Asymptotic Convertibility of Entanglement: A General Approach to Entanglement Concentration and Dilution

（エンタングルメントの漸近的変換可能性：

エンタングルメント蒸留と希釈における一般的アプローチ）

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和文概要

二体系の量子状態として，エンタングルメントと呼ばれる古典系にはない量子力学的相関を有する状態が知られている．エンタングルメントを事前に共有することで，量子テレポーテーションなどの非局所的量子情報処理が現在の実験技術で実現可能となる．中でも，局所量子操作と古典通信による操作はLOCC (Local Operation and Classical Communication) 変換と呼ばれており，エンタングルメントはLOCC変換のリソースと見なされている．

実験室で準備できる状態がリソースとして望ましい状態ではない場合，LOCC変換により望ましい状態に変換する必要が生じる．よく知られている変換の例としてエンタングルメント蒸留と希釈の理論が挙げられる．エンタングルメント蒸留とは多数の弱くエンタングルした状態から，少数の最大エンタンル状態を取り出す操作であり，エンタングルメント希釈とはエンタングルメント蒸留の逆変換のことである．始状態がある状態の複数コピーからなる状態（i.i.d.状態）である場合，エンタングルメント蒸留と希釈の最適な変換レートは漸近的にエンタングルメントエントロピーに等しいことが知られている．

始状態または終状態がi.i.d.状態ではない一般の二体系の状態である場合，情報スペクトル的な方法が有用である．情報スペクトル的な方法は，古典情報理論の分野においてHanらによって提案された一般理論であり，量子系においてはNagaoka-Hayashiによって拡張された．エンタングルメント蒸留と希釈に関する情報スペクトル的方法による研究として，HayashiとBowen-Dattaはエンタングルメント希釈と蒸留の最適レートがエントロピースペクトルレートに等しいことを示している．

本論文では始状態も終状態も最大エンタングル状態に限定せず，情報スペクロル的な方法を用いて，任意に与えられた始状態から任意の終状態への漸近的LOCC変換可能性について符号化定理を与えた．また，エンタングルメント蒸留と希釈の最適レートに関する既知の結果について見通しの良い証明を与えた．

Abstract

A maximally entangled state shared between two distant parties is useful to perform various nonlocal tasks such as quantum teleportation and superdense coding. It should be noted, however, that physically prepared states are not always maximally entangled states. When the prepared states are not maximally entangled states, we may need to transform them to maximally entangled states by local operations and classical communication (LOCC) protocols.

Well-known examples of transforming partially entangled states into maximally entangled states are entanglement concentration and dilution. Entanglement concentration is a task to obtain copies of a maximally entangled state from many copies of a partialy entangled state by LOCC and entanglement dilution is its inverse process. When initial states are independently and identically distributed states (i.i.d. states), the optimal rates of entanglement concentration and dilution are asymptotically equal to the entanglement entropy.

For cases where initial states and target states are not necessarily a tensor power of a bipartite entangled state, the information-spectrum method has been applied to analyze entanglement concentration and dilution. Originally, the information-spectrum method was developed in classical information theory by Han and S. Verdú (1993, 1994) to construct a unified general theorem. Later it has been extended to quantum information theory by Nagaoka and Hayashi (2003, 2007) in the context of quantum hypothesis testing and classical-quantum channel coding. Under the information-spectrum setting, the optimal rates of entanglement concentration and dilution are obtained in terms of inf-/sup-spectral entanglement entropy rates by Hayashi (2006) and Bowen-Datta (2008), respectively.

In this thesis, we consider a more general situation in which an arbitrary sequence of a bipartite pure entangled state is asymptotically converted into another by a sequence of LOCC protocols. We require that the trace distance between the final state and the target state vanishes when the sequences are sufficient large. We seek conditions for such an asymptotic conversion to be possible. Different from the previous approaches, we do not assume that neither the target state nor the initial state is a maximally entangled state. We derive necessary and sufficient conditions for the asymptotic LOCC convertibility of one sequence to another in terms of spectral entropy rates of entanglement of the sequences. Based on these results, we also provide simple proofs for previously known results on the optimal rates of entanglement concentration and dilution of general sequences of bipartite pure states.

Contents

Chapter 1

Introduction

1.1 Backgrounds

1.1.1 Entanglement convertibility

In quantum systems, two distant particles may behave dependently, such physical phenomenon is called entanglement. It was first pointed out by Einstein, Podolsky and Rosen [1] in 1935. Almost fifty years later, entanglement phenomenon was confirmed experimentally by A. Aspect, J. Dalibrad and G.Roger in 1982 [2] by showing the violation of the Bell inequality [3]. In 1990s, lots of quantum protocols using an entanglement were proposed, such as quantum teleportation and super dense coding [4, 5, 6]. After then, the researchers gradually recognized that an entanglement is an useful resource in quantum information theory and the trend of research has been shifted to utilize the entanglement.

It is well known that quantum teleportation and super dense coding protocols require the maximally entangled states as resources. It should be noted that physically prepared states are not always maximally entangled states. Under this situation, we may need to transform the prepared states to maximally entangled states by local operations and classical communication (LOCC). It is mainly because performing global quantum operations between two distant parties are not realistic by today's technology. Well-known examples of converting a state into another by LOCC are entanglement concentration and dilution. *Entanglement concentration* is a task to convert many copies of a non-maximum entangled state into copies of a maximally entangled state by LOCC and *entanglement* *dilution* is its inverse process.

The research on entanglement convertibility in the asymptotic setting began by Bennett *et al.* [7]. They showed that the optimal rates of entanglement concentration and dilution are asymptotically equal to the von-Neumann entropy of the reduced state of initial state when the initial state is independently and identically distributed state (i.i.d. state) [see section 4.1]. After their research, Lo and Popescu [8] showed that for a known bipartite pure entangled state, concerning a two-way LOCC protocol is equivalent to concerning a one-way LOCC protocol. By their result, Nielsen *et al.* [9] derived necessary and sufficient conditions for the possibility of converting a bipartite pure entangled state into another only by LOCC with majorization. Since then the possibility of converting a bipartite pure entangled state into another only by LOCC is called as LOCC convertibility.

After then, the research on entanglement convertibility under the general setting using the informtion-spectrum method started. Hayashi [10] and Bowen-Datta [11] obtained general formulas for entanglement concentration and entanglement dilution, respectively. The optimal rates of entanglement concentration and dilution are obtained in terms of *inf-/sup-spectral entanglement entropy rates* [see Section 4.4].

1.1.2 Information-spectrum methods

In 1948, Shannon [12] established the filed of information theory and demonstrated the source coding theorem and the classical channel coding theorem for stationary and memoryless channels. The source coding theorem states that Shannon entropy is the optimal compression rate of a given information source for many observations. On the other hand, the classical channel coding theorem states that for all communication rates under the Shannon capacity the error probability can be made asymptotically to zero.

In quantum information theory, the source coding theorem was found by Schumacher [13], which states that the von-Neuman entropy is the optimal compression rate of a given information source. On the other hand, the direct part of the classical-quantum channel coding theorem for a stationary and memoryless classical-quantum channel was shown by Holevo [14] and Schumacher-Westmoreland [15] independently in 1990s, while the converse part was shown by Holevo [16, 17] in 1970s. The classical-quantum channel coding theorem states that the Holevo capacity is the maximum achievable rate for transmission of classical information through quantum channels.

It should be noted that in the real world many channels are neither stationary nor memoryless even in the asymptotic setting. Han and S. Verdú [18, 19] developed the information-spectrum method in classical information theory to construct a unified general theorem in which channels may be arbitrary nonstationary and/or nonergodic in 1993. They obtained the general formula for classical channel coding theorem by making no structural assumptions over the source and channel. In [20], Han also gave the general formula for various problems in information theory such as source coding theorem, random number generation, hypothesis testing, and rate distortion theory. Later, the information-spectrum method has been extended to quantum information theory by Nagaoka and Hayashi [21, 22, 23], initially in the context of quantum hypothesis testing (simple hypotheses testing for quantum states) and was used to determine the general expression for the capacity of arbitrary classical-quantum channels. After then, Hayashi [10] and Bowen-Datta [11] obtained general formulas for entanglement concentration and entanglement dilution, respectively, by informtion-spectrum approaches.

1.2 Motivation and approach

1.2.1 Motivation

A maximally entangled state shared between two parties is useful to perform various nonlocal tasks such as quantum teleportation protocol and super dense coding. On the other hand, a secret key shared by two parties is useful to perform private communication over a public channel. Operational equivalences of these two resources have been suggested by Schumacher [24] and Schumacher-Nielsen [25] in one shot scenario through noisy quantum channels, and by Devetak [26] in asymptotic scenario. Given a correlated quantum state as a resource, Devetak-Winter [27] addressed the questions of secret key distillation via one-way public commnication and entanglement distillation via one-way LOCC protocol from quantum states under the i.i.d. assumption.

In information theory, the *i.i.d.* case is just a starting point to solve the problem of various setting such as correlated cases. It is natural to consider general theory developed in the information-spectrum method. Because by using the information-spectrum method, we can obtain general and unified theorems without any assumptions. The motivation of this thesis is to apply the information-spectrum method to obtain general and unified formulas of secret key distillation and entanglement distillation from general sequence of quantum states.

1.2.2 Approach

Let $\mathbb{X} = \{X^n\}_{n=1}^{\infty}$ be an arbitrary sequence of random variables, called a *general source*, taking values in arbitrary countable sets $\mathcal{X}^n$ $(n = 1, 2, \dots)$, and $P_{X^n}(x^n)$ $(x^n \in \mathcal{X}^n)$ be the probability function of X^n for each n. A typical example of the general source is the i.i.d. case, i.e., X^n is written as $X^n = X_1, X_2, \dots, X_n$ $(n = 1, 2, \dots)$ and each X_i $(i = 1, 2, \dots, n)$ is a random variable subject to some identilal distribution independently.

With the information-spectrum method, the source coding theorem under the i.i.d. asumption is usually expressed as follows:

$$R(\mathbb{X}) = \overline{H}(\mathbb{X}) = H(X), \tag{1.1}$$

where $R(\mathbb{X})$ stands for the optimal compression rate of the general source, $\overline{H}(\mathbb{X})$ means *sup-spectral entropy*, and $H(X)$ is the Shannon entropy of the given i.i.d. source X_i $(i = 1, 2, \dots, n)$.

The first formula $R(\mathbb{X}) = \overline{H}(\mathbb{X})$ is entirely of information-theoretic coding aspects, providing the key framework or skeleton of mathematical (or logical) arguments in the world of information-spectrum and apparently has no connection with the assumption on probabilistic structure. We may say the formula $R(\mathbb{X}) = \overline{H}(\mathbb{X})$ is extremely general framework with simplicity and some beauty. However, it should be noted that it is not always easy or rather hard to find the proof of $R(\mathbb{X}) = \overline{H}(\mathbb{X})$. Once an excellent logic has been found to prove the equality, the proof can be transparent and simple with few assumptions, providing a framework or "skeleton" for information theory.

The second formula $\overline{H}(\mathbb{X}) = H(X)$ is entirely of probabilistic or statistical nature, providing a "concrete building" for information theory, and apparently has no connection with information-theoretic coding aspects. Thus, with information spectrum methods, we can divide the problem into two parts: coding problem and probabilistic problem.

For entanglement convertibility, we address a more general situation in which an arbitrary sequence of a bipartite pure state is asymptotically converted into another by a sequence of LOCC protocols. Different from Hayashi [10] and Bowen-Datta's approaches [11], we do not assume that neither the target state nor the initial state is a maximally entangled state. Compared to Hayashi and Bowen-Datta's research, our framework or skeleton is further simple as we removed the assumption on the initial state or on the target state. For the proof, the logical process of our method is much simpler than theirs. Since we assembled framework or skeleton parts such as random number generation [20], majorization [9] and a lemma obtained by Kumagai-Hayashi [28] for direct part. For converse part, we derived generalized properties of the spectral divergence rates such as monotonicity under positive trace preserving (PTP) map and continuity as skeleton parts.

1.3 Related works

We state several other related works beyond i.i.d. approach in this section. Smooth entropies were first introduced in the purely classical case [29] and later for a more general quantum regime [30, 31] by Renner. Datta-Renner [32] have shown that "spectral entropy rates are asymptotically equal to the limit of the smooth entropy rates."

Recently, on shot scenario of entanglement convertibility of a bipartite pure entangled state in infinite-dimensional systems has been studied. By introducing the concept of ε-convertibility and reconstructing Nielsen's theorems in infinite-dimension systems, Owari *et al.* [33] stated that an entangled state is ε-convertible to another, if and only if their Schmidt coefficients (see Lemma 2 for definition) have majorization relations. Using a different approach to [33], Asakura [34] established an infinite dimensional version of Birkoff's theorem with the weakly operator topology to prove LOCC convertibility in infinite dimension systems.

1.4 Contributions

In this thesis, the following results are obtained.

1. We obtained a general formula and unified form of asymptotic convertibility of arbi-

trary sequences of bipartite pure entangled states under the information-spectrum setting (Section 4.3).

2. By applying our results, we gave simple proofs of the previously known results on entanglement concentration [10] and dilution [11] of general sequences of a bipartite pure entangled state. (Section 4.4).

3. As a byproduct of our approach, we addressed asymptotic convertibility of two arbitrary sequences of states by random unitary operations, which is a subclass of unital operations (Section 5.3).

4. It is proved by Bowen-Datta that the spectral divergence rate of two general sequences of states are monotonically nonincreasing under complete positive and trace preserving (CPTP) maps for $\varepsilon = 0$. [35]. We generalized their result to an arbitrary $\varepsilon \in [0, 1]$ under positive trace preserving (PTP) maps (Subsection 6.2.1). We also showed the continuity of spectral divergence rates with respect to states in the asymptotic sense (Subsection 6.2.2).

1.5 Organization

This thesis is organized as follows. First, in Chapter 2 we review several basic preliminaries in quantum information theory that will be used in the later chapters. In Chapter 3, we state several known results on LOCC convertibility and review criterions of one shot scenario of LOCC convertibility obtained by Nielsen. The main results of this thesis are given in Chapter 4. By applying our results, simple proofs for previously known results of entanglement concentration and entanglement dilution will also be stated in Chapter 4. In Chapter 5, we review the information-spectrum method by introducing random number generation and we prove the direct part of main results which is the achievability of asymptotic convertibility. In Chapter 6, the definitions and properties of spectral divergence rates are provided. We also prove the converse part of main results, which is the optimality of LOCC convertibility in Chapter 6. Conclusion is given in Chapter 7.

Chapter 2

Mathematical preliminaries of quantum information theory

In this chapter, we brifely review several basic preliminaries of quantum information theory which will be used in the later chapters. Note that the proofs of the Lemmas and Propositions are omited here. In Section 2.1 we review linear operators on Hilbert spaces. In Section 2.2, we introduce a mathematical formalism to describe quantum states and measurement operations. A composite system of two Hilbert spaces is described in Section 2.3. In Section 2.4, we give a description of completely positive and trace preserving (CPTP) map. In Section 2.5, we state two distance measures of states and a relationship between them. Contents in this chapter are mainly based on [9, 36, 37].

2.1 Linear operators on Hilbert space

A quantum system is described by a Hilbert space $\mathcal{H}$. By Hilbert space we mean finite-dimensional Hilbert space in this thesis. We use $|\phi\rangle$ to stand for a *ket vector*. The symbol $\langle\psi|\phi\rangle$ denotes *inner product* of two vectors $|\phi\rangle$ and $|\psi\rangle$. For every ket vector $|\phi\rangle$ on $\mathcal{H}$, $\langle\phi|$ is defined by the linear functional on $\mathcal{H}$, namely $\langle\phi| : |\psi\rangle \mapsto \langle\phi|\psi\rangle$. If $\langle\psi|\phi\rangle = 0$, we say two vectors $|\psi\rangle$ and $|\phi\rangle$ are *orthogonal* to each other. $\|\psi\|$ denotes the *norm* of a vector $|\psi\rangle \in \mathcal{H}$ which is defined by $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$. If $\|\psi\| = 1$, vector $|\psi\rangle$ is called a *unit vector*.

If the elements of unit vectors $\{|e_i\rangle\}_{i=1}^m$ are orthogonal to each other, $\{|e_i\rangle\}_{i=1}^m$ is called an *orthonormal system*. An orthonormal system can be shown to be linearly independent.

For an n-dimensional vector space, an orthonormal system $\{|e_i\rangle\}_{i=1}^n$ which consists of n vectors forms a *basis* of the vector space. This orthonormal system is called a *complete orthonormal system (CONS).*

Definition 1. *Map A from a vector space V to another W is called a linear operator if it satisfies the following linearity condition.*

$$A(\alpha |\psi\rangle + \beta |\phi\rangle) = \alpha A |\psi\rangle + \beta A |\phi\rangle, \forall \alpha, \beta \in \mathbb{C}, \forall |\psi\rangle, |\phi\rangle \in V.$$

We use $\mathcal{L}(\mathcal{H})$ to stand for the set of linear operators from $\mathcal{H}$ to itself. An operator $A^\dagger \in \mathcal{L}(\mathcal{H})$ is called the *adjoint operator* of A if it satisfies $\langle v|Aw\rangle = \langle A^\dagger v|w\rangle$. If $A = A^\dagger$, A is called *Hermitian.* For any $|v\rangle \in \mathcal{H}$, if $\langle v|Av\rangle \geq 0$, A is called *nonnegative*, especially when $\langle v|Av\rangle > 0$, A is called *positive.* Hereafter we use $A > 0$ to mean that A is positive. If $A = A^\dagger = A^2$, A is called *projection.* A linear operator is called *normal* if $AA^\dagger = A^\dagger A$.

The *trace* of an operator A is given by

$$\mathrm{Tr}A := \sum_{i=1}^{n} \langle f_i|A|f_i\rangle,$$

where $\{|f_i\rangle\}_{i=1}^n$ is an arbitrary CONS of $\mathcal{H}$. Note that this quantity does not depend on the choice of the CONS.

Lemma 1. *For $A \in \mathcal{L}(\mathcal{H})$, any $|v\rangle \neq 0$ and $a \in \mathbb{C}$, if*

$$A|v\rangle = a|v\rangle, \tag{2.1}$$

then a is called an eigenvalue of A and $|v\rangle$ is an eigenvector of A corresponding to eigenvalue a.

2.2 Quantum states and measurement operators

Quantum states are described by density operators on Hilbert space $\mathcal{H}$. By a *density operator* we mean $\rho = \rho^\dagger \geq 0$ and $\mathrm{Tr}\rho = 1$. The set of density operator $\mathcal{S}(\mathcal{H})$ is defined by

$$\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{L}(\mathcal{H}) | \rho \geq 0, \mathrm{Tr}\rho = 1\}.$$

When the rank of quantum state ρ equals to 1, ρ is called a *pure state*, otherwise ρ is a *mixed state*.

Quantum measurement on $\mathcal{H}^A$ (A Hilbert space labeled by A) is described by a set of Hermitian operators $\mathbf{M} := \{M_k\}_{k=1}^d$, satisfying $\sum_k^d M_k^\dagger M_k = I$, where I denotes the identity operator. We call them *measurement operators*. The index k represents a measurement result. Let us perform a measurement on a quantum state ρ by a measurement operator M_k, then the probability that a result k occurs is given by

$$p(k) = \mathrm{Tr}\rho M_k^\dagger M_k \tag{2.2}$$

After the measurement the state ρ changes to the state

$$\frac{M_k \rho M_k^\dagger}{p_k}.$$

The description of measurement process is illustrated by Figure 2.1.

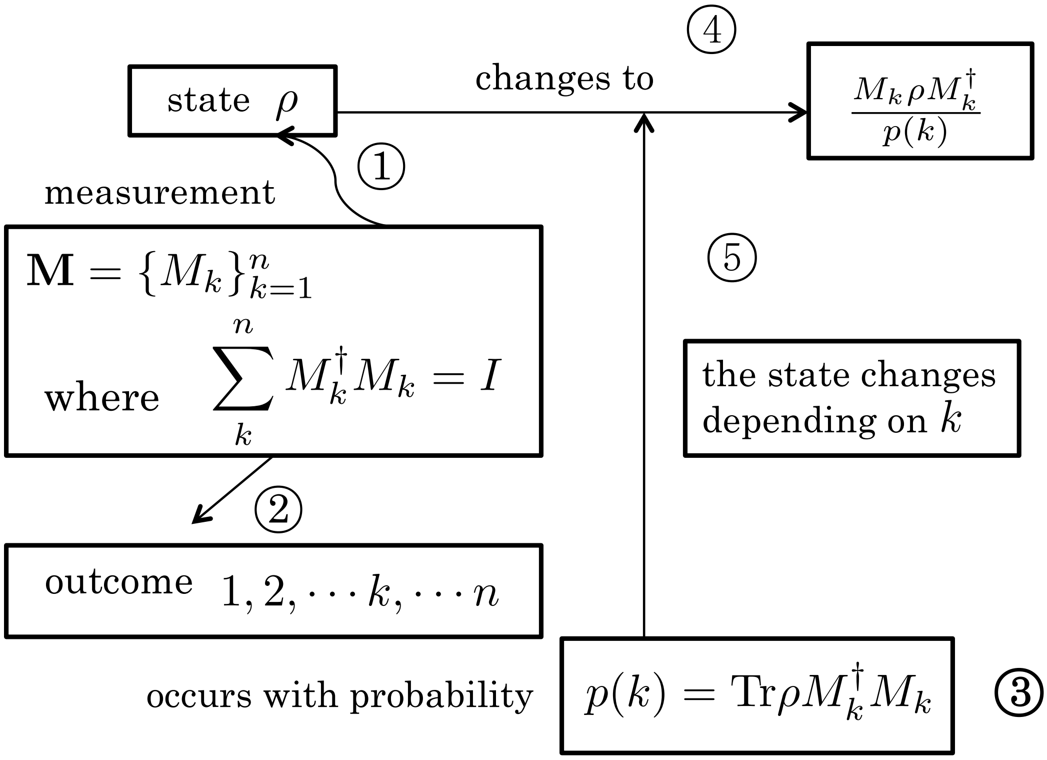


Fig. 2.1: measurement process

2.3 Composite system

2.3.1 Tensor product space

Let $\mathcal{H}^A$ be a d_A-dimensional Hilbert space and $\mathcal{H}^B$ be a d_B-dimensional Hilbert space. The *tensor product* operation " $\otimes$ " of Hilbert space $\mathcal{H}^A$ and $\mathcal{H}^B$ is a bilinear map from $\mathcal{H}^A \times \mathcal{H}^B$ to some $d_A d_B$-dimensional Hilbert space $\mathcal{H}^{AB}$, denoted as $(|\phi\rangle, |\psi\rangle) \mapsto |\phi\rangle \otimes |\psi\rangle$. If we let $\{|e\rangle_i\}_{i=1}^{d_A}$ be a CONS of Hilbert space $\mathcal{H}^A$ and $\{|f\rangle_j\}_{j=1}^{d_B}$ be a CONS of Hilbert space $\mathcal{H}^B$, then $\{e_i \otimes f_j | i = 1, \cdots, d_A, j = 1, \cdots, d_B\}$ is a CONS of $\mathcal{H}^{AB}$. The tensor product Hilbert space $\mathcal{H}^{AB}$ is known to be unique for arbitrary two Hilbert space $\mathcal{H}^A$ and

$\mathcal{H}^B$ up to isomorphism with inner product defined by

$$\langle \psi \otimes \phi | u \otimes v \rangle = \langle \psi | u \rangle \langle \phi | v \rangle \quad (\forall \, |\psi\rangle \; and \, |u\rangle \in \mathcal{H}^A, |\phi\rangle \; and \, |v\rangle \in \mathcal{H}^B). \tag{2.3}$$

Hence, $\mathcal{H}^{AB}$ is written as $\mathcal{H}^A \otimes \mathcal{H}^B$.

Note that any vector of $\mathcal{H}^A \otimes \mathcal{H}^B$ can be written as

$$|\psi\rangle^{AB} = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} x_{ij} \, |e_i\rangle^A \otimes |f_j\rangle^B . \tag{2.4}$$

Lemma 2 (Schmidt decomposition)**.** *For any unit vector* $|\psi\rangle^{AB}$ *in* $\mathcal{H}^A \otimes \mathcal{H}^B$*, there exist* $p_i > 0$ $(i = 1, 2, \cdots, m \leq \min[d_A, d_B])$*, a CONS* $\{|e\rangle_i\}_{i=1}^{d_A}$ *of* $\mathcal{H}^A$ *and a CONS* $\{|f\rangle_j\}_{j=1}^{d_B}$ *of* $\mathcal{H}^B$*, such that*

$$|\psi\rangle^{AB} = \sum_{i=1}^{m} \sqrt{p_i} \, |e_i\rangle^A \otimes |f_i\rangle^B . \tag{2.5}$$

In equation (2.5) m *is called the Schmidt rank and* p_i *is the Schmidt coefficient satisfying* $\sum_{i=1}^m p_i = 1$.

Using this Lemma and by choosing a suitable CONS of $\mathcal{H}^A$ and $\mathcal{H}^B$, any vector $|\psi\rangle^{AB}$ has a diagonal form $|\psi\rangle^{AB} = \sum_i x_{ii} \, |e_i\rangle^A \otimes |f_i\rangle^B$.

2.3.2 Tensor product between linear operators

For a linear operator A on $\mathcal{H}^A$ and a linear operator B on $\mathcal{H}^B$, the linear operator $A \otimes B$ of Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$ is defined by

$$(A \otimes B)(|u\rangle \otimes |w\rangle) := (A \, |u\rangle) \otimes (B \, |w\rangle) \quad (\forall \, |u\rangle \in \mathcal{H}^A, |w\rangle \in \mathcal{H}^B), \tag{2.6}$$

for vectors in tensor form and linearity using (2.4)

$$(A \otimes B) \, |\psi\rangle^{AB} = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} x_{ij} A \, |e_i\rangle^A \otimes B \, |f_j\rangle^B . \tag{2.7}$$

For a linear operator X of Hilbert space $\mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$, it can be uniquely written as

$$X = \sum_{i\ j} \sum_{k\ l} x_{ijkl} \left| e_i \right\rangle \left\langle e_j \right| \otimes \left| f_k \right\rangle \left\langle f_l \right|, \tag{2.8}$$

where $\{|e\rangle_i\}_{i=1}^{d_A}$ and $\{|f\rangle_j\}_{j=1}^{d_B}$ be the CONSs of Hilbert space $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively. The *partial trace* with respect to the system A is defined by

$$\mathrm{Tr}_B X = \sum_{i\ j} \sum_{k\ l} x_{ijkl} \left| e_i \right\rangle \left\langle e_j \right| \mathrm{Tr}(\left| f_k \right\rangle \left\langle f_l \right|) \tag{2.9}$$

$$= \sum_{ij} \sum_{k} x_{ijkk} \left| e_i \right\rangle \left\langle e_j \right|. \tag{2.10}$$

From equation (2.5), the reduced state of $|\psi\rangle^{AB}$ on the subsystem A and B are given by

$$\psi^A = \mathrm{Tr}_B \left|\psi\rangle\langle\psi\right|^{AB} = \sum_i p_i \left| e_i \right\rangle \left\langle e_i \right|, \tag{2.11}$$

$$\psi^B = \mathrm{Tr}_A \left|\psi\rangle\langle\psi\right|^{AB} = \sum_i p_i \left| f_i \right\rangle \left\langle f_i \right|. \tag{2.12}$$

If the reduced state of $|\psi\rangle^{AR}$ is equal to ρ, namely

$$\psi^A = \mathrm{Tr}_R \left|\psi\rangle\langle\psi\right|^{AR} = \rho,$$

then the bipartite pure state $|\psi\rangle^{AR}$ is called a *purification* of $\rho \in \mathcal{L}(\mathcal{H}^A)$.

Proposition 1. *If both $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ are purifications of ρ, i.e.*

$$\rho = \mathrm{Tr}_B \left|\psi\rangle\langle\psi\right| = \mathrm{Tr}_B \left|\phi\rangle\langle\phi\right|,$$

then there exists an unitary operator $U_B \in \mathcal{L}(\mathcal{H}_B)$ such that

$$|\phi\rangle^{AB} = (I^A \otimes U_B) \, |\psi\rangle^{AB}, \tag{2.13}$$

where I^A is an identity operator on $\mathcal{H}^A$.

2.4 TPCP maps

Hereafter, we use $\mathcal{H}^A$ to stand for a Hilbert space labeled by a system A. A quantum operation on $\mathcal{H}^A$ is defined by a super linear operator $\Lambda : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^{A'})$

$$^{\forall}\rho \longrightarrow \boxed{\Lambda} \longrightarrow \rho^{'}$$

Fig. 2.2

that satisfies the following two conditions:

(1) trace preserving

For any $\rho \in \mathcal{L}(\mathcal{H}^A)$, $\mathrm{Tr}(\Lambda(\rho)) = \mathrm{Tr}(\rho') = \mathrm{Tr}\rho$ holds, where $\rho' \in \mathcal{L}(\mathcal{H}^{A'})$. (see Fig. 2.2).

(2) complete positivity

For any system R and $X^{RA} \in \mathcal{L}(\mathcal{H}^R \otimes \mathcal{H}^A)$, $X^{RA} \geq 0 \Rightarrow (\mathcal{I}^R \otimes \Lambda)(X^{RA}) = X^{RA'} \geq 0$ holds, where I^R is an identity operator on $\mathcal{H}^R$ (see Fig. 2.3).

$$^{\forall}X^{RA} \longrightarrow \left\{ \begin{array}{c} \longrightarrow \boxed{\mathcal{I}^R} \longrightarrow \\ \longrightarrow \boxed{\Lambda} \longrightarrow \end{array} \right\} \longrightarrow X^{RA'}$$

Fig. 2.3

Such a quantum operation Λ is called a *completely positive and trace preserving (CPTP)* map. An example of quantum operation is a *random unitary operation* defined by

$$\Lambda(\rho) = \sum_i p_i U_i \rho U_i^{\dagger},$$

where U_is are unitaries and p_i is a probability distribution such that $p_i \geq 0, \sum_i p_i = 1$. By unitary we mean a linear operator $U : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ such that

$$U^{\dagger}U = I_{\mathcal{H}},\ UU^{\dagger} = I_{\mathcal{K}}.$$

2.5 Distance measures of two states

We introduce two distance measures, *trace distance* and *fidelity* to measure "closeness" of two quantum states. The trace distance of two quantum states ρ and σ is given as

$$d(\rho,\sigma) := \frac{1}{2}\mathrm{Tr}|\rho-\sigma|, \tag{2.14}$$

where $|A| := \sqrt{A^\dagger A}$.

Trace distance has the property of the monotonicity.

Lemma 3. *For any CPTP maps $\Lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ and any states ρ, $\sigma \in \mathcal{S}(\mathcal{H})$,*

$$d(\rho,\sigma) \geq d(\Lambda(\rho),\Lambda(\sigma)). \tag{2.15}$$

holds.

The fidelity of any two quantum states ρ and σ is defined as

$$F(\rho,\sigma) := \mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|. \tag{2.16}$$

When $\sigma = |\phi\rangle\langle\phi|$ is a pure state, the following is satisfied.

$$F(\rho,\sigma) = \sqrt{\langle\phi|\rho|\psi\rangle}. \tag{2.17}$$

In particular, when $\rho = |\psi\rangle\langle\psi|, \sigma = |\phi\rangle\langle\phi|$ both are pure states, the fidelity is given by

$$F(\rho,\sigma) = |\langle\psi|\phi\rangle|. \tag{2.18}$$

The fidelity has the following properties.

Lemma 4. *The properties of fidelity of any state ρ and σ are as follows.*

1. *(symmetry). $F(\rho,\sigma) = F(\sigma,\rho)$.*
2. *(positivity). $1 \geq F(\rho,\sigma) \geq 0$, $F(\rho,\sigma) = 1$ if and only if $\rho = \sigma$.*
3. *(monotonicity). For any CPTP $\Lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$, and two states ρ and σ, $F(\rho,\sigma) \leq F(\Lambda(\rho),\Lambda(\sigma))$ holds.*

Note that the trace distance and fidelity is related by the following Lemma.

Lemma 5. *For any ρ and σ, we have the following equalities.*

$$1 - F(\rho, \sigma) \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)^2}, \tag{2.19}$$

if ρ and σ are pure state, the second inequality can be replaced by equality.

Chapter 3

LOCC convertibility

In this chapter, we review several known results on LOCC convertibility. First, we give a description of an entanglement and state definitions of an entanglement and a maximally entangled state. Then in Section 3.2, we introduce the concept of LOCC protocols and state an important result obtained by Lo and Popescu (Proposition 2). Finally we review the research on one shot scenario of LOCC convertibility and state the necessary and sufficient conditions for LOCC convertibility obtained by Nielsen.

3.1 Entanglement

Let two parties A and B share an entanglement (Figure 3.1). If party A observe his part, party B will be affected by the observation of party A, even if they are far away from each other, e.g. A is in Tokyo and B is in Beijing.

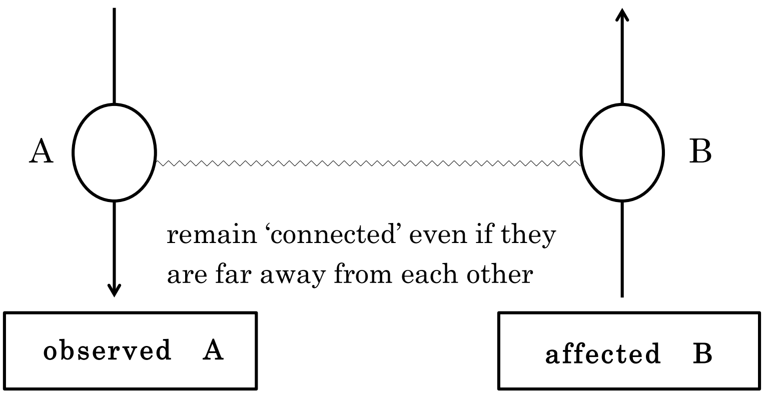


Fig. 3.1: entanglement

The mathematical definition of entanglement is given as follows. Recalling that by Lemma 2, a vector $|\psi\rangle^{AB}$ of composite system $\mathcal{H}^A \otimes \mathcal{H}^B$ can be written as

$$|\psi\rangle^{AB} = \sum_{i}^{m} \sqrt{p_i}\, |e_i\rangle^A \otimes |f_i\rangle^B . \tag{3.1}$$

When $m \geq 2$, the right side of (3.1) cannot be written as a product state. We call such a state as *an entangled state*. In particular, if all p_is are equal to $1/d$ ($d = \min[\dim \mathcal{H}^A, \dim \mathcal{H}^B]$) , we call such a state as a *maximally entangled state*, which is denoted by

$$|\Phi\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}}\, |e_i\rangle \otimes |f_i\rangle . \tag{3.2}$$

3.2 LOCC protocol

First, let us consider a situation where neither global operation nor direct transmission of quantum state between the two parties A and B is not allowed. However, applying physical operations (i.e., measurement, unitary operations) on their individual systems is allowed. These operations are called *local operations*. Next, let us consider another situation that quantum communication is still not allowed but classical communication is allowed between these two parties. Then the following operations become possible: A measures his part as a local operation, then he communicates with B to tell him the result of the measurement. Depending on the information from A, B chooses his subsequent local operations. Such operations are called local operations and classical communication (LOCC).

Now we give an specific description of the LOCC entanglement conversion protocol. The generalized version of LOCC protocol is written in Section 6.4. We assume that two distant parties A and B share a pure bipartite entangled state in advance. The two parties A and B aim to transform the given entangled state into another state only by LOCC protocol. The starting state is called the initial state, and the ending state is called the target state. Let us consider a situation where A and B engage in a multi-round LOCC protocol. Without loss of generality, we may assume that the LOCC protocol starts with A's measurement and ends with A's operation on his system. By rearranging the order of quantum operation and classical communication, the LOCC protocol can then be described as follows.

1. A performs a measurement on his part of the initial state, and obtains an outcome.
2. A communicates a classical message to B.
3. B performs a measurement on his part of the initial state, depending on the information received from A, and obtains an outcome.
4. B communicates a classical message to A.
5. A and B recursively apply 1∼4.
6. A performs an operation on his part.

Such rounds are concatenated until the transformation from the initial state to the target state is accomplished deterministically. Such a LOCC protocol with two-way classical communication from A to B and B to A is called two-way LOCC (Figure 3.2). A LOCC protocol with one-way classical communication from A to B is called one-way LOCC (Figure 3.3).

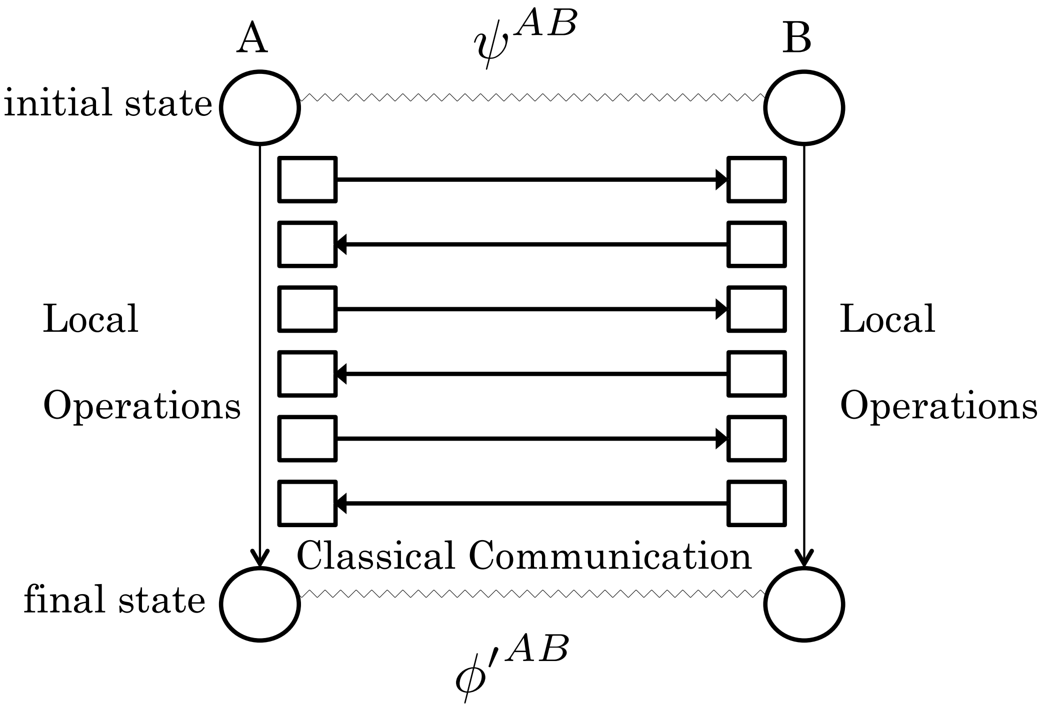


Fig. 3.2: two-way LOCC

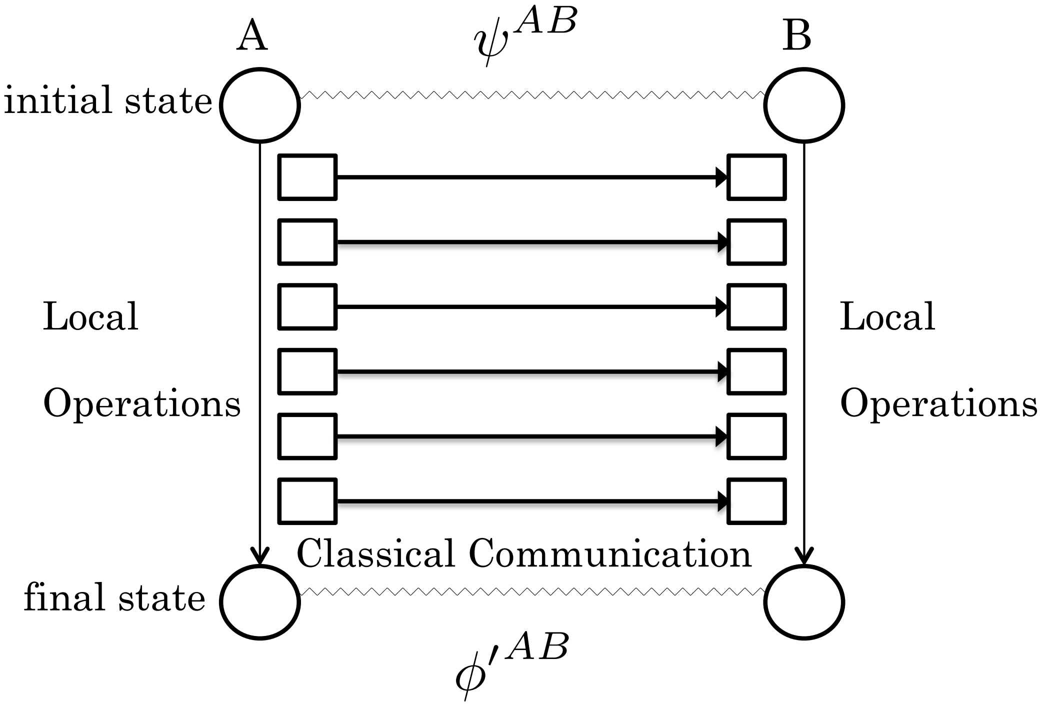


Fig. 3.3: one-way LOCC

The following proposition shows that, for a known bipartite pure state, concerning two-way communication is equivalent to concerning one-way communication.

Proposition 2 (Lo and Popescu [8], Proposition 1)**.** *Let $|\psi\rangle\langle\psi|^{AB}$ be a pure state known by parties A and B. Entanglement transformation of $|\psi\rangle\langle\psi|^{AB}$ by two-way LOCC can be realized by one-way LOCC.*

Proof. By Lemma 2, a bipartite pure state $|\psi\rangle^{AB}$ on a composite system is written as

$$|\psi\rangle^{AB} = \sum_i \sqrt{p_i}\,|i\rangle^A \otimes |i\rangle^B \,. \tag{3.3}$$

Let B performs a measurement on his part of $|\psi\rangle^{AB}$, where measurement operators $\{M_l\}$ are given as

$$M_l^B = \sum_{jk} m_{jk}^l |j\rangle^B \langle k|^B , \tag{3.4}$$

and $\sum_l M_l^\dagger M_l = I$. When B obtains a result l, then the state changes to

$$|\psi_l\rangle^B = (I \otimes M_l^B) |\psi\rangle^{AB} = \sum_{ij} m_{ji}^l \sqrt{p_i} |i\rangle^A \otimes |j\rangle^B , \tag{3.5}$$

up to the normalization constant. On the other hand, let A performs a measurement on his part of $|\psi\rangle^{AB}$, where measurement operators are given as

$$N_l^A = \sum_{jk} m_{jk}^l |j\rangle^A \langle k|^A . \tag{3.6}$$

When he obtains the result l, the state changes to

$$|\phi_l\rangle^A = (N_l^A \otimes I) |\psi\rangle^{AB} = \sum_{ij} m_{ji}^l \sqrt{p_i} |j\rangle^A \otimes |i\rangle^B . \tag{3.7}$$

Noted that since $|\psi_l\rangle^B$ and $|\phi_l\rangle^A$ have the same coefficients and thus have the same Schmidt coefficients, hence they are related by $|\psi\rangle_l^B = (U_l^A \otimes V_l^B) |\phi_l\rangle^A$ where U_l^A is a local unitary on A system and V_l^B is a local unitary on B system. Therefore, the statement that party B performs a measurement described by M_l^B is equivalent to the statement that party A performs a measurement described by $U_l^A N_l^A$, which is followed by party B performing the unitary transformation V_l^B. □

3.3 One shot scenario of LOCC convertibility

Nielsen *et al.* [9] obtained the necessary and sufficient condition that for a bipartite pure entangled state may be converted into another only by LOCC using majorization. The problem setting of one shot of LOCC convertibility is described by Figure 3.4.

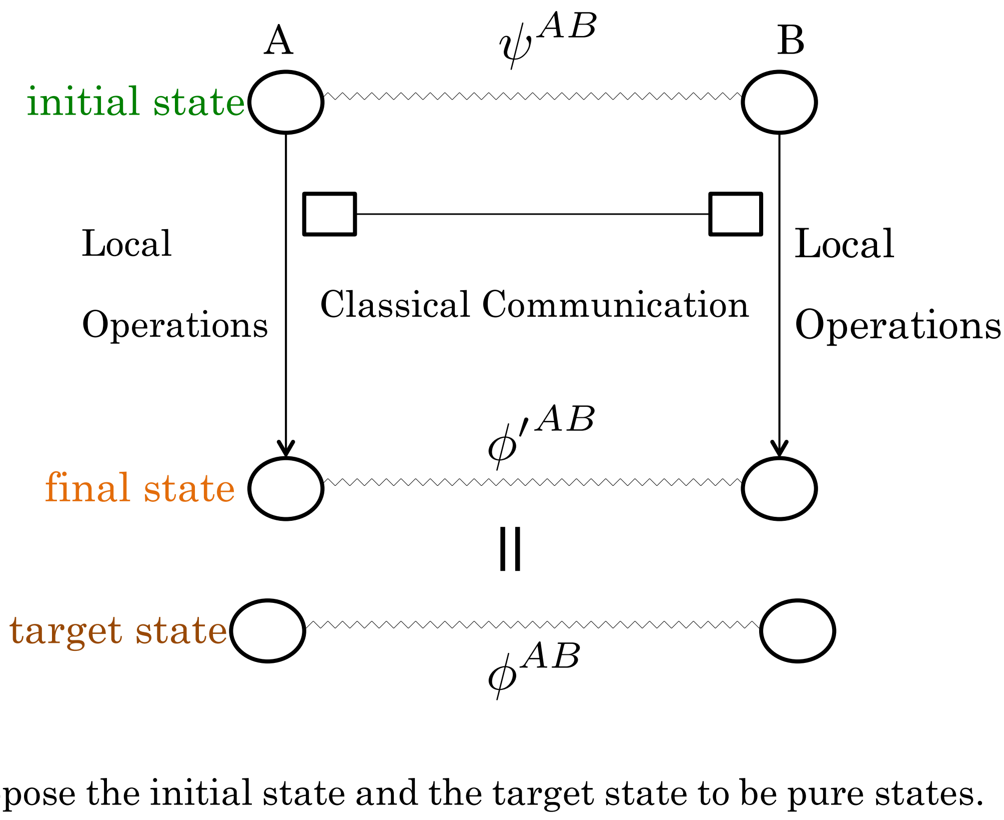


Fig. 3.4: one shot scenario of LOCC convertibility

First we give the definitions of majorization as follows.

Definition 2. *For a sequence of real nonnegative numbers* $a = \{a_i\}_{i=1}^m$ $(m \in \mathbb{N})$*, let* $a^{\downarrow} = \{a_i^{\downarrow}\}_{i=1}^m$ *denotes the sequence rearranged in decreasing order. We say that* $a = \{a_i\}_{i=1}^m$ *is majorized by* $b = \{b_i\}_{i=1}^m$ *and write* $a \prec b$ *if we have*

$$\sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \quad (k = 1, 2, \dots, m) \tag{3.8}$$

and the equality for $k = m$.

Note that the majorization relation $a \prec b$ can be defined even when the numbers of elements in a and b are different, by including zero if necessary. When both $a \prec b$ and

$b \prec a$ hold, or equivalently $a^{\downarrow} = b^{\downarrow}$, we write $a \sim b$.

Example 1. *If $p_i \geq 0$ and $\sum_i p_i = 1$, then*

$$(\frac{1}{m}, \frac{1}{m}, \cdots) \prec (p_1, p_2, \cdots) \prec (1, 0, 0, \cdots).$$

For simplicity of the notation, we denote that $\psi^{AB} = |\psi\rangle\langle\psi|^{AB}$ and $\phi^{AB} = |\phi\rangle\langle\phi|^{AB}$. Recall that their reduced states are denoted as $\rho = \mathrm{Tr}_B \psi^{AB}$ and $\sigma = \mathrm{Tr}_B \phi^{AB} \in \mathcal{L}(\mathcal{H}^A)$, respectively.

Definition 3. *For density operators ρ and σ, let λ_ρ and λ_σ be vectors whose entries are the eigenvalues of ρ and σ, respectively. We say that state ρ is majorized by state σ and write $\rho \prec \sigma$ if $\lambda_\rho \prec \lambda_\sigma$.*

It is well known that the following relations (Proposition 3 and 4) between majorization and doubly stochastic matrices hold (see [38] for proofs).

Definition 4 (doubly stochastic)**.** *An $m \times m$ matrix $A = (a_{ij})$ is called doubly stochastic if*

$$a_{ij} \geq 0 \qquad \text{for all} \quad i, j,$$
$$\sum_i^m a_{ij} = 1 \qquad \text{for all} \quad j,$$
$$\sum_j^m a_{ij} = 1 \qquad \text{for all} \quad i.$$

Proposition 3. *x is majorized by y ($x \prec y$) if and only if $y = Dx$ for some doubly stochastic matrix D.*

Proposition 4 (Birkhoff's theorem)**.** *A $d \times d$ matrix D is doubly stochastic if and only if $D = \sum_j p_j P_j$ for some probability distribution p_j and permutation matrices P_j.*

Proposition 5 (Nielsen [9])**.** *For $\rho = \mathrm{Tr}_B |\psi\rangle\langle\psi|^{AB}$ and $\sigma = \mathrm{Tr}_B |\phi\rangle\langle\phi|^{AB}$, the following conditions are equivalent.*

1. *$\rho \prec \sigma$.*

2. *$|\psi\rangle^{AB} \xrightarrow[LOCC]{} |\phi\rangle^{AB}$ (which means $|\psi\rangle^{AB}$ can be converted into $|\phi\rangle^{AB}$ by LOCC).*

3. There exist matrices U_i and probability p_i, such that $\rho = \sum p_i U_i \sigma U_i^\dagger$ where $p_i \geq 0, \sum_i p_i = 1$.

Proof. 1 $\Rightarrow$ 2. Suppose $\rho \prec \sigma$. By Definition 3, we have $\lambda_\rho \prec \lambda_\sigma$. By Proposition 3 and 4, $\lambda_\rho = \sum_i p_i P_i \lambda_\sigma$ holds with permutation matrices P_i. Let $\Lambda(\rho)$ and $\Lambda(\sigma)$ denote the diagonal matrix whose entries are the eigenvalues of ρ and σ, respectively. Then we have,

$$\Lambda(\rho) = \sum_i p_i P_i \Lambda(\sigma) P_i^\dagger.$$

Since $\rho = V\Lambda(\rho)V^\dagger$ and $\Lambda(\sigma) = W\sigma W^\dagger$ hold for some unitary matrices V and W, we obtain $\rho = \sum p_i U_i \sigma U_i^\dagger$, where $U_i = VP_iW$ is a unitary matrix. Define operators M_i for A system by

$$M_i \sqrt{\rho} := \sqrt{p_i \sigma} U_i^\dagger. \tag{3.9}$$

Then we can check the following relation:

$$\sum_i M_i^\dagger M_i = \rho^{-1/2} (\sum_i p_i U_i \sigma U_i^\dagger) \rho^{-1/2} = I, \tag{3.10}$$

from which $\{M_i\}$ are measurement operators. Suppose A performs a measurement on his part of $|\psi\rangle^{AB}$ by M_i. When he obtained a outcome i, then the state changed to $|\psi_i\rangle = (M_i \otimes I) |\psi\rangle^{AB}$ up to the normalization constant. Note that

$$\begin{aligned} &\mathrm{Tr}_B (M_i \otimes I) |\psi^{AB}\rangle\langle\psi^{AB}| (M_i^\dagger \otimes I) \\ &= M_i \rho M_i^\dagger \\ &= p_i \sigma, \end{aligned} \tag{3.11}$$

where the last equality follows from (3.9). Regarding that

$$p_i = \mathrm{Tr}_A \mathrm{Tr}_B (M_i \otimes I) |\psi^{AB}\rangle\langle\psi^{AB}| (M_i^\dagger \otimes I), \tag{3.12}$$

then the post measurement state is given by

$$|\psi_i\rangle = \frac{(M_i \otimes I)\,|\psi^{AB}\rangle}{\sqrt{p_i}}$$

and

$$\mathrm{Tr}_B\,|\psi_i\rangle\langle\psi_i| = \sigma = \mathrm{Tr}_B\,|\phi^{AB}\rangle\langle\phi^{AB}|\,.$$

By Proposition 1 in Chapter 2, there exists unitary U_i, such that

$$|\phi^{AB}\rangle = (I \otimes U_i)\,|\psi^{AB}\rangle\,.$$

Therefore we have $|\psi\rangle^{AB} \xrightarrow[LOCC]{} |\phi\rangle^{AB}$.

$2 \Rightarrow 3$. Suppose $|\psi\rangle^{AB} \xrightarrow[LOCC]{} |\phi\rangle^{AB}$. Then by Proposition 2, we may assume that the conversion is given by one way LOCC protocol, where A performs a measurement with measurement operator M_i then sending the result to B, who performs a unitary operation U_i. From a point of view that the initial state is ρ and the final state is σ, regardless of the measurement outcome, so we must have

$$\sigma = \frac{M_i \rho M_i^\dagger}{p_i} \tag{3.13}$$

where $p_i = \mathrm{Tr} M_i \rho M_i^\dagger$ is the probability of the outcome i. Polar decomposition of $M_i\sqrt{\rho}$ implies that there exists a unitary V_i such that

$$M_i\sqrt{\rho} = \sqrt{M_i \rho M_i^\dagger}\,U_i = \sqrt{p_i \sigma}\,U_i. \tag{3.14}$$

Multiplying this equation by its adjoint and summing on i gives $\rho = \sum_i p_i U_i \sigma U_i^\dagger$, by using $\sum_i M^\dagger M_i = 1$.

$3 \Rightarrow 1$. Suppose $\rho = \sum p_i U_i \sigma U_i^\dagger$. Let $\rho = \sum_j q_j\,|e_j\rangle\,\langle e_j|$ and $\sigma = \sum_k q_k'\,|f_k\rangle\,\langle f_k|$ be their eigenvalue decompositions. Define a unitary operator as $W = \sum_k |f_k\rangle\,\langle e_k|\,.$ Then $U_i W$ is a unitary. Let u_{ik}^h denote coefficients of $U_i\,|f_k\rangle$, that is $U_i\,|f_k\rangle = U_i W\,|e_k\rangle = \sum_h u_{ik}^h\,|e_h\rangle$

and $\sum_h |u_{ik}^h|^2 = 1$. The equation $\rho = \sum p_i U_i \sigma U_i^\dagger$ can be rewritten as

$$\sum_j q_j \left|e_j\right\rangle \left\langle e_j\right| = \sum_{i,k} p_i q'_k (\sum_h u_{ik}^h \left|e_h\right\rangle (\sum_l u_{ik}^{\dagger\ l} \left\langle e_l\right|) \tag{3.15}$$

Taking the inner product between $|e_m\rangle$,

$$q_m = \sum_i p_i \sum_k q'_k |u_{ik}^m|^2 \tag{3.16}$$

Define a matrix D with entries $D_{mi} = \sum_k q'_k |u_{ik}^m|^2$. So we have $q = Dq'$. The D_{mi} is non-negative by definition, and the rows and columns of D all sum to one because the rows and columns of unitary are unit vectors. So D is doubly stochastic and thus $\rho \prec \sigma$.

$\square$

Chapter 4

Asymptotic LOCC convertibility

In this chapter, we state the main results of this thesis on asymptotic LOCC convertibility between two arbitrary sequences of bipartite pure states $\widehat{\psi}$ and $\widehat{\phi}$. The research on entanglement convertibility in the asymptotic setting began by Bennett *et al.* [7]. First in Section 4.1, we review their setting and results. Then, in Section 4.2 we state the main results of this thesis. Finally, in Section 4.3, we apply main results to give simple proofs of previously known results obtained by Hayashi [10] and Bowen-Datta [11] on entanglement concentration and entanglement dilution.

4.1 i.i.d. case

In this section, we give an intuitive explanation for what is asymptotic entanglement convertibility. Specific definition is given later in the general setting.

A: Entanglement concentration

Entanglement concentration is a task to obtain copies of a maximally entangled state from many copies of a non-maximum entangled state only by LOCC.

Let us suppose that two distant parties A and B share n pairs of a partially entangled pure state $|\psi\rangle^{AB}$ beforehand. By an LOCC protocol (Figure 4.1), they can convert n pairs of a partially entangled pure state $|\psi\rangle^{AB}$ into m $(m < n)$ pairs of a maximally entangled state $|\Phi\rangle^{AB}$.

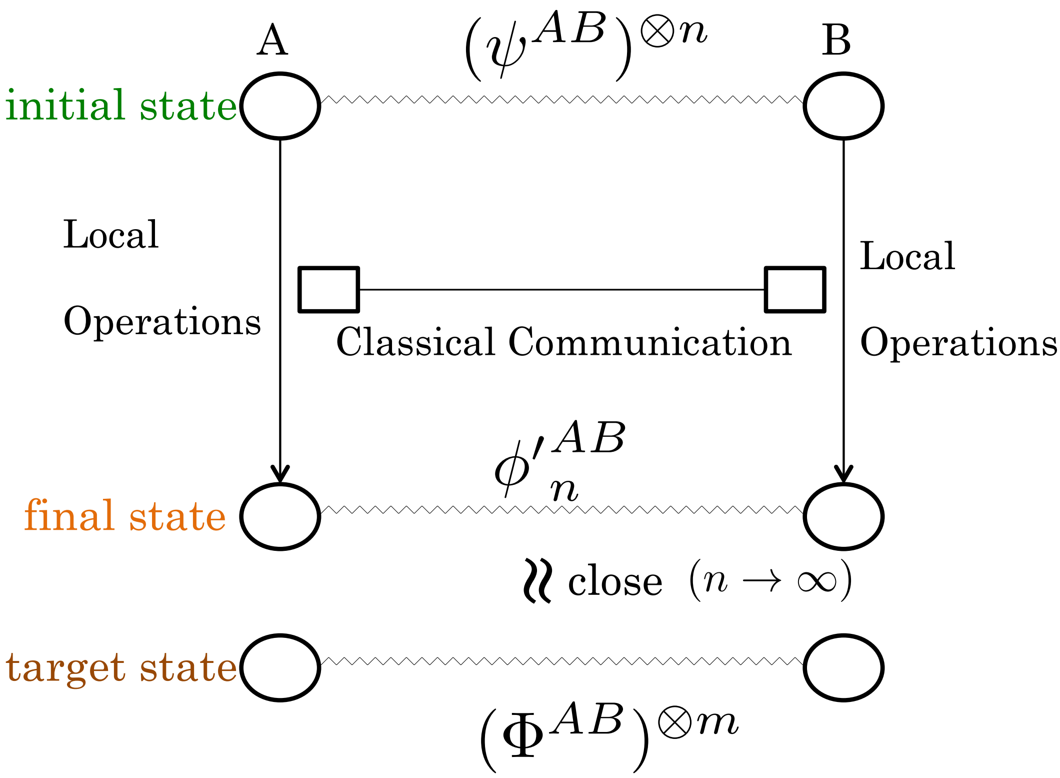


Fig. 4.1: i.i.d. case of entanglement concentration

Proposition 6 (Bennet *et al.* [7]). *Under the i.i.d. assumption, the asymptotic optimal rates* $\frac{m}{n}$ *of entanglement concentration are asymptotically equal to the entanglement entropy of initial state* $(\psi^{AB})^{\otimes n}$ *which is defined by*

$$H(\psi^A) = -\mathrm{Tr}\psi^A \log \psi^A,$$

where $\psi^A = \mathrm{Tr}_B(\psi^{AB})$ *and* $\psi^{AB} = |\psi\rangle\langle\psi|^{AB}$.

B: Entanglement dilution

Entanglement dilution is a task to convert copies of a maximally entangled state asymptotically into many copies of a partially entangled state only by LOCC.

Let us suppose that two parties A and B share m pairs of a maximally entangled state $|\Psi\rangle^{AB}$ beforehand. By an LOCC protocol (Figure 4.2), they can share n pairs of a partially entangled state $|\phi\rangle^{AB}$ instead of m pairs of a maximally entangled state $|\Psi_{\mathrm{M_n}}\rangle$.

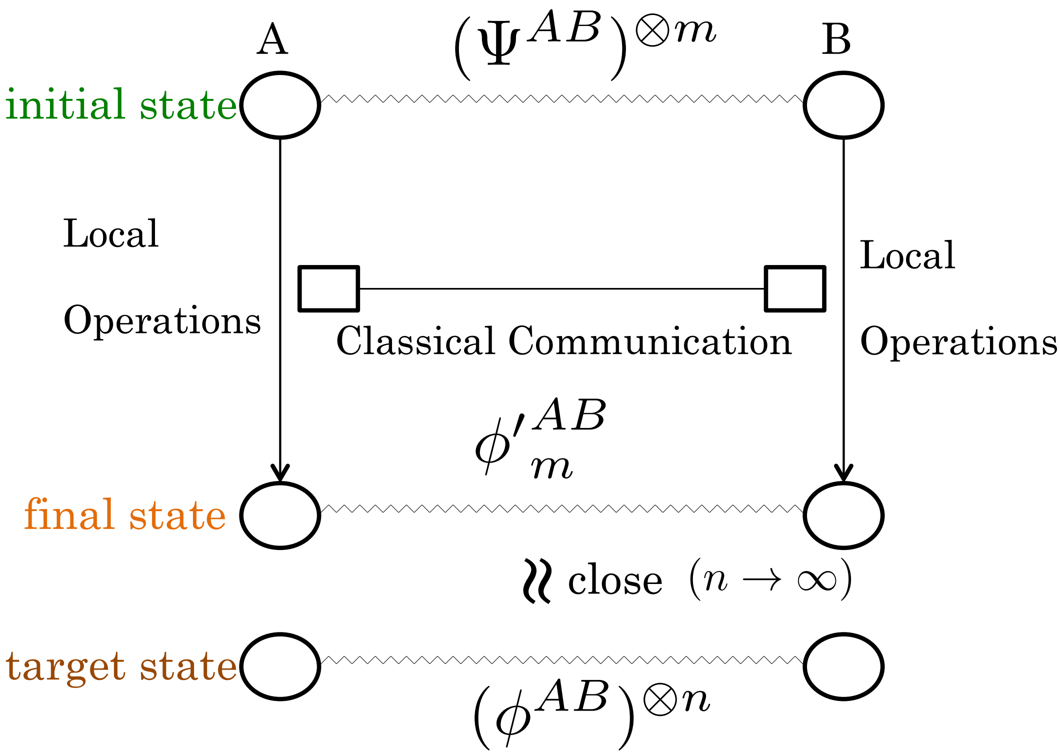


Fig. 4.2: i.i.d. case of entanglement dilution

Proposition 7 (Bennet [7]). *Under the i.i.d. assumption, the asymptotic optimal rate* $\frac{m}{n}$ *of entanglement dilution are asymptotically equal to the entanglement entropy of target* *state* $(\phi^{AB})^{\otimes n}$*, which is defined by*

$$H(\phi^A) = -\mathrm{Tr}\phi^A \log \phi^A,$$

where $\phi^A = \mathrm{Tr}_B(\phi^{AB})$ *and* $\phi^{AB} = |\phi\rangle\langle\phi|^{AB}$*.*

4.2 General setting

Let $\mathcal{H}_n^A$ and $\mathcal{H}_n^B$ $(n = 1, 2, \dots)$ be arbitrary Hilbert spaces. Let us consider a general sequence of bipartite systems $\mathcal{H}_n^{AB} = \mathcal{H}_n^A \otimes \mathcal{H}_n^B$ $(n = 1, 2, \dots)$ composed of them. For simplicity we assume that $\dim \mathcal{H}_n^A < \infty$ and $\dim \mathcal{H}_n^B < \infty$ for each $n \in \mathbb{N}$. Let $|\psi_n\rangle^{AB}$ and $|\phi_n\rangle^{AB}$ be arbitrary pure states in $\mathcal{H}_n^{AB}$ for each $n \in \mathbb{N}$. For simplicity of the notation, we denote these density operators by $\psi_n^{AB} = |\psi_n\rangle\langle\psi_n|^{AB}$ and $\phi_n^{AB} = |\phi_n\rangle\langle\phi_n|^{AB}$.

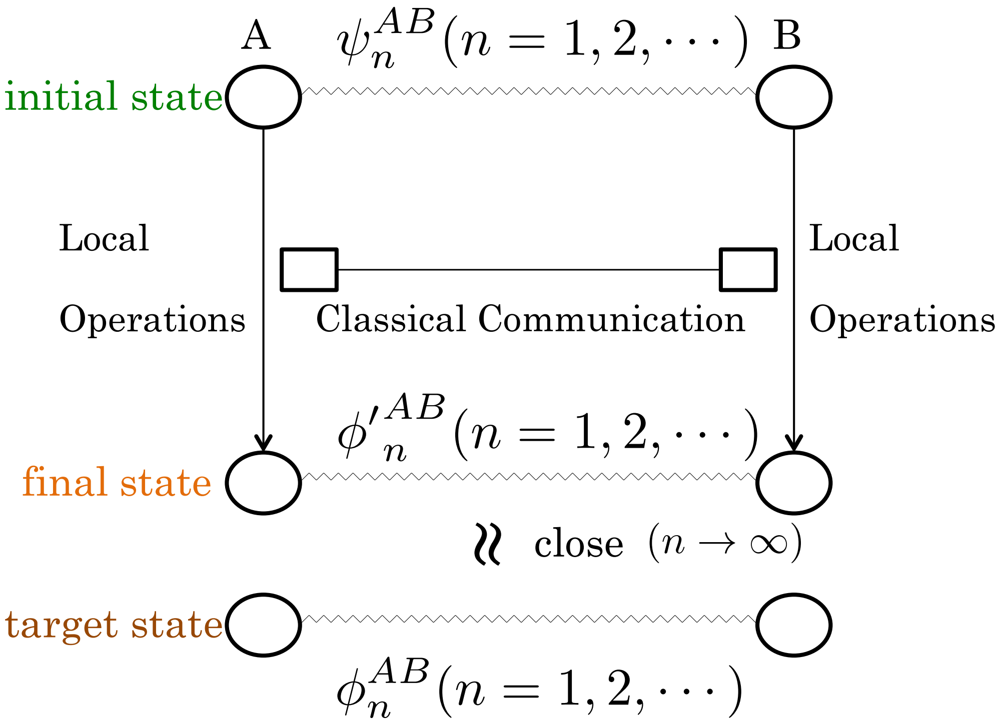
For arbitrary sequences of bipartite pure states $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^{\infty}$ and $\widehat{\phi}^{AB} = \{\phi_n^{AB}\}_{n=1}^{\infty}$, we seek for conditions under which ψ_n^{AB} can be asymptotically converted into ϕ_n^{AB} only by LOCC Protocols (Figure 4.3) for each n, up to a certain error that vanishes in the limit of $n \to \infty$.

We give a definition of asymptotic convertibility here for readers' convenience since it is not so familiar to some readers.

Definition 5 (Asymptotic Convertibility)**.** *We say that an arbitrary sequence of bipartite pure states* $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^{\infty}$ *can be asymptotically converted into another* $\widehat{\phi}^{AB} = \{\phi_n^{AB}\}_{n=1}^{\infty}$ *only by LOCC, if there exists a sequence of LOCC* $\widehat{L}_n = \{L_n\}_{n=1}^{\infty}$ *such that*

$$\lim_{n\to\infty} \|L_n(\psi_n^{AB}) - \phi_n^{AB}\|_1 = 0. \tag{4.1}$$

Note that $\|\cdot\|_1$ *is the trace norm defined by* $\|A\|_1 = \mathrm{Tr}|A|$ *for an operator* A*.*



Suppose the initial state and the target state to be pure states.

Fig. 4.3: general setting

4.3 Main results under the general setting

Let $\widehat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ be an arbitrary sequence of density operators. Then the *inf- and sup-spectral entropy rates* [23, 35] are defined as

$$\underline{H}(\varepsilon|\widehat{\rho}) := \sup \left\{ a \;\middle|\; \limsup_{n\to\infty} \mathrm{Tr}\rho_n \left\{\rho_n > e^{-na} I_n\right\} \le \varepsilon \right\}, \tag{4.2}$$

$$\overline{H}(\varepsilon|\widehat{\rho}) := \inf \left\{ a \;\middle|\; \liminf_{n\to\infty} \mathrm{Tr}\rho_n \left\{\rho_n > e^{-na} I_n\right\} \ge 1 - \varepsilon \right\} \tag{4.3}$$

for $\varepsilon \in [0,1]$, where $\widehat{I} = \{I_n\}_{n=1}^{\infty}$ is the sequence of identity operators and

$$\left\{\rho_n > e^{-na} I_n\right\} = \left\{\rho_n - e^{-na} I_n > 0\right\} \tag{4.4}$$

denotes the spectral projection corresponding to the positive part of the Hermitian operator $\rho_n - e^{-na} I_n$; see Subsection 6.1 for details. Especially, for $\varepsilon = 0$ we write

$$\underline{H}(\widehat{\rho}) := \underline{H}(0|\widehat{\rho}), \quad \overline{H}(\widehat{\rho}) := \overline{H}(0|\widehat{\rho}). \tag{4.5}$$

For a sequence of bipartite pure state $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^{\infty}$, let $\widehat{\psi}^{A} = \{\psi_n^{A}\}_{n=1}^{\infty}$ and $\widehat{\psi}^{B} = \{\psi_n^{B}\}_{n=1}^{\infty}$ be corresponding sequences of reduced states. Note that $\psi_n^{AB} = |\psi_n\rangle\langle\psi_n|^{AB}$. Let the Schmidt decompositions of $|\psi_n\rangle^{AB}$ given by

$$|\psi_n\rangle^{AB} = \sum_{x^n \in \mathcal{X}^n} \sqrt{p_n(x^n)}\, |e_{x^n}\rangle^{A} \otimes |e_{x^n}\rangle^{B}, \tag{4.6}$$

where $\{|e_{x^n}\rangle^A\}_{x^n \in \mathcal{X}^n}$ is a CONS of $\mathcal{H}^A$ and $\{|e_{x^n}\rangle^B\}_{x^n \in \mathcal{X}^n}$ is a CONS of $\mathcal{H}^B$. Then the reduced density operators of subsystem of A and B are given as follows.

$$\psi_n^A = \mathrm{Tr}_B\left[\psi_n^{AB}\right] = \sum_{x^n \in \mathcal{X}^n} p_n(x^n)\, |e_{x^n}\rangle\langle e_{x^n}|^{A}, \tag{4.7}$$

$$\psi_n^B = \mathrm{Tr}_A\left[\psi_n^{AB}\right] = \sum_{x^n \in \mathcal{Y}^n} p_n(x^n)\, |e_{x^n}\rangle\langle e_{x^n}|^{B}. \tag{4.8}$$

Since the reduced density operators ψ_n^A and ψ_n^B have the same eigenvalues and the spectral entropy rates only depend on eigenvalues of density operators (see (4.2) and (4.3)), it is clear that $\widehat{\psi}^A$ and $\widehat{\psi}^B$ have the same spectral entropy rates:

$$\underline{H}(\varepsilon|\widehat{\psi}^A) = \underline{H}(\varepsilon|\widehat{\psi}^B), \quad \overline{H}(\varepsilon|\widehat{\psi}^A) = \overline{H}(\varepsilon|\widehat{\psi}^B), \tag{4.9}$$

which we call the *inf-/sup-spectral entanglement entropy rates* of $\widehat{\psi}^{AB}$. The main results of this thesis are as follows.

Theorem 1 (Direct Part). *Let $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^{\infty}$ and $\widehat{\phi}^{AB} = \{\phi_n^{AB}\}_{n=1}^{\infty}$ be arbitrary sequences of bipartite pure states on $\mathcal{H}_n^{AB}$ $(n = 1, 2, \ldots)$. If $\underline{H}(\widehat{\psi}^A) > \overline{H}(\widehat{\phi}^A)$ holds, then*

$\widehat{\psi}^{AB}$ *can be asymptotically converted into* $\widehat{\phi}^{AB}$ *by LOCC.*

Theorem 2 (Converse Part)**.** *Let* $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^{\infty}$ *and* $\widehat{\phi}^{AB} = \{\phi_n^{AB}\}_{n=1}^{\infty}$ *be arbitrary sequences of bipartite pure states on* $\mathcal{H}_n^{AB}$ $(n = 1, 2, \ldots)$*. If* $\widehat{\psi}^{AB}$ *can be asymptotically converted into* $\widehat{\phi}^{AB}$ *by LOCC, it must hold that* $\overline{H}(\varepsilon|\widehat{\psi}^A) \geq \overline{H}(\varepsilon|\widehat{\phi}^A)$ *and* $\underline{H}(\varepsilon|\widehat{\psi}^A) \geq \underline{H}(\varepsilon|\widehat{\phi}^A)$ *for every* $\varepsilon \in [0, 1]$*.*

Proofs of the above two theorems are given in Section 5.2 and Section 6.5, respectively.

4.4 Applying main results to entanglement concentration and dilution

In this section, we use the above two theorems to provide simple proofs of known results [10, 11] on the optimal rates of entanglement concentration and dilution for general sequences of bipartite pure states.

Let $\{M_n\}_{n=1}^{\infty}$ be an arbitrary sequence of natural numbers, and let $|\Phi_{M_n}\rangle \in \mathcal{H}_n^{AB}$ be a maximally entangled state with Schmidt rank M_n for each n. As a shorthand notation, we write $\Phi_{M_n}^{AB} = |\Phi_{M_n}\rangle\langle\Phi_{M_n}|$. Note that $\Phi_{M_n}^{A} = \mathrm{Tr}_B[\Phi_{M_n}^{AB}]$ and $\Phi_{M_n}^{B} = \mathrm{Tr}_A[\Phi_{M_n}^{AB}]$ are the maximally mixed states with rank M_n, it is straightforward to verify that

$$\underline{H}(\widehat{\Phi}^A) = \liminf_{n\to\infty} \frac{1}{n} \log M_n, \tag{4.10}$$

$$\overline{H}(\widehat{\Phi}^A) = \limsup_{n\to\infty} \frac{1}{n} \log M_n \tag{4.11}$$

for $\widehat{\Phi}^A = \{\Phi_{M_n}^A\}_{n=1}^{\infty}$.

4.4.1 A simple proof of entanglement concentration

Entanglement concentration is a task to obtain a sequence of maximally entangled states $\widehat{\Phi}^{AB}$ asymptotically from a sequence of bipartite pure states $\widehat{\psi}^{AB}$ by LOCC.

The problem setting is described as follows (Figure 4.4).

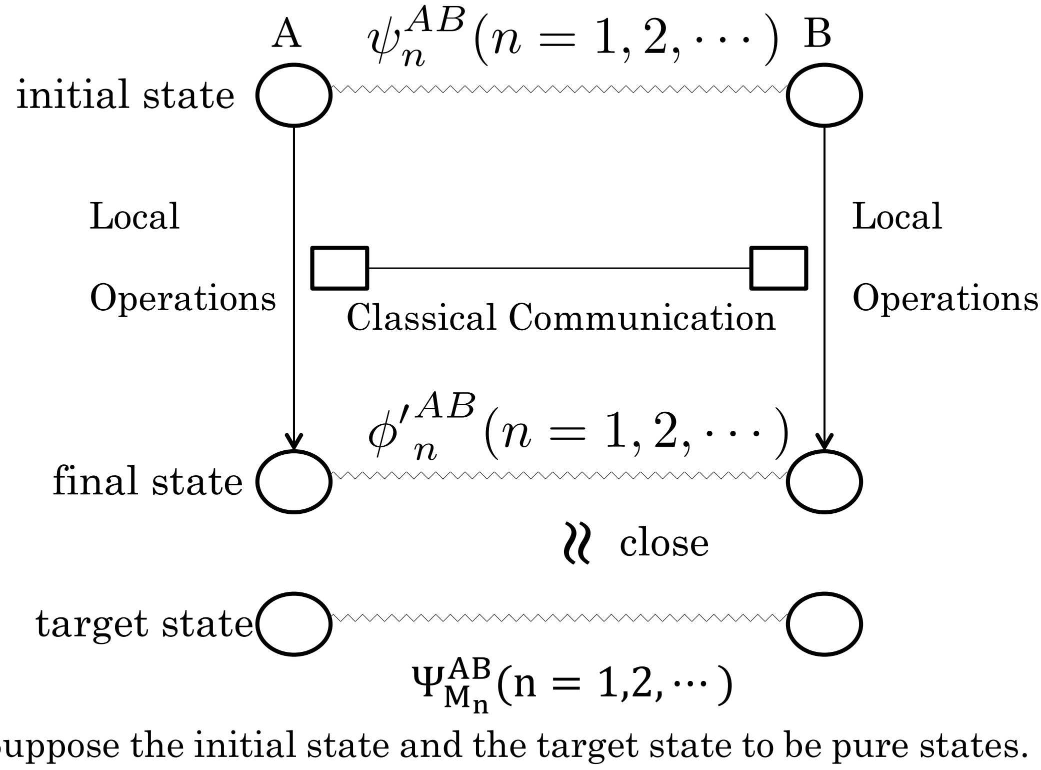


Fig. 4.4: entanglement concentration

Definition 6 (Distillable Entanglement)**.** *For a sequence of bipartite pure state* $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^{\infty}$*, we say a rate* R *is achievable if there exists a sequence of natural numbers* $\{M_n\}_{n=1}^{\infty}$ *such that* $\widehat{\psi}^{AB}$ *can be asymptotically converted into a sequence of maximally entangled states* $\widehat{\Phi}^{AB} = \{\Phi_{M_n}^{AB}\}_{n=1}^{\infty}$ *only by LOCC and the conversion rate* R *satisfies*

$$\liminf_{n\to\infty} \frac{1}{n} \log M_n \geq R. \tag{4.12}$$

The entanglement concentration rate [10], or distillable entanglement [11], of a sequence $\widehat{\psi}^{AB}$ *is defined by*

$$R(\widehat{\psi}^{AB}) := \sup \{\, R \mid R \textit{ is achievable} \,\}. \tag{4.13}$$

Proposition 8 (Hayashi [10, Theorem 1], Bowen-Datta [11, Theorem 3])**.** *For a sequence of bipartite pure states* $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^{\infty}$*, the distillable entanglement of* $\widehat{\psi}^{AB}$ *is given by*

$$R(\widehat{\psi}^{AB}) = \underline{H}(\widehat{\psi}^{A}). \tag{4.14}$$

Proof. Note that the target sequence of states $\widehat{\phi}^{AB}$ should be instead by $\widehat{\Phi}^{AB}$ in Theorem 1 and Theorem 2. First we show $\underline{H}(\widehat{\psi}^{A}) \leq R(\widehat{\psi}^{AB})$. Suppose that $R < \underline{H}(\widehat{\psi}^{A})$. Then, taking $M_n = e^{nR}$ and $\widehat{\Phi}^{AB} = \{\Phi_{M_n}^{AB}\}_{n=1}^{\infty}$, (4.11) yields

$$\overline{H}(\widehat{\Phi}^{A}) = \limsup_{n\to\infty} \frac{1}{n} \log M_n = R < \underline{H}(\widehat{\psi}^{A}). \tag{4.15}$$

Hence, from Theorem 1, we know that $\widehat{\psi}^{AB}$ can be asymptotically converted into $\widehat{\Phi}^{AB}$ only by LOCC, and the conversion rate satisfies (4.12) (with equality). Thus a rate R is achievable if $\underline{H}(\widehat{\psi}^{A}) > R$, which implies $\underline{H}(\widehat{\psi}^{A}) \leq R(\widehat{\psi}^{AB})$.

Next we show $\underline{H}(\widehat{\psi}^{A}) \geq R(\widehat{\psi}^{AB})$. Suppose that a rate R is achievable. From Definition 6, we know that there exists a sequence $\widehat{\Phi}^{AB} = \{\Phi_{M_n}^{AB}\}_{n=1}^{\infty}$ such that $\widehat{\psi}^{AB}$ can be asymptotically converted into $\widehat{\Phi}^{AB}$ and (4.12) holds. Then, from Theorem 2 and (4.10), it must hold that

$$\underline{H}(\widehat{\psi}^{A}) \geq \underline{H}(\widehat{\Phi}^{A}) = \liminf_{n\to\infty} \frac{1}{n} \log M_n \geq R, \tag{4.16}$$

which implies $\underline{H}(\widehat{\psi}^{A}) \geq R(\widehat{\psi}^{AB})$. Therefore, we have (4.14). □

4.4.2 A simple proof of entanglement dilution

Entanglement dilution is a task to convert a sequence of maximally entangled states $\widehat{\Phi}^{AB}$ asymptotically into a sequence of bipartite pure states $\widehat{\phi}^{AB}$ only by LOCC.

The problem setting is described as follows (Figure 4.5).

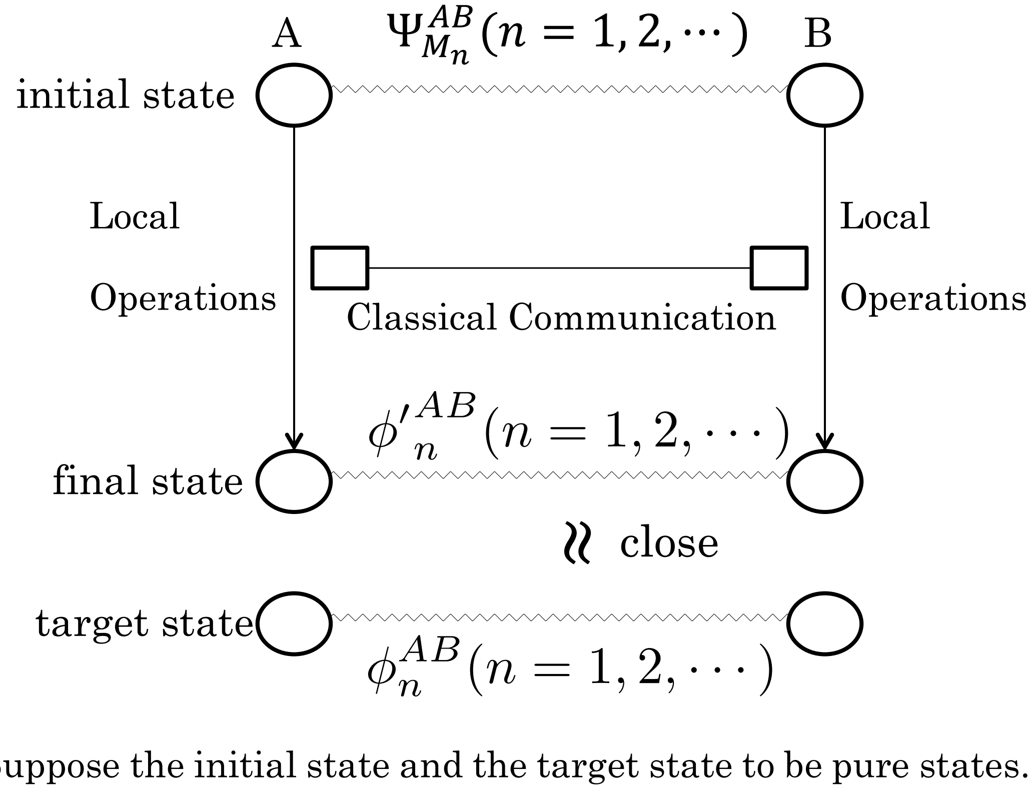


Fig. 4.5: entanglement diluion

Definition 7 (Entanglement Cost)**.** *For a sequence of bipartite pure states $\widehat{\phi}^{AB} = \{\phi_n^{AB}\}_{n=1}^{\infty}$, we say a rate R is achievable if there exists a sequence of natural numbers $\{M_n\}_{n=1}^{\infty}$ such that a sequence of maximally entangled states $\widehat{\Phi}^{AB} = \{\Phi_{M_n}^{AB}\}_{n=1}^{\infty}$ can be asymptotically converted into $\widehat{\phi}^{AB}$ only by LOCC and the rate R satisfies*

$$\limsup_{n\to\infty} \frac{1}{n} \log M_n \leq R. \tag{4.17}$$

The entanglement dilution rate, or entanglement cost [11], of a sequence $\widehat{\phi}^{AB}$ is defined by

$$R^*(\widehat{\phi}^{AB}) := \inf \{\, R \mid R \textit{ is achievable} \,\}. \tag{4.18}$$

Proposition 9 (Bowen-Datta [11, Theorem 4])**.** *For a sequence of bipartite pure states* $\widehat{\phi}^{AB} = \{\phi_n^{AB}\}_{n=1}^{\infty}$*, the entanglement cost of* $\widehat{\phi}^{AB}$ *is given by*

$$R^*(\widehat{\phi}^{AB}) = \overline{H}(\widehat{\phi}^A). \tag{4.19}$$

Proof. Note that the initial sequence of states $\widehat{\psi}^{AB}$ should be instead by $\widehat{\Phi}^{AB}$ in Theorem 1 and Theorem 2. First we show $R^*(\widehat{\phi}^{AB}) \leq \overline{H}(\widehat{\phi}^A)$. Suppose that $R > \overline{H}(\widehat{\phi}^A)$, and let $M_n = e^{nR}$ and $\widehat{\Phi}^{AB} = \{\Phi_{M_n}^{AB}\}_{n=1}^{\infty}$. Then using (4.10) we have

$$\underline{H}(\widehat{\Phi}^A) = \liminf_{n\to\infty} \frac{1}{n} \log M_n = R > \overline{H}(\widehat{\phi}^A). \tag{4.20}$$

Hence, from Theorem 1, we know that $\widehat{\Phi}^{AB}$ can be asymptotically converted into $\widehat{\phi}^{AB}$ only by LOCC and (4.17) holds (with equality). Consequently, a rate R is achievable if $R > \overline{H}(\widehat{\phi}^A)$, which implies $R^*(\widehat{\phi}^{AB}) \leq \overline{H}(\widehat{\phi}^A)$.

Next we show $R^*(\widehat{\phi}^{AB}) \geq \overline{H}(\widehat{\phi}^A)$. Suppose that a rate R is achievable. From Definition 7, we know that there exists a sequence $\widehat{\Phi}^{AB} = \{\Phi_{M_n}^{AB}\}_{n=1}^{\infty}$ such that $\widehat{\Phi}^{AB}$ can be asymptotically converted into $\widehat{\phi}^{AB}$ only by LOCC and (4.17) holds. Then from (4.11) and Theorem 2, it must hold that

$$R \geq \limsup_{n\to\infty} \frac{1}{n} \log M_n = \overline{H}(\widehat{\Phi}^A) \geq \overline{H}(\widehat{\phi}^A), \tag{4.21}$$

which implies $R^*(\widehat{\phi}^{AB}) \geq \overline{H}(\widehat{\phi}^A)$. Therefore, we have (4.19). □

Chapter 5

Achievability of asymptotic LOCC convertibility

In this chapter, first we review the concept of random number generation of the information-spectrum method in classical information theory. Then we state a relation between random number generation and majorization. Next, we prove the achievability of asymptotic convertibility which is the direct part of the main results. Finally we state asymptotic LOCC convertibility between two arbitrary sequences of bipartite pure states by random unitary operations.

5.1 Random number generation

Let $\mathbb{X} = \{X^n\}_{n=1}^{\infty}$ be an arbitrary sequence of random variables, called a *general source*, taking values in arbitrary countable sets $\mathcal{X}^n$ $(n = 1, 2, \dots)$, and $P_{X^n}(x^n)$ $(x^n \in \mathcal{X}^n)$ be the probability function of X^n for each n. Then the *inf- and sup-spectral entropy rates* of $\mathbb{X}$ are defined by

$$\underline{H}(\varepsilon|\mathbb{X}) := \sup\left\{a \;\middle|\; \limsup_{n\to\infty} \Pr\left\{-\frac{1}{n}\log P_{X^n}(X^n) < a\right\} \le \varepsilon\right\}, \tag{5.1}$$

$$\overline{H}(\varepsilon|\mathbb{X}) := \inf\left\{a \;\middle|\; \liminf_{n\to\infty} \Pr\left\{-\frac{1}{n}\log P_{X^n}(X^n) < a\right\} \ge 1-\varepsilon\right\} \tag{5.2}$$

for $\varepsilon \in [0,1]$. Especially, for $\varepsilon = 0$ we write

$$\underline{H}(\mathbb{X}) := \underline{H}(0|\mathbb{X}), \tag{5.3}$$

$$\overline{H}(\mathbb{X}) := \overline{H}(0|\mathbb{X}). \tag{5.4}$$

In the following paragraph, we state the relation between the classical version (Eqs. (5.1) and (5.2)) and the quantum version (Eqs. (4.2) and (4.3)) of inf-/sup-spectral entropy rates.

Let $\widehat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ be a sequence of density operators and let

$$\rho_n = \sum_{x^n \in \mathcal{X}^n} p_n(x^n)\, |e_{x^n}\rangle\langle e_{x^n}| \quad (n = 1, 2, \dots) \tag{5.5}$$

be their eigenvalue decompositions. Then we have

$$\begin{aligned} \mathrm{Tr}\rho_n \left\{\rho_n > e^{-na} I_n\right\} &= \sum_{x^n \in \mathcal{X}^n :\, p_n(x^n) > e^{-na}} p_n(x^n) \\ &= \Pr\left\{ -\frac{1}{n} \log p_n(X^n) < a \right\}. \end{aligned} \tag{5.6}$$

where X^n is the random variable subject to $p_n(x^n)$ for each $n \in \mathbb{N}$. Thus, the quantum inf-/sup-spectral entropy rates of $\widehat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ are regarded as the classical ones with respect to the general source $\mathbb{X} = \{X^n\}_{n=1}^{\infty}$ corresponding to the eigenvalues of density operators.

Note that under the *i.i.d.* assumption where X^n is given by $(X_1, X_2, \cdots, X_n)$, the inf- and sup-spectral entropy rates reduce to Shannon entropy, namely,

$$\overline{H}(\mathbb{X}) = \underline{H}(\mathbb{X}) = H(X), \tag{5.7}$$

which follows from the law of large numbers, i.e.

$$\lim_{n\to\infty} \mathrm{Tr}\rho_n \left\{\rho_n > e^{-na} I_n\right\} = \lim_{n\to\infty} \Pr\left\{ -\frac{1}{n} \log p_n(X^n) < a \right\} = \begin{cases} 1, & if\ a > H(\mathbb{X}), \\ 0, & if\ \ a < H(\mathbb{X}). \end{cases} \tag{5.8}$$

Let Y and $\tilde{Y}$ be random variables taking values in a countable set $\mathcal{Y}$, and let $q(y)$ and $\tilde{q}(y)$ $(y \in \mathcal{Y})$ be the corresponding probability functions, respectively. Then the *variational distance* between Y and $\tilde{Y}$ is defined by

$$d(Y,\tilde{Y}) := \frac{1}{2}\sum_{y\in\mathcal{Y}} |q(y) - \tilde{q}(y)|. \tag{5.9}$$

Proposition 10 (Nagaoka [20, Theorem 2.1.1]). *Let $\mathbb{X} = \{X^n\}_{n=1}^{\infty}$ and $\mathbb{Y} = \{Y^n\}_{n=1}^{\infty}$ be arbitrary two general sources. If $\underline{H}(\mathbb{X}) > \overline{H}(\mathbb{Y})$, then there exists a sequence of maps $\varphi_n : \mathcal{X}^n \to \mathcal{Y}^n$ $(n = 1, 2, \dots)$ such that*

$$\lim_{n\to\infty} d(Y^n, \varphi_n(X^n)) = 0. \tag{5.10}$$

Random number generation and majorization are related as follows. We show a proof here for readers' convenience since we cannot find any proofs of this Lemma in the literature.

Lemma 6 (Kumagai-Hayashi [28, Section 3.2]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be finite sets. Given a map $\varphi : \mathcal{X} \to \mathcal{Y}$ and a probability function $p : x \in \mathcal{X} \mapsto p(x) \in [0,1]$ on $\mathcal{X}$, let*

$$\tilde{q}(y) = p(\varphi^{-1}(\{y\})) = \sum_{x\in\varphi^{-1}(\{y\})} p(x) \tag{5.11}$$

be the induced probability function on $\mathcal{Y}$. Then we have $p \prec \tilde{q}$.

Proof. For each $y \in \mathcal{Y}$, let $n(y) = |\varphi^{-1}(\{y\})|$ and $\varphi^{-1}(\{y\}) = \{x_{y,1}, x_{y,2}, \dots, x_{y,n(y)}\}$. Define real column vectors by

$$\alpha_y := \left(p(x_{y,1}), p(x_{y,2}), \dots, p(x_{y,n(y)})\right)^t, \tag{5.12}$$

$$\beta_y := (\tilde{q}(y), 0, \dots, 0)^t, \tag{5.13}$$

where $(\dots)^t$ denotes the transposition of the vector. Then it obviously holds that $\alpha_y \prec \beta_y$. From Proposition 3, we know that there exists a doubly stochastic matrix $\mathcal{D}_y$ such that

$\alpha_y = \mathcal{D}_y \beta_y$. Indeed, if we let

$$\mathcal{D}_y = \sum_{j=1}^{n(y)} \frac{p(x_{y,j})}{\tilde{q}(y)} U_{\sigma(1,j)} \tag{5.14}$$

then we have the relation $\alpha_y = \mathcal{D}_y \beta_y$, where $\sigma(1,j)$ is the transposition and $U_{\sigma(1,j)}$ is the $n(y)$ dimensional unitary representation, which transpose the 1st and j-th elements. From Proposition 4, we know that if $\mathcal{D}_y$ is a convex combination of permutation matrices, then it is doubly stochastic. Now let us introduce a notation for the direct sum of two vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, and the corresponding direct sum of matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$, by

$$u \oplus v = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{5.15}$$

Then we have $p \sim \bigoplus_{y \in \mathcal{Y}} \alpha_y$ and $\tilde{q} \sim \bigoplus_{y \in \mathcal{Y}} \beta_y =: \tilde{q}'$. Since $\mathcal{D} := \bigoplus_{y \in \mathcal{Y}} \mathcal{D}_y$ is a doubly stochastic matrix, we obtain

$$\begin{aligned} p \sim \bigoplus_{y \in \mathcal{Y}} \alpha_y &= \bigoplus_{y \in \mathcal{Y}} \mathcal{D}_y \beta_y \\ &= \left(\bigoplus_{y \in \mathcal{Y}} \mathcal{D}_y \right) \left(\bigoplus_{y \in \mathcal{Y}} \beta_y \right) \\ &= \mathcal{D} \tilde{q}' \prec \tilde{q}' \sim \tilde{q} \end{aligned} \tag{5.16}$$

as asserted, where the last equality means that $\mathcal{D}\tilde{q}'$ is majorized by $\tilde{q}'$, and $\tilde{q}' \sim \tilde{q}$ means that $\tilde{q}' \prec \tilde{q}$ and $\tilde{q} \prec \tilde{q}'$ hold at the same time (see Definition 2). □

5.2 Proof of the direct part

Let $|\psi_n\rangle^{AB}$ and $|\phi_n\rangle^{AB}$ $(n = 1, 2, \ldots)$ be the initial and target states, respectively. Let their Schmidt decompositions given as follows.

$$|\psi_n\rangle^{AB} = \sum_{x^n \in \mathcal{X}^n} \sqrt{p_n(x^n)}\, |e_{x^n}\rangle^A \otimes |e_{x^n}\rangle^B, \tag{5.17}$$

$$|\phi_n\rangle^{AB} = \sum_{y^n \in \mathcal{Y}^n} \sqrt{q_n(y^n)}\, |f_{y^n}\rangle^A \otimes |f_{y^n}\rangle^B. \tag{5.18}$$

Then their reduced density operators are given by

$$\psi_n^A = \mathrm{Tr}_B\left[\psi_n^{AB}\right] = \sum_{x^n \in \mathcal{X}^n} p_n(x^n)\, |e_{x^n}\rangle\langle e_{x^n}|, \tag{5.19}$$

$$\phi_n^A = \mathrm{Tr}_B\left[\phi_n^{AB}\right] = \sum_{y^n \in \mathcal{Y}^n} q_n(y^n)\, |f_{y^n}\rangle\langle f_{y^n}|. \tag{5.20}$$

From the Schmidt coefficients we can define random variables X^n and Y^n subject to probability functions $p_n(x^n)$ $(x^n \in \mathcal{X}^n)$ and $q_n(y^n)$ $(y^n \in \mathcal{Y}^n)$, and general sources $\mathbb{X} = \{X^n\}_{n=1}^{\infty}$ and $\mathbb{Y} = \{Y^n\}_{n=1}^{\infty}$ composed of them. For sequences of density operators $\widehat{\psi}^A = \{\psi_n^A\}_{n=1}^{\infty}$ and $\widehat{\phi}^A = \{\phi_n^A\}_{n=1}^{\infty}$, it is straightforward to verify that

$$\underline{H}(\mathbb{X}) = \underline{H}(\widehat{\psi}^A), \quad \overline{H}(\mathbb{Y}) = \overline{H}(\widehat{\phi}^A). \tag{5.21}$$

Suppose that $\underline{H}(\widehat{\psi}^A) > \overline{H}(\widehat{\phi}^A)$, or equivalently $\underline{H}(\mathbb{X}) > \overline{H}(\mathbb{Y})$. From Proposition 10, there exists a sequence of maps $\varphi_n : \mathcal{X}^n \to \mathcal{Y}^n$ $(n = 1, 2, \ldots)$ such that the variational distance between $\tilde{q}_n(y^n) = p(\varphi_n^{-1}(\{y^n\}))$ and $q_n(y^n)$ $(y^n \in \mathcal{Y})$ goes to zero asymptotically, i.e.,

$$\lim_{n\to\infty} d(Y^n, \tilde{Y}^n) = 0, \tag{5.22}$$

where $\tilde{Y}^n$ is a random variable subject to the probability function $\tilde{q}_n(y^n)$. Then from Lemma 6 we have $p_n \prec \tilde{q}_n$. Consider a state

$$|\phi'_n\rangle^{AB} := \sum_{y^n \in \mathcal{Y}^n} \sqrt{\tilde{q}_n(y^n)}\, |f_{y^n}\rangle^A \otimes |f_{y^n}\rangle^B. \tag{5.23}$$

Due to Proposition 5 (Nielsen *et al.* [9]), $|\psi_n\rangle^{AB}$ can be deterministically converted to $|\phi'\rangle_n^{AB}$ by LOCC for each n.

To complete the proof, we verify that the state $|\phi_n'\rangle^{AB}$ is asymptotically equal to the target state $|\phi_n\rangle^{AB}$. Let $F(\rho,\sigma) := \mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|$ be the fidelity between states ρ and σ. Noting that

$$\phi_n'^{A} = \mathrm{Tr}_B\left[\phi_n'^{AB}\right] = \sum_{y^n \in \mathcal{Y}^n} \tilde{q}_n(y^n)\,|f_{y^n}\rangle\langle f_{y^n}|\,, \tag{5.24}$$

we have

$$\begin{aligned} F(\phi_n'^{AB}, \phi_n^{AB}) &= |\langle \phi_n', \phi_n\rangle| \\ &= \sum_{y^n \in \mathcal{Y}^n} \sqrt{\tilde{q}_n(y^n) q_n(y^n)} \\ &= F(\phi_n'^{A}, \phi_n^{A}). \end{aligned} \tag{5.25}$$

Using (2.19) for pure states and from (5.25), we have

$$\begin{aligned} \|\phi_n'^{AB} - \phi_n^{AB}\|_1 &= 2\sqrt{1 - F(\phi_n'^{AB}, \phi_n^{AB})^2} \\ &= 2\sqrt{1 - F(\phi_n'^{A}, \phi_n^{A})^2}. \end{aligned} \tag{5.26}$$

Since (5.22) yields

$$\lim_{n\to\infty} \|\phi_n'^{A} - \phi_n^{A}\|_1 = 2 \lim_{n\to\infty} d(Y^n, \tilde{Y}^n) = 0, \tag{5.27}$$

we have

$$\lim_{n\to\infty} F(\phi_n'^{A}, \phi_n^{A}) = 1, \tag{5.28}$$

from the first inequality of (2.19). Combining (5.26) and (5.28) leads to

$$\lim_{n\to\infty} \|\phi_n'^{AB} - \phi_n^{AB}\|_1 = 0. \tag{5.29}$$

5.3 Existence of random unitary

As a byproduct of our approach, we address asymptotic convertibility of two arbitrary sequences of states $\widehat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ and $\widehat{\sigma} = \{\sigma_n\}_{n=1}^{\infty}$ by random unitary operations, which is a subclass of unital operations. The obtained result is applied to a study of quantum thermodynamics in [39].

Along the same line as the proof of the direct part, it can be shown that if two arbitrary sequences of states $\widehat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ and $\widehat{\sigma} = \{\sigma_n\}_{n=1}^{\infty}$ satisfy $\overline{H}(\widehat{\rho}) < \underline{H}(\widehat{\sigma})$, there exists a sequence of random unitary operations $\mathcal{R}_n$ $(n = 1, 2, \cdots)$ such that

$$\lim_{n\to\infty} \|\mathcal{R}_n(\rho_n) - \sigma_n\|_1 = 0. \tag{5.30}$$

To prove this, let

$$\rho_n = \sum_{y^n \in \mathcal{Y}^n} q_n(y^n) \,|f_{y^n}\rangle\langle f_{y^n}| \,,$$
$$\sigma_n = \sum_{x^n \in \mathcal{X}^n} p_n(x^n) \,|e_{x^n}\rangle\langle e_{x^n}|$$

be the eigenvalue decompositions of ρ_n and σ_n for each n. Equivalently to (5.28), the states

$$\rho'_n = \sum_{y^n \in \mathcal{Y}^n} \tilde{q}_n(y^n) \,|f_{y^n}\rangle\langle f_{y^n}| \;\; (n = 1, 2, \cdots) \tag{5.31}$$

satisfies

$$\lim_{n\to\infty} \|\rho_n - \rho'_n\|_1 = 0. \tag{5.32}$$

In addition, for each n, the condition $p_n \prec \tilde{q}_n$ implies the existence of a random unitary operation $\mathcal{R}_n$ such that $\mathcal{R}_n(\rho'_n) = \sigma_n$ (see Proposition 5). Hence, due to the monotonicity of the trace distance, we have

$$\lim_{n\to\infty} \|\mathcal{R}_n(\rho_n) - \sigma_n\|_1 \leq \lim_{n\to\infty} \|\rho_n - \rho'_n\|_1 = 0, \tag{5.33}$$

which completes the proof.

Chapter 6

Optimality of asymptotic LOCC convertibility

In this chapter, we prove Theorem 2 which is optimality of asymptotic LOCC convertibility. The general properties of the spectral divergence rates play an important role in the proofs. In Sections 6.1, we provide mathematical prerequisites for spectral divergence rates. The definitions and properties are given in Section 6.2. Then we prove Theorem 2 under the assumption that the final state $\mathcal{L}_n(\psi_n^{AB})$ is pure in Section 6.3. By a description of a general LOCC protocol (section 6.4), we complete the proof of Theorem 2.

6.1 Mathematical prerequisites

Let $A = \sum_k a_k E_k$ be the spectral decomposition of a Hermitian operator A. Then the positive and negative parts of Hermitian operator A are given by

$$A_+ := \sum_{k:\, a_k > 0} a_k E_k, \quad A_- := \sum_{k:\, a_k \leq 0} (-a_k) E_k,$$

respectively. Following the notations of [21, 23], we denote the corresponding projections as

$$\{A > 0\} := \sum_{k:\, a_k > 0} E_k, \quad \{A \leq 0\} := \sum_{k:\, a_k \leq 0} E_k. \tag{6.1}$$

We have $A_+ = A\{A > 0\}$ and $A_- = -A\{A \leq 0\}$ strightforward. Note that

$$A = A_+ - A_-, \quad |A| = A_+ + A_- \tag{6.2}$$

are the Jordan decomposition and the absolute value of the operator A, respectively.

The following lemma is essential in the information-spectrum method.

Lemma 7. *For any* $0 \leq T \leq I$*, we have*

$$\mathrm{Tr}A_+ = \mathrm{Tr}A\{A > 0\} \geq \mathrm{Tr}AT, \tag{6.3}$$

or equivalently,

$$\mathrm{Tr}A_+ = \max_{T:\, 0 \leq T \leq I} \mathrm{Tr}AT. \tag{6.4}$$

It is useful to note that the following relation with the trace norm follows from (6.2):

$$\mathrm{Tr}A_+ = \frac{1}{2}\{\mathrm{Tr}|A| + \mathrm{Tr}A\},\ \mathrm{Tr}A_- = \frac{1}{2}\{\mathrm{Tr}|A| - \mathrm{Tr}A\}.$$

Especially, if $\mathrm{Tr}A = 0$, we have

$$\mathrm{Tr}|A| = 2\,\mathrm{Tr}A_+ = 2\,\mathrm{Tr}A_- = 2 \max_{T:\, 0 \leq T \leq I} \mathrm{Tr}AT. \tag{6.5}$$

It should also be noted that from $\mathrm{Tr}(A - B)_+ = \mathrm{Tr}(A - B)\{A - B > 0\} \geq 0$, we have

$$\mathrm{Tr}A\{A - B > 0\} \geq \mathrm{Tr}B\{A - B > 0\} \tag{6.6}$$

and

$$\begin{aligned} \mathrm{Tr}(A - B)_+ &= \mathrm{Tr}(A - B)\{A - B > 0\} \\ &\leq \mathrm{Tr}A\{A - B > 0\}. \end{aligned} \tag{6.7}$$

The following lemma regarding the monotonicity of $\mathrm{Tr}A_+$ under trace preserving maps was pointed out by Bowen-Datta [11], based on the additional assumption that the maps are completely positive. It should be noted that the condition of complete positivity is not need; see [40, p. 1620] for example.

Lemma 8. *Let A be a Hermitian operator. For any positive trace preserving (PTP) maps $\mathcal{F}$, we have* $\mathrm{Tr}A_+ \geq \mathrm{Tr}\mathcal{F}(A)_+$.

6.2 Definitions and properties of spectral divergence rates

Let $\widehat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ be an arbitrary sequence of density operators and $\widehat{\sigma} = \{\sigma_n\}_{n=1}^{\infty}$ be an arbitrary sequence of nonnegative Hermitian operators. For each $\varepsilon \in [0,1]$, the spectral divergence rates [23] between the sequences $\widehat{\rho}$ and $\widehat{\sigma}$ are defined by

$$\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) := \sup\left\{ a \,\middle|\, \liminf_{n\to\infty} \mathrm{Tr}\rho_n\{\rho_n - e^{na}\sigma_n > 0\} \geq 1-\varepsilon \right\}, \tag{6.8}$$

$$\overline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) := \inf\left\{ a \,\middle|\, \limsup_{n\to\infty} \mathrm{Tr}\rho_n\{\rho_n - e^{na}\sigma_n > 0\} \leq \varepsilon \right\}, \tag{6.9}$$

where $\{A > 0\}$ denotes the spectral projection corresponding to the positive part of a Hermitian operator A (see (6.1)). It is straightforward to verify that the spectral entropy rates defined by Eqs. (4.2) and (4.3) can be rewritten as

$$\underline{H}(\varepsilon|\widehat{\rho}) = -\overline{D}(\varepsilon|\widehat{\rho}||\widehat{I}), \tag{6.10}$$

$$\overline{H}(\varepsilon|\widehat{\rho}) = -\underline{D}(\varepsilon|\widehat{\rho}||\widehat{I}), \tag{6.11}$$

where $\widehat{I} = \{I_n\}_{n=1}^{\infty}$ is the sequence of identity operators.

6.2.1 Monotonicity under PTP maps

It was proved by Bowen-Datta [35, Proposition 4] that the spectral divergence rates between two general sequences of states are monotonically nonincreasing under completely positive and trace preserving (CPTP) maps for $\varepsilon = 0$. In the following, we generalize the monotonicity to an arbitrary $\varepsilon \in [0,1]$ and positive trace preserving (PTP) maps.

Proposition 11. *For any sequences of states $\widehat{\rho}$, $\widehat{\sigma}$, and for any sequence of PTP maps $\widehat{\mathcal{F}} = \{\mathcal{F}_n\}_{n=1}^{\infty}$, we have*

$$\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq \underline{D}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})||\widehat{\mathcal{F}}(\widehat{\sigma})\big), \tag{6.12}$$

$$\overline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq \overline{D}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})||\widehat{\mathcal{F}}(\widehat{\sigma})\big), \tag{6.13}$$

where we defined $\widehat{\mathcal{F}}(\widehat{\rho}) = \{\mathcal{F}_n(\rho_n)\}_{n=1}^{\infty}$ and $\widehat{\mathcal{F}}(\widehat{\sigma}) = \{\mathcal{F}_n(\sigma_n)\}_{n=1}^{\infty}$.

To prove the above proposition, we use an alternative expression for the spectral divergence rates introduced by Bowen-Datta [35]. For each $\varepsilon \in [0,1]$, define

$$\underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) := \sup\left\{a \,\middle|\, \liminf_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ \geq 1-\varepsilon\right\}, \tag{6.14}$$

$$\overline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) := \inf\left\{a \,\middle|\, \limsup_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ \leq \varepsilon\right\}. \tag{6.15}$$

It can be shown that the spectral divergence rates defined by the above expressions coincide with those defined by Eqs. (6.8) and (6.9), that is:

Lemma 9. *For any $\varepsilon \in [0,1]$, we have*

$$\underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) = \underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}), \tag{6.16}$$

$$\overline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) = \overline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}). \tag{6.17}$$

Eqs. (6.16) and (6.17) were proved in [35] for $\varepsilon = 0$. A simple proof for the case of an arbitrary $\varepsilon \in [0,1]$ is provided as follows.

Proof. Recall that (6.7) gives

$$\begin{aligned}\mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ &= \mathrm{Tr}(\rho_n - e^{na}\sigma_n)\{\rho_n - e^{na}\sigma_n > 0\} \\ &\leq \mathrm{Tr}\rho_n\{\rho_n - e^{na}\sigma_n > 0\}.\end{aligned} \tag{6.18}$$

Let $\gamma > 0$ be arbitrary and $a = \underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) - \gamma$. From the definition of $\underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma})$, we have

$$\liminf_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ \geq 1-\varepsilon.$$

Thus, taking $\liminf_{n\to\infty}$ in the both sides of (6.18), we have

$$\liminf_{n\to\infty} \mathrm{Tr}\rho_n\{\rho_n - e^{na}\sigma_n > 0\} \geq \liminf_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ \geq 1-\varepsilon, \tag{6.19}$$

which implies

$$\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq a = \underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) - \gamma. \tag{6.20}$$

Since $\gamma > 0$ can be arbitrary, we have

$$\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq \underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}). \tag{6.21}$$

We show the converse inequality. For any real number a and b, (6.3) yields

$$\begin{aligned}\mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ &\geq \mathrm{Tr}(\rho_n - e^{na}\sigma_n)\{\rho_n - e^{nb}\sigma_n > 0\} \\ &= \mathrm{Tr}\rho_n\{\rho_n - e^{nb}\sigma_n > 0\} - e^{na}\mathrm{Tr}\sigma_n\{\rho_n - e^{nb}\sigma_n > 0\} \\ &\geq \mathrm{Tr}\rho_n\{\rho_n - e^{nb}\sigma_n > 0\} - e^{na}e^{-nb}\mathrm{Tr}\rho_n\{\rho_n - e^{nb}\sigma_n > 0\} \\ &\geq \mathrm{Tr}\rho_n\{\rho_n - e^{nb}\sigma_n > 0\} - e^{na}e^{-nb},\end{aligned} \tag{6.22}$$

where the fourth line follows from (6.6). Letting $a = \underline{D}(\widehat{\rho}||\widehat{\sigma}) - 2\gamma$ and $b = \underline{D}(\widehat{\rho}||\widehat{\sigma}) - \gamma$ ($\gamma > 0$), we have

$$\begin{aligned}\liminf_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ &\geq \liminf_{n\to\infty} \left[\mathrm{Tr}\rho_n\{\rho_n - e^{nb}\sigma_n > 0\} - e^{-n\gamma}\right] \\ &= \liminf_{n\to\infty} \mathrm{Tr}\rho_n\{\rho_n - e^{nb}\sigma_n > 0\} \\ &\geq 1-\varepsilon,\end{aligned} \tag{A6}$$

which implies

$$\underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq a = \underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) - 2\gamma. \tag{6.23}$$

Since $\gamma > 0$ can be arbitrary, we have

$$\underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq \underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}). \tag{6.24}$$

Thus we have (6.16). In the same way, we have (6.17). □

In the following, we prove Proposition 11.

Proof of Proposition 11: Due to Lemma 9, it is sufficient to prove that Eqs. (6.14) and (6.15) are monotonically nonincreasing under PTP maps, i.e., that we have

$$\underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq \underline{C}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})||\widehat{\mathcal{F}}(\widehat{\sigma})\big), \tag{6.25}$$

$$\overline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq \overline{C}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})||\widehat{\mathcal{F}}(\widehat{\sigma})\big) \tag{6.26}$$

for any sequences of states $\widehat{\rho}$, $\widehat{\sigma}$ and for any sequence of PTP maps $\widehat{\mathcal{F}}$.

For any $\gamma > 0$, let $a = \underline{C}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})||\widehat{\mathcal{F}}(\widehat{\sigma})\big) - \gamma$. From the definition in (6.14) and Lemma 8, we have

$$1 - \varepsilon \leq \liminf_{n\to\infty} \mathrm{Tr}\big(\mathcal{F}_n(\rho_n) - e^{na}\mathcal{F}_n(\sigma_n)\big)_+ \leq \liminf_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+. \tag{6.27}$$

Thus we obtain $\underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq a = \underline{C}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})||\widehat{\mathcal{F}}(\widehat{\sigma})\big) - \gamma$ for any $\gamma > 0$, which implies (6.25).

Similarly, let $a = \overline{C}(\widehat{\rho}||\widehat{\sigma}) + \gamma$. From the definition in (6.15) and Lemma 8, we have

$$\limsup_{n\to\infty} \mathrm{Tr}\big(\mathcal{F}_n(\rho_n) - e^{na}\mathcal{F}_n(\sigma_n)\big)_+ \leq \limsup_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ \leq \varepsilon. \tag{6.28}$$

Hence we have $\overline{C}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})||\widehat{\mathcal{F}}(\widehat{\sigma})\big) \leq a = \overline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) + \gamma$ for any $\gamma > 0$, which leads to (6.26).

□

The monotonicity of the spectral entropy rates immediately follows from Proposition 11 and Eqs. (6.10) and (6.11):

Corollary 1. *For any sequence of unital TP maps* $\widehat{\mathcal{F}} = \{\mathcal{F}_n\}_{n=1}^{\infty}$ *and for any* $\varepsilon \in [0,1]$, *we have*

$$\overline{H}(\varepsilon|\widehat{\rho}) \le \overline{H}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})\big), \tag{6.29}$$

$$\underline{H}(\varepsilon|\widehat{\rho}) \le \underline{H}\big(\varepsilon|\widehat{\mathcal{F}}(\widehat{\rho})\big). \tag{6.30}$$

6.2.2 Continuity

The spectral divergence rates are "continuous" with respect to the sequences of density operators in the first argument, that is, the spectral divergence rates of two sequences coincide if the sequences are asymptotically equal.

Lemma 10. *Let* $\widehat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ *and* $\widehat{\rho}' = \{\rho'_n\}_{n=1}^{\infty}$ *be sequences of density operators. If*

$$\lim_{n\to\infty} ||\rho_n - \rho'_n||_1 = 0, \tag{6.31}$$

then

$$\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \ = \ \underline{D}(\varepsilon|\widehat{\rho}'||\widehat{\sigma}), \tag{6.32}$$

$$\overline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \ = \ \overline{D}(\varepsilon|\widehat{\rho}'||\widehat{\sigma}) \tag{6.33}$$

hold for any $\varepsilon \in [0,1]$ *and any sequence* $\widehat{\sigma} = \{\sigma_n\}_{n=1}^{\infty}$ *of nonnegative Hermitian operators.*

Proof. From (6.5), we have

$$\begin{aligned} \|\rho_n - \rho'_n\|_1 &\ge 2\mathrm{Tr}(\rho_n - \rho'_n)\{\rho_n - e^{na}\sigma_n > 0\} \\ &= 2\mathrm{Tr}(\rho_n - e^{na}\sigma_n)\{\rho_n - e^{na}\sigma_n > 0\} \\ &\quad - 2\mathrm{Tr}(\rho'_n - e^{na}\sigma_n)\{\rho_n - e^{na}\sigma_n > 0\} \\ &\ge 2\mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ - 2\mathrm{Tr}(\rho'_n - e^{na}\sigma_n)_+, \end{aligned} \tag{6.34}$$

where the last inequality follows from (6.3). Hence

$$\mathrm{Tr}(\rho'_n - e^{na}\sigma_n)_+ + \frac{1}{2}||\rho_n - \rho'_n||_1 \ge \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+. \tag{6.35}$$

For any $\gamma > 0$, let $a = \underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) - \gamma$. From $\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) = \underline{C}(\varepsilon|\widehat{\rho}||\widehat{\sigma})$, we have

$$\liminf_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ \geq 1 - \varepsilon. \tag{6.36}$$

Thus taking the limit infimum of (6.35) gives

$$\begin{aligned}\liminf_{n\to\infty} \mathrm{Tr}(\rho'_n - e^{na}\sigma_n)_+ &\geq \liminf_{n\to\infty} \mathrm{Tr}(\rho_n - e^{na}\sigma_n)_+ \\ &\geq 1 - \varepsilon,\end{aligned}$$

which implies $\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) - \gamma = a \leq \underline{C}(\varepsilon|\widehat{\rho}'||\widehat{\sigma}) = \underline{D}(\varepsilon|\widehat{\rho}'||\widehat{\sigma})$. Since $\gamma > 0$ can be arbitrary, we have $\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \leq \underline{D}(\varepsilon|\widehat{\rho}'||\widehat{\sigma})$. Interchanging the role of $\widehat{\rho}$ and $\widehat{\rho}'$, we have the converse inequality $\underline{D}(\varepsilon|\widehat{\rho}||\widehat{\sigma}) \geq \underline{D}(\varepsilon|\widehat{\rho}'||\widehat{\sigma})$. Thus we obtain (6.32). (6.33) is obtained along the same line. □

The following corollary immediately follows from Eqs. (6.10) and (6.11).

Corollary 2. *Let $\widehat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ and $\widehat{\rho}' = \{\rho'_n\}_{n=1}^{\infty}$ be sequences of density operators. If*

$$\lim_{n\to\infty} \|\rho_n - \rho'_n\|_1 = 0, \tag{6.37}$$

then

$$\underline{H}(\varepsilon|\widehat{\rho}) = \underline{H}(\varepsilon|\widehat{\rho}'), \tag{6.38}$$

$$\overline{H}(\varepsilon|\widehat{\rho}) = \overline{H}(\varepsilon|\widehat{\rho}') \tag{6.39}$$

hold for any $\varepsilon \in [0, 1]$.

6.3 Proof of the converse part for pure final states

Suppose that $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^{\infty}$ can be asymptotically converted into $\widehat{\phi}^{AB} = \{\phi_n^{AB}\}_{n=1}^{\infty}$ by LOCC. By Definition 5, there exists a sequence of LOCC $\widehat{L}_n = \{L_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \|L_n(\psi_n^{AB}) - \phi_n^{AB}\|_1 = 0. \tag{6.40}$$

Let $\phi_n'^{AB} := L_n(\psi_n^{AB})$ be final states, which we assume to be pure states, and recall that the reduced density operators are written as $\phi_n^A = \mathrm{Tr}_B[\phi_n^{AB}]$. Then by the monotonicity of trace distance, we have

$$\lim_{n\to\infty} \|\phi_n'^A - \phi_n^A\|_1 \le \lim_{n\to\infty} \|\phi_n'^{AB} - \phi_n^{AB}\|_1 = 0, \tag{6.41}$$

which leads to

$$\underline{H}(\varepsilon|\widehat{\phi'}^A) = \underline{H}(\varepsilon|\widehat{\phi}^A), \tag{6.42}$$

$$\overline{H}(\varepsilon|\widehat{\phi'}^A) = \overline{H}(\varepsilon|\widehat{\phi}^A), \tag{6.43}$$

due to the continuity (Corollary 2). From Nielsen's theorem [41] (see also Remark 5.3 and [9, proof of Theorem 12.15]), there exists a unital CPTP map that transforms $\phi_n'^A$ to ψ_n^A for each n. Applying the monotonicity of spectral inf-/sup-entropy rates (Corollary 1), we have

$$\underline{H}(\varepsilon|\widehat{\phi'}^A) \le \underline{H}(\varepsilon|\widehat{\psi}^A), \tag{6.44}$$

$$\overline{H}(\varepsilon|\widehat{\phi'}^A) \le \overline{H}(\varepsilon|\widehat{\psi}^A). \tag{6.45}$$

Combining the above relations yields

$$\underline{H}(\varepsilon|\widehat{\phi}^A) \le \underline{H}(\varepsilon|\widehat{\psi}^A), \tag{6.46}$$

$$\overline{H}(\varepsilon|\widehat{\phi}^A) \le \overline{H}(\varepsilon|\widehat{\psi}^A) \tag{6.47}$$

for any $\varepsilon \in [0, 1]$.

6.4 Description of a general LOCC protocol

Note that the final states need not always be pure states in general even if they are close to the target states. To address the cases where the final states can be mixed states, we've introduced a method to describe a multi-round LOCC protocol by two distant parties in a "purified" picture, which simplifies an analysis of LOCC protocols.

Let us consider a situation where A and B engage in a multi-round LOCC protocol. Without loss of generality, we assume that an LOCC protocol starts with A's measurement and end up with A's operation on his system. Due to the Naimark extension theorem ([42], see also Theorem 4.5 in [36]), such a protocol (Figure 3.2) can in general be described as follows:

1. A and B recursively apply 1∼6 for $\gamma = 1, \cdots, \Gamma$, where $\Gamma \in \mathbb{N}$ is the number of rounds of the protocol.
2. A performs an isometry operation $V_\gamma : A \to AE_{A,\gamma}^1 E_{A,\gamma}^2$.
3. A performs a projective measurement on $E_{A,\gamma}^1$, and obtains an outcome.
4. A communicates a classical message to B.
5. B performs an isometry operation $W_\gamma : B \to BE_{B,\gamma}^1 E_{B,\gamma}^2$.
6. B performs a projective measurement on $E_{B,\gamma}^1$, and obtains an outcome.
7. B communicates a classical message to A.
8. A performs an isometry operation $V^* : A \to AE_A^*$, where E_A^* is an ancillary system.
9. A and B discard ancillary systems $E_{A,1}^2 \cdots E_{A,\Gamma}^2 E_A^*$ and $E_{B,1}^2 \cdots E_{B,\Gamma}^2$, respectively.

An advantage of introducing such a description is that, if the initial state is pure, the whole state remains pure until the last step in which A and B discard ancillary systems $E_{A,1}^2 \cdots E_{A,\Gamma}^2 E_A^*$ and $E_{B,1}^2 \cdots E_{B,\Gamma}^2$. (Step 9 above).

6.5 Proof of converse part for mixed final states

Theorem 2 for mixed final states is proved as follows. Suppose $\widehat{\psi}^{AB} = \{\psi_n^{AB}\}_{n=1}^\infty$ can be asymptotically converted into $\widehat{\phi}^{AB} = \{\phi_n^{AB}\}_{n=1}^\infty$ by LOCC. By Definition 5, there exists a sequence of LOCC $\mathcal{L}_n$ ($n = 1, 2, \cdots$) such that

$$\lim_{n\to\infty} \|\mathcal{L}_n(\psi_n^{AB}) - \phi_n^{AB}\|_1 = 0. \tag{6.48}$$

Due to (2.19), the above equality implies

$$\lim_{n\to\infty} F(\mathcal{L}_n(\psi_n^{AB}), \phi_n^{AB}) = 1. \tag{6.49}$$

Let $\mathcal{L}'_n$ be an LOCC protocol corresponding to Step 1 to 8 of $\mathcal{L}_n$ (Subsection 6.4) for each n, and denote ancillary systems $E_{A,1}^2 \cdots E_{A,\Gamma}^2 E_A^*$ and $E_{B,1}^2 \cdots E_{B,\Gamma}^2$ simply by E_A and E_B, respectively. Define a pure state $\phi_n'^{ABE_AE_B}$ by

$$\phi_n'^{ABE_AE_B} = \mathcal{L}'_n(\psi_n^{AB}). \tag{6.50}$$

The final state of the protocol is then given by

$$\mathcal{L}_n(\psi_n^{AB}) = \mathrm{Tr}_{E_AE_B}[\mathcal{L}'_n(\psi_n^{AB})] = \mathrm{Tr}_{E_AE_B}[\phi_n'^{ABE_AE_B}]. \tag{6.51}$$

Due to Eqs. (6.49), (6.51) and Uhlmann's theorem [43], there exists a sequence of pure states $\widehat{\xi}^{E_AE_B} = \{\xi_n^{E_AE_B}\}_{n=1}^{\infty}$ such that $\phi_n'^{ABE_AE_B}$ is asymptotically equal to $\phi_n^{AB} \otimes \xi_n^{E_AE_B}$, i.e.,

$$\lim_{n\to\infty} F(\phi_n'^{ABE_AE_B}, \phi_n^{AB} \otimes \xi_n^{E_AE_B}) = 1, \tag{6.52}$$

which implies

$$\lim_{n\to\infty} \|\phi_n'^{ABE_AE_B} - \phi_n^{AB} \otimes \xi_n^{E_AE_B}\|_1 = 0 \tag{6.53}$$

from (2.19). Thus, there exists a sequence of LOCC $\mathcal{L}'_n$ such that

$$\lim_{n\to\infty} \|\mathcal{L}'_n(\psi_n^{AB}) - \phi_n^{AB} \otimes \xi_n^{E_AE_B}\|_1 = 0. \tag{6.54}$$

Since the final state $\phi_n'^{ABE_AE_B} = \mathcal{L}'_n(\psi_n^{AB})$ are pure states, and applying the Theorem 2 for pure final states (see Section 6.3), we have

$$\underline{H}(\varepsilon|\widehat{\phi}^A \otimes \widehat{\xi}^{E_A}) \le \underline{H}(\varepsilon|\widehat{\psi}^A), \tag{6.55}$$

$$\overline{H}(\varepsilon|\widehat{\phi}^A \otimes \widehat{\xi}^{E_A}) \le \overline{H}(\varepsilon|\widehat{\psi}^A). \tag{6.56}$$

To prove inequalities Eqs. (6.67) and (6.68), we first prove similar relations for classical general sources.

Lemma 11. *Let* $(\mathbb{X}, \mathbb{Y}) = \{(X^n, Y^n)\}_{n=1}^{\infty}$ *be an arbitrary sequence of a pair of random variables, taking values in* $\mathcal{X}^n \times \mathcal{Y}^n$ $(n = 1, 2, \dots)$ *according to a joint distribution* $P_{X^nY^n}$ *for each* n*. For any* $\varepsilon \in [0, 1]$*, we have*

$$\underline{H}(\varepsilon|\mathbb{XY}) \geq \underline{H}(\varepsilon|\mathbb{X}), \tag{6.57}$$

$$\overline{H}(\varepsilon|\mathbb{XY}) \geq \overline{H}(\varepsilon|\mathbb{X}). \tag{6.58}$$

Proof. Since we have

$$\begin{aligned} -\log P_{X^nY^n}(x^n, y^n) \quad &= -\log P_{X^n}(x^n) - \log P_{X^n|Y^n}(y^n|x^n) \\ &\geq -\log P_{X^n}(x^n) \end{aligned} \tag{6.59}$$

for any $x^n \in \mathcal{X}^n$ and $y^n \in \mathcal{Y}^n$, we obtain

$$\{(x^n, y^n)| - \log P_{X^nY^n}(x^n, y^n) < a\} \quad \subseteq \{(x^n, y^n)| - \log P_{X^n}(x^n) < a\} \tag{6.60}$$

for any real number a. Consequently we have

$$\Pr\left\{-\frac{1}{n}\log P_{X^nY^n}(X^n, Y^n) < a\right\} \leq \Pr\left\{-\frac{1}{n}\log P_{X^n}(X^n) < a\right\}, \tag{6.61}$$

and hence

$$\left\{a \left| \lim_{n\to\infty} \Pr\left\{-\frac{1}{n}\log P_{X^nY^n}(X^n, Y^n) < a\right\} \leq \varepsilon \right.\right\} \supseteq \left\{a \left| \lim_{n\to\infty} \Pr\left\{-\frac{1}{n}\log P_{X^n}(X^n) < a\right\} \leq \varepsilon \right.\right\}.$$

Therefore, from the definition (5.1) we obtain (6.57). The inequality (6.58) is obtained along the same line. □

The following lemma is obtained as a corollary of Lemma 11, and immediately leads to Eqs. (6.67) and (6.68). We remark that Lemma 12 below was proved by Bowen-Datta for the case of $\varepsilon = 0$ [35, Corollary 7].

Lemma 12. *For arbitrary sequences* $\widehat{\rho}^A = \{\rho_n^A\}_{n=1}^{\infty}$ *and* $\widehat{\sigma}^B = \{\sigma_n^B\}_{n=1}^{\infty}$*, and for any* $\varepsilon \in [0,1]$*, we have*

$$\underline{H}(\varepsilon|\widehat{\rho}^A \otimes \widehat{\sigma}^B) \geq \underline{H}(\varepsilon|\widehat{\rho}^A), \tag{6.62}$$

$$\overline{H}(\varepsilon|\widehat{\rho}^A \otimes \widehat{\sigma}^B) \geq \overline{H}(\varepsilon|\widehat{\rho}^A). \tag{6.63}$$

Proof. For density operators ρ_n and σ_n ($n \in \mathbb{N}$), let

$$\rho_n = \sum_{x^n \in \mathcal{X}^n} p_n(x^n) \left|e_{x^n}\right\rangle\!\left\langle e_{x^n}\right|,$$

$$\sigma_n = \sum_{y^n \in \mathcal{Y}^n} q_n(y^n) \left|f_{y^n}\right\rangle\!\left\langle f_{y^n}\right|,$$

be their eigenvalue decompositions, where $p_n(x^n)$ and $q_n(y^n)$ are the eigenvalues of ρ_n and σ_n corresponding to eigenvectors $|e_{x^n}\rangle$ and $|f_{y^n}\rangle$, respectively. Here, $\mathcal{X}^n$ and $\mathcal{Y}^n$ are appropriate finite sets indicating eigenvalues. Let X^n and Y^n be random variables that takes values in $\mathcal{X}^n$ and $\mathcal{Y}^n$, respectively, according to a joint distribution $P_{X^nY^n}$ defined by

$$P_{X^nY^n}(x^n, y^n) = p_n(x^n) q_n(y^n) \tag{6.64}$$

for each n. Consider general sources $(\mathbb{X}, \mathbb{Y}) = \{(X^n, Y^n)\}_{n=1}^{\infty}$ composed of them. It is straightforward to verify that for any $\varepsilon \in [0,1]$ we have

$$\overline{H}(\varepsilon|\widehat{\rho}) = \overline{H}(\varepsilon|\mathbb{X}),\ \underline{H}(\varepsilon|\widehat{\rho}) = \underline{H}(\varepsilon|\mathbb{X}) \tag{6.65}$$

and

$$\overline{H}(\varepsilon|\widehat{\rho} \otimes \widehat{\sigma}) = \overline{H}(\varepsilon|\mathbb{X}\mathbb{Y}),\ \underline{H}(\varepsilon|\widehat{\rho} \otimes \widehat{\sigma}) = \underline{H}(\varepsilon|\mathbb{X}\mathbb{Y}). \tag{6.66}$$

Hence Eqs. (6.62) and (6.63) follow from Eqs. (6.57) and (6.58), respectively. □

By Lemma 12, we also have

$$\underline{H}(\varepsilon|\widehat{\phi}^A) \leq \underline{H}(\varepsilon|\widehat{\phi}^A \otimes \widehat{\xi}^{E_A}), \tag{6.67}$$

$$\overline{H}(\varepsilon|\widehat{\phi}^A) \leq \overline{H}(\varepsilon|\widehat{\phi}^A \otimes \widehat{\xi}^{E_A}). \tag{6.68}$$

Combining all the inequalities above finishes the proof.

Chapter 7

Conclusion

In this thesis, we applied an information-spectrum approach to analyze asymptotic LOCC convertibility between two arbitrary sequences of bipartite pure states. In the following paragraphs, we state the obtained results of this thesis and their applications.

(1) We obtained a general formula and unified form of asymptotic LOCC convertibility between two arbitrary sequences of bipartite pure entangled states by an information-spectrum approach (Theorem 1 and Theorem 2). Applying our results, we can provide simple proofs for known results on LOCC conversion of a sequence of maximally entangled states. (Section 4.4.1 and Section 4.4.2). The information-spectrum method has applications in analyzing physical states, such as Gibbs states [44] and finitely correlated states [45, 46, 47]. By a similar approach as [48] and this thesis, it is possible to analyze asymptotic LOCC convertibility between two arbitrary sequence of finitely correlated states.

(2) We obtained asymptotic LOCC convertibility between two arbitrary sequences of bipartite pure states by random unitary operations as a byproduct of our approach. These random unitary operations are a subclass of unital operations. Recently, the existence of unital operations which converts a particular state into another plays an key role in several studies of quantum thermodynamics [49, 50, 51, 52]. This byproduct has been applied to study quantum thermodynamics in [39].

(3) We obtained the general result that the spectral divergence rate of two arbitrary sequences of bipartite pure states are monotonically nonincreasing for an arbitrary $\varepsilon \in [0, 1]$ under positive trace preserving (PTP) maps (Section 6.2.1). We also showed the

continuity of spectral divergence rates with respect to states in the asymptotic sense. In this thesis, we applied the properties of spectral entropy rates, which follows from those of spectral divergence rates, to proof the converse part of our main results. The generalized properties of spectral divergence rates may become useful tools in information-spectrum methods for analyzing general source.

As written in Chapter 1, next study following this thesis would be adopting information-spectrum methods to analyze asymptotic LOCC convertibility between two arbitrary sequence including mixed entangled states [53]. The information-spectrum approach used in this thesis and obtained results would play important roles in many applications.

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