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THE IWASAWA DECOMPOSITION AND THE BRUHAT DECOMPOSITION OF THE AUTOMORPHISM GROUP ON CERTAIN EXCEPTIONAL JORDAN ALGEBRA

By

Akihiro Nishio

Abstract. Let $\mathcal{J}^1$ be the real form of a complex simple Jordan algebra such that the automorphism group is $F_{4(-20)}$. By using some orbit types of $F_{4(-20)}$ on $\mathcal{J}^1$, for $F_{4(-20)}$, explicitly, we give the Iwasawa decomposition, the Oshima–Sekiguchi's K_ε-Iwasawa decomposition, the Matsuki decomposition, and the Bruhat and Gauss decompositions.

This article is a continuation of [13].

Contents

9. Overview

Let G be a connected non-compact semisimple $\mathbf{R}$-Lie group of which the center $Z(G)$ is finite. We denote its $\mathbf{R}$-Lie algebra by $\mathfrak{g} = Lie(G)$. Let θ be a Cartan involution of $\mathfrak{g}$ and its Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} := \{X \in \mathfrak{g} \,|\, \theta X = X\}$ and $\mathfrak{p} := \{X \in \mathfrak{g} \,|\, \theta X = -X\}$. Let $\mathfrak{a}$ be a maximal abelian

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subspace of $\mathfrak{p}$, $\mathfrak{a}^*$ the dual space of $\mathfrak{a}$, and $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ of $\mathfrak{k}$. For each $\lambda \in \mathfrak{a}^*$, let $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \,|\, [H, X] = \lambda(H)X$ for all $H \in \mathfrak{a}\}$. λ is called a *root* of $(\mathfrak{g}, \mathfrak{a})$ if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$. We denote the set of roots of $(\mathfrak{g}, \mathfrak{a})$ by Σ. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda$, $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$, $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, and $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ (cf. [8, Ch V]). We introduce an ordering in $\mathfrak{a}^*$, and this ordering single out the set Σ^+ of positive roots. We denote $\Sigma^- := \{-\lambda \,|\, \lambda \in \Sigma^+\}$, $\mathfrak{n}^+ := \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$, and $\mathfrak{n}^- := \sum_{\lambda \in \Sigma^-} \mathfrak{g}_\lambda$. Then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent subalgebras such that $\theta\mathfrak{n}^\pm = \mathfrak{n}^\mp$ (*resp*) and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$. For each involutive automorphism φ on G, we denote the subgroup $G^\varphi = \{g \in G \,|\, \varphi(g) = g\}$ of G. Let Θ be an involutive automorphism on G, of which the differential at the identity element is the Cartan involution θ of $\mathfrak{g}$: $d\Theta = \theta$, and $K := G^\Theta$. Note that $Lie(K) = \mathfrak{k}$, K is connected and closed, and that K is a maximal compact subgroup of G (cf. [7, Ch VI, Theorem 1.1]). We denote the subgroups $A := \exp \mathfrak{a}$, $N^\pm := \exp \mathfrak{n}^\pm$ (*resp*), and $M := Z_K(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ of K, respectively. Then the identity connected component M^0 of M is a connected Lie subgroup corresponding to $\mathfrak{m}$, and $\Theta(N^\pm) = N^\mp$ (*resp*). We denote the normalizer of $\mathfrak{a}$ of K by $M^* := N_K(\mathfrak{a})$, and the finite factor group $W := M^*/M$. For all $w \in W$, we fix a representative $\tilde{w} \in M^*$. Then

(1) $G = KAN^+$ (Iwasawa decomposition),

(2) $G = \coprod_{w \in W} N^- \tilde{w} MAN^+$ (Bruhat decomposition),

(2)$'$ $G = \overline{N^- MAN^+}$ (Gauss decomposition).

(cf. [7], [11]). For any $g \in G$, there exist unique elements $k(g) \in K$, $H(g) \in \mathfrak{a}$, and $n_I(g) \in N^+$ such that

$$g = k(g)(\exp H(g))n_I(g).$$

In (2)$'$, the submanifold $N^- MAN^+$ is open dense in G, and for any $g \in N^- MAN^+$, there exist unique elements $n_G^-(g) \in N^-$, $m_G(g) \in M$, $a_G(g) \in A$, and $n_G^+(g) \in N^+$ such that

$$g = n_G^-(g) m_G(g) a_G(g) n_G^+(g).$$

However, in this article, the existence and uniqueness of factors of Iwasawa and Gauss decompositions for the Lie group $F_{4(-20)}$ will be shown by using concrete $F_{4(-20)}$-orbits and stabilizers of $F_{4(-20)}$ in [13].

According to [14, Definition 1.1], a *signature of roots* is defined by the mapping ε of Σ to $\{-1, 1\}$ such that ε satisfies the following conditions:

(i) $\varepsilon(\lambda) = \varepsilon(-\lambda)$ for any $\lambda \in \Sigma$,

(ii) $\varepsilon(\lambda + \mu) = \varepsilon(\lambda)\varepsilon(\mu)$ if λ, μ and $\lambda + \mu \in \Sigma$.

According to [14, Definition 1.2], for any signature ε of roots with respect to the Cartan involution θ, the involutive automorphism θ_ε of $\mathfrak{g}$ is defined as

(i) $\theta_\varepsilon(X) := \varepsilon(\lambda)\theta(X)$ for any $\lambda \in \Sigma$ and $X \in \mathfrak{g}_\lambda$,

(ii) $\theta_\varepsilon(X) := \theta(X)$ for any $X \in \mathfrak{a} \oplus \mathfrak{m}$.

Setting $\mathfrak{k}_\varepsilon := \{X \in \mathfrak{g} \mid \theta_\varepsilon X = X\}$ and $\mathfrak{p}_\varepsilon := \{X \in \mathfrak{g} \mid \theta_\varepsilon X = -X\}$, $\mathfrak{g} = \mathfrak{k}_\varepsilon \oplus \mathfrak{p}_\varepsilon$. We denote the connected Lie subgroup having the Lie algebra $\mathfrak{k}_\varepsilon$ by $(K_\varepsilon)^0$. We define the subgroup K_ε by

$$K_\varepsilon := (K_\varepsilon)^0 M.$$

In fact, since all elements of M normalize $(K_\varepsilon)^0$ from [14, Lemma 1.4(i)], K_ε is a subgroup of G. We denote

$$M_\varepsilon^* := K_\varepsilon \cap M^*, \quad W_\varepsilon := M_\varepsilon^*/M.$$

Proposition 9.1 (T. Oshima and J. Sekiguchi [14, Proposition 1.10]). *Let the factor set $W_\varepsilon \backslash W = \{w_1 = 1, w_2, \ldots, w_r\}$ where $r = [W : W_\varepsilon]$. Fix representatives $\tilde{w}_1 = 1, \tilde{w}_2, \ldots, \tilde{w}_r \in M_\varepsilon^* = K_\varepsilon \cap M^*$ for $w_1 = 1, w_2, \ldots, w_r$. Then the decomposition*

$$G \supset \bigcup_{i=1}^r K_\varepsilon \tilde{w}_i A N^+$$

has the following properties.

(1) *If $k\tilde{w}_i an = k'\tilde{w}_j a'n'$ with $k, k' \in K_\varepsilon$, $a, a' \in A$, and $n, n' \in N^+$, then $k = k'$, $i = j$, $a = a'$, and $n = n'$.*

(2) *The map $(k, a, n) \mapsto k\tilde{w}_i an$ defines an analytic diffeomorphism of the product manifold $K_\varepsilon \times A \times N^+$ onto the open submanifold $K_\varepsilon \tilde{w}_i A N^+$ of G $(i = 1, \ldots r)$.*

(3) *The submanifold $\bigcup_{i=1}^r K_\varepsilon \tilde{w}_i A N^+$ is open dense in G.*

The decomposition $G = \overline{\bigcup_{i=1}^r K_\varepsilon \tilde{w}_i A N^+}$ is called the *K_ε-Iwasawa decomposition of G*.

If a group G acts on a set S, we denote the pointwise stabilizer of finite set $\{x_1, \ldots, x_n\}$ of S by $G_{x_1, \ldots, x_n} := \{g \in G \mid gx_i = x_i \text{ for } i = 1, \ldots, n\}$, and the G-orbit of $x \in S$ by $G \cdot x := \{gx \mid g \in G\}$. We denote the Kronecker delta by $\delta_{i,j}$.

Let $\mathbf{O}$ be the octonions having the conjugation $\bar{x}$ and inner product $(x|y)$ for $x, y \in \mathbf{O}$. We denote the natural unit octonions: $\{1(=e_0), e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Set

$$h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) := \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & \sqrt{-1}\overline{x_2} \\ \sqrt{-1}\overline{x_3} & \xi_2 & x_1 \\ \sqrt{-1}x_2 & \overline{x_1} & \xi_3 \end{pmatrix}$$

with $\xi_i \in \mathbf{R}$, $x_i \in \mathbf{O}$. In [13, §1], the *exceptional Jordan algebra* $\mathscr{J}^1$ is given by

$$\mathscr{J}^1 := \{h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) \mid \xi_i \in \mathbf{R}, x_i \in \mathbf{O}\}$$

with the *Jordan product* $X \circ Y = 2^{-1}(XY + YX)$ for $X, Y \in \mathscr{J}^1$. Put $E = h^1(1, 1, 1; 0, 0, 0)$, $E_i := h^1(\delta_{i,1}, \delta_{i,2}, \delta_{i,3}; 0, 0, 0)$, and $F_i^1(x) := h^1(0, 0, 0; \delta_{i,1}x, \delta_{i,2}x, \delta_{i,3}x)$. Then $h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = \sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i))$. We recall that $\mathscr{J}^1$ has the *trace* $\operatorname{tr}(X) := \sum_{i=1}^3 \xi_i$ where $X = \sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i))$, the *inner product* $(X|Y) := \operatorname{tr}(X \circ Y)$, the *cross product* $X \times Y$ by

$$X \times Y := 2^{-1}(2X \circ Y - \operatorname{tr}(X)Y - \operatorname{tr}(Y)X + (\operatorname{tr}(X)\operatorname{tr}(Y) - (X|Y))E)$$

as well as $X^{\times 2} := X \times X$, and the *determinant* $\det(X) := 3^{-1}(X|X^{\times 2})$, respectively. By [13, Lemma 1.6],

$$(9.1) \qquad (X|Y) = \left(\sum_{i=1}^3 \xi_i \eta_i\right) + 2(x_1|y_1) - 2(x_2|y_2) - 2(x_3|y_3),$$

$$\det(X) = \xi_1\xi_2\xi_3 - 2(1 \mid (x_1x_2)x_3) - \xi_1(x_1|x_1) + \xi_2(x_2|x_2) + \xi_3(x_3|x_3),$$

$$(9.2) \qquad X^{\times 2} = (\xi_2\xi_3 - (x_1|x_1))E_1 + (\xi_3\xi_1 + (x_2|x_2))E_2 + (\xi_1\xi_2 + (x_3|x_3))E_3$$
$$+ F_1^1(-\overline{x_2x_3} - \xi_1 x_1) + F_2^1(\overline{x_3x_1} - \xi_2 x_2) + F_3^1(\overline{x_1x_2} - \xi_3 x_3)$$

where $X = \sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i))$ and $Y = \sum_{i=1}^3 (\eta_i E_i + F_i^1(y_i))$. We recall that $\mathscr{J}^1$ has the *exceptional hyperbolic planes* $\mathscr{H}$, $\mathscr{H}'$ and the *exceptional null cones* $\mathscr{N}_1^+$, $\mathscr{N}_1^-$ as

$$\mathscr{H} := \{X \in \mathscr{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 1, (E_1|X) \geq 1\},$$

$$\mathscr{H}' := \{X \in \mathscr{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 1, (E_1|X) \leq 0\},$$

$$\mathscr{N}_1^+ := \{X \in \mathscr{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 0, (E_1|X) > 0\},$$

$$\mathscr{N}_1^- := \{X \in \mathscr{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 0, (E_1|X) < 0\}.$$

respectively. In Lemma 10.17, we will show the following equations:

$$(9.3)\qquad \begin{cases} \text{(i)} \ \{X \in \mathscr{H} \mid (E_1|X) = 1\} = \{E_1\}, \\ \text{(ii)} \ \{X \in \mathscr{H}' \mid (E_1|X) = 0\} = 2^{-1}(S^8 + (E_2 + E_3)) \supset \{E_2, E_3\} \end{cases}$$

where $S^8 = \{\xi(E_2 - E_3) + F_1^1(x) \mid \xi^2 + (x|x) = 1\}$.

The *exceptional Lie group* $F_{4(-20)}$ is given by

$$F_{4(-20)} := \{g \in \mathrm{GL}_{\mathbf{R}}(\mathscr{J}^1) \mid g(X \circ Y) = gX \circ gY\}.$$

Then

$$\mathrm{tr}(gX) = \mathrm{tr}(X), \quad gE = E, \quad (gX|gY) = (X|Y),$$
$$g(X \times Y) = gX \times gY, \quad \det(gX) = \det(X)$$

for all $g \in F_{4(-20)}$ and $X, Y \in \mathscr{J}^1$, from [13, Proposition 1.8]. In [21, Theorem 2.2.2] and [22, Theorem 2.14.1], I. Yokota has proved that $F_{4(-20)}$ is connected and a simply connected semisimple Lie group of type $\mathbf{F}_{4(-20)}$, by showing the polar decomposition $F_{4(-20)} \simeq \mathrm{Spin}(9) \times \mathbf{R}^{16}$ with the center $Z(F_{4(-20)}) = \{1\}$ ([21, Theorem 2.14.2]). We denote the elements $P^+, P^- \in \mathscr{J}^1$ by $P^+ := h^1(1, -1, 0; 0, 0, 1)$ and $P^- := h^1(-1, 1, 0; 0, 0, 1)$ respectively. From [13, Proposition 0.1], we recall that the exceptional hyperbolic planes and the exceptional null cones are $F_{4(-20)}$-orbits in $\mathscr{J}^1$:

$$(9.4)\qquad \mathscr{H} = F_{4(-20)} \cdot E_1,$$

$$(9.5)\qquad \mathscr{H}' = F_{4(-20)} \cdot E_2 = F_{4(-20)} \cdot E_3,$$

$$(9.6)\qquad \mathscr{N}_1^+ = F_{4(-20)} \cdot P^+,$$

$$(9.7)\qquad \mathscr{N}_1^- = F_{4(-20)} \cdot P^-.$$

For $i \in \{1, 2, 3\}$, we denote the element $\sigma_i \in F_{4(-20)}$ by

$$\sigma_i\Big(\sum_{j=1}^3 (\xi_j E_j + F_j^1(x_j))\Big) := \sum_{j=1}^3 (\xi_j E_j + F_j^1((-1)^{1-\delta_{i,j}} x_j)).$$

(see [13, §4]), and the involutive inner automorphism $\tilde{\sigma}_i$; $\tilde{\sigma}_i(g) := \sigma_i g \sigma_i^{-1} = \sigma_i g \sigma_i$ for $g \in F_{4(-20)}$. We simply write σ and $\tilde{\sigma}$ for σ_1 and $\tilde{\sigma}_1$, respectively. Set $(G, \Theta) = (F_{4(-20)}, \tilde{\sigma})$ and $K := (F_{4(-20)})^{\tilde{\sigma}}$. From [13, Proposition 4.8] (note $(F_{4(-20)})_{E_2} \cong (F_{4(-20)})_{E_3}$), the stabilizers $(F_{4(-20)})_{E_1}$ and $(F_{4(-20)})_{E_2}$ are connected two-hold covering groups of $SO(9)$ and $SO^0(8,1)$, respectively. So we denote $\mathrm{Spin}(9) := (F_{4(-20)})_{E_1}$ and $\mathrm{Spin}^0(8,1) := (F_{4(-20)})_{E_2}$, respectively. By [13, Proposition 4.14],

$$K = (F_{4(-20)})_{E_1} = \mathrm{Spin}(9). \tag{9.8}$$

$$(F_{4(-20)})^{\tilde{\sigma}_2} = (F_{4(-20)})_{E_2} = \mathrm{Spin}^0(8,1). \tag{9.9}$$

Then

$$\mathscr{H} \simeq F_{4(-20)}/\mathrm{Spin}(9), \quad \mathscr{H}' \simeq F_{4(-20)}/\mathrm{Spin}^0(8,1).$$

We denote $D_4 := (F_{4(-20)})_{E_1,E_2,E_3} (\subset K)$. From [13, Lemma 3.2(1) and Proposition 2.6(1)], D_4 is a connected two-hold covering group of $SO(8)$, and set $\mathrm{Spin}(8) := D_4$. We denote the Lie algebras $\mathfrak{f}_{4(-20)} := Lie(F_{4(-20)})$ and $\mathfrak{d}_4 := Lie(D_4) = \{D \in \mathfrak{f}_{4(-20)} \mid DE_i = 0,\ i = 1,2,3\}$, respectively. From [13, Lemma 3.9], $\mathfrak{f}_{4(-20)}$ has the decomposition

$$\mathfrak{f}_{4(-20)} = \mathfrak{d}_4 \oplus \tilde{\mathfrak{u}}_1^1 \oplus \tilde{\mathfrak{u}}_2^1 \oplus \tilde{\mathfrak{u}}_3^1 \quad \text{where } \tilde{\mathfrak{u}}_i^1 := \{\tilde{A}_i^1(a) \mid a \in \mathbf{O}\}$$

(see [13, §3]). The differential $d\tilde{\sigma}$ of $\tilde{\sigma}$ at the identity is often denoted by $\tilde{\sigma}$. From [13, Lemma 7.2(2)], $d\tilde{\sigma}$ is a Cartan involution with a Cartan decomposition $\mathfrak{f}_{4(-20)} = \mathfrak{k} \oplus \mathfrak{p}$. We denote $a_t := \exp(t\tilde{A}_3^1(1))$ with $t \in \mathbf{R}$, the one-parameter subgroup $A := \{a_t \mid t \in \mathbf{R}\}$, the Lie algebra $\mathfrak{a} := \{t\tilde{A}_3^1(1) \mid t \in \mathbf{R}\}$ of A, and the linear functional α on $\mathfrak{a}$ such that $\alpha(\tilde{A}_3^1(1)) = 1$. Set the centralizer $M := \{k \in K \mid k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(1)\}$ of $\mathfrak{a}$ of K, and its Lie subalgebra $\mathfrak{m} := \{\phi \in \mathfrak{k} \mid [\phi, \tilde{A}_3^1(1)] = 0\}$. Then

$$ma = am \quad \text{for all } m \in M \text{ and } a \in A. \tag{9.10}$$

From [13, Lemma 3.2(2) and Proposition 2.6(2)], $(F_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)}$ is a connected two-hold covering group of $SO(7)$, and set $\mathrm{Spin}(7) := (F_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)}$. By [13, Proposition 7.4],

$$M = \mathrm{Spin}(7) = (F_{4(-20)})_{E_1,E_2,E_3,F_3^1(1)} = (F_{4(-20)})_{E_j,F_3^1(1)} \tag{9.11}$$

with $j \in \{1,2\}$. In particular, M is connected. From [13, Lemma 7.5], $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ with the following root space decomposition of $(\mathfrak{f}_{4(-20)}, \mathfrak{a})$:

$$\mathfrak{f}_{4(-20)} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}, \tag{9.12}$$

the set of roots $\Sigma = \{\pm\alpha, \pm 2\alpha\}$, and $\mathfrak{n}^{\pm} = \mathfrak{g}_{\pm\alpha} \oplus \mathfrak{g}_{\pm 2\alpha}$ (*resp*). Then $\mathfrak{g}_\alpha$ (*resp.* $\mathfrak{g}_{-\alpha}$) is parameterized by the octonions $\mathbf{O}$:

$$\mathfrak{g}_\alpha = \{\mathscr{G}_1(x) \mid x \in \mathbf{O}\} \quad (\textit{resp. } \mathfrak{g}_{-\alpha} = \{\mathscr{G}_{-1}(x) \mid x \in \mathbf{O}\}) \tag{9.13}$$

where $\mathscr{G}_{\pm 1}(x) := \tilde{A}_1^1(x) + \tilde{A}_2^1(\mp\bar{x})$ (*resp*) and $\mathfrak{g}_{2\alpha}$ (*resp.* $\mathfrak{g}_{-2\alpha}$) is parameterized by the vector parts $\mathrm{Im}\,\mathbf{O} := \{\sum_{i=1}^7 r_i e_i \mid r_i \in \mathbf{R}\}$ of octonions:

$$\mathfrak{g}_{2\alpha} = \{\mathscr{G}_2(p) \mid p \in \mathrm{Im}\,\mathbf{O}\} \quad (\textit{resp.}\ \mathfrak{g}_{-2\alpha} = \{\mathscr{G}_{-2}(p) \mid p \in \mathrm{Im}\,\mathbf{O}\}) \tag{9.14}$$

where $\mathscr{G}_{\pm 2}(p) := \tilde{A}_3^1(\mp p) - \delta(p)$ (*resp*) and $\delta(p) \in \mathfrak{m} \subset \mathfrak{d}_4$ (see [13, §7]). Set $N^{\pm} := \exp \mathfrak{n}^{\pm} = \{\exp(\mathscr{G}_{\pm 1}(x) + \mathscr{G}_{\pm 2}(p)) \mid x \in \mathbf{O}, p \in \mathrm{Im}\,\mathbf{O}\}$ (*resp*). Because of $[\mathscr{G}_{\pm 1}(x), \mathscr{G}_{\pm 2}(p)] = 0$ (*resp*),

$$\begin{aligned} \exp \mathscr{G}_{\pm 2}(p) \exp \mathscr{G}_{\pm 1}(x) &= \exp(\mathscr{G}_{\pm 1}(x) + \mathscr{G}_{\pm 2}(p)) \\ &= \exp \mathscr{G}_{\pm 1}(x) \exp \mathscr{G}_{\pm 2}(p) \quad (\textit{resp}). \end{aligned} \tag{9.15}$$

By [13, Lemma 7.1], for any $D \in \mathfrak{d}_4$ and $a \in \mathbf{O}$,

$$\begin{cases} \text{(i)}\ \ d\tilde{\sigma}_i D = D, \quad \text{(ii)}\ \ d\tilde{\sigma}_i \tilde{A}_i^1(a) = \tilde{A}_i^1(a), \\ \text{(iii)}\ \ d\tilde{\sigma}_i \tilde{A}_j^1(a) = -\tilde{A}_j^1(a) \quad \text{for } j = i+1, i+2 \end{cases} \tag{9.16}$$

where indexes i, $i+1$, $i+2$, j are counted modulo 3. Then we get

$$d\tilde{\sigma}\mathscr{G}_{\pm 1}(x) = \mathscr{G}_{\mp 1}(x), \quad d\tilde{\sigma}\mathscr{G}_{\pm 2}(p) = \mathscr{G}_{\mp 2}(p) \quad (\textit{resp}), \tag{9.17}$$

$$\tilde{\sigma} \exp(\mathscr{G}_{\pm 1}(x) + \mathscr{G}_{\pm 2}(p)) = \exp(\mathscr{G}_{\mp 1}(x) + \mathscr{G}_{\mp 2}(p)) \quad (\textit{resp}) \tag{9.18}$$

with $x \in \mathbf{O}$ and $p \in \mathrm{Im}\,\mathbf{O}$. Especially, $\tilde{\sigma}(N^{\pm}) = N^{\mp}$ (*resp*). By [13, Corollary 8.9],

$$(F_{4(-20)})_{P^-} = N^+ M = M N^+. \tag{9.19}$$

Then from (9.7),

$$\mathscr{N}_1^- \simeq F_{4(-20)}/MN^+.$$

Fix the Cartan involution $\theta := d\tilde{\sigma}$ and set $\varepsilon(\alpha) = \varepsilon(-\alpha) := -1$ and $\varepsilon(2\alpha) = \varepsilon(-2\alpha) := 1$ on Σ. Then ε satisfies conditions (i) and (ii) of the signature of roots, and we consider the involutive automorphism θ_ε. We use same notations $\mathfrak{k}_\varepsilon$, $(K_\varepsilon)^0$, K_ε, M^*, M_ε^*, W, and W_ε corresponding to notations of given for general G, respectively.

PROPOSITION 9.2. (1) *$\theta_\varepsilon = d\tilde{\sigma}_2$ on $\mathfrak{f}_{4(-20)}$.*
(2) *θ_ε can be lifted on the group $F_{4(-20)}$ as $\tilde{\sigma}_2$ and*

$$K_\varepsilon = (F_{4(-20)})^{\tilde{\sigma}_2} = (F_{4(-20)})_{E_2} = \mathrm{Spin}^0(8,1). \tag{9.20}$$

PROOF. Since $M \subset \mathrm{D}_4$ by (9.11), $\mathfrak{m} \subset \mathfrak{d}_4$. Let $t \in \mathbf{R}$, $D \in \mathfrak{m}$, $x \in \mathbf{O}$, and $p \in \operatorname{Im} \mathbf{O}$. Then using (9.16), (9.17), and the definition of ε,

$$d\tilde{\sigma}_2(t\tilde{A}_3^1(1) + D) = -t\tilde{A}_3^1(1) + D = \theta(t\tilde{A}_3^1(1) + D) = \theta_\varepsilon(t\tilde{A}_3^1(1) + D),$$

$$d\tilde{\sigma}_2\mathscr{G}_{\pm 1}(x) = -\mathscr{G}_{\mp 1}(x) = \varepsilon(\pm\alpha)\theta\mathscr{G}_{\pm 1}(x) = \theta_\varepsilon\mathscr{G}_{\pm 1}(x),$$

$$d\tilde{\sigma}_2\mathscr{G}_{\pm 2}(p) = \mathscr{G}_{\mp 2}(p) = \varepsilon(\pm 2\alpha)\theta\mathscr{G}_{\pm 2}(p) = \theta_\varepsilon\mathscr{G}_{\pm 2}(p).$$

Thus it follows from (9.12), (9.13), and (9.14) that $d\tilde{\sigma}_2 = \theta_\varepsilon$ on $\mathfrak{f}_{4(-20)}$. Then θ_ε can be lifted on $\mathrm{F}_{4(-20)}$ as $\tilde{\sigma}_2$. From (9.9), we see $(K_\varepsilon)^0 = \mathrm{Spin}^0(8,1) = (\mathrm{F}_{4(-20)})^{\tilde{\sigma}_2} = (\mathrm{F}_{4(-20)})_{E_2}$, and $M \subset (\mathrm{F}_{4(-20)})_{E_2}$ by (9.11). Therefore $K_\varepsilon = (K_\varepsilon)^0 M = (\mathrm{F}_{4(-20)})_{E_2}$. □

PROPOSITION 9.3. (1) $M^* = M \coprod \sigma M$. *Especially,*

$$W = \{M, \sigma M\} \cong \{1, \sigma\} \cong \mathbf{Z}_2.$$

(2) $M_\varepsilon^* = M^* = M \coprod \sigma M$. *Especially,*

$$W_\varepsilon = \{M, \sigma M\} \cong \{1, \sigma\} \cong \mathbf{Z}_2, \quad [W : W_\varepsilon] = 1.$$

PROOF. (1) Fix $k \in M^*$. Then $k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(t)$ for some $t \in \mathbf{R}$. We set B as the Killing form of $\mathfrak{f}_{4(-20)}$, and a negative definite inner product $B_{\tilde{\sigma}}(\phi, \phi') := B(\phi, \tilde{\sigma}\phi')$ for $\phi, \phi' \in \mathfrak{f}_{4(-20)}$. Then $B_{\tilde{\sigma}}(\tilde{A}_3^1(1), \tilde{A}_3^1(1)) = B_{\tilde{\sigma}}(k\tilde{A}_3^1(1)k^{-1}, k\tilde{A}_3^1(1)k^{-1}) = t^2 B_{\tilde{\sigma}}(\tilde{A}_3^1(1), \tilde{A}_3^1(1))$. Thus $t = \pm 1$, so that $M^* = \{k \in K \mid k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(\pm 1)\}$. Put $L = \{k \in K \mid k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(-1)\}$. Then $M^* = M \coprod L$. Now, $\sigma \in (\mathrm{F}_{4(-20)})_{E_1} = K$ by (9.8), and $\sigma\tilde{A}_3^1(1)\sigma^{-1} = \tilde{\sigma}\tilde{A}_3^1(1) = \tilde{A}_3^1(-1)$ by (9.16). Therefor $\sigma \in M^*$, and since $\sigma k \in M$ for all $k \in L$, we get $L = \sigma M$. Hence (1) follows.

(2) Because of $\sigma E_2 = E_2$ and (9.20), we see $\sigma \in (\mathrm{F}_{4(-20)})_{E_2} = K_\varepsilon$. Then $\sigma \in K_\varepsilon \cap M^* = M_\varepsilon^*$. Therefore, because M is a subgroup of M_ε^* and (1), $M^* = M \coprod \sigma M \subset M_\varepsilon^* \subset M^*$, and so (2) follows. □

From $[W : W_\varepsilon] = 1$ and Proposition 9.1, the submanifold $K_\varepsilon A N^+$ is open dense in $\mathrm{F}_{4(-20)}$, and for any $g \in K_\varepsilon A N^+$, there exist unique elements $k_\varepsilon(g) \in K_\varepsilon$, $H_\varepsilon(g) \in \mathfrak{a}$, and $n_\varepsilon(g) \in N^+$ such that

$$g = k_\varepsilon(g)\exp(H_\varepsilon(g))n_\varepsilon(g).$$

However, this fact will be actually shown in this article.

For $x \in \mathbf{O}$, we denote $Q^+(x) := h^1(0,0,0;x,\bar{x},0)$ and $Q^-(x) := h^1(0,0,0;x,-\bar{x},0)$. We will prove the following main-theorem in §11.

MAIN THEOREM 9.4 (The explicit Iwasawa decomposition of $F_{4(-20)}$). *For any $g \in F_{4(-20)}$, there exist unique $k(g) \in K$, $H(g) \in \mathfrak{a}$, and $n_I(g) \in N^+$ such that*

$$g = k(\exp H(g))n_I(g)$$

where

(i) $$H(g) = 2^{-1}\log(-(gP^- \mid E_1))\tilde{A}_3^1(1) \in \mathfrak{a},$$

(ii) $$n_I(g) = \exp\Big(\mathscr{G}_1\Big(2^{-1}\Big(\sum\nolimits_{i=0}^{7}(gQ^+(e_i) \mid E_1)e_i\Big)\Big/(gP^- \mid E_1)\Big) + \mathscr{G}_2\Big(-2^{-1}\Big(\sum\nolimits_{i=1}^{7}(gF_3^1(e_i) \mid E_1)e_i\Big)\Big/(gP^- \mid E_1)\Big)\Big) \in N^+,$$

(iii) $$k(g) = gn_I(g)^{-1}\exp(-H(g)) \in K.$$

We define the equivalence relation $\sim$ on $\mathscr{N}_1^-$ by

$$X \sim Y \overset{\text{def}}{\iff} Y = rX \quad \text{for some } r > 0$$

where $X, Y \in \mathscr{N}_1^-$. We denote the quotient set

$$\mathscr{F} := \mathscr{N}_1^-/\sim,$$

and the equivalence class of $X \in \mathscr{N}_1^-$ by $[X]$. From (9.7), $F_{4(-20)}$ acts on $\mathscr{F}$:

$$g[X] := [gX] \quad \text{for } g \in F_{4(-20)} \text{ and } X \in \mathscr{N}_1^-.$$

We will prove the following theorem in §11.

THEOREM 9.5.

(1) $$(F_{4(-20)})_{[P^-]} = MAN^+.$$

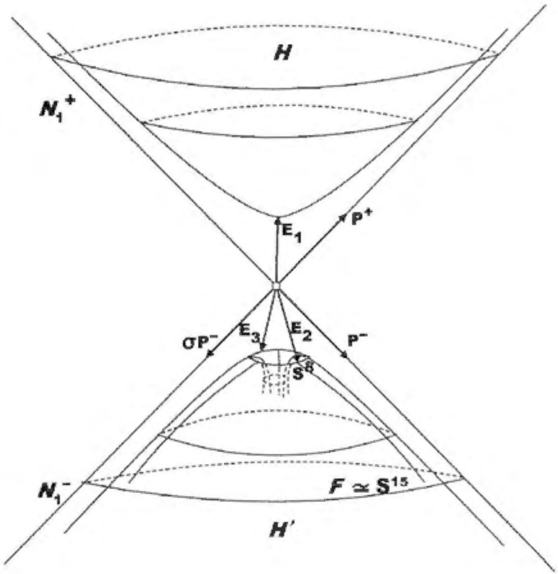
(2) $$F_{4(-20)}/MAN^+ \simeq \mathscr{F}.$$

(3) $$\mathscr{F} = K \cdot [P^-].$$

(4) $$\mathscr{F} \simeq \mathrm{Spin}(9)/\mathrm{Spin}(7).$$

Since $rX \in \mathscr{N}_1^-$ for all $r > 0$ and $X \in \mathscr{N}_1^-$, $\mathscr{N}_1^-$ is a cone in $\mathscr{J}^1$. Setting $-\mathscr{N}_1^+ := \{-X \mid X \in \mathscr{N}_1^+\}$, we see that $\mathscr{N}_1^- = -\mathscr{N}_1^+$ from the definitions of $\mathscr{N}_1^+$

and $\mathcal{N}_1^-$, and that $\sigma P^- = -P^+$. And noting that $\mathcal{F} = \mathcal{N}_1^-/\sim$, (9.3), and $\mathcal{F} \simeq S^{15}$ (see Proposition 11.2), we draw the following figure.



We will prove the following main-theorem in §12.

MAIN THEOREM 9.6 (The explicit K_ε-Iwasawa decomposition of $F_{4(-20)}$).

$$K_\varepsilon AN^+ = \{g \in F_{4(-20)} \mid (gP^- \mid E_2) \neq 0\}$$
$$= \{g \in F_{4(-20)} \mid (gP^- \mid E_2) > 0\}.$$

Furthermore, the submanifold $K_\varepsilon AN^+$ is open dense in $F_{4(-20)}$.

For any $g \in K_\varepsilon AN^+$, there exist unique $k_\varepsilon(g) \in K_\varepsilon$, $H_\varepsilon(g) \in \mathfrak{a}$, and $n_\varepsilon(g) \in N^+$ such that

$$g = k_\varepsilon(\exp H_\varepsilon(g))n_\varepsilon(g)$$

where

(i) $$H_\varepsilon(g) = 2^{-1} \log((gP^- \mid E_2))\tilde{A}_3^1(1) \in \mathfrak{a},$$

(ii) $$n_\varepsilon(g) = \exp\Big(\mathcal{G}_1\Big(2^{-1}\Big(\sum_{i=0}^7 (gQ^+(e_i) \mid E_2)e_i\Big)\Big/(gP^- \mid E_2)\Big) + \mathcal{G}_2\Big(-2^{-1}\Big(\sum_{i=1}^7 (gF_3^1(e_i) \mid E_2)e_i\Big)\Big/(gP^- \mid E_2)\Big)\Big) \in N^+,$$

(iii) $$k_\varepsilon(g) = gn_\varepsilon(g)^{-1} \exp(-H_\varepsilon(g)).$$

We denote the elements $P_{12}^-, P_{13}^- \in \mathcal{J}^1$ by $P_{12}^- := h^1(-1,1,0;0,0,1) = P^-$ and $P_{13}^- := h^1(-1,0,1;0,1,0)$, respectively. We will prove the following theorems in §13.

THEOREM 9.7. *$\mathcal{F}$ decomposes into the following two K_ε-orbits:*

$$\mathcal{F} = \coprod_{i=2}^3 K_\varepsilon \cdot [P_{1i}^-]$$

where

$$K_\varepsilon \cdot [P_{12}^-] = \{[X] \in \mathcal{F} \mid (X|E_2) \neq 0\} = \{[X] \in \mathcal{F} \mid (X|E_2) > 0\},$$

$$K_\varepsilon \cdot [P_{13}^-] = \{[X] \in \mathcal{F} \mid (X|E_2) = 0\}.$$

MAIN THEOREM 9.8 (The explicit Matsuki decomposition of $F_{4(-20)}$).

$$F_{4(-20)} = K_\varepsilon MAN^+ \coprod K_\varepsilon \exp(-2^{-1}\pi \tilde{A}_1^1(1))MAN^+$$

where $K_\varepsilon MAN^+ = K_\varepsilon AN^+$ and

$$K_\varepsilon \exp(-2^{-1}\pi \tilde{A}_1^1(1))MAN^+ = \{g \in F_{4(-20)} \mid (gP^- \mid E_2) = 0\}.$$

Theorems 9.7 and 9.8 are special cases of general theory [10, Theorems 1-Corollary and 3].

Since the Bruhat decomposition is associated with the N^--orbits on $F_{4(-20)}/MAN^+$, we will show the following theorem in §14.

THEOREM 9.9. *$\mathcal{F}$ decomposes into the following two N^--orbits:*

$$\mathcal{F} = N^- \cdot [P^-] \coprod N^- \cdot [\sigma P^-]$$

where

$$N^- \cdot [P^-] = \{[X] \in \mathcal{F} \mid (X \mid \sigma P^-) > 0\} = \{[X] \in \mathcal{F} \mid (X \mid \sigma P^-) \neq 0\},$$

$$N^- \cdot [\sigma P^-] = \{[X] \in \mathcal{F} \mid (X \mid \sigma P^-) = 0\} = \{[\sigma P^-]\}.$$

We will prove the following main-theorem in §14.

MAIN THEOREM 9.10. (1) (The explicit Bruhat decomposition of $F_{4(-20)}$).

$$F_{4(-20)} = N^- MAN^+ \coprod \sigma MAN^+$$

where

$$N^-MAN^+ = \{g \in F_{4(-20)} \mid (gP^- \mid \sigma P^-) \neq 0\}$$
$$= \{g \in F_{4(-20)} \mid (gP^- \mid \sigma P^-) > 0\},$$
$$\sigma MAN^+ = N^-\sigma MAN^+$$
$$= \{g \in F_{4(-20)} \mid (gP^- \mid \sigma P^-) = 0\}$$
$$= \{g \in F_{4(-20)} \mid g[P^-] = [\sigma P^-]\}.$$

Furthermore, the submanifold N^-MAN^+ *is open dense in* $F_{4(-20)}$.

(2) (The explicit Gauss decomposition of $F_{4(-20)}$).

For any $g \in N^-MAN^+$, *there exist unique* $n_G^-(g) \in N^-$, $m_G(g) \in M$, $a_G(g) \in A$, *and* $n_G^+(g) \in N^+$ *such that*

$$g = n_G^-(g)m_G(g)a_G(g)n_G^+(g)$$

where

(i) $$a_G(g) = \exp(2^{-1}\log(4^{-1}(gP^- \mid \sigma P^-))\tilde{A}_3^1(1)) \in A,$$

(ii) $$n_G^-(g) = \exp\Big(\mathscr{G}_{-1}\Big(-2^{-1}\Big(\sum\nolimits_{i=0}^7 (Q^-(e_i) \mid gP^-)e_i\Big)\Big/(gP^- \mid \sigma P^-)\Big)$$
$$+ \mathscr{G}_{-2}\Big(-2^{-1}\Big(\sum\nolimits_{i=1}^7 (F_3^1(e_i) \mid gP^-)e_i\Big)\Big/(gP^- \mid \sigma P^-)\Big)\Big) \in N^-,$$

(iii) $$n_G^+(g) = n_I(n_G^-(g)^{-1}g) \in N^+,$$

(iv) $$m_G(g) = n_G^-(g)^{-1}gn_G^+(g)^{-1}a_G(g)^{-1} \in M.$$

Here $n_I : F_{4(-20)} \to N^+$ *is the map used in the Iwasawa decomposition.*

REMARK 9.11. In Main Theorems 9.4, 9.6, 9.8, and 9.10, it appears that the Iwasawa decomposition, the K_ε-Iwasawa decomposition, the Matsuki decomposition, and the Bruhat and Gauss decompositions of $F_{4(-20)}$ can be explicitly described by using the geometric quantities $(gP^- \mid E_1)$, $(gP^- \mid E_2)$, and $(gP^- \mid \sigma P^-)$ with $g \in F_{4(-20)}$.

REMARK 9.12. The Iwasawa decomposition of the exceptional Lie group $F_{4(-20)}$ has been studied by R. Takahashi [18, Theorem 1]. He showed that AN^+ transitively and freely acts on the hyperbolic plane $\mathscr{H} = F_{4(-20)}/K$. Thereby, he gave the existence and uniqueness of the factors of the Iwasawa decomposition for $F_{4(-20)}$. In Main-Theorem 9.4, we give explicit formulas of $H(g)$ and $n_I(g)$.

10. Preliminaries

If $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)) \in \mathscr{J}^1$, then we denote $(X)_{E_i} := \xi_i \in \mathbf{R}$ and $(X)_{F_i^1} = x_i \in \mathbf{O}$. Set $F_3^1(\mathrm{Im}\,\mathbf{O}) := \{F_3^1(p) \mid p \in \mathrm{Im}\,\mathbf{O}\}$, $Q^+(\mathbf{O}) := \{Q^+(x) \mid x \in \mathbf{O}\}$, and $Q^-(\mathbf{O}) := \{Q^-(x) \mid x \in \mathbf{O}\}$ in $\mathscr{J}^1$. Then

$$\mathscr{J}^1 = \mathbf{R}(-E_1 + E_2) \oplus \mathbf{R}P^- \oplus \mathbf{R}E \oplus \mathbf{R}E_3 \oplus F_3^1(\mathrm{Im}\,\mathbf{O}) \oplus Q^+(\mathbf{O}) \oplus Q^-(\mathbf{O}). \tag{10.1}$$

So, for any $X \in \mathscr{J}^1$, we can uniquely write

$$X = r(-E_1 + E_2) + sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$$
$$= \begin{pmatrix} -r - s + u & \sqrt{-1}(s+p) & \sqrt{-1}(x-y) \\ \sqrt{-1}(s-p) & r + s + u & x + y \\ \sqrt{-1}(\bar{x} - \bar{y}) & \bar{x} + \bar{y} & u + v \end{pmatrix}$$

with $r, s, u, v \in \mathbf{R}$, $p \in \mathrm{Im}\,\mathbf{O}$, and $x, y \in \mathbf{O}$, and set

$$\{X\}_{-E_1+E_2} := r, \quad \{X\}_{P^-} := s, \quad \{X\}_E := u, \quad \{X\}_{E_3} := v,$$
$$\{X\}_{\mathrm{Im}\,F_3^1} := p, \quad \{X\}_{Q^+} := x, \quad \{X\}_{Q^-} := y.$$

LEMMA 10.1.

(1) $$\{X\}_{-E_1+E_2} = 2^{-1}(P^- | X).$$

(2) $$\{X\}_{Q^-} = 2^{-1}((X)_{F_1^1} - \overline{(X)_{F_2^1}}) = 4^{-1}\sum_{i=0}^{7}(Q^+(e_i) \mid X)e_i.$$

(3) $$\{X\}_{\mathrm{Im}\,F_3^1} = \mathrm{Im}((X)_{F_3^1}) = -2^{-1}\sum_{i=1}^{7}(F_3^1(e_i) \mid X)e_i.$$

PROOF. Let $X = r(-E_1 + E_2) + sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$ with $r, s, u, v \in \mathbf{R}$, $p \in \mathrm{Im}\,\mathbf{O}$, and $x, y \in \mathbf{O}$. Then $(P^- | X) = 2r$, and so (1) follows. Because of $(X)_{F_1^1} = x + y$ and $(X)_{F_2^1} = \bar{x} - \bar{y}$, $\{X\}_{Q^-} = y = 2^{-1}((X)_{F_1^1} - \overline{(X)_{F_2^1}})$. Now, set $(X)_{F_1^1} = \sum_{i=0}^{7} p_i e_i$ and $(X)_{F_2^1} = \sum_{i=0}^{7} q_i e_i$ with $p_i, q_i \in \mathbf{R}$. From (9.1), $p_i = 2^{-1}(F_1^1(e_i) \mid X)$ and $q_i = -2^{-1}(F_2^1(e_i) \mid X)$. Then $(X)_{F_1^1} - \overline{(X)_{F_2^1}} = 2^{-1}\sum_{i=0}^{7}(Q^+(e_i) \mid X)e_i$, and so (2) follows. Last, obviously $\{X\}_{\mathrm{Im}\,F_3^1} = p = \mathrm{Im}((X)_{F_3^1})$. Set $(X)_{F_3^1} = \sum_{i=0}^{7} r_i e_i$ with $r_i \in \mathbf{R}$. From (9.1), $r_i = -2^{-1}(F_3^1(e_i) \mid X)$, and so (3) follows. □

We denote $\mathscr{J}^1(2;\mathbf{K}) := \{\xi_1 E_1 + \xi_2 E_2 + F_3^1(x) \mid \xi_i \in \mathbf{R}, x \in \mathbf{K}\}$ with $\mathbf{K} = \mathbf{O}$ or $\mathbf{R}$.

LEMMA 10.2.

(1) $$\mathscr{J}^1 = \mathscr{J}^1(2;\mathbf{O}) \oplus \mathbf{R}E_3 \oplus Q^+(\mathbf{O}) \oplus Q^-(\mathbf{O}).$$

(2) $$\mathscr{J}^1(2;\mathbf{O}) = \mathbf{R}(-E_1 + E_2) \oplus \mathbf{R}P^- \oplus \mathbf{R}(E - E_3) \oplus F_3^1(\operatorname{Im}\mathbf{O}).$$

(3) $$\mathscr{J}^1(2;\mathbf{R}) = \mathbf{R}(-E_1 + E_2) \oplus \mathbf{R}P^- \oplus \mathbf{R}(E - E_3).$$

Let $p, q \in \operatorname{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$. From [13, Lemma 7.11],

$$(10.2)\quad \begin{cases} \text{(i)} & \exp\mathscr{G}_2(p)(-E_1 + E_2) = (-E_1 + E_2) + F_3^1(-2p) + 2(p|p)P^-, \\ \text{(ii)} & \exp\mathscr{G}_2(p)P^- = P^-, \quad \text{(iii)}\ \exp\mathscr{G}_2(p)E = E, \\ \text{(iv)} & \exp\mathscr{G}_2(p)E_3 = E_3, \\ \text{(v)} & \exp\mathscr{G}_2(p)F_3^1(q) = F_3^1(q) - 2(p|q)P^-, \\ \text{(vi)} & \exp\mathscr{G}_2(p)Q^+(y) = Q^+(y), \\ \text{(vii)} & \exp\mathscr{G}_2(p)Q^-(y) = Q^-(y) + Q^+(-2py), \end{cases}$$

$$(10.3)\quad \begin{cases} \text{(i)} & \exp\mathscr{G}_1(x)(-E_1 + E_2) = (-E_1 + E_2) + Q^-(-x) \\ & \qquad - (x|x)(E - 3E_3) + Q^+((x|x)x) + 2^{-1}(x|x)^2P^-, \\ \text{(ii)} & \exp\mathscr{G}_1(x)P^- = P^-, \quad \text{(iii)}\ \exp\mathscr{G}_1(x)E = E, \\ \text{(iv)} & \exp\mathscr{G}_1(x)E_3 = E_3 + Q^+(x) + (x|x)P^-, \\ \text{(v)} & \exp\mathscr{G}_1(x)F_3^1(q) = F_3^1(q) + Q^+(-qx), \\ \text{(vi)} & \exp\mathscr{G}_1(x)Q^+(y) = Q^+(y) + 2(x|y)P^-, \\ \text{(vii)} & \exp\mathscr{G}_1(x)Q^-(y) = Q^-(y) + 2(x|y)(E - 3E_3) + F_3^1(2\operatorname{Im}(x\bar{y})) \\ & \qquad + Q^+(-3(x|y)x - \operatorname{Im}(x\bar{y})x) - 2(x|y)(x|x)P^-. \end{cases}$$

We denote the subset $\mathfrak{P}$ of $\mathscr{J}^1$ by $\mathfrak{P} := \{X \in \mathscr{J}^1 \mid X^{\times 2} = 0, X \neq 0\}$. Then $\mathfrak{P}$ contains the exceptional hyperbolic planes $\mathscr{H}$, $\mathscr{H}'$ and the exceptional null cones $\mathscr{N}_1^+$, $\mathscr{N}_1^-$. Since the action of $F_{4(-20)}$ preserves the cross product, $F_{4(-20)}$ acts on $\mathfrak{P}$. For any subset $S \subset \mathscr{J}^1$ and $Z \in \mathscr{J}^1$, we denote

$$S^Z_{>0} := \{X \in S \mid (Z|X) > 0\}, \quad S^Z_{<0} := \{X \in S \mid (Z|X) < 0\},$$

$$S^Z_{=0} := \{X \in S \mid (Z|X) = 0\}, \quad S^Z_{\neq 0} := \{X \in S \mid (Z|X) \neq 0\}.$$

We recall Lemma 10.1. For any $X \in (\mathscr{J}^1)^{P^-}_{\neq 0}$, we define the elements $n_1(X) \in \exp\mathfrak{g}_\alpha \subset N^+$ and $n_2(X) \in \exp\mathfrak{g}_{2\alpha} \subset N^+$ by

$$n_1(X) := \exp \mathscr{G}_1(\{X\}_{Q^-}/\{X\}_{-E_1+E_2})$$
$$= \exp \mathscr{G}_1\Big(2^{-1}\Big(\sum\nolimits_{i=0}^{7}(Q^+(e_i)\,|\,X)e_i\Big)\Big/(P^-|X)\Big),$$
$$n_2(X) := \exp \mathscr{G}_2(\{X\}_{\mathrm{Im}\,F_3^1}/(P^-|X))$$
$$= \exp \mathscr{G}_2\Big(-2^{-1}\Big(\sum\nolimits_{i=1}^{7}(F_3^1(e_i)\,|\,X)e_i\Big)\Big/(P^-|X)\Big)$$

respectively, and $n_X := n_1(X)n_2(X) = n_2(X)n_1(X) \in N^+$ (see (9.15)).

LEMMA 10.3. (1) *For any* $n \in N^+$ *and* $X \in \mathscr{J}^1$, $(P^-\,|\,nX) = (P^-|X)$. *Especially,* N^+ *acts on* $(\mathscr{J}^1)^{P^-}_{\neq 0}$ *and* $\mathfrak{P}^{P^-}_{\neq 0}$, *respectively.*

(2) *For any* $X \in (\mathscr{J}^1)^{P^-}_{\neq 0}$,

(i) $$n_1(X)X \in (\mathscr{J}^1(2;\mathbf{O}) \oplus \mathbf{R}E_3 \oplus Q^+(\mathbf{O})) \cap (\mathscr{J}^1)^{P^-}_{\neq 0},$$

(ii) $$\{n_1(X)X\}_{\mathrm{Im}\,F_3^1} = \{X\}_{\mathrm{Im}\,F_3^1}.$$

(3) *If* $X \in \mathscr{J}^1(2;\mathbf{O}) \cap (\mathscr{J}^1)^{P^-}_{\neq 0}$, *then*

$$n_2(X)X \in \mathscr{J}^1(2;\mathbf{R}) \cap (\mathscr{J}^1)^{P^-}_{\neq 0}.$$

PROOF. (1) From (9.19), $(P^-\,|\,nX) = (n^{-1}P^-\,|\,X) = (P^-|X)$ and so on.

(2) We can write $X = r(-E_1+E_2) + sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$ for some $r,s,u,v \in \mathbf{R}$, $p \in \mathrm{Im}\,\mathbf{O}$, and $x, y \in \mathbf{O}$. From Lemma 10.1(1), $r \neq 0$ and put $n_1' = n_1(X) = \exp \mathscr{G}_1(r^{-1}y)$. In (10.3), we notice that the equations (10.3)(i) and (10.3)(vii) have terms of $Q^-(\cdot)$ and the other equations have not terms of $Q^-(\cdot)$, and that the equations (10.3)(v) and (10.3)(vii) have terms of $F_3^1(\cdot)$ and the other equations have not terms of $F_3^1(\cdot)$. Therefore

$$\{n_1' \cdot X\}_{Q^-} = \{n_1' \cdot (r(-E_1+E_2) + Q^-(y) + (\textit{other terms}))\}_{Q^-}$$
$$= -r(r^{-1}y) + y + 0 = 0.$$

Thus $\{n_1(X)X\}_{Q^-} = 0$, so that $n_1(X)X \in \mathscr{J}^1(2;\mathbf{O}) \oplus \mathbf{R}E_3 \oplus Q^+(\mathbf{O})$. Then $(P^-|n_1' \cdot X) = (P^-|X) \neq 0$ by (1), and

$$\{n_1' \cdot X\}_{\mathrm{Im}\,F_3^1} = \{n_1' \cdot (F_3^1(p) + Q^-(y) + (\textit{other terms}))\}_{\mathrm{Im}\,F_3^1}$$
$$= p + 2\,\mathrm{Im}((r^{-1}y)\bar{y}) + 0 = p = \{X\}_{\mathrm{Im}\,F_3^1}.$$

Hence we obtain (2).

(3) We can write $X = r(-E_1 + E_2) + sP^- + u(E - E_3) + F_3^1(p)$ for some $r, s, u \in \mathbf{R}$ and $p \in \operatorname{Im} \mathbf{O}$. From Lemma 10.1(1), $r \neq 0$ and put $n_2' = n_2(X) = \exp \mathscr{G}_2((2r)^{-1}p)$. Using (10.2), we calculate that

$$n_2' \cdot X = r(-E_1 + E_2) + (s - (2r)^{-1}(p|p))P^- + u(E - E_3).$$

Then $n_2' \cdot X \in \mathscr{J}^1(2; \mathbf{R})$; and $(P^-|n_2' \cdot X) = (P^-|X) \neq 0$. Hence we obtain (3). □

LEMMA 10.4.

$$(\mathscr{J}^1(2; \mathbf{O}) \oplus \mathbf{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathfrak{P}_{\neq 0}^{P^-} = \mathscr{J}^1(2; \mathbf{O}) \cap \mathfrak{P}_{\neq 0}^{P^-}.$$

PROOF. Obviously, $\mathscr{J}^1(2; \mathbf{O}) \cap \mathfrak{P}_{\neq 0}^{P^-} \subset (\mathscr{J}^1(2; \mathbf{O}) \oplus \mathbf{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathfrak{P}_{\neq 0}^{P^-}$. Conversely, take $X \in (\mathscr{J}^1(2; \mathbf{O}) \oplus \mathbf{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathfrak{P}_{\neq 0}^{P^-}$ and set $X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1^1(x) + F_2^1(\bar{x}) + F_3^1(y)$ with $\xi_i \in \mathbf{R}$ and $x, y \in \mathbf{O}$. Suppose that $\xi_3 \neq 0$. Because of $X \in \mathfrak{P}$ and (9.2),

$$\text{(i)} \quad \xi_2\xi_3 - (x|x) = (X^{\times 2})_{E_1} = 0, \quad \text{(ii)} \ \xi_3\xi_1 + (x|x) = (X^{\times 2})_{E_2} = 0,$$

$$\text{(iii)} \ (x|x) - \xi_3 y = (X^{\times 2})_{F_3^1} = 0.$$

From (i), (ii), and (iii), $X = -((x|x)/\xi_3)E_1 + ((x|x)/\xi_3)E_2 + \eta E_3 + F_1^1(x) + F_2^1(\bar{x}) + F_3^1((x|x)/\xi_3)$. Then $(P^-|X) = 0$, and it contradicts with $X \in \mathfrak{P}_{\neq 0}^{P^-}$. Thus $\xi_3 = 0$. Then $(x|x) = (X^{\times 2})_{E_2} = 0$, so that $x = 0$. Thus $X = \xi_1 E_1 + \xi_2 E_2 + F_3^1(y) \in \mathscr{J}^1(2; \mathbf{O}) \cap \mathfrak{P}_{\neq 0}^{P^-}$, and so $(\mathscr{J}^1(2; \mathbf{O}) \oplus \mathbf{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathfrak{P}_{\neq 0}^{P^-} \subset \mathscr{J}^1(2; \mathbf{O}) \cap \mathfrak{P}_{\neq 0}^{P^-}$. Hence the result follows. □

LEMMA 10.5. *For any $X \in \mathfrak{P}_{\neq 0}^{P^-}$, $n_X X \in \mathscr{J}^1(2; \mathbf{R}) \cap \mathfrak{P}_{\neq 0}^{P^-}$. Further,*

$$\begin{aligned} n_X X &= 4^{-1}(2 \operatorname{tr}(X) - (P^-|X)^{-1} \operatorname{tr}(X)^2 - (P^-|X))E_1 \\ &\quad + 4^{-1}(2 \operatorname{tr}(X) + (P^-|X)^{-1} \operatorname{tr}(X)^2 + (P^-|X))E_2 \\ &\quad + F_3^1(4^{-1}((P^-|X)^{-1} \operatorname{tr}(X)^2 - (P^-|X))) \\ &= 2^{-1}(P^-|X)(-E_1 + E_2) + 4^{-1}((P^-|X)^{-1} \operatorname{tr}(X)^2 - (P^-|X))P^- \\ &\quad + 2^{-1} \operatorname{tr}(X)(E - E_3). \end{aligned}$$

PROOF. N^+ acts on $\mathfrak{P}_{\neq 0}^{P^-}$, and $n_i(X) \in N^+$. Put $X' = n_1(X)X \in \mathfrak{P}_{\neq 0}^{P^-}$. By Lemma 10.3(2),

$$X' \in (\mathscr{J}(2; \mathbf{O}) \oplus \mathbf{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathfrak{P}_{\neq 0}^{P^-}$$

where $(P^-|X') = (P^-|X) \neq 0$ and $\{X'\}_{\mathrm{Im}\, F_3^1} = \{X\}_{\mathrm{Im}\, F_3^1}$. Applying Lemma 10.4.

$$X' \in \mathscr{J}(2;\mathbf{O}) \cap \mathfrak{P}^{P^-}_{\neq 0}.$$

Applying Lemma 10.3(3),

$$n_2(X')X' \in \mathscr{J}^1(2;\mathbf{R}) \cap \mathfrak{P}^{P^-}_{\neq 0}.$$

Then, since $\exp \mathscr{G}_2(\{X\}_{\mathrm{Im}\, F_3^1}/(P^-|X)) = \exp \mathscr{G}_2(\{X'\}_{\mathrm{Im}\, F_3^1}/(P^-|X'))$, we see $n_2(X) = n_2(X')$. Therefore

$$n_X X = n_2(X)n_1(X)X = n_2(X')X' \in \mathscr{J}^1(2;\mathbf{R}) \cap \mathfrak{P}^{P^-}_{\neq 0}.$$

Set $n_X X = \xi_1 E_1 + \xi_2 E_2 + F_3^1(x) \in \mathscr{J}^1(2;\mathbf{R}) \cap \mathfrak{P}^{P^-}_{\neq 0}$ with $\xi_1, \xi_2, x \in \mathbf{R}$. Then $\mathrm{tr}(X) = \xi_1 + \xi_2$ and $(0 \neq)(P^-|X) = (P^-\,|\,n_X X) = -\xi_1 + \xi_2 - 2x$, so that $\xi_1 = 2^{-1}\,\mathrm{tr}(X) - x - 2^{-1}(P^-|X)$ and $\xi_2 = 2^{-1}\,\mathrm{tr}(X) + x + 2^{-1}(P^-|X)$. From $(n_X X)^{\times 2} = 0$, $0 = ((n_X X)^{\times 2})_{E_3} = \xi_1\xi_2 + x^2 = 4^{-1}\,\mathrm{tr}(X)^2 - 4^{-1}(P^-|X)^2 - x(P^-|X)$. Thus $x = 4^{-1}((P^-|X)^{-1}\,\mathrm{tr}(X)^2 - (P^-|X))$, $\xi_1 = 4^{-1}(2\,\mathrm{tr}(X) - (P^-|X)^{-1}\,\mathrm{tr}(X)^2 - (P^-|X))$, and $\xi_2 = 4^{-1}(2\,\mathrm{tr}(X) + (P^-|X)^{-1}\,\mathrm{tr}(X)^2 + (P^-|X))$. Moreover, the last equation follows from direct calculations. □

Let $i \in \{1,2,3\}$, $t \in \mathbf{R}$, and $a \in \mathbf{O}$ with $(a|a) = 1$. From [13, Lemma 3.10], we recall the operation of $\exp(t\tilde{A}_i^1(a))$. Set

$$h^1(\eta_1,\eta_2,\eta_3;y_1,y_2,y_3) := \exp(t\tilde{A}_i^1(a))h^1(\xi_1,\xi_2,\xi_3;x_1,x_2,x_3)$$

with $\xi_i, \eta_i \in \mathbf{R}$ and $x_i, y_i \in \mathbf{O}$. When $i = 1$,

$$(10.4)\qquad \begin{cases} \eta_1 = \xi_1, \\ \eta_2 = 2^{-1}((\xi_2+\xi_3) + (\xi_2-\xi_3)\cos 2t) + (a|x_1)\sin 2t, \\ \eta_3 = 2^{-1}((\xi_2+\xi_3) - (\xi_2-\xi_3)\cos 2t) - (a|x_1)\sin 2t, \\ y_1 = x_1 - 2^{-1}(\xi_2-\xi_3)a\sin 2t - 2(a|x_1)a\sin^2 t, \\ y_2 = x_2\cos t - \overline{x_3 a}\sin t, \\ y_3 = x_3\cos t + \overline{a x_2}\sin t \end{cases}$$

and when $i \in \{2,3\}$,

$$(10.5)\qquad \begin{cases} \eta_i = \xi_i, \\ \eta_{i+1} = 2^{-1}((\xi_{i+1}+\xi_{i+2}) + (\xi_{i+1}-\xi_{i+2})\cosh 2t) - (a|x_i)\sinh 2t, \\ \eta_{i+2} = 2^{-1}((\xi_{i+1}+\xi_{i+2}) - (\xi_{i+1}-\xi_{i+2})\cosh 2t) + (a|x_i)\sinh 2t, \\ y_i = x_i - 2^{-1}(\xi_{i+1}-\xi_{i+2})a\sinh 2t + 2(a|x_i)a\sinh^2 t. \\ y_{i+1} = x_{i+1}\cosh t + \overline{x_{i+2}a}\sinh t, \\ y_{i+2} = x_{i+2}\cosh t + \overline{a x_i}\sinh t \end{cases}$$

where indexes i, $i+1$, $i+2$ are counted modulo 3. In particular,

(10.6) $$\exp(2^{-1}\pi\tilde{A}_1^1(1))h^1(\xi_1,\xi_2,\xi_3;x_1,x_2,x_3) = h^1(\xi_1,\xi_3,\xi_2;-\overline{x_1},-\overline{x_3},\overline{x_2})$$

with $\xi_i \in \mathbf{R}$ and $x_i \in \mathbf{O}$. Using (10.5), we have the following lemma.

LEMMA 10.6. *For any t, r, s, $u \in \mathbf{R}$,*

$$a_t(r(-E_1+E_2)+sP^- + u(E-E_3))$$
$$= re^{-2t}(-E_1+E_2)+(r\sinh 2t + se^{2t})P^- + u(E-E_3).$$

LEMMA 10.7. *For any $m \in M$, $t \in \mathbf{R}$, and $n \in N^+$,*

$$ma_tnP^- = e^{2t}P^-.$$

Furthermore, $A \cap MN^+ = \{1\}$ and $M \cap AN^+ = \{1\}$.

PROOF. From (9.19) and Lemma 10.6, $ma_tnP^- = e^{2t}P^-$. Suppose $a_t = mn$ for some $t \in \mathbf{R}$, $m \in M$, and $n \in N^+$. From the above equation and (9.19), $e^{2t}P^- = a_tP^- = mnP^- = P^-$. Thus $t = 0$, and $A \cap MN^+ = \{1\}$. Similarly, suppose $m = a_tn$ for some $m \in M$, $t \in \mathbf{R}$, and $n \in N^+$. Then $P^- = mP^- = a_tnP^- = e^{2t}P^-$. Thus $t = 0$, and $M \cap AN^+ = \{1\}$. □

LEMMA 10.8. $(\mathrm{F}_{4(-20)})_{\sigma P^-} = MN^-$.

PROOF. Because of $M \subset K = (\mathrm{F}_{4(-20)})^{\tilde{\sigma}}$, $\sigma M = M\sigma$. Using (9.19), $(\mathrm{F}_{4(-20)})_{\sigma P^-} = \sigma(\mathrm{F}_{4(-20)})_{P^-}\sigma^{-1} = \sigma MN^+\sigma^{-1} = M\tilde{\sigma}(N^+) = MN^-$. □

LEMMA 10.9. (1) *For any $t \in \mathbf{R}$, $x \in \mathbf{O}$, and $p \in \mathrm{Im}\,\mathbf{O}$,*

$$a_t(\mathscr{G}_1(x)+\mathscr{G}_2(p))a_t^{-1} = \mathscr{G}_1(e^tx)+\mathscr{G}_2(e^{2t}p).$$

(2) *$AN^+ = N^+A$. Furthermore, AN^+ is a subgroup of $\mathrm{F}_{4(-20)}$.*
(3) *MAN^+ is a subgroup of $\mathrm{F}_{4(-20)}$.*

PROOF. (1) Set $T(t) \in \mathrm{GL}_{\mathbf{R}}(\mathfrak{f}_{4(-20)})$ and $\mathrm{ad}_{\tilde{A}_3^1(1)} \in \mathrm{End}_{\mathbf{R}}(\mathfrak{f}_{4(-20)})$ as $T(t)\phi := a_t\phi a_t^{-1}$ and $\mathrm{ad}_{\tilde{A}_3^1(1)}\phi := [\tilde{A}_3^1(1),\phi]$ for $\phi \in \mathfrak{f}_{4(-20)}$, respectively. Then $T(t) = \exp(t\,\mathrm{ad}_{\tilde{A}_3^1(1)})$, and using (9.13) and (9.14), $T(t)\mathscr{G}_1(x) = (\sum(t\,\mathrm{ad}_{\tilde{A}_3^1(1)})^n/n!)\mathscr{G}_1(x)$

$= \mathscr{G}_1((\sum(1/n!)t^n)x) = \mathscr{G}_1(e^t x)$ and $T(t)\mathscr{G}_2(p) = (\sum(t\,\mathrm{ad}_{\tilde{A}_3^1(1)})^n/n!)\mathscr{G}_2(p) = \mathscr{G}_1((\sum(1/n!)(2t)^n)p) = \mathscr{G}_2(e^{2t}p)$. Thus we obtain (1).

(2) From (1), $a_t n = a_t n a_t^{-1} a_t = \exp(a_t(\mathscr{G}_1(x) + \mathscr{G}_2(p))a_t^{-1})a_t = \exp(\mathscr{G}_1(e^t x) + \mathscr{G}_2(e^{2t}p))a_t$ and $na_t = a_t a_t^{-1} n a_t = a_t \exp(\mathscr{G}_1(e^{-t}x) + \mathscr{G}_2(e^{-2t}p))$. This implies that $AN^+ = NA^+$. Therefore $(a_t n)^{-1}(a_s n') \in AN^+$ for all $s, t \in \mathbf{R}$ and $n, n' \in N^+$, so that AN^+ is a subgroup.

(3) Because of (9.10), $MN^+ = N^+M$, and $AN^+ = N^+A$, we get $(ma_t n)^{-1}(m'a_s n') \in MAN^+$ for all $m, m' \in M$, $s, t \in \mathbf{R}$, and $n, n' \in N^+$. Thus MAN^+ is a subgroup of $F_{4(-20)}$. □

LEMMA 10.10. *Let* $k \in K$, $k_\varepsilon \in K_\varepsilon$, $m \in M$, $t \in \mathbf{R}$, $n \in N^+$, *and* $z \in N^-$.

$$(ka_t nP^- \mid E_1) = -e^{2t}. \tag{1}$$

$$(k_\varepsilon a_t nP^- \mid E_2) = e^{2t}. \tag{2}$$

$$(zma_t nP^- \mid \sigma P^-) = 4e^{2t}. \tag{3}$$

PROOF. From (9.8), (9.20), Lemmas 10.7, and 10.8, it follows that

$$(ka_t nP^- \mid E_1) = (a_t nP^- \mid k^{-1}E_1) = e^{2t}(P^- | E_1) = -e^{2t},$$

$$(k_\varepsilon a_t nP^- \mid E_2) = (a_t nP^- \mid k_\varepsilon^{-1}E_2) = e^{2t}(P^- | E_2) = e^{2t},$$

$$(zma_t nP^- \mid \sigma P^-) = (ma_t nP^- \mid z^{-1}\sigma P^-) = e^{2t}(P^- \mid \sigma P^-) = 4e^{2t}. \qquad \square$$

LEMMA 10.11. $M = (F_{4(-20)})_{P^-, E_j} = (F_{4(-20)})_{P^-, \sigma P^-}$ *with* $j \in \{1, 2\}$.

PROOF. Note $P^- = -E_1 + E_2 + F_3^1(1)$, $\sigma P^- = -E_1 + E_2 + F_3^1(-1)$, and $M = (F_{4(-20)})_{E_1, E_2, E_3, F_3^1(1)}$. Obviously, $M \subset (F_{4(-20)})_{P^-, E_j}$. Conversely, fix $g \in (F_{4(-20)})_{E_j, P^-}$. Now $((-1)^{j+1}E_j + P^-)^{\times 2} = E_3$. Then $gE_3 = g((-1)^{j+1}E_j + P^-)^{\times 2} = (g((-1)^{j+1}E_j + P^-))^{\times 2} = E_3$, and $gE_k = g(E - E_j - E_3) = E - E_j - E_3 = E_k$ where $(j, k) = (1, 2), (2, 1)$. Therefore $gE_i = E_i$ for all $i \in \{1, 2, 3\}$, and $gF_3^1(1) = g(P^+ + E_1 - E_2) = P^- + E_1 - E_2 = F_3^1(1)$. Then $g \in M$, so that $(F_{4(-20)})_{P^-, E_j} \subset M$. Thus $M = (F_{4(-20)})_{P^-, E_j}$.

Obviously $M \subset (F_{4(-20)})_{P^-, \sigma P^-}$. Conversely, fix $g \in (F_{4(-20)})_{P^-, \sigma P^-}$. Because of $-E_1 + E_2 = 2^{-1}(P^- - \sigma P^-)$, $(-E_1 + E_2)^{\times 2} = -E_3$, $F_3^1(1) = P^- - (-E_1 + E_2)$, $E_1 = 2^{-1}(E - (-E_1 + E_2) - E_3)$, and $E_2 = 2^{-1}(E + (-E_1 + E_2) - E_3)$, we sequentially get $g(-E_1 + E_2) = -E_1 + E_2$, $gE_3 = E_3$, $gF_3^1(1) = F_3^1(1)$, $gE_1 = E_1$, and $gE_2 = E_2$. Thus $g \in M$, and so $(F_{4(-20)})_{P^-, \sigma P^-} \subset M$. Hence $M = (F_{4(-20)})_{P^-, \sigma P^-}$. □

LEMMA 10.12. *Let $K' = K$ or K_ε.*

(1) $$D_4 \cap N^\pm = \{1\} \quad (resp),$$

(2) $$K' \cap AN^+ = \{1\},$$

(3) $$N^- \cap MAN^+ = \{1\}.$$

PROOF. (1) Fix $n \in D_4 \cap N^+$. Then $n \in D_4 \subset (F_{4(-20)})_{E_3, -E_1+E_2}$. Now, $n = \exp \mathscr{G}_1(x) \exp \mathscr{G}_2(p)$ for some $x \in \mathbf{O}$ and $p \in \mathrm{Im}\, \mathbf{O}$. Using (10.2) and (10.3), $E_3 = nE_3 = \exp \mathscr{G}_1(x) E_3 = E_3 + Q^+(x) + (x|x)P^-$. Then $x = 0$ by (10.1). Therefore from (10.2), $-E_1 + E_2 = n(-E_1 + E_2) = \exp \mathscr{G}_2(p)(-E_1 + E_2) = (-E_1 + E_2) + F_3^1(-2p) + (p|p)P^-$. Then $p = 0$. Thus $n = 1$, and $D_4 \cap N^+ = \{1\}$. Because of $D_4 \subset K = (F_{4(-20)})^{\tilde{\sigma}}$, $\tilde{\sigma}(D_4) = D_4$. Then from $\tilde{\sigma}(N^+) = N^-$, $D_4 \cap N^- = \tilde{\sigma}(D_4 \cap N^+) = \{1\}$.

(2) Take $j = 1$ if $K' = K$, and $j = 2$ if $K' = K_\varepsilon$. Suppose $k' = a_t n$ for some $k \in K'$, $t \in \mathbf{R}$, and $n \in N^+$. Using Lemma 10.10(1)(2), (9.8), and (9.20), $(-1)^j e^{2t} = (a_t n P^- \mid E_j) = (P^- \mid k'^{-1} E_j) = (P^- | E_j) = (-1)^j$. Therefore $t = 0$, and $K' \cap AN^+ \subset K' \cap N^+$. Next, using (9.8), (9.20), and (9.19), $K' \cap N^+ \subset (F_{4(-20)})_{E_j, P^-}$, and from Lemma 10.11, $K' \cap N^+ \subset (F_{4(-20)})_{E_j, P^-} \cap N^+ = M \cap N^+$. Therefore because of $M \subset D_4$ and (1), $\{1\} \subset K' \cap AN^+ \subset K' \cap N^+ \subset M \cap N^+ \subset D_4 \cap N^+ = \{1\}$. Hence $K' \cap AN^+ = \{1\}$.

(3) Suppose $z = m a_t n$ for some $z \in N^-$, $m \in M$, $t \in \mathbf{R}$, and $n \in N^+$. Using Lemmas 10.10(3) and 10.8, $4e^{2t} = (m a_t n P^- \mid \sigma P^-) = (P^- \mid z^{-1} \sigma P^-) = (P^- \mid \sigma P^-) = 4$. Therefore $t = 0$, so that $N^- \cap MAN^+ \subset N^- \cap MN^+$. Next, using (9.19) and Lemma 10.8, $N^- \cap MN^+ \subset (F_{4(-20)})_{P^-, \sigma P^-}$, and from Lemma 10.11, $N^- \cap MN^+ \subset N^- \cap (F_{4(-20)})_{P^-, \sigma P^-} = N^- \cap M$. Therefore because of $M \subset D_4$ and (1), $\{1\} \subset N^- \cap MAN^+ \subset N^- \cap MN^+ \subset M \cap N^- \subset D_4 \cap N^- = \{1\}$. Hence $N^- \cap MAN^+ = \{1\}$. □

LEMMA 10.13. (1) *If $k a_t n = k' a_s n'$ with $k, k' \in K$, $t, s \in \mathbf{R}$, and $n, n' \in N^+$ then $k = k'$, $t = s$, and $n = n'$.*

(2) *If $k_\varepsilon a_t n = k'_\varepsilon a_s n'$ with $k_\varepsilon, k'_\varepsilon \in K_\varepsilon$, $t, s \in \mathbf{R}$, and $n, n' \in N^+$ then $k_\varepsilon = k'_\varepsilon$, $t = s$, and $n = n'$.*

(3) *If $z m a_t n = z' m' a_s n'$ with $z, z' \in N^-$, $m, m' \in M$, $t, s \in \mathbf{R}$, and $n, n' \in N^+$ then $z = z'$, $m = m'$, $t = s$, and $n = n'$.*

PROOF. (1) From Lemma 10.9(2), $(a_s n')(a_t n)^{-1} \in AN^+$, so that $k'^{-1} k = (a_s n')(a_t n)^{-1} \in K \cap AN^+$. Using Lemma 10.12(2), $k = k'$ and $a_t n = a_s n'$. Next, because of $a_s^{-1}a_t = nn'^{-1} \in A \cap N^+$ and Lemma 10.7, $a_t = a_s \Leftrightarrow t = s$ and $n = n'$. Hence we obtain (1). Similarly, substituting K for K_ε, we obtain (2).

(3) By Lemma 10.9(3), $(m'a_sn')(ma_tn)^{-1} \in MAN^+$, so that $z'^{-1}z = (m'a_sn')(ma_tn)^{-1} \in N^- \cap MAN^+$. Using Lemma 10.12(3), $z = z'$ and $ma_tn = m'a_sn'$. Next, by Lemma 10.9(2), $(a_sn')(a_tn)^{-1} \in AN^+$, so that $m'^{-1}m = (a_sn')(a_tn)^{-1} \in M \cap AN^+$. Using Lemma 10.7, $m = m'$ and $a_tn = a_sn'$. Last, because of $a_s^{-1}a_t = nn'^{-1} \in A \cap N^+$ and Lemma 10.7, $a_t = a_s \Leftrightarrow t = s$ and $n = n'$. Hence we obtain (3). □

LEMMA 10.14. (1) *For any* $X \in \mathscr{H}$ *and* $Y \in \mathscr{N}_1^-$, $(X|Y) < 0$.

(2) *For any* $X \in \mathscr{H}'$ *and* $Y \in \mathscr{N}_1^-$, $(X|Y) \geq 0$.

(3) *For any* $X, Y \in \mathscr{N}_1^-$, $(X|Y) \geq 0$. *Moreover*, $(X|Y) = 0$ *if and only if* $X = sY$ *for some* $s > 0$.

PROOF. (1) Using (9.4), $X = gE_1$ for some $g \in F_{4(-20)}$. Then from (9.7), $g^{-1}Y \in \mathscr{N}_1^-$, and from the definition of $\mathscr{N}_1^-$, we obtain that $(X|Y) = (gE_1 \mid Y) = (E_1 \mid g^{-1}Y) < 0$.

(2) Suppose that $c = (X|Y) < 0$. Using (9.5), $X = gE_2$ for some $g \in F_{4(-20)}$. Put $Z = g^{-1}Y$. From (9.7), $Z \in \mathscr{N}_1^-$. Now, because of $c = (gE_2 \mid Y) = (E_2|Z)$, $Z = h^1(\xi_1, c, \xi_3; x_1, x_2, x_3)$ for some $\xi_i \in \mathbf{R}$ and $x_i \in \mathbf{O}$. Because of $Z \in \mathscr{N}_1^-$, $\xi_1 = (E_1|Z) < 0$ and $\xi_1 c + (x_3|x_3) = (Z^{\times 2})_{E_3} = 0$. Then $0 = \xi_1 c + (x_3|x_3) > 0$, and it is a contradiction. Thus $c \geq 0$, and so (2) follows.

(3) Suppose that $(X|Y) < 0$. Using (9.7), $Y = gP^-$ for some $g \in F_{4(-20)}$. Put $Z = g^{-1}X$. From (9.7), $Z \in \mathscr{N}_1^-$. Set $Z = \sum_{i=1}^3(\eta_iE_i + F_i^1(y_i))$ with $\eta_i \in \mathbf{R}$ and $y_i \in \mathbf{O}$, and put $r = (y_3|1)$. Then $-\eta_1 + \eta_2 - 2r = (Z|P^-) = (X|Y) < 0$. Because of $Z \in \mathscr{N}_1^-$, $\eta_1 = (E_1|Z) < 0$ and $\eta_1\eta_2 + (y_3|y_3) = (Z^{\times 2})_{E_3} = 0$. Then $\eta_1\eta_2 = -(y_3|y_3) \leq 0$. Therefore from $\eta_1 < 0$, $\eta_2 \geq 0$, so that $2r > \eta_2 - \eta_1 > 0$. Now, using Schwarz inequality, $r^2 = (y_3|1)^2 \leq (y_3|y_3)(1|1) = (y_3|y_3)$. Therefore because of $\eta_1\eta_2 + (y_3|y_3) = 0$, $4r^2 > (\eta_2 - \eta_1)^2 = (\eta_2 - \eta_1)^2 + 4(\eta_1\eta_2 + (y_3|y_3)) = (\eta_2 + \eta_1)^2 + 4(y_3|y_3) \geq (\eta_2 + \eta_1)^2 + 4r^2 \geq 4r^2$. It is a contradiction, and so $(X|Y) \geq 0$.

If $X = sY$ then $(X|Y) = 0$. Conversely, suppose that $(X|Y) = 0$. Using (9.7), $Y = gP^-$ for some $g \in F_{4(-20)}$. Put $Z = g^{-1}X$. From (9.7), $Z \in \mathscr{N}_1^-$. Because of $(Z|P^-) = (X|Y) = 0$ and Lemma 10.1(1), $\{Z\}_{-E_1+E_2} = 0$. Then by (10.1), $Z = sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$ for some $u, v \in \mathbf{R}$, $p \in \operatorname{Im}\mathbf{O}$, and $x, y \in \mathbf{O}$. Setting $z = x + y$ and $w = \bar{x} - \bar{y}$, $Z = sP^- + uE + vE_3 + F_3^1(p) + F_1^1(z) + F_2^1(w)$. Now, because of $Z \in \mathscr{N}_1^-$, $u^2 + (p|p) = (Z^{\times 2})_{E_3} = 0$ and $3u + v = \operatorname{tr}(Z) = 0$. Then $u = p = v = 0$, and $Z = sP^- + F_1^1(z) + F_2^1(w)$. Again, because of

$Z \in \mathcal{N}_1^-$, $-s = (Z|E_1) < 0$, $-(z|z) = (Z^{\times 2})_{E_1} = 0$, and $(w|w) = (Z^{\times 2})_{E_2} = 0$. Thus $Z = sP^-$ with $s > 0$. Therefore, multiplying g from left, $X = sY$. Hence we obtain (3). □

LEMMA 10.15. (1) $\mathcal{H} = \mathcal{H}_{<0}^{P^-} = \mathcal{H}_{\neq 0}^{P^-}$.

(2) $\mathcal{H}' = \mathcal{H}'^{P^-}_{>0} \coprod \mathcal{H}'^{P^-}_{=0}$. *Especially*, $\mathcal{H}'^{P^-}_{>0} = \mathcal{H}'^{P^-}_{\neq 0}$.

(3) $\mathcal{N}_1^- = (\mathcal{N}_1^-)_{<0}^{E_1} = (\mathcal{N}_1^-)_{\neq 0}^{E_1}$.

(4) $\mathcal{N}_1^- = (\mathcal{N}_1^-)_{>0}^{E_2} \coprod (\mathcal{N}_1^-)_{=0}^{E_2}$. *Especially*, $(\mathcal{N}_1^-)_{>0}^{E_2} = (\mathcal{N}_1^-)_{\neq 0}^{E_2}$.

(5) $\mathcal{N}_1^- = (\mathcal{N}_1^-)_{>0}^{\sigma P^-} \coprod (\mathcal{N}_1^-)_{=0}^{\sigma P^-}$. *Especially*, $(\mathcal{N}_1^-)_{>0}^{\sigma P^-} = (\mathcal{N}_1^-)_{\neq 0}^{\sigma P^-}$. *Furthermore*, $(\mathcal{N}_1^-)_{=0}^{\sigma P^-} = \{s(\sigma P^-) \mid s > 0\}$.

PROOF. (1) Because of $P^- \in \mathcal{N}_1^-$ and Lemma 10.14(1), $(X|P^-) < 0$ for all $X \in \mathcal{H}$, and so (1) follows.

(2) Because of $P^- \in \mathcal{N}_1^-$ and Lemma 10.14(2), $(X|P^-) \geq 0$ for all $X \in \mathcal{H}'$, and so (2) follows.

(3) Because of $E_1 \in \mathcal{H}$ and Lemma 10.14(1), $(X|E_1) < 0$ for all $X \in \mathcal{N}_1^-$, and so (3) follows.

(4) Because of $E_2 \in \mathcal{H}'$ and Lemma 10.14(2), $(X|E_2) \geq 0$ for all $X \in \mathcal{N}_1^-$, and so (4) follows.

(5) Because of $\sigma P^- \in \mathcal{N}_1^-$ and Lemma 10.14(3), we obtain that $(X \mid \sigma P^-) \geq 0$ for all $X \in \mathcal{N}_1^-$, and that $Y \in \mathcal{N}_1^-$ and $(Y \mid \sigma P^-) = 0$ if and only if $Y = s(\sigma P^-)$ for some $s > 0$. Thus (5) follows. □

LEMMA 10.16. *For* $X, Y \in \mathcal{J}^1$, *let* $\mathcal{D}_{X,Y} = \{g \in F_{4(-20)} \mid (gX \mid Y) = 0\}$. *Assume that there exists* $g_0 \in F_{4(-20)}$ *such that* $(g_0 X \mid Y) \neq 0$. *Then* $\mathcal{D}_{X,Y}$ *has no interior points in* $F_{4(-20)}$, *and the complement set* $(\mathcal{D}_{X,Y})^c$ *of* $\mathcal{D}_{X,Y}$ *is an open dense submanifold of* $F_{4(-20)}$.

PROOF. Set the function $f(g) = (gX \mid Y)$ for $g \in F_{4(-20)}$. Note that $F_{4(-20)}$ is a connected real analytic manifold, and that f is a real analytic function. Therefore, if the set $f^{-1}(0)$ has some interior points then $f \equiv 0$ on $F_{4(-20)}$. Since $f(g_0) \neq 0$ for some $g_0 \in F_{4(-20)}$, $f^{-1}(0)$ has no interior points. Therefore $(\mathcal{D}_{X,Y})^c$ is dense, and since $\mathcal{D}_{X,Y}$ is a closed set, $(\mathcal{D}_{X,Y})^c$ is an open set. □

LEMMA 10.17. *The equations* (9.3) *hold.*

PROOF. Put $S_0 = \{X \in \mathcal{H} \mid (X|E_1) = 1\}$. Obviously, $\{E_1\} \subset S_0$. Fix $X \in S_0$. Because of $\mathrm{tr}(X) = 1$ and $(X|E_1) = 1$, we can write $X = h^1(1, \xi, -\xi; x_1, x_2, x_3)$ for some $\xi \in \mathbf{R}$ and $x_i \in \mathbf{O}$. Because of $X^{\times 2} = 0$, $-\xi^2 - (x_1|x_1) = (X^{\times 2})_{E_1} = 0$, so that $\xi = x_1 = 0$. Then $(x_i|x_i) = (X^{\times 2})_{E_i} = 0$ for $i \in \{2, 3\}$, so that $x_i = 0$. Thus $X = E_1$, and so (9.3)(i) follows.

Put $S_1 = \{X \in \mathscr{H}' \,|\, (X|E_1) = 0\}$, and $S_2 = \{h^1(0, 1/2 - \xi, 1/2 + \xi; x, 0, 0) \in \mathscr{J}^1 | \xi^2 + (x|x) = 1/4\}$. Taking $x = 0$ and $\xi = \pm 1/2$, we see $\{E_2, E_3\} \subset S_2$. From direct calculations, $S_2 \subset S_1$. Conversely, fix $X \in S_1$. Because of $\mathrm{tr}(X) = 1$ and $(X|E_1) = 0$, we can write $X = h^1(0, 1/2 + \xi, 1/2 - \xi; x_1, x_2, x_3)$ for some $\xi \in \mathbf{R}$ and $x_i \in \mathbf{O}$. Then $1/4 - \xi^2 - (x_1|x_1) = (X^{\times 2})_{E_1} = 0$ and $(x_i|x_i) = (X^{\times 2})_{E_i} = 0$ with $i \in \{2, 3\}$. Therefore $X = h^1(0, 1/2 + \xi, 1/2 - \xi; x_1, 0, 0)$ with $\xi^2 + (x_1|x_1) = 1/4$, and $X \in S_2$. Thus $S_1 \subset S_2$, and so $S_1 = S_2$. □

11. The Iwasawa Decomposition of $F_{4(-20)}$

Because of $\mathscr{H} \simeq F_{4(-20)}/K$, we consider AN^+-orbits on $\mathscr{H}$ to give the Iwasawa decomposition of $F_{4(-20)}$.

LEMMA 11.1. *For all* $X \in \mathscr{H}$,

$$a_{2^{-1}\log(-(P^-|X))} n_X X = E_1.$$

PROOF. Put $t = 2^{-1}\log(-(P^-|X))$. By Lemma 10.15(1), $\mathscr{H} = \mathscr{H}^{P^-}_{<0} = \mathscr{H}^{P^-}_{\neq 0} \subset \mathfrak{P}^{P^-}_{\neq 0}$. Then $(P^-|X) < 0$, and $\log(-(P^-|X))$ is well-defined. Using $\mathrm{tr}(X) = 1$ and Lemma 10.5, $n_X X = r(-E_1 + E_2) + sP^- + 2^{-1}(E - E_3)$ where $r = 2^{-1}(P^-|X)$ and $s = 4^{-1}((P^-|X)^{-1} - (P^-|X))$. Because of $re^{-2t} = -2^{-1}$, $r \sinh 2t + se^{2t} = 0$, and Lemma 10.6, we get $a_t n_X X = -2^{-1}(-E_1 + E_2) + 2^{-1}(E_1 + E_2) = E_1$. □

PROOF OF MAIN-THEOREM 9.4. Using (9.4), $g^{-1}E_1 \in \mathscr{H}$. Then using Lemma 11.1 and $a_{2^{-1}\log(-(gP^-|E_1))} = a_{2^{-1}\log(-(P^-|g^{-1}E_1))}$,

$$a_{2^{-1}\log(-(gP^-|E_1))} n_{g^{-1}E_1} g^{-1} E_1 = E_1.$$

Put $k = a_{2^{-1}\log(-(gP^-|E_1))} n_{g^{-1}E_1} g^{-1}$. Then $k \in (F_{4(-20)})_{E_1} = K$ by (9.8), and

$$g = k^{-1} a_{2^{-1}\log(-(gP^-|E_1))} n_{g^{-1}E_1} \in KAN^+.$$

Set $H(g) = 2^{-1}\log(-(gP^- \,|\, E_1))\tilde{A}^1_3(1) \in \mathfrak{a}$, $n_I(g) = n_{g^{-1}E_1} \in N^+$, and $k(g) = k^{-1} \in K$, respectively. Then $g = k(g)\exp(H(g)) n_I(g)$, and it follows from Lemma 10.13(1) that $H(g)$, $n_I(g)$, and $k(g)$ are uniquely determined. Because of $(P^- \,|\, g^{-1}E_1) = (gP^- \,|\, E_1)$, $(Q^+(e_i) \,|\, g^{-1}E_1) = (gQ^+(e_i) \,|\, E_1)$, and $(F^1_3(e_i) \,|\, g^{-1}E_1)$

$= (gF_3^1(e_i) \mid E_1)$, we see

$$n_{g^{-1}E_1} = \exp\Big(\mathscr{G}_1\Big(2^{-1}\Big(\sum_{i=0}^{7}(gQ^+(e_i) \mid E_1)e_i\Big)\Big/(gP^- \mid E_1)\Big) + \mathscr{G}_2\Big(-2^{-1}\Big(\sum_{i=1}^{7}(gF_3^1(e_i) \mid E_1)e_i\Big)\Big/(gP^- \mid E_1)\Big)\Big).$$

Moreover, $k(g) = gn_t(g)^{-1}\exp(-H(g))$. Hence the result follows. □

Set $\tilde{D}_4 := \{(g_1, g_2, g_3) \in \mathrm{SO}(8)^3 \mid (g_1x)(g_2y) = \overline{g_3(\overline{xy})}$ for $x, y \in \mathbf{O}\}$. From [13, Lemma 3.2(1)], the following map $\varphi_0 : \tilde{D}_4 \to \mathrm{D}_4$ is a group isomorphism; $\varphi_0(g_1, g_2, g_3)(\sum_{i=1}^{3}(\xi_iE_i + F_i^1(x_i))) = \sum_{i=1}^{3}(\xi_iE_i + F_i^1(g_ix_i))$. From [13, (4.5)], for $j \in \{1, 2, 3\}$ and $X = \sum_{i=1}^{3}(\xi_iE_i + F_i^1(x_i))$, there exists $g_0 = \varphi_0(g_1, g_2, g_3) \in \mathrm{D}_4$ such that

$$g_0X = \Big(\sum_{i=1}^{3}\xi_iE_i\Big) + F_j^1(r_0) + \sum_{k=1}^{2}F_{j+k}^1(g_{j+k}x_{j+k}) \tag{11.1}$$

$$\text{with } r_0 = \sqrt{(x_j|x_j)} \in \mathbf{R}$$

where the index $j+k$ is counted modulo 3.

PROOF OF THEOREM 9.5. For all $m \in M$, $t \in \mathbf{R}$, and $n \in N^+$, using Lemma 10.7, $ma_tn[P^-] = [e^{2t}P^-] = [P^-]$, so that $MAN^+ \subset (\mathrm{F}_{4(-20)})_{[P^-]}$. Conversely, fix $g \in (\mathrm{F}_{4(-20)})_{[P^-]}$. Then $gP^- = sP^-$ for some $s > 0$. Because of $\mathrm{F}_{4(-20)} = KAN^+$, g can be expressed by $g = ka_tn$ with $k \in K$, $t \in \mathbf{R}$, and $n \in N^+$. From Lemma 10.7, $sP^- = gP^- = k(a_tnP^-) = e^{2t}kP^-$. Now, using Lemma 10.10(1), $-s = (sP^- \mid E_1) = (gP^- \mid E_1) = (ka_tnP^- \mid E_1) = -e^{2t}$, so that $s = e^{2t}$. Then $kP^- = P^-$, and from (9.8) and Lemma 10.11, $k \in K \cap (\mathrm{F}_{4(-20)})_{P^-} = (\mathrm{F}_{4(-20)})_{E_1, P^-} = M$. Thus $g = ka_tn \in MAN^+$, and $(\mathrm{F}_{4(-20)})_{[P^-]} \subset MAN^+$. Hence $(\mathrm{F}_{4(-20)})_{[P^-]} = MAN^+$, and it follows from (9.7) and $\mathscr{F} = \mathscr{N}_1^-/\sim$ that

$$\mathscr{F} = \mathrm{F}_{4(-20)} \cdot [P^-] \simeq \mathrm{F}_{4(-20)}/(\mathrm{F}_{4(-20)})_{[P^-]} = \mathrm{F}_{4(-20)}/MAN^+.$$

Next, let us show that K transitively acts on $\mathscr{F}$. Obviously K acts on $\mathscr{F}$. Fix $[X] \in \mathscr{F}$ with $X \in \mathscr{N}_1^-$. Using [13, Lemma 5.2(4)], there exists $k_1 \in K$ such that $k_1X = h^1(-\xi, \xi, 0; 0, 0, x)$ where $\xi > 0$, $x \in \mathbf{O}$, and $\xi^2 - (x|x) = 0$. Using (11.1), there exists $k_2 \in \mathrm{D}_4 \subset K$ such that $k_2k_1X = h^1(-\xi, \xi, 0; 0, 0, \xi) = \xi P^-$. Thus $k_2k_1[X] = [\xi P^-] = [P^-]$, and so $\mathscr{F} = K \cdot [P^-]$. Last, from $(\mathrm{F}_{4(-20)})_{[P^-]} = MAN^+$ and Lemma 10.13(1), $K_{[P^-]} = (\mathrm{F}_{4(-20)})_{[P^-]} \cap K = (MAN^+) \cap K = M$. Thus from $\mathscr{F} = K \cdot [P^-]$, (9.8), and (9.11), it follows that

$$\mathscr{F} \simeq K/K_{[P^-]} = K/M = \mathrm{Spin}(9)/\mathrm{Spin}(7).$$

□

We define the quadratic space $(\mathbf{O}^2, Q)$ by the normal linear space $\mathbf{O}^2 = \mathbf{O} \times \mathbf{O}$ and $Q(x, y) := (x|x) + (y|y)$ for $x, y \in \mathbf{O}$, and $S^{15} := \{(x, y) \in \mathbf{O}^2 \mid Q(x, y) = 1\}$.

PROPOSITION 11.2.

$$\mathscr{F} \simeq S^{15}.$$

Furthermore, $K/M \simeq S^{15}$.

PROOF. Set the map $f : S^{15} \to \mathscr{F}$ as

$$f(x, y) := [h^1(-1, (y|y), (x|x); \overline{xy}, x, y)] \quad \text{for } (x, y) \in S^{15}.$$

Put $X = h^1(-1, (y|y), (x|x); \overline{xy}, x, y)$. From direct calculations, we get $X \in \mathscr{N}_1^-$. Therefore f is well-defined. On the other hand, the map $g : \mathscr{F} \to S^{15}$ set as

$$g([h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3)]) := (-\xi_1^{-1}x_2, -\xi_1^{-1}x_3)$$

for $h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) \in \mathscr{N}_1^-$. Put $X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3)$, $x = -\xi_1^{-1}x_2$, and $y = -\xi_1^{-1}x_3$, respectively. Because of $\xi_3\xi_1 + (x_2|x_2) = (X^{\times 2})_{E_2} = 0$, $\xi_1\xi_2 + (x_3|x_3) = (X^{\times 2})_{E_3} = 0$, and $\xi_1 + \xi_2 + \xi_3 = \mathrm{tr}(X) = 0$, we get $Q(x, y) = \xi_1^{-2}((x_2|x_2) + (x_3|x_3)) = \xi_1^{-2}(-\xi_3\xi_1 - \xi_1\xi_2) = \xi_1^{-1}(-\xi_2 - \xi_3) = \xi_1^{-1}\xi_1 = 1$. Therefore g is well-defined.

Now, it follows that $g \circ f(x, y) = (x, y)$ for all $(x, y) \in S^{15}$, so that $g \circ f = id$. On the other hand, for all $X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) \in \mathscr{N}_1^-$, because $\mathscr{F} = \mathscr{N}_1^-/\!\sim$, $\xi_1 = (X|E_1) < 0$, $x_1 = -\xi_1^{-1}(\overline{x_2x_3})$ from $-\overline{x_2x_3} - \xi_1x_1 = (X^{\times 2})_{F_1} = 0$, $\xi_2 = -\xi_1^{-1}(x_3|x_3)$ from $\xi_1\xi_2 + (x_3|x_3) = (X^{\times 2})_{E_3} = 0$, and $\xi_3 = -\xi_1^{-1}(x_2|x_2)$ from $\xi_3\xi_1 + (x_2|x_2) = (X^{\times 2})_{E_2} = 0$, we see

$$\begin{aligned}
&f \circ g([h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3)]) \\
&\quad = [h^1(-1, \xi_1^{-2}(x_3|x_3), \xi_1^{-2}(x_2|x_2); \xi_1^{-2}(\overline{x_2x_3}), -\xi_1^{-1}x_2, -\xi_1^{-1}x_3)] \\
&\quad = [h^1(\xi_1, -\xi_1^{-1}(x_3|x_3), -\xi_1^{-1}(x_2|x_2); -\xi_1^{-1}(\overline{x_2x_3}), x_2, x_3)] \\
&\quad = [h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3)].
\end{aligned}$$

Therefore $f \circ g = id$. Hence $\mathscr{F} \simeq S^{15}$, and from Theorem 9.5(4), $K/M \simeq S^{15}$. □

REMARK 11.3. I. Yokota has proved $\mathrm{Spin}(9)/\mathrm{Spin}(7) \simeq S^{15}$ ([20, Example 5.6], [19]) by realizing Spin(9) and Spin(7) as stabilizers of finite points in the compact exceptional Lie group $F_4 := Aut_{\mathbf{R}}(\mathscr{J})$ where $\mathscr{J}$ is an exceptional Jordan algebra, and showing that Spin(9) transitively acts on S^{15} embed in $\mathscr{J}$. In Proposition 11.2, we give the other proof by using $\mathscr{F} = \mathscr{N}_1^-/\sim$ where $\mathscr{N}_1^-$ is an exceptional null cone.

12. The K_ε-Iwasawa Decomposition of $F_{4(-20)}$

Because of $\mathscr{H}' \simeq F_{4(-20)}/K_\varepsilon$, we consider AN^+-orbits on $\mathscr{H}'$ to give the K_ε-Iwasawa decomposition of $F_{4(-20)}$.

LEMMA 12.1. *Assume that* $X \in \mathscr{H}'^{P^-}_{\neq 0}$. *Then*

$$a_{2^{-1}\log((P^-|X))} n_X X = E_2.$$

PROOF. Put $t = 2^{-1}\log((P^-|X))$. By Lemma 10.15(2), $\mathscr{H}'^{P^-}_{>0} = \mathscr{H}'^{P^-}_{\neq 0} \subset \mathfrak{P}^{P^-}_{\neq 0}$. Then $(P^-|X) > 0$, and $\log((P^-|X))$ is well-defined. Using $\mathrm{tr}(X) = 1$ and Lemma 10.5, $n_X X = r(-E_1 + E_2) + sP^- + 2^{-1}(E - E_3)$ where $r = 2^{-1}(P^-|X)$ and $s = 4^{-1}((P^-|X)^{-1} - (P^-|X))$. Because of $re^{-2t} = 2^{-1}$, $r\sinh 2t + se^{2t} = 0$, and Lemma 10.6, we get $a_t n_X X = 2^{-1}(-E_1 + E_2) + 2^{-1}(E_1 + E_2) = E_2$. □

PROOF OF MAIN THEOREM 9.6. Put $\mathscr{D} = \{g \in F_{4(-20)} \mid (gP^- \mid E_2) > 0\}$. From (9.7) and Lemma 10.15(4), we see $\mathscr{D} = \{g \in F_{4(-20)} \mid gP^- \in (\mathscr{N}_1^-)^{E_2}_{>0}\} = \{g \in F_{4(-20)} \mid gP^- \in (\mathscr{N}_1^-)^{E_2}_{\neq 0}\} = \{g \in F_{4(-20)} \mid (gP^- \mid E_2) \neq 0\}$. Now, from Lemma 10.10(2), $K_\varepsilon AN^+ \subset \mathscr{D}$. Conversely, fix $g \in \mathscr{D}$. From (9.5), $g^{-1}E_2 \in \mathscr{H}'$, and $(P^- \mid g^{-1}E_2) = (gP^- \mid E_2) > 0$, so that $g^{-1}E_2 \in \mathscr{H}'^{P^-}_{>0}$. Using Lemma 12.1 and $a_{2^{-1}\log((gP^-|E_2))} = a_{2^{-1}\log((P^-|g^{-1}E_2))}$,

$$a_{2^{-1}\log((gP^-|E_2))} n_{g^{-1}E_2} g^{-1} E_2 = E_2.$$

Put $k' = a_{2^{-1}\log((gP^-|E_2))} n_{g^{-1}E_2} g^{-1}$. Then $k' \in (F_{4(-20)})_{E_2} = K_\varepsilon$ by (9.20), and

$$(*) \qquad g = k'^{-1} a_{2^{-1}\log((gP^-|E_2))} n_{g^{-1}E_2} \in K_\varepsilon AN^+.$$

Thus $\mathscr{D} \subset K_\varepsilon AN^+$, and so $\mathscr{D} = K_\varepsilon AN^+$. Since the identity element $1 \in F_{4(-20)}$ is in $\mathscr{D}$ and the complement set $\mathscr{D}^c$ is given by $\mathscr{D}^c = \{g \in F_{4(-20)} \mid (gP^- \mid E_2) = 0\}$, applying Lemma 10.16, $\mathscr{D} = K_\varepsilon AN^+$ is an open dense submanifold of $F_{4(-20)}$.

From $(*)$, we set $H_\varepsilon(g) = 2^{-1}\log((gP^- \mid E_2))\tilde{A}_3^1(1) \in \mathfrak{a}$, $n_\varepsilon(g) = n_{g^{-1}E_2} \in N^+$, and $k_\varepsilon(g) = k'^{-1} \in K_\varepsilon$, respectively. Then we get $g = k_\varepsilon(g)\exp(H_\varepsilon(g))n_\varepsilon(g)$, and it follows from Lemma 10.13(2) that $H_\varepsilon(g)$, $n_\varepsilon(g)$, and $k_\varepsilon(g)$ are uniquely determined. Since $(X \mid g^{-1}Y) = (gX \mid Y)$ for all $X, Y \in \mathcal{J}^1$, we see

$$n_{g^{-1}E_2} = \exp\Big(\mathcal{G}_1\Big(2^{-1}\Big(\sum_{i=0}^{7}(gQ^+(e_i) \mid E_2)e_i\Big)\Big/(gP^- \mid E_2)\Big) + \mathcal{G}_2\Big(-2^{-1}\Big(\sum_{i=1}^{7}(gF_3^1(e_i) \mid E_2)e_i\Big)\Big/(gP^- \mid E_2)\Big)\Big).$$

Moreover, $k_\varepsilon(g) = gn_\varepsilon(g)^{-1}\exp(-H_\varepsilon(g))$. Hence the result follows. □

13. The Matsuki Decomposition of $F_{4(-20)}$

For $X \in \mathcal{J}^1$, we denote $L^\times(X) \in \mathrm{End}_{\mathbf{R}}(\mathcal{J}^1)$ by $L^\times(X)Y = X \times Y$ for $Y \in \mathcal{J}^1$. For $j \in \{1,2,3\}$ and $p, q \in \mathbf{R}$, we denote the subspace $(\mathcal{J}^1)^j_{p,q}$ of $\mathcal{J}^1$ by

$$(\mathcal{J}^1)^j_{p,q} := \{X \in \mathcal{J}^1 \mid \sigma_j X = pX, L^\times(2E_j)X = qX\}.$$

LEMMA 13.1. *Let $j \in \{1,2,3\}$ and $p, q \in \mathbf{R}$.*
(1) *For all $k \in (F_{4(-20)})_{E_j}$,*

$$L^\times(2E_j) \cdot k = k \cdot L^\times(2E_j).$$

(2) *The stabilizer $(F_{4(-20)})_{E_j}$ invariants the space $(\mathcal{J}^1)^j_{p,q}$.*

PROOF. (1) It follows from $L^\times(2E_j)(kX) = 2E_j \times (kX) = k(2E_j \times X) = k(L^\times(2E_j)X)$ for all $X \in \mathcal{J}^1$.

(2) From [13, Proposition 4.14] and (1), we see that $k \cdot \sigma_j = \sigma_j \cdot k$ and $L^\times(2E_j) \cdot k = k \cdot L^\times(2E_j)$ for all $k \in (F_{4(-20)})_{E_j}$. Hence (2) follows. □

By direct calculations, we have the following lemma.

LEMMA 13.2. *Let $j \in \{1,2,3\}$.*

$$\mathcal{J}^1 = (\mathcal{J}^1)^j_{-1,0} \oplus (\mathcal{J}^1)^j_{1,0} \oplus (\mathcal{J}^1)^j_{1,1} \oplus (\mathcal{J}^1)^j_{1,-1}$$

where

$$(\mathcal{J}^1)^j_{-1,0} = \{F^1_{j+1}(x_{j+1}) + F^1_{j+2}(x_{j+2}) \mid x_{j+1}, x_{j+2} \in \mathbf{O}\},$$

$$(\mathcal{J}^1)^j_{1,0} = \{pE_j \mid p \in \mathbf{R}\}, \quad (\mathcal{J}^1)^j_{1,1} = \{q(E - E_j) \mid q \in \mathbf{R}\},$$

$$(\mathcal{J}^1)^j_{1,-1} = \{\xi(E_{j+1} - E_{j+2}) + F^1_j(x_j) \mid \xi \in \mathbf{R}, x_j \in \mathbf{O}\}$$

and indexes j, $j+1$, $j+2$ are counted modulo 3.

Let $j \in \{2,3\}$. For $X \in \mathscr{J}^1$, we denote the quadratic form Q by $Q(Y) := -\mathrm{tr}(Y^{\times 2})$ for $Y \in \mathscr{J}^1$, and $\mathscr{S}_j^{8,1} := \{X \in (\mathscr{J}^1)^j_{1,-1} \mid Q(X) = 1\} = \{\xi(E_{j+1} - E_{j+2}) + F_j^1(x) \mid \xi \in \mathbf{R}, x \in \mathbf{O}, \xi^2 - (x|x) = 1\}$.

LEMMA 13.3. *Let $j \in \{2,3\}$ and indexes j, $j+1$, $j+2$ be counted modulo 3. $\mathscr{S}_j^{8,1}$ decomposes into the following two $(\mathrm{F}_{4(-20)})_{E_j}$-orbits:*

$$\mathscr{S}_j^{8,1} = (\mathrm{F}_{4(-20)})_{E_j} \cdot (E_{j+1} - E_{j+2}) \coprod (\mathrm{F}_{4(-20)})_{E_j} \cdot (-E_{j+1} + E_{j+2}).$$

PROOF. From [13, Lemmas 4.2 and 4.6],

$$\mathscr{S}_3^{8,1} = (\mathrm{F}_{4(-20)})_{E_3} \cdot (E_1 - E_2) \coprod (\mathrm{F}_{4(-20)})_{E_3} \cdot (-E_1 + E_2).$$

Put $g_0 = \exp(2^{-1}\pi \tilde{A}_1^1(1))$. Multiplying g_0 from the left, we have

$$\mathscr{S}_2^{8,1} = (\mathrm{F}_{4(-20)})_{E_2} \cdot (E_1 - E_3) \coprod (\mathrm{F}_{4(-20)})_{E_2} \cdot (-E_1 + E_3).$$

Here, using $g_0\sigma_3 g_0^{-1} = \sigma_2$, $g_0\mathscr{S}_3^{8,1} = \mathscr{S}_2^{8,1}$. □

LEMMA 13.4. *Let $X \in \mathscr{N}_1^-$.*

(1) *If $(X|E_2) \neq 0$, then there exists $k_\varepsilon \in K_\varepsilon$ such that $k_\varepsilon X = rP_{12}^-$ for some $r > 0$.*

(2) *If $(X|E_2) = 0$, then there exists $k_\varepsilon \in K_\varepsilon$ such that $k_\varepsilon X = rP_{13}^-$ for some $r > 0$.*

PROOF. (1) From Lemma 13.2, X can be expressed by $X = (F_3^1(x_3) + F_1^1(x_1)) + pE_2 + q(E - E_2) + (\xi(E_3 - E_1) + F_2^1(x_2))$ where $F_3^1(x_3) + F_1^1(x_1) \in (\mathscr{J}^1)^2_{-1,0}$, $pE_2 \in (\mathscr{J}^1)^2_{1,0}$, $q(E - E_2) \in (\mathscr{J}^1)^2_{1,1}$, $\xi(E_3 - E_1) + F_2^1(x_2) \in (\mathscr{J}^1)^2_{1,-1}$, and $p = (X|E_2) \neq 0$ with $p, q, \xi \in \mathbf{R}$ and $x_i \in \mathbf{O}$. Because of $X \in \mathscr{N}_1^-$, we see $p + 2q = \mathrm{tr}(X) = 0$ and $q^2 - \xi^2 + (x_2|x_2) = (X^{\times 2})_{E_2} = 0$. Then $\xi^2 - (x_2|x_2) = 4^{-1}p^2 > 0$. Setting $r = 2^{-1}|p|$, we can write $\xi(E_3 - E_1) + F_2^1(x_2) = rW$ for some $W \in \mathscr{S}_2^{8,1}$. From Lemma 13.3 and (9.20), there exists $k_0 \in K_\varepsilon = (\mathrm{F}_{4(-20)})_{E_2}$ such that $k_0 W = \varepsilon(E_3 - E_1)$ with $\varepsilon = \pm 1$. Because of $K_\varepsilon = (\mathrm{F}_{4(-20)})_{E_2}$, we get $k_0(pE_2) = pE_2$ and $k_0(q(E - E_2)) = q(E - E_2)$. And because of $F_3^1(x_3) + F_1^1(x_1) \in (\mathscr{J}^1)^2_{-1,0}$ and Lemma 13.1(2), we get $k_0(F_3^1(x_3) + F_1^1(x_1)) = F_3^1(y_3) + F_1^1(y_1)$ for some $y_i \in \mathbf{O}$. Therefore $k_0 X = h^1(\eta_1, p, \eta_3; y_1, 0, y_3)$ where $\eta_1 = q - \varepsilon r$ and

$\eta_3 = q + \varepsilon r$. Put $X' = k_0 X$. Because of $X' \in \mathcal{N}_1^-$ by (9.7),

(i) $\eta_1 = (E_1|X') < 0$, (ii) $\eta_1 + p + \eta_3 = \mathrm{tr}(X') = 0$,

(iii) $\eta_3\eta_1 = (X'^{\times 2})_{E_2} = 0$, (iv) $p\eta_3 - (y_1|y_1) = (X'^{\times 2})_{E_1} = 0$,

(v) $\eta_1 p + (y_3|y_3) = (X'^{\times 2})_{E_3} = 0$.

Form (i), (ii), and (iii), we get $\eta_3 = 0$, $\eta_1 = -p$, and $p > 0$. And by (iv) and (v), we get $y_1 = 0$ and $p = \sqrt{(y_3|y_3)}$. Consequently $X' = h^1(-p, p, 0; 0, 0, y_3)$ with $p = \sqrt{(y_3|y_3)}$. Using (11.1), there exists $k_1 \in D_4 \subset K_\varepsilon$ such that $k_1 k_0 X = k_1 X' = h^1(-p, p, 0; 0, 0, p) = pP_{12}^-$.

(2) Because of $\mathrm{tr}(X) = 0$ and $(X|E_2) = 0$, $X = h^1(-r, 0, r; x_1, x_2, x_3)$ for some $r \in \mathbf{R}$ and $x_i \in \mathbf{O}$. Because of $X \in \mathcal{N}_1^-$, we get $-r = (E_1|X) < 0$, $-(x_1|x_1) = (X^{\times 2})_{E_1} = 0$, $-r^2 + (x_2|x_2) = (X^{\times 2})_{E_2} = 0$, and $(x_3|x_3) = (X^{\times 2})_{E_3} = 0$. Then $X = h^1(-r, 0, r; 0, x_2, 0)$ with $r = \sqrt{(x_2|x_2)}$. Using (11.1), there exists $k' \in D_4 \subset K_\varepsilon$ such that $k'X = h^1(-r, 0, r; 0, r, 0) = rP_{13}^-$. □

PROOF OF THEOREM 9.7. Set $\mathcal{O} = \{[X] \in \mathcal{F} \mid (X|E_2) \neq 0\}$, and $\mathcal{O}' = \{[X] \in \mathcal{F} \mid (X|E_2) = 0\}$. Then $\mathcal{F} = \mathcal{O} \coprod \mathcal{O}'$. Because of $\mathcal{F} = \mathcal{N}_1^-/\sim$ and Lemma 10.15(4), $\mathcal{O} = \{[X] \in \mathcal{F} \mid (X|E_2) > 0\}$. For any $k \in K_\varepsilon$, $(kX \mid E_2) = (X \mid k^{-1}E_2) = (X|E_2)$ by (9.20), so that K_ε acts on $\mathcal{O}$ and $\mathcal{O}'$, respectively. When $[X] \in \mathcal{O}$, by Lemma 13.4(1), there exists $k \in K_\varepsilon$ such that $k[X] = [kX] = [P_{12}^-]$. And when $[X] \in \mathcal{O}'$, by Lemma 13.4(2), there exists $k' \in K_\varepsilon$ such that $k'[X] = [k'X] = [P_{13}^-]$. Hence the result follows. □

PROOF OF THEOREM 9.8. Put $g_0 = \exp(-2^{-1}\pi\tilde{A}_1^1(1))$, and $\mathcal{D} = \{g \in F_{4(-20)} \mid (gP_{12}^- \mid E_2) = 0\}$. Then $g_0^{-1} = \exp(2^{-1}\pi\tilde{A}_1^1(1))$, and from (10.6), $g_0^{-1}P_{13}^- = P_{12}^-$ and $g_0^{-1}E_2 = E_3$. Fix $g \in \mathcal{D}$. By (9.7), $gP_{12}^- \in \mathcal{N}_1^-$, and applying Theorem 9.7, $[gP_{12}^-] \in K_\varepsilon \cdot [P_{13}^-]$. Therefore $k[gP_{12}^-] = [P_{13}^-]$ for some $k \in K_\varepsilon$. Then $g_0^{-1}kg[P_{12}^-] = [g_0^{-1}P_{13}^-] = [P_{12}^-]$, so that $g_0^{-1}kg \in (F_{4(-20)})_{[P_{12}^-]}$. Using Theorem 9.5(1), $g_0^{-1}kg = ma_t n$ for some $m \in M$, $t \in \mathbf{R}$, and $n \in N^+$. Thus $g = k^{-1}g_0 man \in K_\varepsilon g_0 MAN^+$, and so $\mathcal{D} \subset K_\varepsilon g_0 MAN^+$. Conversely, take $g = kg_0 a_t mn \in K_\varepsilon g_0 MAN^+$ with $k \in K_\varepsilon$, $t \in \mathbf{R}$, and $n \in N^+$. Because of Lemma 10.7, (9.20), and $g_0^{-1}E_2 = E_3$, we see $(gP_{12}^- \mid E_2) = (ma_t nP_{12}^- \mid g_0^{-1}k^{-1}E_2) = e^{2t}(P_{12}^-|E_3) = 0$. Thus $g \in \mathcal{D}$, and so $K_\varepsilon g_0 MAN^+ \subset \mathcal{D}$. Hence $K_\varepsilon g_0 MAN^+ = \mathcal{D}$. Last, from $M \subset K_\varepsilon$ and Main Theorem 9.6, $K_\varepsilon MAN^+ = K_\varepsilon AN^+ = \{g \in F_{4(-20)} \mid (gP_{12}^- \mid E_2) \neq 0\}$. Thus $F_{4(-20)} = \{g \in F_{4(-20)} \mid (gP_{12}^- \mid E_2) \neq 0\} \coprod \{g \in F_{4(-20)} \mid (gP_{12}^- \mid E_2) = 0\} = K_\varepsilon MAN^+ \coprod K_\varepsilon g_0 MAN^+$. □

14. The Bruhat and Gauss Decompositions of $F_{4(-20)}$

Because of $\mathscr{F} \simeq F_{4(-20)}/MAN^+$, we consider N^--orbits on $\mathscr{F}$ to give the Bruhat and Gauss decompositions of $F_{4(-20)}$. For any $X \in (\mathscr{N}_1^-)^{\sigma P^-}_{\neq 0}$, denote $z_X := \tilde{\sigma}(n_{\sigma X}) \in N^-$.

LEMMA 14.1. *Assume that* $X \in (\mathscr{N}_1^-)^{\sigma P^-}_{\neq 0}$. *Then*

$$z_X X = 4^{-1}(X \mid \sigma P^-)P^-.$$

PROOF. Since $(\sigma X \mid P^-) = (X \mid \sigma P^-) \neq 0$ and $\mathrm{tr}(\sigma X) = \mathrm{tr}(X) = 0$, applying Lemma 10.5,

$$n_{\sigma X}(\sigma X) = 4^{-1}(\sigma X \mid P^-)(-E_1 + E_2 + F_3^1(-1)) = 4^{-1}(X \mid \sigma P^-)\sigma P^-.$$

Thus $z_X X = (\sigma n_{\sigma X} \sigma)X = 4^{-1}(X \mid \sigma P^-)P^-$. □

PROOF OF THEOREM 9.9. Set $\mathscr{O} = \{[X] \in \mathscr{F} \mid (X \mid \sigma P^-) > 0\}$, and $\mathscr{O}' = \{[X] \in \mathscr{F} \mid (X \mid \sigma P^-) = 0\}$. Using Lemma 10.15(5), $\mathscr{O} = \{[X] \in \mathscr{F} \mid X \in (\mathscr{N}_1^-)^{\sigma P^-}_{>0}\}$ $= \{[X] \in \mathscr{F} \mid X \in (\mathscr{N}_1^-)^{\sigma P^-}_{\neq 0}\} = \{[X] \in \mathscr{F} \mid (X \mid \sigma P^-) \neq 0\}$ and $\mathscr{O}' = \{[X] \in \mathscr{F} \mid X \in (\mathscr{N}_1^-)^{\sigma P^-}_{=0}\} = \{[\sigma P^-]\}$. Then $\mathscr{F} = \mathscr{O} \coprod \mathscr{O}'$. For any $z \in N^-$ and $[X] \in \mathscr{O}$, using Lemma 10.8, $(zX \mid \sigma P^-) = (X \mid z^{-1}(\sigma P^-)) = (X \mid \sigma P^-) > 0$, and N^- acts on $\mathscr{O}$. Fix $[X] \in \mathscr{O}$. Taking $z_X \in N^-$, from Lemma 14.1, $z_X[X] = [4^{-1}(X \mid \sigma P^-)P^-] = [P^-]$, and $\mathscr{O} = N^- \cdot [P^-]$. Next, using Lemma 10.8, $N^- \cdot [\sigma P^-] = \{[\sigma P^-]\} = \mathscr{O}'$. Therefore $\mathscr{F} = \mathscr{O} \coprod \mathscr{O}' = N^- \cdot [P^-] \coprod \{[\sigma P^-]\} = N^- \cdot [P^-] \coprod N^- \cdot [\sigma P^-]$. □

PROOF OF MAIN THEOREM 9.10. Put $\mathscr{D} = \{g \in F_{4(-20)} \mid (gP^- \mid \sigma P^-) > 0\}$. From (9.7), $gP^- \in \mathscr{N}_1^-$, and using Lemma 10.15(5), $\mathscr{D} = \{g \in F_{4(-20)} \mid gP^- \in (\mathscr{N}_1^-)^{\sigma P^-}_{>0}\} = \{g \in F_{4(-20)} \mid gP^- \in (\mathscr{N}_1^-)^{\sigma P^-}_{\neq 0}\} = \{g \in F_{4(-20)} \mid (gP^- \mid \sigma P^-) \neq 0\}$ and the complement set $\mathscr{D}^c$ of $\mathscr{D}$ is given by $\mathscr{D}^c = \{g \in F_{4(-20)} \mid (gP^- \mid \sigma P^-) = 0\} = \{g \in F_{4(-20)} \mid gP^- \in (\mathscr{N}_1^-)^{\sigma P^-}_{=0}\} = \{g \in F_{4(-20)} \mid g[P^-] = [\sigma P^-]\}$. First, let us show $\mathscr{D} = N^- MAN^+$. From Lemma 10.10(3), $N^- MAN^+ \subset \mathscr{D}$. Conversely, fix $g \in \mathscr{D}$. Then $gP^- \in (\mathscr{N}_1^-)^{\sigma P^-}_{>0}$, and from Lemma 14.1, $(z_{gP^-})gP^- = 4^{-1}(gP^- \mid \sigma P^-)P^-$ and $(gP^- \mid \sigma P^-) > 0$. Therefore $(z_{gP^-})g[P^-] = [4^{-1}(gP^- \mid \sigma P^-)P^-] = [P^-]$. Using Theorem 9.5(1), $(z_{gP^-})g = ma_t n$ for some $t \in \mathbf{R}$, $m \in M$, and $n \in N^+$. Thus

$$(*) \qquad g = (z_{gP^-})^{-1} m a_t n \in N^- MAN^+,$$

and so $\mathscr{D} \subset N^- MAN^+$. Hence $\mathscr{D} = N^- MAN^+$. Since the identity element $1 \in F_{4(-20)}$ is in $\mathscr{D}$, applying Lemma 10.16, $\mathscr{D} = N^- MAN^+$ is an open dense submanifold of $F_{4(-20)}$.

Second, let us show $\mathscr{D}^c = \sigma MAN$. Fix $\sigma m a_t n \in \sigma MAN$ with $m \in M$, $t \in \mathbf{R}$, and $n \in N^+$. By Lemma 10.7, $\sigma m a_t n P^- = e^{2t}(\sigma P^-)$, so that $\sigma m a_t n[P^-] = [\sigma P^-]$. Thus $\sigma MAN \subset \mathscr{D}^c$. Conversely, fix $g \in \mathscr{D}^c$. Because of $g[P^-] = [\sigma P^-]$, $\sigma g[P^-] = [P^-]$. Using Theorem 9.5(1), $\sigma g \in MAN^+$. Thus $g \in \sigma MAN^+$, and so $\mathscr{D}^c \subset \sigma MAN^+$. Hence $\mathscr{D}^c = \sigma MAN^+$, and $F_{4(-20)} = \mathscr{D} \coprod \mathscr{D}^c = N^- MAN^+ \coprod \sigma MAN^+$. Now, from $N^- = \tilde{\sigma}(N^+) = \sigma N^+ \sigma$ and Lemma 10.9(3), it follows that $N^- \sigma MAN^+ = \sigma N^+ MAN^+ = \sigma MAN^+$.

Next, from (*), set $n_G^-(g) = (z_{gP^-})^{-1}$, $a_G(g) = a_t$, $n_G^+(g) = n$, and $m_G(g) = m$, respectively. Then $g = n_G^-(g) m_G(g) a_G(g) n_G^+(g)$, and it follows from Lemma 10.13(3) that $a_G(g)$, $n_G^-(g)$, $n_G^+(g)$, and $m_G(g)$ are uniquely determined. Now, since $(z_{gP^-})g = m a_t n$ and the uniqueness of factors of the Iwasawa decomposition of $F_{4(20)}$, $a_G(g)$, $n_G^+(g)$, and $m_G(g)$ are given by $a_G(g) = \exp(H((z_{gP^-})g)\tilde{A}_3^1(1))$, $n_G^+(g) = n_I((z_{gP^-})g)$, and $m_G(g) = k((z_{gP^-})g)$, respectively. Then these equations imply that (i), (ii), (iii), and (iv). Indeed, using Lemma 14.1,

$$-((z_{gP^-})gP^- \mid E_1) = -4^{-1}(gP^- \mid \sigma P^-)(P^- | E_1) = 4^{-1}(gP^- \mid \sigma P^-),$$

so that $t = 2^{-1}\log(4^{-1}(gP^- \mid \sigma P^-))$. Because of $\sigma Q^+(e_i) = Q^-(e_i)$, $\sigma F_3^1(e_i) = -F_3^1(e_i)$, and (9.18), we see

$$\begin{aligned}(z_{gP^-})^{-1} &= \tilde{\sigma}\Big(\exp\Big(\mathscr{G}_1\Big(-2^{-1}\Big(\sum\nolimits_{i=0}^{7}(Q^+(e_i) \mid \sigma g P^-)e_i\Big)\Big/(P^- \mid \sigma g P^-)\Big)\\ &\qquad + \mathscr{G}_2\Big(2^{-1}\Big(\sum\nolimits_{i=1}^{7}(F_3^1(e_i) \mid \sigma g P^-)e_i\Big)\Big/(P^- \mid \sigma g P^-)\Big)\Big)\Big)\\ &= \exp\Big(\mathscr{G}_{-1}\Big(-2^{-1}\Big(\sum\nolimits_{i=0}^{7}(Q^-(e_i) \mid gP^-)e_i\Big)\Big/(gP^- \mid \sigma P^-)\Big)\\ &\qquad + \mathscr{G}_{-2}\Big(-2^{-1}\Big(\sum\nolimits_{i=1}^{7}(F_3^1(e_i) \mid gP^-)e_i\Big)\Big/(gP^- \mid \sigma P^-)\Big)\Big).\end{aligned}$$

Moreover, we get $n_G^+(g) = n_I((z_{gP^-})g) = n_I(n_G^-(g)^{-1}g)$ and $m_G(g) = (z_{gP^-})gn^{-1}a_t^{-1} = n_G^-(g)^{-1} g n_G^+(g)^{-1} a_G(g)^{-1}$. Hence the result follows. □

Appendix A. The Explicit Formula c-Function of $F_{4(-20)}$

Recall $\mathfrak{a} = \{t\tilde{A}_3^1(1) \mid t \in \mathbf{R}\}$. Let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$, and $\mathfrak{a}_{\mathbf{C}}^*$ the complexification of $\mathfrak{a}^*$, and recall $\alpha \in \Sigma \subset \mathfrak{a}^* \subset \mathfrak{a}_{\mathbf{C}}^*$ satisfies $\alpha(\tilde{A}_3^1(1)) = 1$. Let $B(\cdot,\cdot)$ be the Killing form of $\mathfrak{f}_{4(-20)}$. For $\lambda \in \mathfrak{a}^*$, we define the element $H_\lambda \in \mathfrak{a}$ by $B(H_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{a}$, and the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}_{\mathbf{C}}^*$ by setting $\langle\lambda_1, \lambda_2\rangle := B(H_{\lambda_1}, H_{\lambda_2})$ and extending it to the whole of $\mathfrak{a}_{\mathbf{C}}^*$ by linearity. For any $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, we define $\lambda_\alpha \in \mathbf{C}$ by

$$\lambda_\alpha := (2\langle\lambda, \alpha\rangle)/\langle\alpha, \alpha\rangle.$$

Because of $\dim \mathfrak{a}_{\mathbf{C}}^* = \dim \mathfrak{a} = 1$, $\lambda = 2^{-1}\lambda_\alpha \alpha$. We denote $m_\alpha := \dim \mathfrak{g}_\alpha = \dim \mathbf{O} = 8$ and $m_{2\alpha} := \dim \mathfrak{g}_{2\alpha} = \dim(\mathrm{Im}\,\mathbf{O}) = 7$, and we define $\rho \in \mathfrak{a}_{\mathbf{C}}^*$ by

$$\rho := 2^{-1}((\dim \mathfrak{g}_\alpha)\alpha + (\dim \mathfrak{g}_{2\alpha})2\alpha) = 2^{-1}(m_\alpha + 2m_{2\alpha})\alpha.$$

For $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, we consider the *spherical function* φ_λ on $\mathrm{F}_{4(-20)}$ and the *c-function of Harish-Chandra* on $\mathfrak{a}_{\mathbf{C}}^*$. From [5] (cf. [14], [15], [16]), φ_λ is given by

$$\varphi_\lambda(g) := \int_K e^{(\lambda-\rho)(H(gk))}\,dk = \int_{N^-} e^{(\lambda-\rho)(H(gz))}e^{-(\lambda+\rho)(H(z))}\,dz$$

for $g \in \mathrm{F}_{4(-20)}$, and the function c is given by

$$c(\lambda) := \int_{N^-} e^{-(\lambda+\rho)(H(z))}\,dz.$$

Here the measure dk on compact group K is normalized such that the total measure is 1, and the Haar measures of nilpotent groups N^+ and N^- are normalized such that

$$\tilde{\sigma}(dn) = dz \quad \text{and} \quad \int_{N^-} e^{-2\rho(H(z))}\,dz = 1.$$

LEMMA A.1. *Let* $t \in \mathbf{R}$, $p \in \mathrm{Im}\,\mathbf{O}$, $x \in \mathbf{O}$, *and* $t \in \mathbf{R}$. *Assume that* $z = \exp(\mathscr{G}_{-2}(p) + \mathscr{G}_{-1}(x)) \in N^-$. *Then*

(1) $$H(a_t z) = 2^{-1}\log(e^{-2t}((e^{2t} + (x|x))^2 + 4(p|p)))\tilde{A}_3^1(1),$$

(2) $$H(z) = 2^{-1}\log((1 + (x|x))^2 + 4(p|p))\tilde{A}_3^1(1).$$

PROOF. From (9.18) and (9.15), $z = \sigma \exp \mathscr{G}_2(p) \exp \mathscr{G}_1(x)\sigma$. Put $X = \sigma \exp \mathscr{G}_2(p) \exp \mathscr{G}_1(x)\sigma P^-$. Using $\sigma P^- = 2(-E_1 + E_2) - P^-$, (10.2), and (10.3), we calculate that

$$\begin{aligned} X = &-(((x|x)+1)^2 + 4(p|p))E_1 + (((x|x)-1)^2 + 4(p|p))E_2 \\ &+ 4(x|x)E_3 + F_1^1(2((x|x) + 2p - 1)x) \\ &+ F_2^1(-2\bar{x}((x|x) - 2p + 1)) + F_3^1(-(x|x)^2 - 4(p|p) + 1 + 4p). \end{aligned}$$

Set $X = h^1(\eta_1, \eta_2, \eta_3; y_1, y_2, y_3)$. Using (10.5), we get $(a_t z P^- \mid E_1) = (a_t X)_{E_1} = 2^{-1}((\eta_1 + \eta_2) + (\eta_1 - \eta_2)\cosh(2t)) - (1|y_3)\sinh(2t)$. Because of $2^{-1}(\eta_1 + \eta_2) = -2(x|x)$, $2^{-1}(\eta_1 - \eta_2) = -(x|x)^2 - 4(p|p) - 1$, and $(1|y_3) = -(x|x)^2 - 4(p|p) + 1$, we calculate that

$$\begin{aligned}(a_t z P^- \mid E_1) &= -e^{-2t}((x|x)^2 + 2e^{2t}(x|x) + e^{4t} + 4(p|p)) \\ &= -e^{-2t}((e^{2t} + (x|x))^2 + 4(p|p)).\end{aligned}$$

Thus (1) follows from Main Theorem 9.4(i), and substituting $t = 0$ in (1), we obtain (2). □

PROPOSITION A.2. *Assume* $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$,

$$a = 4^{-1}(m_\alpha + 2m_{2\alpha} + \lambda_\alpha), \quad b = 4^{-1}(m_\alpha + 2m_{2\alpha} - \lambda_\alpha).$$

Then there exists the constant $C_0 \in \mathbf{R}$ *such that*

(1) $$c(\lambda) = C_0 \int_{\mathbf{R}^{m_\alpha} \times \mathbf{R}^{m_{2\alpha}}} ((1 + (x|x))^2 + 4(p|p))^{-a}\, dx dp,$$

(2) $$\varphi_\lambda(a_t) = C_0 \int_{\mathbf{R}^{m_\alpha} \times \mathbf{R}^{m_{2\alpha}}} e^{2bt}((e^{2t} + (x|x))^2 + 4(p|p))^{-b} \cdot ((1 + (x|x))^2 + 4(p|p))^{-a}\, dx dp$$

where the measure dx and dp are the Euclidean measure.

PROOF. It follows from Lemma A.1. □

From [13, Lemma 7.2],

$$B(\phi, \tilde{\sigma}\phi) = -3\Big(\sum_{i=1}^{3}\Big(\Big(\sum_{j=0}^{7}(D_i e_j \mid D_i e_j)\Big) + 24(a_i|a_i)\Big)\Big)$$

where $\phi = d\varphi_0(D_1, D_2, D_3) + \sum_{i=1}^{3} \tilde{A}_i^1(a_i)$ with $d\varphi_0(D_1, D_2, D_3) \in \mathfrak{d}_4$ and $a_i \in \mathbf{O}$. We denote $Q(\phi) := -\langle \alpha, \alpha \rangle B(\phi, \tilde{\sigma}\phi)$ for $\phi \in \mathfrak{f}_{4(-20)}$. Then from direct calculations, we have the following proposition.

PROPOSITION A.3. *Assume that* $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, $p \in \operatorname{Im} \mathbf{O}$ *and* $x \in \mathbf{O}$. *Then* $Q(\mathscr{G}_{-1}(x)) = 2(x|x)$ *and* $Q(\mathscr{G}_{-2}(p)) = 2(p|p)$.

COROLLARY A.4 ([6], [17], cf. [14, Lemma 4.12 and (4.27)]). *Assume that* $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$. *Then for* $X \in \mathfrak{g}_{-\alpha}$ *and* $Y \in \mathfrak{g}_{-2\alpha}$,

$$e^{\lambda(H(\exp(X+Y)))} = ((1 + 2^{-1}Q(X))^2 + 2Q(Y))^{4^{-1}\lambda_\alpha}.$$

REMARK A.5 ([6], [17], cf. [15], [16]). From Proposition A.2(1), changing variables to polar coordinates, up to the constant multiple, $c(\lambda)$ is equal to

$$\int_0^\infty \int_0^\infty t^{m_\alpha-1} s^{m_{2\alpha}-1}((1+t^2)^2+s^2)^{-4^{-1}(\lambda_\alpha+m_\alpha+2m_{2\alpha})}\,dsdt$$

$$= \int_0^\infty \int_0^\infty (s/(1+t^2))^{m_{2\alpha}-1}(1+(s/(1+t^2))^2)^{-4^{-1}(\lambda_\alpha+m_\alpha+2m_{2\alpha})}$$

$$\cdot\, t^{m_\alpha-1}(1+t^2)^{-2^{-1}(\lambda_\alpha+m_\alpha)+1}\,dsdt$$

$$= \int_0^\infty u^{m_{2\alpha}-1}(1+u^2)^{-4^{-1}(\lambda_\alpha+m_\alpha+2m_{2\alpha})}\,du$$

$$\cdot \int_0^\infty t^{m_\alpha-1}(1+t^2)^{-2^{-1}(\lambda_\alpha+m_\alpha)}\,dt.$$

By using the integral formula

$$\int_0^\infty x^a(1+x^c)^{-(b+1)}\,dx = c^{-1}\Gamma[(a+1)/c]\Gamma[b-((a-c+1)/c)]/\Gamma(1+b)$$

$(\mathrm{Re}(c)>0; \mathrm{Re}(a), \mathrm{Re}(b)>-1; \mathrm{Re}(b)>\mathrm{Re}((a-c+a)/c))$, up to the constant multiple, this integral is equal to

$$(\Gamma(\lambda_\alpha/2)\Gamma((\lambda_\alpha+m_\alpha)/4))/(\Gamma((\lambda_\alpha+m_\alpha)/2)\Gamma((\lambda_\alpha+m_\alpha+2m_{2\alpha})/4)).$$

These calculations imply the Gindikin and Karpelevich formula of the semisimple Lie group $F_{4(-20)}$ which is known [2] (cf. [15, (4.3)], [9]).

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Graduate School of Engineering
University of Fukui
Fukui-shi, 910-8507
Japan
E-mail: nishio@quantum.apphy.u-fukui.ac.jp
Recent address: aknishio@tea.ocn.ne.jp