\lambda^\pm)\right] = \theta^2\left[3\frac{\lambda^2}{1-\theta^2\lambda} + \frac{(3\lambda)^2}{1+3\theta^2\lambda}\right]$$
$$= \frac{12\theta^2\lambda^2}{(1-\theta^2\lambda)(1+3\theta^2\lambda)}.$$

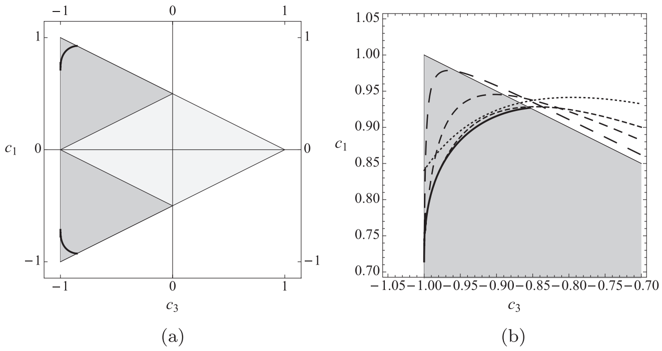

FIG. 3. Entangled regions detected by the depolarizing channels for Bell-diagonal states (56). (a) The threshold curves detecting entangled states, $c_1 = \pm\sqrt{f^*(c_3)}$, corresponding to criterion (19). (b) Entanglement detection curves [Eq. (78)] for four different values of $\theta = 0.3, 0.5, 0.7, 0.9$ (from the dotted curve to the dashed ones).

Thus, a sufficient condition for entanglement obtained from Theorem 1 is $g_\theta[\Gamma_\theta^2(\rho_\lambda^\pm)] > 2g_\theta^*[\Gamma_\theta]$, or, equivalently,

$$3\theta^2(2-\theta^2)\lambda^2 - 2\theta^2\lambda - 1 > 0. \tag{79}$$

This inequality then gives the entangled region for λ as

$$r_{\text{ent}}(\theta) = \left(\frac{\theta + \sqrt{2(3-\theta^2)}}{3\theta(2-\theta^2)}, 1\right), \tag{80}$$

which depends explicitly on the value of the channel parameter θ. As before, an important remark is that the parameter θ needs to satisfy $\theta \in [\theta_c, 1]$ in order for the depolarizing channel to detect entanglement successfully. Otherwise, the criterion cannot tell if the states are entangled or not. Here the threshold is found analytically as $\theta_c = 1/\sqrt{3}$.

Since this sufficient condition holds for any $\theta \in (\theta_c, 1)$, the most useful one is given by the union as

$$R_{\text{ent}} = \bigcup_{\theta\in\Theta} r_{\text{ent}}(\theta) = (\lambda_{DP}, 1), \tag{81}$$

where λ_{DP} is the minimum of the function appearing in expression (80) and is readily calculated as

$$\lambda_{DP} := \min_{\theta\in\Theta} \frac{\theta + \sqrt{2(3-\theta^2)}}{3\theta(2-\theta^2)} = \frac{1+\sqrt{3}}{3}, \tag{82}$$

which is attained with $\theta_* = [(3-\sqrt{3})/2]^{1/2}$. The numerical value $\lambda_{DP} \simeq 0.910$ is larger than the one from a rotation around the x axis.

We note that the authors of Ref. [44] analyzed a parameter estimation problem of the depolarizing channel based on a specific measurement and an estimator. They observed that entangled states are superior to separable states for a certain subfamily of the Werner state. The numerical value found in Ref. [44] is close to the value reported in this paper, yet they differ by the nature of problem.

D. Transpose channel

In this last example, we shall analyze a rather unusual channel defined in terms of a transpose operation. It is known that transposition operations are not completely positive, only 1-positive. Here a key point is that the derivation of entanglement criteria does not rely on complete positivity but on monotonicity of SLD quantum Fisher information, which is true for arbitrary 1-positive maps. Thus, trace preserving and a 1-positive map are also capable of detecting entanglement.

We consider the following channel from $\mathcal{S}(\mathbb{C}^2)$ to itself:

$$\Gamma_\theta^{TP}(\rho) := \frac{1+\theta}{2}\rho + \frac{1-\theta}{2}\rho^T, \tag{83}$$

with T being the transpose operation. Here the parameter θ takes values in $\Theta = (-1, 1)$. We call the above channel a transpose channel in this paper. The maximum value of the SLD quantum Fisher information when one uses a single-qubit input state is

$$g_\theta^*[\Gamma_\theta^{TP}] = \frac{1}{1-\theta^2}, \tag{84}$$

which is attained with the eigenstates of σ_y. The SLD quantum Fisher information for the Bell-diagonal input state is

$$\begin{aligned} g_\theta\big[\Gamma_\theta^{TP}(\rho_{BD})\big] &= \theta^2 c_2^2 \bigg(\frac{1}{1-c_1-\theta^2 c_2 - c_3} + \frac{1}{1-c_1+\theta^2 c_2 + c_3} \\ &\quad \times \frac{1}{1+c_1-\theta^2 c_2 + c_3} + \frac{1}{1+c_1+\theta^2 c_2 - c_3}\bigg). \end{aligned} \tag{85}$$

In contrast to the previous examples, θ dependence only appears in front of the state parameter c_2.

To understand the criterion $g_\theta[\Gamma_\theta^{TP}(\rho_{BD})] > 2g_\theta^*[\Gamma_\theta^{TP}]$, let us consider case (a) ($c_1 = c_2$). A straightforward calculation gives

$$\begin{aligned} c_1^2 &> \frac{\beta_\theta(c_3) - [\beta_\theta(c_3)^2 - \alpha_\theta(c_3)\gamma(c_3)]^{1/2}}{\alpha_\theta(c_3)} =: F(\theta, c_3), \\ \alpha_\theta(c_3) &= (1-\theta^2)[(1+\theta^2)^3 - 4\theta^2(1-c_3)], \\ \beta_\theta(c_3) &= (1+\theta^2)(1+c_3^2) + 4\theta^2 c_3, \\ \gamma(c_3) &= (1-c_3^2)^2. \end{aligned} \tag{86}$$

One can show that $\alpha_\theta(c_3), \beta_\theta(c_3) > 0$ within our parameter regions $[\theta \in (0,1), |c_1| < 1, |c_3| < 1]$ and the right-hand side of inequality (86) is in the range $(0,1)$. We next optimize inequality (86) by changing the channel parameter θ. In a manner similar to what we did before, we obtain the following result:

$$c_1^2 > \begin{cases} 1 - |c_3| & \text{if } -1 < c_3 \leqslant -\frac{1}{3}, \\ \frac{1}{2}(1-c_3) & \text{if } -\frac{1}{3} < c_3 < 1. \end{cases} \tag{87}$$

Thus, the entanglement criterion is useful only when the state parameter satisfies $-1 < c_3 \leqslant -1/3$. Figure 4(a) shows the entangled region detected by the transpose channel (entangled regions are outside of the lines). In Fig. 4(b), we plot several curves $c_1 = \sqrt{F(\theta, c_3)}$, which appear in inequality (86), for four different values of θ, $\theta = 0.1, 0.4, 0.7, 0.9$ (from the dotted curve to the dashed ones).

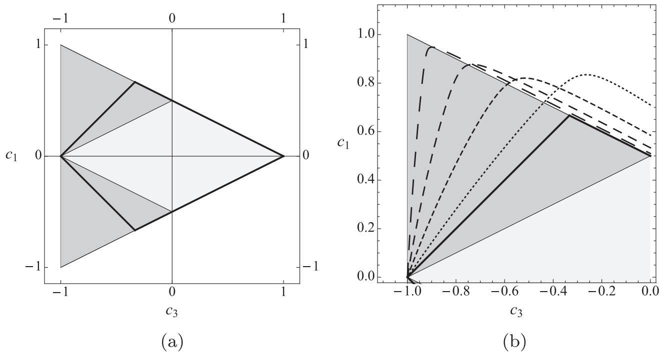

FIG. 4. Entangled regions detected by the transpose channels for Bell-diagonal states (56). (a) Entanglement detection lines and (b) the curves $c_1 = \sqrt{F(\theta,c_3)}$ [Eq. (86)] for fixed values of $\theta = 0.1, 0.4, 0.7, 0.9$ (from the dotted curve to the dashed ones).

Last, we consider the one-parameter family of input states ρ_λ^{pm}. The SLD quantum Fisher information is the same for the both states, and it is given by

$$g_\theta\big[\Gamma_\theta^{TP}(\rho_\lambda^{\pm})\big] = \theta^2\lambda^2\left[2\frac{1}{1-\theta^2\lambda} + \frac{1}{1+(2+\theta^2)\lambda} + \frac{1}{1-(2-\theta^2)\lambda}\right] = 2\theta^2\lambda^2\frac{1+2\theta^2\lambda-4\lambda^2}{(1-\theta^2\lambda)[(1+\theta^2\lambda)^2-4\lambda^2]}.$$

A sufficient condition for entangled states $g_\theta[\Gamma_\theta^{PT}(\rho_\lambda^+)] > 2g_\theta^*[\Gamma_\theta^{TP}]$ is expressed after some calculation as

$$H_\theta(\lambda) := 4\theta^2(1-\theta^2)\lambda^4 + \theta^2(\theta^4-2\theta^2+4)\lambda^3 + (\theta^4-2\theta^2-4)\lambda^2 + \theta^2\lambda + 1 < 0. \quad (88)$$

A detailed analysis of the quartic equation $H_\theta(\lambda) = 0$ shows that there are four real roots for all values of $\theta \in \Theta$.

The relevant entangled region is then found as

$$r_{\text{ent}}(\theta) = \left(\lambda_2(\theta), \frac{1}{2-\theta^2}\right). \quad (89)$$

Here $\lambda_2(\theta)$ is the second largest solution to the quartic equation $H_\theta(\lambda) = 0$, whose explicit form is omitted because of its lengthy expression. We see that $\lambda_2(\theta)$ varies from 1/2 to 1 depending on the value of θ. As in the depolarizing channel, we take the union of $r_{\text{ent}}(\theta)$ to get the most useful criterion:

$$R_{\text{ent}} = \bigcup_{\theta\in\Theta} r_{\text{ent}}(\theta) = (\lambda_{TP}, 1), \quad (90)$$

where $\lambda_{TP} = \min_{\theta\in\Theta}\lambda_2(\theta) = 1/2$ is calculated analytically.

E. Comparison and discussion

In this section, we compare the five different channels studied in the previous section and discuss our result. They are (1) rotation around the z axis (U_z) Γ_θ^z, (2) rotation around the x axis (U_x) Γ_θ^x, (3) the dephasing channel (Dph) Γ_θ^{Dph}, (4) the depolarizing channel (DP) Γ_θ^{DP}, and (5) the transpose channel (TP) Γ_θ^{TP}. Mathematically, their actions are

$$\Gamma_\theta^z(\rho) = e^{i\theta\sigma_z/2}\rho\, e^{-i\theta\sigma_z/2},$$
$$\Gamma_\theta^x(\rho) = e^{i\theta\sigma_x/2}\rho\, e^{-i\theta\sigma_x/2},$$
$$\Gamma_\theta^{Dph}(\rho) = \frac{1+\theta}{2}\rho + \frac{1-\theta}{2}\sigma_z\rho\sigma_z,$$
$$\Gamma_\theta^{DP}(\rho) = \theta\rho + \frac{1-\theta}{2}\text{tr}(\rho)I,$$
$$\Gamma_\theta^{TP}(\rho) = \frac{1+\theta}{2}\rho + \frac{1-\theta}{2}\rho^T.$$

From the entanglement detection perspective, first of all, unitary channels detect entangled states without any dependence on rotational angles θ. This is because the SLD quantum Fisher information is independent of parameter values in general (see Lemma 1). In contrast, nonunitary channels do depend on the parameter values θ. In particular, it was demonstrated that some parameter regions are only useful for entanglement detection. From examples studied in this paper, noise level should not be too high; otherwise, output states do not carry useful information about input states. Comparing the five channels indicates that TP works better than other channels and Dph does not perform any function. However, one cannot say which channel is superior among U_z, U_x, and DP for the Bell-diagonal states in general. It always depends on the nature of the states under study. An interesting question might be the reverse-engineering question: For a given entangled state ρ, is there a quantum channel Γ_θ such that the SLD quantum Fisher information $g_\theta[\Gamma_\theta(\rho)] > Ng_\theta^*$ if and only if state ρ is entangled? The example of the transposed channel and preliminary analysis suggest that another unphysical channel (not completely positive) could be a candidate. This line of research may open up another approach on how to quantify the entanglement from state estimation perspectives. From the channel-parameter estimation perspective, as demonstrated by these examples, there are several differences between unitary channels and nonunitary ones. We first note these nonunitary channels, except for the transpose channel, are typical noise models. Estimating the value of noise is also an important issue for any practical quantum information processing task. It is rather unexpected that all Bell-diagonal states are useless upon estimating the dephasing channel. This means separable states have the best performance when compared to the Bell-diagonal states. It also happens that most of the Bell-diagonal states do not provide an advantage for the depolarizing channel since the entangled region in Fig. 3(a) is rather small. However, as an important reminder, a specific entangled state is more useful for a particular channel but not others. For example, the singlet state, which is a maximally entangled state, is located in the bottom left corner of Fig. 1(a), and its neighborhood states are completely useless for unitary channels but are good for the depolarizing channel.

Next, let us compare the one-parameter family ρ_λ^+. The results for ρ_λ^+ are summarized in Table I. In Table I, the symbol Ø indicates that a channel cannot be used to detect

TABLE I. Summary of entanglement detection for state $\rho^+(\lambda)$ for five different channels: two unitary channels (U_z, U_x), the dephasing channel (Dph), the depolarizing channel (DPC), and the transpose channel (TPC). Entanglement region $r_{\text{ent}}(\theta)$ is defined in Eq. (17), and its union is denoted by R_{ent}, defined by Eq. (18). The symbol $\emptyset$ represents the empty set. Numerical values are $(1+\sqrt{3})/3 \simeq 0.91 > (\sqrt{17}+1)/8 \simeq 0.64 > 1/2$.

	θ Dependence	$r_{\text{ent}}(\theta)$	R_{ent}
U_z	No	$\emptyset$	$\emptyset$
U_x	No	$(\frac{\sqrt{17}+1}{8},1)$	$(\frac{\sqrt{17}+1}{8},1)$
Dph	Yes	$\emptyset$	$\emptyset$
DPC	Yes	$(\frac{\theta+\sqrt{2(3-\theta^2)}}{3\theta(2-\theta^2)},1)$	$(\frac{1+\sqrt{3}}{3},1)$
TPC	Yes	$(\lambda_2(\theta),\frac{1}{2-(1-2\theta)^2})$	$(\frac{1}{2},1)$

entangled states. $\lambda_2(\theta)$ is the second largest solution to the quartic equation $H_\theta(\lambda)=0$ [Eq. (88)]. Numerically, $\lambda_2(\theta)$ varies from $1/2$ to 1 depending on the value of θ.

As noted before, the rotation around any axis is not useful for the Werner state ρ_λ^- since SLD quantum Fisher information about the output states is always zero. As we can see from Table I, for ρ_λ^+, the rotation around the x axis can be used to detect entanglement which performs better than the depolarizing channel. Interestingly, the (unphysical) transpose channel can detect entangled states better than other examples analyzed in this paper.

The main difference between unitary channels and nonunitary channels is that SLD quantum Fisher information is θ independent for the unitary case. This might be an advantage in a realistic situation if one wishes to detect entangled states with an unknown unitary channel. From our point of view, however, this is not a problem since we are willing to detect entangled states by engineering appropriate quantum channels.

Experimentally, we prepare a family of quantum channels Γ_θ with a controllable parameter θ. We next apply this family of channels to an unknown multipartite state and perform a good measurement on the output state. The measurement results then give probability distributions depending on the value of the parameter θ. We can then calculate classical Fisher information, which coincides with the SLD quantum Fisher information if the measurement is chosen to be the optimal one. By comparing the value of Fisher information for multipartite states with the optimal Fisher information for a single-input state, which is exploited in advance, one can tell if the states are entangled or not based on the criterion given in Theorem 1.

Last, we show that for a certain parameter range (low-noise regime), the depolarizing cannel can be estimated more efficiently if we use entangled input states. Although we cannot get the full benefit from entanglement to attain quantum metrological enhancement, entanglement indeed brings the estimation error lower than the separable input states. Whether this effect is significantly important depends on how accurately one wishes to estimate the value of a parameter of a given channel. More analysis of other quantum channels as well as various entangled input states is needed to make any general statement.

V. EXTENSION TO NON-i.i.d. QUANTUM CHANNELS: AN APPLICATION TO OPEN QUANTUM SYSTEMS

So far we have mainly been concerned with only the i.i.d. extension of quantum channels. In this section, we shall extend the proposed criterion to the non-i.i.d. case first and then discuss briefly how to apply it to open quantum systems. The open quantum system is, in general, described by the dynamics of a subsystem of a large quantum system, typically a small system coupled with another large quantum system (bath). Mathematically, it can be described by the Lindblad-type master equation. Many such dynamics in open quantum systems cannot be written as the i.i.d.-extended form of a single channel, but rather take a more general expression. It is clear that if a given quantum noise is capable of generating entangled states, one cannot use this channel to detect entanglement contained in an input state. Therefore, we study a restricted class of quantum noises that does not create any entangled states from any separable state in the following.

Let Γ_θ be a quantum channel from quantum states on $\mathcal{H}^N := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ to itself. We say that a quantum channel is separable if all separable states remain separable under the action of this channel. We consider a further restricted class of separable channels such that all product states remain product states. We call these channels *completely separable*, meaning that they do not create any classical correlation. Mathematically, a completely separable quantum channel Γ^{sep} satisfies the following condition: For all possible states $\rho^{(j)} \in \mathcal{S}(\mathcal{H}_i)$, there exist some output states $\sigma^{(j)} \in \mathcal{S}(\mathcal{H}_i)$ such that

$$\Gamma^{\text{sep}}(\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(N)}) = \sigma^{(1)} \otimes \sigma^{(2)} \otimes \cdots \otimes \sigma^{(N)} \tag{91}$$

holds.

When considering completely separable channels, we have the following theorem.

Theorem 3. Consider a completely separable channel $\Gamma_\theta : \mathcal{S}(\mathcal{H}^N) \to \mathcal{S}(\mathcal{H}^N)$ parametrized by a single parameter θ. Let $g_\theta^*(\Gamma_\theta)$ be the largest value of SLD quantum Fisher information defined by

$$g_\theta^*(\Gamma_\theta) := \max_i \max_{\rho^{(i)} \in \mathcal{S}(\mathcal{H}_i)} \operatorname{Tr}_{j\neq i} \times \left\{\Gamma_\theta\left(\rho_{CM}^{(1)} \otimes \cdots \otimes \rho^{(i)} \otimes \cdots \otimes \rho_{CM}^{(N)}\right)\right\}, \tag{92}$$

with $\rho_{CM}^{(i)}$ being the completely mixed state on $\mathcal{H}_i$. For each value θ, if a density operator ρ on $\mathcal{S}(\mathcal{H}^N)$ is separable, then the SLD quantum Fisher information $g_\theta[\Gamma_\theta(\rho)]$ is smaller than or equal to the value $N g_\theta^*(\Gamma_\theta)$.

The proof of this theorem goes exactly the same as that for Theorem 1 as follows.

Proof. Consider arbitrary separable states on $\mathcal{S}(\mathcal{H}^N)$ of the form

$$\rho_{\text{sep}} = \sum_i p_j \rho_i^{(1)} \otimes \rho_i^{(2)} \otimes \cdots \otimes \rho_i^{(N)}. \tag{93}$$

Then, we have the following inequalities:

$$\begin{aligned} g_\theta[\Gamma_\theta(\rho_{\rm sep})] &= g_\theta\left[\sum_i p_i \Gamma_\theta\big(\rho_i^{(1)}\otimes\cdots\otimes\rho_i^{(N)}\big)\right] \\ &= g_\theta\left[\sum_i p_i \sigma_{\theta,i}^{(1)}(\theta)\otimes\cdots\otimes\rho_{\theta,i}^{(N)}(\theta)\right] \\ &\leqslant \sum_i p_i g_\theta\big[\sigma_{\theta,i}^{(1)}(\theta)\otimes\cdots\otimes\rho_{\theta,i}^{(N)}(\theta)\big] \\ &= \sum_i p_i \sum_k g_\theta\big[\sigma_{\theta,i}^{(k)}(\theta)\big] \\ &\leqslant \sum_i p_i \sum_k g_\theta^*[\Gamma_\theta] \\ &= N g_\theta^*[\Gamma_\theta]. \end{aligned}$$

The first inequality follows from the convexity of SLD quantum Fisher information. The second inequality follows from the definition of $g_\theta^*[\Gamma_\theta]$ and the identification $g_\theta^*(\Gamma_\theta) = \max_k \max_{\rho^{(i)}} g_\theta[\sigma_{\theta,i}^{(k)}(\theta)]$. ■

As a note, it is possible to generalize the above theorem to any separable channel with additional terms. Details of the general formalism will be presented elsewhere together with examples.

Example: N-qubit master equation

To gain insight into the result of Theorem 3, let us consider an open quantum system of N qubits described by the following Lindblad master equation:

$$\frac{\partial}{\partial t}\rho(t) = i[\rho(t)\,,H_\theta] - \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{3}\gamma_j\big[\big[\rho(t)\,,\sigma_j^{(i)}\big]\,,\sigma_j^{(i)}\big], \tag{94}$$

where $H_\theta = \frac{1}{2}\sum_j \theta\sigma_3^{(i)}$ is the free Hamiltonian describing a global rotation about an angle θ, γ_j are the damping parameters that may depend on the parameter of interest θ, and $\sigma_j^{(i)} = I\otimes\cdots\otimes\sigma_j\otimes\cdots\otimes I$ is the jth Pauli matrix for the ith qubit system. This kind of master equation has been investigated by several authors under the name of noisy quantum metrology (see, for example, Ref. [45]). It is straightforward to see that the solution to this master equation is regarded as a completely separable channel for a given initial state. Thus, we can apply Theorem 3 to detect entanglement even in the presence of quantum noises described by the above master equation. The quantity $g_\theta^*[\Gamma_\theta]$ for this channel is calculated by

$$g_\theta^*[\Gamma_\theta] = \max_{\rho_0\in\mathcal{S}(\mathbb{C}^2)} g_\theta[\rho_\theta(t)], \tag{95}$$

where $\rho_\theta(t)$ is the solution to the master equation for the single-qubit system:

$$\frac{\partial}{\partial t}\rho_\theta(t) = i\left[\rho_\theta(t)\,,\frac{1}{2}\theta\sigma_3\right] - \frac{1}{4}\sum_{j=1}^{3}\gamma_j[[\rho_\theta(t)\,,\sigma_j]\,,\sigma_j], \tag{96}$$

with the initial state $\rho(t=0)=\rho_0$.

The master equation (96) can be solved analytically, but the resulting SLD quantum Fisher information gets complicated in general, in particular when the damping coefficients depend on θ. Below we consider an isotropic noise $\gamma_1=\gamma_2=\gamma_3=\gamma$, and γ is independent of θ to simplify the result. In this case, the obtained maximum SLD quantum Fisher information over all possible initial states is

$$g_\theta^*[\Gamma_\theta] = t^2\,e^{-2\gamma t} \tag{97}$$

at some later time t. Thus, the proposed criterion to detect entangled states is as follows. For a given initial state ρ_0 on $\mathcal{S}(\mathcal{H}^{\otimes N})$ with $\mathcal{H}=\mathbb{C}^2$, ρ_0 is entangled if the inequality

$$g_\theta[\rho_\theta(t)] > Nt^2\,e^{-2\gamma t} \tag{98}$$

holds for later time t. Here $\rho_\theta(t)$ is the solution to the master equation (94) with the initial state ρ_0. Here two remarks on this result are in order.

First, the above criterion seems counterintuitive at first sight. Since the right hand becomes exponentially small for fixed N as the time t increases, this criterion states that almost all states with nonzero SLD quantum Fisher information at the later time are entangled. A simple explanation for this observation is that as time grows, the solution to the master equation (96) approaches the θ-independent state, typically to the completely mixed state, for any initial state. It is then clear that the amount of SLD quantum Fisher information decreases in time as well. Therefore, the inequality (98) still provides useful information to detect entangled states.

The second remark is that the above criterion can be weakened by replacing $\exp(-2\gamma t)$ by 1 [note $\exp(-2\gamma t)<1$ holds for all $t>0$]. This provides a valid criterion by applying the approximation criterion discussed in Sec. III C 2. Hence, if the simplified inequality

$$g_\theta[\rho_\theta(t)] > Nt^2 \tag{99}$$

holds, then state ρ_0 is entangled. This reduction is, of course, not surprising because of the monotonicity of quantum Fisher information. That is, the effect of the above master equation is equivalent to applying a channel Γ_γ. It is then clear that all separable states satisfy the following inequality:

$$g_\theta\big(\Gamma_\gamma\big\{U_z^2(\theta)\rho_{\rm sep}\big[U_z^2(\theta)\big]^\dagger\big\}\big) \leqslant g_\theta\big\{U_z^2(\theta)\rho_{\rm sep}\big[U_z^2(\theta)\big]^\dagger\big\}. \tag{100}$$

Since the right-hand side is bounded by 2, we can verify the criterion (99) after rescaling the phase parameter $\theta\to\theta t$. Criterion (99) is certainly simple; in particular, it is independent of the external noise parameter γ. However, it is obvious that this weaker version becomes useless for the large-t regime. More importantly, the monotonicity argument cannot be applied if quantum noise depends on the parameter θ itself, whereas our Theorem 3 is still valid in such a case.

As an example, we now study the time evolution of the N-qubit Greenberger-Horne-Zeilinger state,

$$|\psi_N\rangle := \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N}+|1\rangle^{\otimes N}) \tag{101}$$

under the master equation (94). In the absence of quantum noise, it is well known that this state is an optimal input

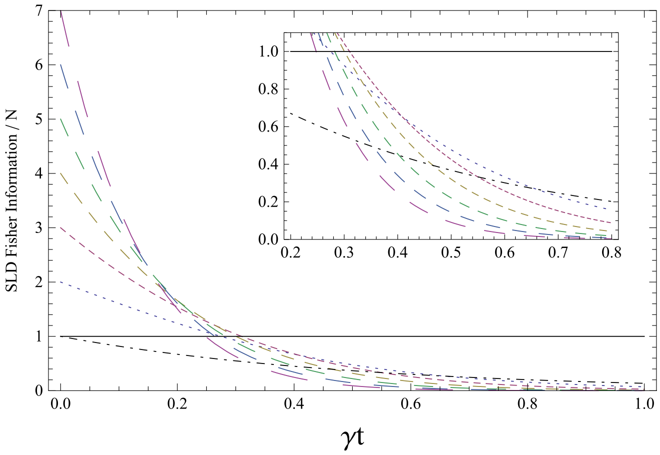

FIG. 5. The SLD quantum Fisher information per number of qubits (in units of t^2) as a function of time (in units of γ^{-1}) for $N = 2,3,4,5,6,7$ qubits (from the dotted line to the dashed lines). The black solid line shows the entanglement criterion due to inequality (99), and the dash-dotted curve shows the one from inequality (98). The inset magnifies these curves near crossing points.

state upon estimating the rotation angle θ, and its SLD quantum Fisher information is $g_\theta[\psi_N] = (Nt)^2$. Solutions for the isotropic noise case can be found analytically, and in the following we only show the SLD quantum Fisher information for up to seven qubits. It is independent of the parameter θ and is given by

$$g_\theta^N = \frac{(Nt)^2}{p_N(e^{-4\gamma t})} e^{-2\gamma N t}, \tag{102}$$

where the functions $p_N(s)$ are

$$p_2(s) = \tfrac{1}{2}(1+s),$$
$$p_3(s) = \tfrac{1}{4}(1+3s),$$
$$p_4(s) = \tfrac{1}{8}(1+6s+s^2),$$
$$p_5(s) = \tfrac{1}{16}(1+10s+5s^2),$$
$$p_6(s) = \tfrac{1}{32}(1+15s+15s^2+s^3),$$
$$p_7(s) = \tfrac{1}{64}(1+21s+35s^2+7s^3).$$

The physical meaning of $p_N(e^{-4\gamma t})$ is a residual component of the subspace spanned by $|0\rangle^{\otimes N}, |1\rangle^{\otimes N}$ at the later time. Thus, $p_N(1) = 1$ $(t = 0)$ and $\lim_{t\to\infty} p_N(e^{-4\gamma t}) = 2^{-(N-1)}$ hold. Figure 5 provides numerical plots of the above SLD quantum Fisher information together with two entanglement detection criteria. The first one is our contribution (98), and the second is the approximated one (99). We plot the SLD quantum Fisher information per number of qubits (in units of t^2) as a function of time (in units of γ^{-1}) for $N = 2,3,4,5,6,7$ qubits (from the dotted line to the dashed lines). The dash-dotted line corresponds to the criterion (98), and the solid line shows (99). The inset in Fig. 5 magnifies these curves near the crossing points.

It is clear that useful entanglement vanishes after a typical decoherence time $\sim O(\gamma^{-1})$. For the two-qubit case, the threshold time detected by the criterion (98) is $\gamma t = -\ln(2-\sqrt{3})/2 \simeq 0.658$. However, a gap exists between the two criteria, and the approximated one failed to detect entanglement even though there is still useful entanglement. Another observation is a tendency for this gap to become smaller as the number of qubits increases. This is because the more qubits the state is composed of, the faster states decohere in the noise described by Eq. (94) in general.

In the experiment reported in Ref. [7], the authors applied the weaker version of the entanglement criterion even though non-negligible decoherence effects are present. The above simple example implies that a stronger criterion can be applied to their experimental data by analyzing the effects of quantum noises to detect entangled states faithfully.

VI. CONCLUSION

We have derived a general criterion to detect entanglement based on the SLD quantum Fisher information for any one-parameter family of quantum channels. This criterion includes all previously known criteria based on the SLD quantum Fisher information or other variants. One of the main messages is that our formulation is free of specific channels such as unitary channels. The second important finding is the revelation of the general structure behind this sort of entanglement criterion based on quantum Fisher information, which was presented in Sec. III C 4.

We have applied our criterion to detect entanglement in the Bell-diagonal states based on the unitary channel, dephasing channel, depolarizing channel, and transpose channel and have analyzed them in detail. Our result shows that even the depolarizing channel can be used to detect entangled states for a certain parameter range. To put it differently, entanglement is still useful for lowering the estimation errors even though channels cannot be estimated with quantum metrological enhancement.

In the last section, we derived a more general criterion that can be applied to the setting of estimation of the channel parameter in a certain class of open quantum systems, that is, to detect entanglement in the input state even though states are affected by some unavoidable quantum noise. The only requirement is that a given quantum noise does not create any entanglement for any separable states. Then, the formalism can be straightforwardly extended to this class of open quantum systems, which is described by a quantum master equation. As an example, we briefly discussed how to apply it to the multiqubit case (from two to seven qubits) in the presence of a coupling to an environment, which is described by a simple Lindblad equation. It was demonstrated that our criterion provides a stronger one than the previous one. A more detailed discussion of entanglement detection in open quantum systems deserves further studies, and it will be analyzed in due course.

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