The number of arrows in the quiver of tilting modules over a path algebra of Dynkin type

著者（英）	Ryoichi KASE
journal or publication title	Tsukuba Journal of Mathematics
volume	37
number	1
page range	153-177
year	2013-07
URL	http://hdl.handle.net/2241/00154306

TSUKUBA J. MATH.
Vol. 37 No. 1 (2013), 153–177

THE NUMBER OF ARROWS IN THE QUIVER OF TILTING MODULES OVER A PATH ALGEBRA OF DYNKIN TYPE

By

Ryoichi KASE

Abstract. Happel and Unger defined a partial order on the set of basic tilting modules. The tilting quiver is the Hasse diagram of the poset of basic tilting modules. We determine the number of arrows in the tilting quiver over a path algebra of Dynkin type.

Contents

Introduction

In this paper we use the following notations. Let A be a finite dimensional algebra over an algebraically closed field k, and let mod-A be the category of finite dimensional right A-modules. For $M \in$ mod-A we denote by $\mathrm{pd}_A M$ the projective dimension of M, and by add M the full subcategory of direct sums of direct summands of M. Let $Q = (Q_0, Q_1)$ be a finite connected quiver without loops and cycles, and Q_0 (resp. Q_1) be the set of vertices (resp. arrows) of Q (we

2000 *Mathematics Subject Classification*: Primary 16G20; Secondary 16D80.
Key words and phrases: Tilting modules, representations of Dynkin quivers.
Received September 11, 2012.

use this notation for an arbitrary quiver). We denote by kQ the path algebra of Q over k, and by rep Q the category of finite dimensional representations of the quiver Q which is category equivalent to mod-kQ. We note that for any two paths,

$$w : x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} x_r, \quad w' : y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_s} y_s,$$

$$w \cdot w' = \begin{cases} x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_r} x_r = y_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_s} y_s & \text{if } x_r = y_0, \\ 0 & \text{if } x_r \neq y_0, \end{cases}$$

in kQ. For $M \in$ rep Q, denote by M_a the vector space of M associated to a vertex a, and denote by $M_{a \to b}$ the linear map $M_a \to M_b$ of M. For a vertex a of Q, let $\sigma_a Q$ be the quiver obtained from Q by reversing all arrows starting at a or ending at a. A module $T \in$ mod-A is called a tilting module provided the following three conditions are satisfied:

(a) pd $T < \infty$,

(b) $\mathrm{Ext}^i(T, T) = 0$ for all $i > 0$,

(c) there exists an exact sequence

$$0 \to A \to T_0 \to T_1 \to \cdots \to T_r \to 0 \quad (T_i \in \text{add } T)$$

in mod-A. In the hereditary case the tilting condition above is equivalent to the following:

(a) $\mathrm{Ext}^1(T, T) = 0$,

(b) the number of indecomposable direct summands of T (up to isomorphism) is equal to the number of simple modules.

In Section 1, following [8], [9], [10], [15], we define a partial order on the set $Tilt(A)$ of all basic tilting modules (up to isomorphism) over A and define the quiver of tilting modules $\vec{\mathscr{K}}(A)$. In Section 2, we explain results from [11]. In Section 3, we first show that the number of arrows of $\vec{\mathscr{K}}(kQ)$ is equal to the number of arrows of $\vec{\mathscr{K}}(kQ')$ if Q and Q' share the same underlying graph by applying the results from Section 2. Then we determine the number of arrows of $\vec{\mathscr{K}}(kQ)$ for any Dynkin quiver Q. Note that the underlying graph of $\vec{\mathscr{K}}(kQ)$ may be embedded into the exchange graph, or the cluster complex, of the corresponding cluster algebra of finite type:the tilting modules of kQ correspond to positive clusters [3] and [12]. The number of positive clusters when the orientation is alternating is given in [5, prop. 3.9]. However, according to experts, the number of edges of this subdiagram of positive clusters is not known in the cluster tilting theory. Note also that if we consider the similar problem for the exchange graph, it is not interesting, because the number of edges is $\frac{n}{2} \times$ {the number of clusters}, and the number of vertices is given in [5, Proposition 3.8]. The following is known ([5, Proposition 3.9]).

type	A_n	D_n	E_6	E_7	E_8
$\#\vec{\mathscr{K}}(kQ)_0$	$\frac{1}{n+1}\binom{2n}{n}$	$\frac{3n-4}{2n}\binom{2(n-1)}{n-1}$	418	2431	17342

The main result of this paper is as follows.

THEOREM 0.1. (1) *Let Q be a quiver without loops and cycles. Then $\#\vec{\mathscr{K}}(kQ)_1$ is independent of the orientation.*

(2) *$\#\vec{\mathscr{K}}(kQ)_1$ is given by the following table,*

type	A_n	D_n	E_6	E_7	E_8
$\#\vec{\mathscr{K}}(kQ)_1$	$\binom{2n-1}{n+1}$	$(3n-4)\binom{2(n-2)}{n-3}$	1140	8008	66976

Now we note that above number is equal to

$$\frac{n}{2}\left(1-\frac{1}{h-1}\right)\times\{\text{the number of positive clusters}\}\cdots(*),$$

where h is the Coxeter number. In this paper we provide separate proof about each type, but $(*)$ suggests that it should be possible to provide a uniform proof.

1. Preliminaries

In this section we define a partial order on tilting modules. First, for a tilting module T, we define the right perpendicular category

$$T^{\perp}=\{X\in \text{mod-}A\mid \operatorname{Ext}_A^{>0}(T,X)=0\}.$$

LEMMA 1.1 (c.f. [9, Lemma 2.1 (a)]). *For tilting modules T, T' the following conditions are equivalent,*

(1) $T^{\perp}\subset T'^{\perp}$;

(2) $T\in T'^{\perp}$.

Recall that $Tilt(A)$ is the set of basic tilting modules over A.

DEFINITION 1.2. We define a partial order on $Tilt(A)$ by

$$T \leq T' \overset{def}{\Longleftrightarrow} T^{\perp} \subset T'^{\perp} \Longleftrightarrow T \in T'^{\perp},$$

for $T, T' \in Tilt(A)$.

REMARK 1.3. By definition, A_A is the unique maximal element of $(Tilt(A), \leq)$. On the other hand, $(Tilt(A), \leq)$ does not always admit a minimal element (c.f. [8]).

Next we define the *tilting quiver* $\vec{\mathscr{K}}(A)$, and recall some of its properties. Let ind A be a category of indecomposable modules in mod-A.

DEFINITION 1.4. The *tilting quiver* $\vec{\mathscr{K}}(A) = (\vec{\mathscr{K}}(A)_0, \vec{\mathscr{K}}(A)_1)$ is defined as follows:

(1) $\vec{\mathscr{K}}(A)_0 = Tilt(A)$,

(2) $T' \to T$ in $\vec{\mathscr{K}}(A)$, for $T, T' \in Tilt(A)$, if $T' = M \oplus X$, $T = M \oplus Y$ with $X, Y \in \operatorname{ind} A$ and there is a non-split short exact sequence

$$0 \to X \to \tilde{M} \to Y \to 0$$

with $\tilde{M} \in \operatorname{add} M$.

THEOREM 1.5 (c.f. [8, Theorem 2.1]). *$\vec{\mathscr{K}}(A)$ is the Hasse-diagram of $(Tilt(A), \leq)$ (i.e. if $T \to T' \in \vec{\mathscr{K}}(A)_1$ and $T \geq T'' \geq T'$ then $T'' = T$ or $T'' = T'$).*

PROPOSITION 1.6 (c.f. [8, Corollary 2.2]). *If $\vec{\mathscr{K}}(A)$ has a finite component $\mathscr{C}$, then $\vec{\mathscr{K}}(A) = \mathscr{C}$.*

Let $Q = (Q_0, Q_1)$ be a quiver without loops and cycles and $A = kQ$. For $T \in Tilt(A)$, let

$$s(T) = \#\{T' \in Tilt(A) \mid T \to T' \text{ in } \vec{\mathscr{K}}(kQ)\}$$

$$e(T) = \#\{T' \in Tilt(A) \mid T' \to T \text{ in } \vec{\mathscr{K}}(kQ)\}$$

and define $\delta(T) = s(T) + e(T)$.

PROPOSITION 1.7 (c.f. [10, Proposition 3.2]). *$\delta(T) = n - \#\{a \in Q_0 \mid (\underline{\dim}\, T)_a = 1\}$, where $n = \#Q_0$.*

2. A Theorem of Ladkani

In this section, we review [11]. Let Q be a quiver without loops and cycles and let x be a source of Q. Let $Tilt(Q) := Tilt(kQ)$ and define

$$Tilt(Q)^x := \{T \in Tilt(Q) \mid S(x) \mid T\},$$

where $S(x)$ is the simple module associated to x.

DEFINITION 2.1. Let $(X, \leq_X)$, $(Y, \leq_Y)$ be posets and $f : X \to Y$ an order-preserving function. Then we define the partial-orders $\leq_+^f$, $\leq_-^f$ of $X \sqcup Y$ as follows:

$$a \leq_+^f b \Leftrightarrow \begin{cases} a \leq_X b & \text{if } a, b \in X, \\ a \leq_Y b & \text{if } a, b \in Y, \\ f(a) \leq_Y b & \text{if } a \in X \text{ and } b \in Y. \end{cases}$$

$$a \leq_-^f b \Leftrightarrow \begin{cases} a \leq_X b & \text{if } a, b \in X, \\ a \leq_Y b & \text{if } a, b \in Y, \\ a \leq_Y f(b) & \text{if } a \in Y \text{ and } b \in X. \end{cases}$$

LEMMA 2.2. *Define the functors*

$$j^{-1} : \text{rep } Q \to \text{rep}(Q \backslash \{x\})$$

and

$$j_* : \text{rep}(Q \backslash \{x\}) \to \text{rep } Q,$$

by

$$(j^{-1}M)_a = M_a, \quad (j^{-1}M)_{a \to b} = M_{a \to b}$$

and

$$(j_*N)_a = \begin{cases} N_a & (a \neq x) \\ \bigoplus_{x \to y} N(y) & (a = x) \end{cases}, \quad (j_*N)_{a \to b} = \begin{cases} N_{a \to b} & (a \neq x) \\ (j_*N)_x \xrightarrow{\text{projection}} N_b & (a = x) \end{cases}.$$

Then j^{-1} and j_* are exact and j_* is right adjoint to j^{-1}.

Denote by $\mathscr{D}^b(Q)$ the bounded derived category $\mathscr{D}^b(\text{rep } Q)$.

LEMMA 2.3. *The functors j^{-1} and j_* induce functors*

$$j^{-1} : \mathscr{D}^b(Q) \to \mathscr{D}^b(Q \backslash \{x\}), \quad j_* : \mathscr{D}^b(Q \backslash \{x\}) \to \mathscr{D}^b(Q)$$

with

$$\mathrm{Hom}_{\mathscr{D}^b(Q\setminus\{x\})}(j^{-1}M, N) \simeq \mathrm{Hom}_{\mathscr{D}^b(Q)}(M, j_*N),$$

for all $M \in \mathscr{D}^b(Q)$, $N \in \mathscr{D}^b(Q\setminus\{x\})$.

LEMMA 2.4. *The functors* j^{-1} *and* j_* *identify* $\mathrm{rep}(Q\setminus\{x\})$ *with the right perpendicular subcategory*

$$S(x)^{\perp} = \{M \in \mathrm{rep}\, Q \mid \mathrm{Ext}^i(S(x), M) = 0 \ \textit{for all} \ i \geq 0\}$$

of $\mathrm{rep}\, Q$.

LEMMA 2.5. *The functor* j_* *takes indecomposables of* $\mathrm{rep}(Q\setminus\{x\})$ *to indecomposables of* $\mathrm{rep}\, Q$.

PROPOSITION 2.6. *Let* T *be a tilting module in* $\mathrm{rep}\, Q$. *Then* $j^{-1}T$ *is a tilting module in* $\mathrm{rep}(Q\setminus\{x\})$.

For $M = \bigoplus_{i=1}^{m} N_i^{r_i}$ (where $N_i \in \mathrm{ind}\, Q$, $r_i > 0$), let $\mathrm{basic}(M) = \bigoplus_{i=1}^{m} N_i$.

COROLLARY 2.7. *The map* $\pi_x : T \mapsto \mathrm{basic}(j^{-1}T)$ *is an order-preserving function*

$$(Tilt(Q), \leq) \to (Tilt(Q\setminus\{x\}), \leq).$$

PROPOSITION 2.8. *Let* $T \in Tilt(Q\setminus\{x\})$. *Then* $S(x) \oplus j_*T \in Tilt(Q)$.

COROLLARY 2.9. *The map* $\iota_x : T \mapsto S(x) \oplus j_*T$ *is an order-preserving function*

$$(Tilt(Q\setminus\{x\}), \leq) \to (Tilt(Q), \leq).$$

PROPOSITION 2.10. *We have*

$$\pi_x \iota_x(T) = T,$$

for all $T \in Tilt(Q\setminus\{x\})$. *In addition,*

$$T \geq \iota_x \pi_x(T),$$

for all $T \in Tilt(Q)$, *with equality if and only if* $T \in Tilt(Q)^x$.

In particular, π_x and ι_x induce an isomorphism of posets between $Tilt(Q)^x$ and $Tilt(Q\backslash\{x\})$.

COROLLARY 2.11. *Let $X = Tilt(Q)\backslash Tilt(Q)^x$ and $Y = Tilt(Q)^x$. Define $f : X \to Y$ by $f = \iota_x \pi_x$. Then*

$$Tilt(Q) \simeq (X \sqcup Y, \leq_-^f).$$

Now let $Q' = \sigma_x Q$. Then x is a sink of Q' and, by arguing in the similar way, we obtain the dual results by replacing

$$(j^{-1}, j_*, \pi_x, \iota_x, X, Y, f, \leq_-^f)$$

with

$$(i^{-1}, i_!, \pi'_x, \iota'_x, X', Y', f', \leq_+^{f'}).$$

In particular we get

$$Tilt(Q')^x \simeq Tilt(Q\backslash\{x\}),$$

and

$$Tilt(Q') \simeq (X' \sqcup Y', \leq_+^{f'}),$$

where $X' = Tilt(Q')\backslash Tilt(Q')^x$ and $Y' = Tilt(Q')^x$.

THEOREM 2.12. *There exists an isomorphism of posets*

$$\rho : Tilt(Q)\backslash Tilt(Q)^x \to Tilt(Q')\backslash Tilt(Q')^x$$

such that the following diagram commutes.

$$\begin{array}{ccccccc} & & Tilt(Q)\backslash Tilt(Q)^x & \xrightarrow[\sim]{\rho_x} & Tilt(Q')\backslash Tilt(Q')^x & & \\ & \swarrow^{f} & & \searrow_{\pi_x} \quad \swarrow^{\pi'_x} & & \searrow^{f'} & \\ Tilt(Q)^x & & \xleftarrow[\sim]{\iota_x} & Tilt(Q\backslash\{x\}) & \xrightarrow[\sim]{\iota'_x} & & Tilt(Q')^x \end{array}$$

COROLLARY 2.13. $\#Tilt(Q) = \#Tilt(Q')$.

REMARK 2.14. In [11] the partial order on $Tilt(A)$ is defined by

$$T \geq T' \Leftrightarrow T^\perp \subset T'^\perp \quad \text{(opposite to our definition).}$$

3. Main Results

In this section we determine the number of arrows in $\vec{\mathscr{K}}(kQ)$ in the case Q is a Dynkin quiver. Let

$$\mathrm{Gen}(M) := \{N \in \text{mod-}A \mid M' \xrightarrow{\text{surjection}} N \text{ for some } M' \in \mathrm{add}\, M\}$$

$$\mathrm{Cogen}(M) := \{N \in \text{mod-}A \mid N \xrightarrow{\text{injection}} M' \text{ for some } M' \in \mathrm{add}\, M\}$$

LEMMA 3.1 (c.f. [4, Proposition 1.3]). *Let A be hereditary, $T = M \oplus Y \in Tilt(A)$ with $Y \in \mathrm{ind}\, A$. If $Y \in \mathrm{Gen}(M)$, then there exists a unique (up to isomorphism) indecomposable module X which is not isomorphic to Y s.t. $M \oplus X \in Tilt(A)$ and there exists an exact sequence*

$$0 \to X \to E \to Y \to 0$$

with $E \in \mathrm{add}\, M$.

Dually, if $Y \in \mathrm{Cogen}(M)$ then there exists a unique (up to isomorphism) indecomposable module X which is not isomorphic to Y s.t. $M \oplus X \in Tilt(A)$ and there exists an exact sequence

$$0 \to Y \to E \to X \to 0$$

with $E \in \mathrm{add}\, M$.

LEMMA 3.2. *Let Q be a quiver without loops and cycles. If x is a sink, then for all $T = M \oplus S(x) \in Tilt(Q)$, $S(x)$ is in $\mathrm{Cogen}(M)$. If x is a source, then for all $T = M \oplus S(x) \in Tilt(Q)$, $S(x)$ is in $\mathrm{Gen}(M)$.*

PROOF. For any $T \in Tilt(Q \backslash \{x\})$, we define $F(T) \in \text{mod-}kQ$ as follows,

$$F(T)_a = \begin{cases} T_a & \text{if } a \neq x, \\ \bigoplus_{y \to x} T_y & \text{if } a = x \text{ and } x \text{ is a sink,} \\ \bigoplus_{x \to y} T_y & \text{if } a = x \text{ and } x \text{ is a source.} \end{cases}$$

$$F(T)_{a \to b} = \begin{cases} T_{a \to b} & \text{if } a, b \neq x, \\ T_y \xrightarrow{\text{injection}} \bigoplus_{y' \to x} T_{y'} & \text{if } a = y \text{ with } y \to x \text{ and} \\ & \text{if } b = x \text{ and } x \text{ is a sink,} \\ \bigoplus_{x \to y'} T_{y'} \xrightarrow{\text{projection}} T_y & \text{if } b = y \text{ with } x \to y \text{ and} \\ & \text{if } a = x \text{ and } x \text{ is a source.} \end{cases}$$

Then, by Proposition 2.10, $T \mapsto F(T) \oplus S(x)$ induces a bijection

$$Tilt(Q \backslash \{x\}) \overset{1:1}{\leftrightarrow} Tilt(Q)^x.$$

Now if x is a sink then

$$S(x) \in \mathrm{Cogen}(M) \Leftrightarrow M_x \neq 0,$$

and if x is a source then

$$S(x) \in \mathrm{Gen}(M) \Leftrightarrow M_x \neq 0.$$

So this Lemma follows from the fact that if $T \in Tilt(Q)$ then $(\underline{\dim}\, T)_a \geq 1$, for all a. □

LEMMA 3.3. *If x is a sink then*

$$\{\alpha \in \vec{\mathscr{K}}(Q)_1 \mid s(\alpha) \in Tilt(Q)^x, t(\alpha) \in Tilt(Q) \backslash Tilt(Q)^x\} \overset{1:1}{\leftrightarrow} Tilt(Q)^x.$$

If x is a source then

$$\{\alpha \in \vec{\mathscr{K}}(Q)_1 \mid t(\alpha) \in Tilt(Q)^x, s(\alpha) \in Tilt(Q) \backslash Tilt(Q)^x\} \overset{1:1}{\leftrightarrow} Tilt(Q)^x.$$

Where, for $T \overset{\alpha}{\to} T'$, $s(\alpha) = T$ and $t(\alpha) = T'$.

PROOF. Suppose x is a sink, and let $T \in Tilt(Q)^x$. Then there exists a unique $T' \in Tilt(Q) \backslash Tilt(Q)^x$ s.t. $T \to T'$ in $\vec{\mathscr{K}}(Q)$ (by Lemma 3.1, 3.2).

On the other hand, let $T' \in Tilt(Q) \backslash Tilt(Q)^x$ and suppose that there exists $T_1, T_2 \in Tilt(Q)^x$ s.t. $T_1 \to T'$, $T_2 \to T'$, for $T' \in Tilt(Q) \backslash Tilt(Q)^x$, in $\vec{\mathscr{K}}(Q)$. Write $T_i = M \oplus S(x) \oplus Y_i$ with $Y_i \in \mathrm{ind}\, kQ$ $(i = 1, 2)$ then $Y_i | T'$; $\mathrm{Ext}(Y_i, Y_j) = 0$ $(i, j = 1, 2)$. Thus $\mathrm{Ext}(T_1 \oplus Y_2, T_1 \oplus Y_2) = 0$ and $Y_1 = Y_2$ follows. □

COROLLARY 3.4.

$$\#\vec{\mathscr{K}}(Q)_1 = \#\vec{\mathscr{K}}(\sigma_x Q)_1.$$

In particular, if Q is a Dynkin quiver then $\#\vec{\mathscr{K}}(Q)_1$ depends only on the underlying graph of Q.

PROOF. By Corollary 2.11 and Lemma 3.3 we get,

$$\begin{aligned}\#\vec{\mathscr{K}}(Q)_1 &= \#\vec{\mathscr{K}}(Q \backslash \{x\})_1 + \#\vec{\mathscr{K}}(Tilt(Q) \backslash Tilt(Q)^x)_1 + \#Tilt(Q)^x \\ &= \#\vec{\mathscr{K}}(\sigma_x Q)_1.\end{aligned}$$

□

3.1. case A. In this subsection we consider the quiver,

$$Q = \overset{1}{\circ} \to \overset{2}{\circ} \to \cdots \to \overset{n}{\circ}.$$

By Gabriel's Theorem, $\operatorname{ind} kQ = \{L(i,j) \mid 0 \le i < j \le n\}$ where

$$L(i,j) = \begin{cases} k & (i < a \le j), \\ 0 & \text{otherwise}, \end{cases} \quad L(i,j)_{a \to b} = \begin{cases} 1 & (i < a, b \le j), \\ 0 & \text{otherwise}. \end{cases}$$

And

$$\tau L(i,j) = \begin{cases} L(i+1, j+1) & (j < n), \\ 0 & (j = n), \end{cases}$$

where τ is a Auslander-Reiten translation.

DEFINITION 3.5. A pair of intervals $([i,j],[i',j'])$ is *compatible* if

$$[i,j] \cap [i',j'] = \emptyset \quad \text{or} \quad [i,j] \subset [i',j'] \quad \text{or} \quad [i',j'] \subset [i,j].$$

Applying Auslander-Reiten duality,

$$\mathrm{DExt}(M,N) \cong \mathrm{Hom}(N, \tau M) \quad (\mathrm{D} = \mathrm{Hom}_k(-,k)),$$

we get the following Lemma.

LEMMA 3.6. *We have*

$$\mathrm{Ext}(L(i,j), L(i',j')) = 0 = \mathrm{Ext}(L(i',j'), L(i,j))$$

if and only if $([i,j],[i',j'])$ *is compatible.*

PROOF. It is obvious that $\mathrm{Hom}(L(i,j), L(i',j')) \neq 0$ if and only if $i' \le i \le j' \le j$. So the lemma follows from this fact and the AR-duality. □

LEMMA 3.7. *For any* $T \in Tilt(Q)$, *we get*

$$\delta(T) = n - 1.$$

PROOF. Let $T \in Tilt(Q)$ then the projective-injective module $L(0,n)$ is a direct summand of T. From this fact, we get $\delta(T) < n$.

Denote by X the set of indecomposable direct summands of T not isomorphic to $L(0,n)$ and define

$$a := \begin{cases} max\{i \mid L(0,i) \in X\} & \text{if } L(0,i) \in X \text{ for some } i, \\ 0 & \text{otherwise}. \end{cases}$$

Then, by Lemma 3.6, we get

$$\mathrm{Ext}(T, L(a+1,n)) = 0 = \mathrm{Ext}(L(a+1,n), T).$$

By $\mathrm{Ext}=0$ condition, we can see $L(a+1,n)$ is a direct summand of T. In particular

$$(\underline{\dim}\ T)_i = 1 \Leftrightarrow i = a+1.$$

This Lemma follows from this fact and Proposition 1.7. □

Now it is easy to check the number of arrows in $\vec{\mathscr{K}}(Q)$, because it is equal to

$$\frac{1}{2}\sum_{T\in\mathrm{Tilt}(Q)} \delta(T)$$

COROLLARY 3.8. $\#\vec{\mathscr{K}}(Q)_1 = \frac{n-1}{2(n+1)}\binom{2n}{n} = \binom{2n-1}{n-2}$.

3.2. case *D*. Through this subsection, we consider the quiver

$$Q = Q_n = \overset{1}{\circ} \to \overset{2}{\circ} \to \cdots \to \overset{n-1}{\circ} \begin{array}{l} \nearrow \circ\ n^+ \\ \searrow \circ\ n^- \end{array}$$

Then

$$\begin{aligned}\mathrm{ind}\ kQ = \{L(a,b)\,|\,0 \le a < b \le n-1\} &\cup \{L^{\pm}(a,n)\,|\,0 \le a \le n-1\} \\ &\cup \{M(a,b)\,|\,0 \le a < b \le n-1\}\end{aligned}$$

where

$$L(a,b)_i = \begin{cases} k & \text{if } a < i \le b, \\ 0 & \text{otherwise,}\end{cases}$$

$$L(a,b)_{i\to j} = \begin{cases} 1 & \text{if } a < i < b, \\ 0 & \text{otherwise,}\end{cases}$$

$$L(a,n)^{\pm}_i = \begin{cases} k & \text{if } a < i \le n-1 \text{ or } i = n^{\pm}, \\ 0 & \text{otherwise,}\end{cases}$$

$$L(a,n)^{\pm}_{i\to j} = \begin{cases} 1 & \text{if } a < i < n-1 \text{ or } i = n-1,\ j = n^{\pm}, \\ 0 & \text{otherwise,}\end{cases}$$

$$M(a,b)_i = \begin{cases} k & \text{if } a < i \le b \text{ or } i = n^{\pm}, \\ k^2 & \text{if } b < i \le n-1, \\ 0 & \text{otherwise,}\end{cases}$$

$$M(a,b)_{i\to j}=\begin{cases}1 & \text{if } a<i<b,\\ \begin{pmatrix}1\\1\end{pmatrix} & \text{if } i=b,\\ (1,0) & \text{if } i=n-1, j=n^+,\\ (0,1) & \text{if } i=n-1,\ j=n^-,\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \text{if } b<i<n-1,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\tau L(a,b)=\begin{cases}L(a+1,b+1) & \text{if } b<n-1,\\ M(0,a+1) & \text{if } b=n-1,\end{cases}$$

$$\tau L^+(a,n)=\begin{cases}L^-(a+1,n) & \text{if } a<n-1,\\ 0 & \text{if } a=n-1,\end{cases}$$

$$\tau L^-(a,n)=\begin{cases}L^+(a+1,n) & \text{if } a<n-1,\\ 0 & \text{if } a=n-1,\end{cases}$$

$$\tau M(a,b)=\begin{cases}M(a+1,b+1) & \text{if } b<n-1,\\ 0 & \text{if } b=n-1.\end{cases}$$

LEMMA 3.9.

(1) $\mathrm{Ext}(L(a,b),L(a',b'))=0=\mathrm{Ext}(L(a',b'),L(a,b))$
$\Leftrightarrow ([a,b],[a',b'])$: compatible.

(2) $\mathrm{Ext}(L(a,b),L^{\pm}(a',n))=0=\mathrm{Ext}(L^{\pm}(a',n),L(a,b))$
$\Leftrightarrow ([a,b],[a',n])$: compatible.

(3) $\mathrm{Ext}(L(a,b),M(a',b'))=0=\mathrm{Ext}(M(a',b'),L(a,b))$
$\Leftrightarrow ([a,b],[a',n]),([a,b],[b',n])$: compatible.

(4) $\mathrm{Ext}(M(a,b),L^{\pm}(a',n))=0=\mathrm{Ext}(L^{\pm}(a',n),M(a,b))\Leftrightarrow a\le a'\le b$.

(5) $\mathrm{Ext}(L^{\pm}(a,n),L^{\pm}(a',n))=0=\mathrm{Ext}(L^{\pm}(a',n),L^{\pm}(a,n))$ for all a,a'.

(6) $\mathrm{Ext}(L^{+}(a,n),L^{-}(a',n))=0=\mathrm{Ext}(L^{-}(a',n),L^{+}(a,n))\Leftrightarrow a=a'$.

(7) $\mathrm{Ext}(M(a,b),M(a',b'))=0=\mathrm{Ext}(M(a',b'),M(a,b))$
$\Leftrightarrow [a,b]\subset[a',b']$ *or* $[a',b']\subset[a,b]$.

PROOF. (1) and (2) follow from a case A and (5), (6) are obvious. (3) (case $b<a'$) It is obvious that

$$\mathrm{Ext}(L(a,b),M(a',b'))=0=\mathrm{Ext}(M(a',b'),L(a,b)).$$

(case $a < a' \leq b < b'$) In this case we claim that

$$\mathrm{Hom}(M(a',b'),\tau L(a,b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a',b'),\tau L(a,b))$ where

$$f_i = \begin{cases} 1 & \text{if } a' < i \leq b+1, \\ 0 & \text{otherwise.} \end{cases}$$

(case $a < a' < b' \leq b < n-1$) In this case we claim that

$$\mathrm{Hom}(M(a',b'),\tau L(a,b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a',b'),\tau L(a,b))$ where

$$f_i = \begin{cases} 1 & \text{if } a' < i \leq b', \\ (0,1) & \text{if } b' < i \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(case $a < a' < b' \leq b = n-1$) In this case we also claim that

$$\mathrm{Hom}(M(a',b'),\tau L(a,b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a',b'),\tau L(a,n-1))$ where

$$f_i = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } a' < i \leq b', \\ 1 & \text{if } b' < i \leq n-1 \text{ or } i = n^{\pm}, \\ 0 & \text{otherwise.} \end{cases}$$

(case $a' \leq a < b < b' < n-1$) In this case we claim that

$$\mathrm{Hom}(M(a',b'),\tau L(a,b)) = 0 = \mathrm{Hom}(L(a,b),\tau M(a',b')).$$

Let $f = (f_i)_i \in \mathrm{Hom}(M(a',b'),\tau L(a,b))$. If $i \leq a+1$ or $b+1 < i \leq n-1$ or $i = n^{\pm}$ then $(\underline{\dim}\, \tau L(a,b))_i = 0$ and this implies $f_i = 0$. Note that

$$f_{a+2} = f_{a+3} = \cdots = f_{b+1}.$$

Now the commutative square for f_{a+1}, f_{a+2} shows $f_{a+2} = 0$. So

$$\mathrm{Hom}(M(a',b'),\tau L(a,b)) = 0.$$

And similarly

$$\mathrm{Hom}(L(a,b),\tau M(a',b')) = 0.$$

(case $a' \leq a < b < b' = n-1$) Similar to the case ($a' \leq a < b < b' < n-1$) we can get

$$\mathrm{Hom}(M(a', n-1), \tau L(a,b)) = 0.$$

And since $M(a', n-1)$ is projective, we have

$$\mathrm{Hom}(L(a,b), \tau M(a', b')) = 0.$$

(case $a' \leq a < b' \leq b < n-1$) In this case we claim that

$$\mathrm{Hom}(M(a', b'), \tau L(a,b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a', b'), \tau L(a,b))$ where

$$f_i = \begin{cases} (1, -1) & \text{if } b' < i \leq b+1, \\ 0 & \text{otherwise.} \end{cases}$$

(case $a' \leq a < b' \leq b = n-1$) In this case we also claim that

$$\mathrm{Hom}(M(a', b'), \tau L(a,b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a', b'), \tau L(a, n-1))$ where

$$f_i = \begin{cases} 1 & \text{if } a' < i \leq a+1 \text{ or } i = n^{\pm}, \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } a+1 < i \leq b', \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } b' < i \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

(case $b' \leq a$) Similar to the case $a' \leq a < b < b'$, we get

$$\mathrm{Hom}(M(a', b'), \tau L(a,b)) = 0 = \mathrm{Hom}(L(a,b), \tau M(a', b')).$$

So we have proved (3).

(4),(7) The proofs are similar to (3). □

LEMMA 3.10. *Let $T \in Tilt(Q)$.*

(1) *$L(0, n-1) \mid T$ implies $L^{\pm}(0,n) \mid T$.*

(2) *If $L^{+}(0,n) \mid T$ (resp. $L^{-}(0,n) \mid T$) and all indecomposable direct summands of T are insincere, then*

$$L^{-}(0,n) \mid T \quad (\textit{resp. } L^{+}(0,n) \mid T).$$

PROOF. (1) Suppose $L(0,n-1) \mid T$. Then

$$\mathrm{Ext}(T, L(0,n-1)) = 0 = \mathrm{Ext}(L(0,n-1), T)$$

and there exist injections

$$\tau L^{\pm}(0,n) \to \tau L(0,n-1).$$

So we get

$$\mathrm{Ext}(L^{\pm}(0,n), T) \simeq \mathrm{Hom}(T, \tau L^{\pm}(0,n)) = 0.$$

Since $L^{\pm}(0,n)$ is injective, we also get

$$\mathrm{Ext}(T, L^{\pm}(0,n)) = 0.$$

Therefore, $L^{\pm}(0,n) \mid T$.

(2) Suppose $L^{+}(0,n) \mid T$ and that all indecomposable direct summands of T are insincere. Now $(\underline{\dim}\, T)_{n^-} \neq 0$, so there exists some indecomposable direct summand N s.t.

$$(\underline{\dim}\, N)_{n^-} \neq 0.$$

If $N = M(a,b)$ then $\mathrm{Ext}(M(a,b), L^{+}(0,n)) = 0 = \mathrm{Ext}(L^{+}(0,n), M(a,b))$ so $a = 0$ and N is sincere. This is a contradiction. So $N = L^{-}(a,n)$ and $a = 0$ by $L^{+}(0,n) \mid T$. □

LEMMA 3.11. *For all $T \in Tilt(Q)$ there exists some indecomposable direct summand N of T s.t.*

$$(\underline{\dim}\, N)_i \geq 1, \quad \text{for all } i \leq n-1.$$

Thus, $N = L(0,n-1)$, $L^{\pm}(0,n)$ or $M(0,b)$, for some b.

PROOF. For an indecomposable direct summand N of T s.t. $(\underline{\dim}\, N)_1 = 1$, define

$$a(N) \overset{def}{=} \sup\{i \mid 1 \leq i \leq n-1, (\underline{\dim}\, N)_i \geq 1\}.$$

Suppose that $\sup a(N) = a < n-1$, then $L(0,a) \mid T$. So indecomposable direct summands of T are of the following form

$$L(a',b') \quad \text{for } b' \leq a \text{ or } a+1 \leq a',$$

$$L^{+}(a',n) \quad \text{for } a+1 \leq a',$$

$$M(a',b') \quad \text{for } a+1 \leq a'.$$

So $(\underline{\dim}\, T)_{a+1} = 0$. This is a contradiction. □

LEMMA 3.12. *We have*

$$\#\{i \mid 1 \leq i \leq n-1, (\underline{\dim}\ T)_i = 1\} \leq 1.$$

In particular, $\delta(T) \geq n-2$.

PROOF. Let $i \neq n^{\pm}$ s.t. $(\underline{\dim}\ T)_i = 1$. Then we claim that

$$L(0, i-1) \mid T.$$

By Lemma 3.11 there exists a unique indecomposable direct summand N of T s.t.

$$(\underline{\dim}\ N)_j \geq 1 \quad \text{for all } j \leq n-1.$$

So, by Lemma 3.10, $N = M(0, b)$ for some $i \leq b \leq n-1$ and any indecomposable direct summand of T not isomorphic to N is one of the following,

$$\begin{array}{ll} L(a,b) & \text{for } b \leq i-1 \text{ or } i \leq a, \\ L^{\pm}(a,n) & \text{for } i \leq a, \\ M(a,b) & \text{for } i \leq a. \end{array}$$

It implies

$$\mathrm{Ext}(T, L(0, i-1)) = 0 = \mathrm{Ext}(L(0, i-1), T),$$

so that

$$L(0, i-1) \mid T. \qquad \square$$

COROLLARY 3.13. *Let* $T \in \mathit{Tilt}(Q)$ *then* $\delta(T) \geq n-1$, *and* $\delta(T) = n-1$ *if and only if* $L^{\pm}(0,n) \mid T$ *and other indecomposable direct summands of* T *have the form* $L(a,b)$ $(0 \leq a < b \leq n-1)$. *In particular,*

$$\#\{T \in \mathit{Tilt}(kQ) \mid \delta(T) = n-1\} = \frac{1}{n}\binom{2(n-1)}{n-1} = \frac{1}{n-1}\binom{2(n-1)}{n-2}.$$

PROOF. Suppose that all indecomposable direct summands of T are insincere. Then, by Lemma 3.10 and Lemma 3.11, $L^{+}(0,n)$ and $L^{-}(0,n)$ are both direct summands of T. So $(\underline{\dim}\ T)_i = 1$ if and only if $i = n^{\pm}$. We have $\delta(T) \geq n-1$. If the equality holds then indecomposable direct summands of T not isomorphic to $L^{\pm}(0,n)$ are of the form $L(a,b)$.

Next we suppose there is a sincere indecomposable direct summand N of T. If $\delta(T) = n-2$ then, by Lemma 3.12, there is a unique $i \leq n-1$ s.t.

$$(\underline{\dim}\, T)_i = (\underline{\dim}\, T)_{n^\pm} = 1.$$

So all indecomposable direct summands of T not isomorphic to N are of the form $L(a,b)$ $(b < i$ or $i \leq a)$. As their direct sum may be viewed as a rigid module in type $A_{i-1} \times A_{n-i-1}$, we get

$$\#\{L(a,b) \mid L(a,b) \mid T\} \leq (i-1) + (n-1-i) = n-2,$$

which is a contradiction. Next we consider the case $\delta(T) = n-1$.

(a) $(\underline{\dim}\, T)_i = (\underline{\dim}\, T)_{n^+} = 1$, for a unique $i (\leq n-1)$. Then indecomposable direct summands of T not isomorphic to N are of the following form:

$$L(a,b) \quad \text{for } b < i \text{ or } i \leq a,$$

$$L^-(a,n) \quad \text{for } i \leq a.$$

We get by the same argument that

$$\#\{L \in \operatorname{ind} kQ \mid L \mid T, L \neq N\} \leq (i-1) + (n-i) = n-1,$$

which is a contradiction.

(b) $(\underline{\dim}\, T)_i = (\underline{\dim}\, T)_{n^-} = 1$, for a unique $i(\leq n-1)$. Then, similar to (a), we reach a contradiction.

(c) $(\underline{\dim}\, T)_{n^\pm} = 1$. Then indecomposable direct summands of T not isomorphic to N are of the form $L(a,b)$. Thus

$$\#\{L(a,b) \mid L(a,b) \mid T\} \leq n-1.$$

It is a contradiction. So we get $\delta(T) \geq n$ and $\delta(T) = n-1$ does not occur in this case.

Thus we have proved that if $\delta(T) = n-1$ then $L^{\pm}(0,n) \mid T$ and the other indecomposable direct summands of T has the form $L(a,b)$. The converse implication is clear. □

Now we define subsets of $Tilt(Q)$ by

$$\mathcal{T}_0 := \{T \in Tilt(Q) \mid \delta(T) = n+1\},$$

$$\mathcal{T}_1 := \{T \in Tilt(Q) \mid \delta(T) = n\},$$

$$\mathcal{T}_2 := \{T \in Tilt(Q) \mid \delta(T) = n-1\}.$$

LEMMA 3.14. *Fix* $1 \le i \le n-1$, *then*

$$\{T \in \mathcal{T}_1 \mid (\underline{\dim}\, T)_i = 1\} \overset{1:1}{\leftrightarrow} Tilt(\circ \to \circ \to \cdots \to \overset{i-1}{\circ}) \times \{T \in Tilt(Q_{n-i+1}) \mid (\underline{\dim}\, T)_1 = 1, \delta(T) = n-i+1\}.$$

PROOF. Let $T \in \mathcal{T}_1$ s.t. $(\underline{\dim}\, T)_i = 1$, for a unique $i(\le n-1)$. By Lemma 3.10 and Lemma 3.11 there exists a unique $j = j(T)(\ge i)$ s.t. $M(0,j) \mid T$. Now let

$$X(T) = \{L(a,b) \mid L(a,b) \mid T, b < i\}$$

and

$$Y(T) = \{N \in \text{ind } kQ \mid N \mid T\} \backslash \{X(T) \cup \{M(0,j)\}\}.$$

We define the maps

$$\varphi_T : X(T) \to \text{ind } k(\circ \to \circ \to \cdots \to \overset{i-1}{\circ})$$

and

$$\psi_T : Y(T) \to \text{ind } kQ_{n-i+1},$$

by

$$(\varphi_T(N))_a = (N)_a \quad (1 \le a < i),$$

$$(\psi_T(N))_a = (N)_{a+i-1} \ (\text{let } (n-i+1)^{\pm} + i - 1 = n^{\pm}).$$

Then

$$T \mapsto \left(\bigoplus_{x \in X(T)} \varphi_T(x), \bigoplus_{y \in Y(T)} \psi_T(y) \oplus M(0, j(T) - i + 1) \right)$$

induces a bijection between

$$\{T \in \mathcal{T}_1 \mid (\underline{\dim}\, T)_i = 1\}$$

and

$$Tilt(\circ \to \circ \to \cdots \to \overset{i-1}{\circ}) \times \{T \in Tilt(Q_{n-i+1}) \mid (\underline{\dim}\, T)_1 = 1, \delta(T) = n-i+1\}. \quad \square$$

Let us define the following subsets of $\mathcal{T}_1$:

$$\mathcal{A}_{\pm} := \left\{ T \in \mathcal{T}_1 \;\middle|\; \begin{array}{l} \text{all indecomposable direct summands of } T \text{ are insincere} \\ \text{and } (\underline{\dim}\, T)_{n^{\pm}} = 1 \end{array} \right\},$$

$$\mathcal{B}_{\pm} := \{T \in \mathcal{T}_1 \mid (\underline{\dim}\, T)_{n^{\pm}} = 1, \text{ there exists some } j \text{ s.t. } M(0,j) \mid T\},$$

$$\mathscr{B}_\pm(j) := \{T \in \mathscr{B}_\pm \mid M(0,j) \mid T\},$$
$$\mathscr{C} := \{T \in \mathscr{T}_1 \mid \delta(T) = n, (\underline{\dim}\ T)_1 = 1\},$$
$$\mathscr{C}(j) := \{T \in \mathscr{C} \mid M(0,j) \mid T\}.$$

THEOREM 3.15. (1) $\mathscr{A}_\pm = \varnothing$.

(2) $\mathscr{B}_\pm(j) \overset{1:1}{\leftrightarrow} \{T' \in Tilt(\circ \to \cdots \to \overset{n}{\circ}) \mid min\{j' \mid L(j', n-1) \mid T'\} = j\}$. *In particular,*

$$\mathscr{B}_\pm \overset{1:1}{\leftrightarrow} Tilt(\circ \to \cdots \to \overset{n}{\circ}) \backslash \{T' \in Tilt(\circ \to \cdots \to \overset{n}{\circ}) \mid L(0, n-1) \mid T'\},$$

and we have

$$\#\mathscr{B}_\pm = \frac{1}{n+1}\binom{2n}{n} - \frac{1}{n}\binom{2(n-1)}{n-1}.$$

(3) $\mathscr{C}(j) \overset{1:1}{\leftrightarrow} \{T' \in Tilt(Q_{n-1}) \mid j = j'(T') + 1\}$
where

$$j'(T') = \sup\{b \mid L^+(b, n-1) \text{ or } L^-(b, n-1) \text{ or } M(a,b) \mid T \text{ for some } a\}.$$

In particular,

$$\mathscr{C} \overset{1:1}{\leftrightarrow} Tilt(Q_{n-1}),$$

and we have

$$\#\mathscr{C} = \frac{3n-4}{2n}\binom{2(n-1)}{n-1}.$$

PROOF. (1) Suppose that there exists some $T \in \mathscr{A}_+$. Then, by Lemma 3.11, we have $L^\pm(0,n) \mid T$. Now there exists some indecomposable direct summand N of T not isomorphic to $L^-(0,n)$ s.t. $(\underline{\dim}\ N)_{n^-} = 1$.

If $N = M(a,b)$ or $L^-(a,n)$ then $a = 0$. This is a contradiction because $L^\pm(0,n) \mid T$. So $\mathscr{A}_+ = \varnothing$ and similarly we have $A_- = \varnothing$.

(2) Define the maps

$$\varphi : \{L(a,b) \mid 0 \le a < b \le n-1\} \cup \{L^-(a,n) \mid 0 \le a \le n-1\}$$
$$\to \text{ind}\ k(\circ \to \circ \to \cdots \to \overset{n}{\circ})$$

and

$$\psi : \text{ind}\ k(\circ \to \circ \to \cdots \to \overset{n}{\circ})$$
$$\to \{L(a,b) \mid 0 \le a < b \le n-1\} \cup \{L^-(a,n) \mid 0 \le a \le n-1\}$$

by

$$(\varphi(L))_a = \begin{cases} L_a & \text{if } 0 \leq a \leq n-1, \\ L_{n^-} & \text{if } a = n, \end{cases}$$

$$(\psi(L'))_a = \begin{cases} L'_a & \text{if } 0 \leq a \leq n-1, \\ L'_n & \text{if } a = n^-, \\ 0 & \text{if } a = n^+. \end{cases}$$

Then $\varphi \circ \psi = 1 = \psi \circ \varphi$. Define

$$Z(T) := \{N \in \operatorname{ind} kQ \mid N \mid T, N \not\simeq M(0,j)\}$$

and

$$Y(T') := \{N \in \operatorname{ind} k(\circ \to \circ \to \cdots \to \overset{n}{\circ}) \mid N \mid T'\}.$$

Then it is easy to see that the maps induce a bijection

$$\mathscr{B}_+(j) \overset{1:1}{\leftrightarrow} \{T' \in \mathit{Tilt}(\circ \to \cdots \to \overset{n}{\circ}) \mid \mathit{min}\{j' \mid L(j', n-1) \mid T'\} = j\}$$

by

$$T \mapsto \bigoplus_{L \in Z(T)} \varphi(L).$$

The inverse map is

$$T' \mapsto \left(\bigoplus_{L' \in Y(T')} \psi(L') \right) \oplus M(0,j).$$

In fact, if $T \in \mathscr{B}_+(j)$ then all indecomposable direct summands of T not isomorphic to $M(0,j)$ are either

$$L(a,b) \ (a \geq j \text{ or } b < j) \quad \text{or} \quad L^-(a,n) \ (a \leq j),$$

which implies $L(j, n-1), L^-(j,n) \mid T$. It follows

$$\mathit{min}\left\{ j' \mid L(j', n-1) \,\middle|\, \bigoplus_{L \in Z(T)} \varphi(L) \right\} = j.$$

Conversely, if

$$T' \in \{T' \in \mathit{Tilt}(\circ \to \cdots \to \overset{n}{\circ}) \mid \mathit{min}\{j' \mid L(j', n-1) \mid T'\} = j\}$$

then $(\bigoplus_{L' \in Y(T')} \psi(L')) \oplus M(0,j) \in \mathscr{B}_+(j)$.

(3) Define the maps

$$\varphi : \{N \in \operatorname{ind} kQ_n \mid (\underline{\dim}\, N)_1 = 0\} \to \operatorname{ind} kQ_{n-1}$$

and

$$\psi : \operatorname{ind} kQ_{n-1} \to \{N \in \operatorname{ind} kQ_n \mid (\underline{\dim}\, N)_1 = 0\}$$

by the obvious way. Then $\varphi \circ \psi = 1 = \psi \circ \varphi$. Define

$$Z(T) := \{N \in \operatorname{ind} kQ_n \mid N \mid T, N \not\cong M(0, j)\}$$

and

$$Y(T') := \{N \in \operatorname{ind} kQ_{n-1} \mid N \mid T'\}.$$

Then they induce a bijection

$$\mathscr{C}(j) \overset{1:1}{\leftrightarrow} \{T' \in \mathit{Tilt}(Q_{n-1}) \mid j = j'(T') + 1\}$$

by

$$T \mapsto \bigoplus_{N \in Z(T)} \varphi(N).$$

The inverse map is

$$T' \mapsto \left(\bigoplus_{N' \in Y(T')} \psi(N') \right) \oplus M(0, j+1).$$

In fact, if $T \in \mathscr{C}(j)$ then

$$Z(T) \subset \{L(a,b) \mid 1 \leq a < b < j \text{ or } j \leq a\} \cup \{L^{\pm}(b,n) \mid 1 \leq b \leq j\}$$
$$\cup \{M(a,b) \mid 1 \leq a < b \leq j\}.$$

It implies $M(1,j) \mid T$ and $j'(\bigoplus_{N \in Z(T)} \varphi(N)) = j - 1$. Conversely, if $j = j'(T') + 1$ then

$$(\underline{\dim} \oplus_{N' \in Y(T')} \psi(N'))_a \begin{cases} \geq 1 & (a \geq 2) \\ = 0 & (a = 1). \end{cases}$$

It implies

$$\left(\bigoplus_{N' \in Y(T')} \psi(N') \right) \oplus M(0, j) \in \mathscr{C}(j). \qquad \square$$

COROLLARY 3.16.

$$\#\mathscr{T}_1 = 3\binom{2(n-1)}{n-2}.$$

PROOF. First we claim that

$$\sum_{i=1}^{n} \frac{1}{i(n+1-i)} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i} = \frac{1}{n+1} \binom{2n}{n}.$$

This follows from the fact that

$$Tilt(\circ \to \cdots \to \overset{n}{\circ}) = \bigsqcup \{T \in Tilt(\circ \to \cdots \to \overset{n}{\circ}) \mid min\{i' \mid L(i',n) \mid T, i' > 0\} = i\}.$$

Thus, by Lemma 3.14 and Theorem 3.15, $\#\mathscr{T}_1$ is equal to

$$2\left(\frac{1}{n+1}\binom{2n}{n} - \frac{1}{n}\binom{2(n-1)}{n-1}\right) + \sum_{i=1}^{n-1} \frac{3(n-i)-1}{2i(n-i+1)} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i}$$

$$= 2\left\{\left(\frac{1}{n+1}\binom{2n}{n} - \frac{1}{n}\binom{2(n-1)}{n-1}\right) - \sum_{i=1}^{n-1} \frac{1}{i(n-i+1)} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i}\right\}$$

$$+ \sum_{i=1}^{n-1} \frac{3}{2i} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i}$$

$$= \frac{3}{2} \sum_{i=1}^{n-1} \frac{1}{i} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i}.$$

Now let

$$a_n = \sum_{i=1}^{n} \frac{1}{i} \binom{2(i-1)}{i-1} \binom{2(n+1-i)}{n+1-i}$$

and

$$f(X) = \left(\sum_{i=1}^{n} \frac{1}{i} \binom{2(i-1)}{i-1} X^i\right)^2.$$

Then the coefficient of X^{n+1} in $f'(X)$ is equal to

$$2a_n - 2\binom{2n}{n}.$$

On the other hand, the coefficient of X^{n+2} in $f(X)$ is equal to

$$\sum_{i=1}^{n+1} \frac{1}{i(n-i+2)} \binom{2(i-1)}{i-1} \binom{2(n-i+1)}{n-i+1} - \frac{2}{n+1}\binom{2n}{n}$$
$$= \frac{1}{n+2}\binom{2(n+1)}{n+1} - \frac{2}{n+1}\binom{2n}{n}.$$

So

$$2a_n = \binom{2(n+1)}{n+1} - \frac{2}{n+1}\binom{2n}{n} = 4\binom{2n}{n-1}.$$

We conclude that

$$\#\mathscr{T}_1 = \frac{3}{2}a_{n-1} = 3\binom{2(n-1)}{n-2}. \qquad \square$$

COROLLARY 3.17. *We have*

$$\#\mathscr{T}_0 = \frac{3(n-1)}{n+1}\binom{2(n-1)}{n-2}.$$

PROOF. In fact,

$$\#\mathscr{T}_0 = \frac{3n-1}{2(n+1)}\binom{2n}{n} - 3\binom{2(n-1)}{n-2} - \frac{1}{n}\binom{2(n-1)}{n-1}.$$
$$= \frac{3(n-1)}{n+1}\binom{2(n-1)}{n-2}. \qquad \square$$

THEOREM 3.18.

$$\#\vec{\mathscr{K}}(Q)_1 = (3n-1)\binom{2(n-1)}{n-2}.$$

PROOF. In fact, $\#\vec{\mathscr{K}}(Q)_1$ is equal to

$$\frac{1}{2}\left\{\frac{n-1}{n-1}\binom{2(n-1)}{n-2} + 3n\binom{2(n-1)}{n-2} + 3(n-1)\binom{2(n-1)}{n-2}\right\}$$
$$= (3n-1)\binom{2(n-1)}{n-2}. \qquad \square$$

3.3. case E_6, E_7, E_8. By using AR-sequences, it is possible to establish the dimension vector of any indecomposable modules and the dimension of the space of the morphism between any two indecomposable modules. So, by using AR-duality and Proposition 1.7, we can calculate the number of arrows in tilting quiver. Now it is clear that to calculate the number of arrows in tilting quiver could be left to a computer. A reader can download a source code from following address, http://rkase.web.fc2.com/source.pdf

Acknowledgement

The author would like to express his gratitude to Professor Susumu Ariki for his mathematical supports, careful reading of this paper and warm encouragements. The author also thanks Professor Syu Kato for his mathematical advices and warm encouragements.

References

[1] M. Auslander, I. Reiten and S. Smalø, Representation theory of artin algebras, Cambridge University Press, 1995.

[2] I. Assem, D. Simson and A. Skowroński, Elements of the representation theory of associative algebras Vol. 1, London Mathematical Society Student Texts **65**, Cambridge University Press, 2006.

[3] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. **204** (2006), no. 2, 572–618.

[4] F. Coelho, D. Happel and L. Unger, Complements to partial tilting modules, J. Algebra **170** (1994), no. 3, 184–205.

[5] S. Fomin and A. Zelevinsky, *Y*-systems and generalized associahedra, Ann. of Math. (2) **158**, (2003), no. 3, 977–1018.

[6] P. Gabriel, Auslander-Reiten sequences and representation-finite algebras. Representation theory I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), pp. 1–71, Lecture Notes in Math., **831**, Springer, Berlin, 1980.

[7] D. Happel and C. M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. **274** (1982), no. 2, 399–443.

[8] D. Happel and L. Unger, On a partial order of tilting modules, Algebr. Represent. Theory **8** (2005), no. 2, 147–156.

[9] D. Happel and L. Unger, On the quiver of tilting modules, J. Algebra **284** (2005), no. 2, 857–868.

[10] D. Happel and L. Unger, Reconstruction of path algebras from their posets of tilting modules, Trans. Amer . Math. Soc **361** (2009), no. 7, 3633–3660.

[11] S. Ladkani, Universal derived equivalences of posets of tilting modules, arXiv:0708.1287v1.

[12] R. Marsh, M. Reineke and A. Zelevinsky, Generalized associahedra via quiver representations, Trans. Amer. Math. Soc **355** (2003), no. 10, 4171–4186.

[13] I. Reiten, Tilting theory and homologically finite subcategories, Handbook of tilting theory, L. Angeleri Hügel, D. Happel, H. Krause, eds., London Mathematical Society Lecture Note Series **332**, Cambridge University Press, 2007.

[14] C. Riedtmann and A. Schofield, On a simplicial complex associated with tilting modules, Comment. Math. Helv **66** (1991), no. 1, 70–78.

[15] L. Unger, Combinatorial aspects of the set of tilting modules, Handbook of tilting theory, L. Angeleri Hügel, D. Happel, H. Krause, eds., London Mathematical Society Lecture Note Series 332, Cambridge University Press, 2007.

Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University
Toyonaka, Osaka 560-0043
Japan
E-mail: r-kase@cr.math.sci.osaka-u.ac.jp