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SOME REMARKS ON CONNECTORS AND GROUPOIDS IN GOURSAT CATEGORIES

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Dedicated to Jiří Adámek on the occasion of his seventieth birthday

Abstract. We prove that connectors are stable under quotients in any (regular) Goursat category. As a consequence, the category $\mathrm{Conn}(\mathbb{C})$ of connectors in $\mathbb{C}$ is a Goursat category whenever $\mathbb{C}$ is. This implies that Goursat categories can be characterised in terms of a simple property of internal groupoids.

Over the last twenty years the property of n-permutability of congruences in a variety of universal algebras has been investigated from a categorical perspective (see [7, 13, 19], for instance, and references therein). When $\mathbb{C}$ is a regular category, the 2-permutability property, usually called the *Mal'tsev* property, is a concept giving rise to a beautiful theory, whose main features are collected in [1]. Many important results still hold when a regular category $\mathbb{C}$ satisfies the strictly weaker property of 3-permutability, namely the *Goursat* property. A nice feature of a (regular) Goursat category $\mathbb{C}$ is that the lattice of equivalence relations on any object in $\mathbb{C}$ is a modular lattice [7], a property that plays a crucial role in commutator theory [10, 17].

The aim of this paper is twofold: first of all we establish some basic properties of Goursat categories in terms of connectors [4], as it was done in [4] for the case of Mal'tsev categories. These results have turned out to be useful to develop a monoidal approach to internal structures [12]. We then give a new characterisation of Goursat categories in terms of properties of (internal) groupoids, on the model of what was done in [11] in the case of Mal'tsev categories.

In the first section, we recall the main properties of Goursat categories that will be needed throughout the paper. In Section 2 we prove that for any Goursat category $\mathbb{C}$, the category $\mathrm{Equiv}(\mathbb{C})$ of equivalence relations in $\mathbb{C}$ is also a Goursat category (Proposition 2.2, see also [2]). We use this result to give some properties of Goursat categories in terms

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of connectors in Section 3. More precisely, we show that, when $\mathbb{C}$ is a Goursat category, then connectors are stable under quotients in $\mathbb{C}$ (Proposition 3.8), and this implies that the category $\mathrm{Conn}(\mathbb{C})$ of connectors in $\mathbb{C}$ is again a Goursat category (Theorem 3.10).

We conclude the paper by giving a new characterisation of Goursat categories in terms of properties of groupoids and internal categories (Theorem 3.11). It turns out that a regular category $\mathbb{C}$ is a Goursat category if and only if the category $\mathrm{Grpd}(\mathbb{C})$ of groupoids (equivalently, the category $\mathrm{Cat}(\mathbb{C})$ of internal categories) in $\mathbb{C}$ is closed under quotients in the category $\mathrm{RG}(\mathbb{C})$ of reflexive graphs in $\mathbb{C}$.

1. Preliminaries

In this section we recall some basic definitions and properties of (regular) Goursat categories, needed throughout the article. We shall always assume that the category $\mathbb{C}$ in which we are working is a **regular category**: this means that $\mathbb{C}$ is finitely complete, regular epimorphisms are stable under pullbacks, and kernel pairs have coequalisers. Equivalently, any arrow $f : A \longrightarrow B$ has a unique factorisation $f = i \circ r$ (up to isomorphism), where r is a regular epimorphism and i is a monomorphism and this factorisation is pullback stable; the subobject corresponding to i is called the **image** of f.

A **relation** R from X to Y is a subobject $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$. The opposite relation of R, denoted R^o, is the relation from Y to X given by the subobject $\langle r_2, r_1\rangle : R \rightarrowtail Y \times X$. A relation R from X to X is called a relation on X. We shall identify a morphism $f : X \longrightarrow Y$ with the relation $\langle 1_X, f\rangle : X \rightarrowtail X \times Y$ and write f^o for its opposite relation. Given another relation $\langle s_1, s_2\rangle : S \rightarrowtail Y \times Z$ from Y to Z, one can define the composite relation SR of R and S as the image of the arrow $(r_1 \circ p_1, s_2 \circ p_2) : R \times_Y S \longrightarrow X \times Z$, where $(R \times_Y S, p_1, p_2)$ is the pullback of $r_2 : R \longrightarrow Y$ along $s_1 : S \longrightarrow Y$. With the above notations, any relation $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ can be seen as the relational composite $r_2 r_1^o$.

The following properties are well known and easy to prove. We collect them in the following lemma:

Lemma 1.1. *Let $f : X \longrightarrow Y$ be an arrow in a regular category $\mathbb{C}$, and let $i \circ r$ be its (regular epimorphism, monomorphism) factorisation. Then:*

(1) *$f^o f$ is the kernel pair of f, thus $1_X \leqslant f^o f$; moreover, $1_X = f^o f$ if and only if f is a monomorphism;*
(2) *$f f^o$ is (i, i), thus $f f^o \leqslant 1_Y$; moreover, $f f^o = 1_Y$ if and only if f is a regular epimorphism;*
(3) *$f f^o f = f$ and $f^o f f^o = f^o$.*

Definition 1.2. A relation (R, r_1, r_2) on an object X is said to be :

- **reflexive** when there is an arrow $r : X \longrightarrow R$ such that $r_1 \circ r = 1_X = r_2 \circ r$;
- **symmetric** when there is an arrow $\sigma : R \longrightarrow R$ such that $r_2 = r_1 \circ \sigma$ and $r_1 = r_2 \circ \sigma$;
- **transitive** when, by considering the following pullback

$$\begin{array}{ccc} R \times_X R & \xrightarrow{p_2} & R \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X, \end{array}$$

there is an arrow $t : R \times_X R \longrightarrow R$ such that $r_1 \circ t = r_1 \circ p_1$ and $r_2 \circ t = r_2 \circ p_2$.

• an **equivalence relation** if R is reflexive, symmetric and transitive.

In particular, a kernel pair $\langle f_1, f_2\rangle : \mathrm{Eq}(f) \rightarrowtail X \times X$ of a morphism $f : X \longrightarrow Y$ is an equivalence relation. The equivalence relations that occur as kernel pairs of some morphism in $\mathbb{C}$ are called **effective**. Let $\mathrm{Equiv}(\mathbb{C})$ be the category whose objects are equivalence relations in $\mathbb{C}$ and arrows from $\langle r_1, r_2\rangle : R \rightarrowtail X \times X$ to $\langle s_1, s_2\rangle : S \rightarrowtail Y \times Y$ are pairs (f, g) of arrows in $\mathbb{C}$ making the following diagram commute

$$\begin{array}{ccc} R & \xrightarrow{\;g\;} & S \\ {\scriptstyle r_1}\downdownarrows{\scriptstyle r_2} & & {\scriptstyle s_1}\downdownarrows{\scriptstyle s_2} \\ X & \xrightarrow[\;f\;]{} & Y. \end{array}$$

When $\mathbb{C}$ is a regular category, (R, r_1, r_2) is an equivalence relation on X and $f : X \twoheadrightarrow Y$ a regular epimorphism, we define the **regular image of** R **along** f to be the relation $f(R)$ on Y induced by the (regular epimorphism, monomorphism) factorisation $\langle s_1, s_2\rangle \circ \psi$ of the composite $(f \times f) \circ \langle r_1, r_2\rangle$:

$$\begin{array}{ccc} R & \overset{\psi}{\twoheadrightarrow} & f(R) \\ {\scriptstyle \langle r_1, r_2\rangle}\downarrow & & \downarrow{\scriptstyle \langle s_1, s_2\rangle} \\ X \times X & \underset{f\times f}{\twoheadrightarrow} & Y \times Y. \end{array}$$

Note that the regular image $f(R)$ can be obtained as the relational composite $f(R) = fRf^o = fr_2r_1^of^o$. When R is an equivalence relation, $f(R)$ is also reflexive and symmetric. In a general regular category $f(R)$ is not necessarily an equivalence relation. This is the case in a *Goursat category* (Theorem 1.4).

Definition 1.3 [8, 7]**.** A regular category $\mathbb{C}$ is called a **Goursat category** when the equivalence relations in $\mathbb{C}$ are 3-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.

The following characterisation will be useful in the sequel:

Theorem 1.4 [7]**.** *A regular category $\mathbb{C}$ is a Goursat category if and only if for any regular epimorphism $f : X \twoheadrightarrow Y$ and any equivalence relation R on X, the regular image $f(R) = fRf^o$ of R along f is an equivalence relation.*

There are many important algebraic examples of Goursat categories. Indeed, by a classical theorem in [18], a variety of universal algebras is a Goursat category precisely when its theory has two ternary operations r and s such that the identities $r(x, y, y) = x$, $r(x, x, y) = s(x, y, y)$ and $s(x, x, y) = y$ hold. Accordingly, the categories of groups, abelian groups, modules over some fixed ring, crossed modules, quasi-groups, rings, associative algebras, Heyting algebras and implication algebras are all Goursat categories.

Any regular *Mal'tsev* category is a Goursat category, thus, in particular, so is any semi-abelian category.

Many interesting properties of Goursat categories can be found in the literature (see [7, 15, 16] and references therein). In particular, the following characterisations will be useful for the development of this work:

Theorem 1.5 [15]**.** *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) $\mathbb{C}$ *is a Goursat category;*

(ii) *any commutative diagram where α and β are regular epimorphisms and f and g are split epimorphisms in $\mathbb{C}$*

$$\begin{array}{ccc} X & \overset{\alpha}{\twoheadrightarrow} & U \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle t}\uparrow\downarrow{\scriptstyle g} \\ Y & \underset{\beta}{\twoheadrightarrow} & W \end{array}$$

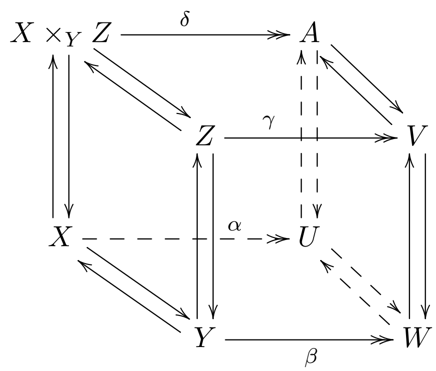
(which is necessarily a pushout) is a ***Goursat pushout****: the morphism $\lambda\colon \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$ induced by the universal property of kernel pair $\mathrm{Eq}(g)$ of g is a regular epimorphism.*

We recall part of Theorem 1.3 in [16]:

Theorem 1.6 [16]**.** *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) $\mathbb{C}$ *is a Goursat category;*

(ii) *for any commutative cube*



where the left square is a pullback of split epimorphisms, the right square is a commutative square of split epimorphisms and the horizontal arrows α, β, γ and δ are regular epimorphisms (commuting also with the splittings), then the right square is a pullback.

2. Equivalence relations in Goursat categories

In this section we prove that $\mathrm{Equiv}(\mathbb{C})$ is a Goursat category for any Goursat category $\mathbb{C}$.

The category $\mathrm{Equiv}(\mathbb{C})$ is finitely complete whenever $\mathbb{C}$ is: the terminal object in $\mathrm{Equiv}(\mathbb{C})$ is the discrete equivalence relation

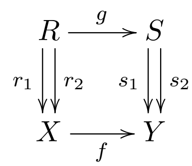
$$1 \rightrightarrows 1$$

on the terminal object 1 of $\mathbb{C}$, and pullbacks are computed "levelwise". In particular, the kernel pair of a morphism (f, g) in $\mathrm{Equiv}(\mathbb{C})$ is given by the kernel pairs $\mathrm{Eq}(f)$ of f and $\mathrm{Eq}(g)$ of g in $\mathbb{C}$

$$\begin{array}{ccccc} \mathrm{Eq}(g) & \underset{g_2}{\overset{g_1}{\rightrightarrows}} & R & \overset{g}{\longrightarrow} & S \\ {\scriptstyle \bar{r_1}}\downarrow\downarrow{\scriptstyle \bar{r_2}} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ \mathrm{Eq}(f) & \underset{f_2}{\overset{f_1}{\rightrightarrows}} & X & \underset{f}{\longrightarrow} & Y. \end{array} \tag{2.1}$$

Consequently, a morphism (f, g) is a monomorphism in $\mathrm{Equiv}(\mathbb{C})$ if and only if f and g are monomorphisms in $\mathbb{C}$. When $\mathbb{C}$ is a Goursat category, a similar property holds with respect to regular epimorphisms:

Lemma 2.1. *Let R and S be two equivalence relations in a Goursat category $\mathbb{C}$ and $(f, g) : R \to S$ a morphism*

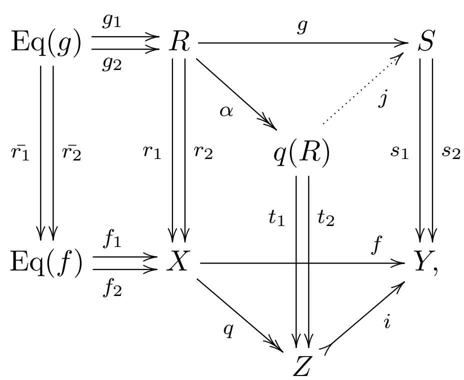


(2.2)

in $\mathrm{Equiv}(\mathbb{C})$*. Then (f, g) is a regular epimorphism in* $\mathrm{Equiv}(\mathbb{C})$ *if and only if f and g are regular epimorphisms in $\mathbb{C}$.*

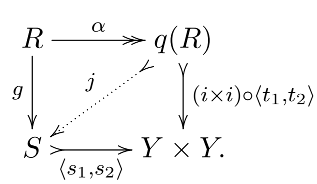
Proof. When f and g are regular epimorphisms in $\mathbb{C}$, it is not difficult to check that (f, g) is necessarily the coequaliser of its kernel pair in $\mathrm{Equiv}(\mathbb{C})$ given in (2.1) (one uses the fact that $g = coeq(g_1, g_2)$ and $f = coeq(f_1, f_2)$ in $\mathbb{C}$).

Conversely, let (f, g) be a morphism in $\mathrm{Equiv}(\mathbb{C})$ as in (2.2) that is a regular epimorphism in $\mathrm{Equiv}(\mathbb{C})$. Consider the kernel pairs of f and g, the (regular epimorphism, monomorphism) factorisation $f = i \circ q$ of f, and the regular image $(q(R), t_1, t_2)$ of (R, r_1, r_2) along q. We obtain the following commutative diagram



(2.3)

where $(q(R), t_1, t_2) \in \mathrm{Equiv}(\mathbb{C})$ (by Theorem 1.4) and (i, j) is the morphism in $\mathrm{Equiv}(\mathbb{C})$ such that $(i, j) \circ (q, \alpha) = (f, g)$. Note that j is induced from the fact that $(i \times i) \circ \langle t_1, t_2 \rangle \circ \alpha$ is the (regular epimorphism, monomorphism) factorisation of $\langle s_1, s_2 \rangle \circ g$, thus it is a monomorphism



From the fact that (f, g) is the coequaliser of its kernel pair in $\mathrm{Equiv}(\mathbb{C})$ it easily follows that (i, j) is an isomorphism in $\mathrm{Equiv}(\mathbb{C})$. This implies that f and g are regular epimorphisms in $\mathbb{C}$. □

Proposition 2.2. Equiv($\mathbb{C}$) *is a Goursat category whenever* $\mathbb{C}$ *is.*

Proof. As mentioned above, the category Equiv($\mathbb{C}$) is finitely complete because $\mathbb{C}$ is so. Lemma 2.1 implies that regular epimorphisms in Equiv($\mathbb{C}$) are stable under pullbacks since regular epimorphisms are stable in $\mathbb{C}$, and regular epimorphisms in Equiv($\mathbb{C}$) are "levelwise" regular epimorphisms. The existence of the (regular epimorphism, monomorphism) factorisation of a morphism (f, g) as in (2.2) in the category Equiv($\mathbb{C}$) follows from the construction of diagram (2.3): the (regular epimorphism, monomorphism) factorisation $f = i \circ q$ of f in $\mathbb{C}$ gives rise to the (regular epimorphism, monomorphism) factorisation $g = j \circ \alpha$ of g in $\mathbb{C}$. Thus $(q, \alpha) \circ (i, j)$ is the (regular epimorphism, monomorphism) factorisation of (f, g) in Equiv($\mathbb{C}$). To see that Equiv($\mathbb{C}$) has the Goursat property one uses Theorem 1.4: to check that the regular image of an equivalence relation in the category Equiv($\mathbb{C}$) is again an equivalence in Equiv($\mathbb{C}$) one mainly uses the same ("levelwise") property in the category $\mathbb{C}$. □

3. Connectors and groupoids in Goursat categories

In this section we prove that connectors are stable under quotients in any Goursat category $\mathbb{C}$. We then define the category Conn($\mathbb{C}$) of connectors in $\mathbb{C}$ whose objects are pairs of equivalence relations equipped with a connector, and prove that Conn($\mathbb{C}$) is a Goursat category whenever the base category $\mathbb{C}$ is. We conclude by giving a new characterisation of Goursat categories in terms of properties of groupoids and internal categories.

Definition 3.1. Let (R, r_1, r_2) and (S, s_1, s_2) be two equivalence relations on an object X and $R \times_X S$ the pullback of r_2 along s_1. A **connector** [4] between R and S is an arrow $p : R \times_X S \longrightarrow X$ in $\mathbb{C}$ such that

(1) $xSp(x, y, z)Rz$;
(2) $p(x, x, y) = y$;
(3) $p(x, y, y) = x$;
(4) $p(x, y, p(z, u, v)) = p(p(x, y, z), u, v)$,

when each term is defined.

Remark 3.2. Given two regular epimorphisms $d \colon X \twoheadrightarrow Y$ and $c \colon X \twoheadrightarrow Z$, a connector on the effective equivalence relations Eq(d) and Eq(c) is the same thing as an internal *pregroupoid* in the sense of Kock [20, 21] (see also the introduction of [5], for instance, for a comparison between these two related notions and some additional references). In the context of Mal'tsev or Goursat categories connectors are useful to develop a centrality theory of non-effective equivalence relations.

Example 3.3. *If* ∇_X *is the largest equivalence relation on an object* X*, then an associative Mal'tsev operation*

$$p : X \times X \times X \longrightarrow X$$

is precisely a connector between ∇_X *and* ∇_X*.*

Connectors provide a way to distinguish groupoids amongst reflexive graphs:

Proposition 3.4 [9]**.** *Given a reflexive graph*

$$X_1 \underset{c}{\overset{d}{\rightrightarrows}} X_0 \quad (e \colon X_0 \to X_1)$$

in a finitely complete category $\mathcal{C}$ *(i.e.* $d \circ e = 1_{X_0} = c \circ e$*) then the connectors between* $\mathrm{Eq}(c)$ *and* $\mathrm{Eq}(d)$ *are in bijections with the groupoid structures on this reflexive graph.*

It is well known that Goursat categories satisfy the so-called *Shifting Property* [17, 6]. In this context connectors are unique when they exist (Theorem 2.13 and Proposition 5.1 in [6]): accordingly, for a given pair of equivalence relations on the same object the fact of having a connector becomes a *property*.

Definition 3.5. Let R and S be two equivalence relations on an object X. A **double equivalence relation** on R and S is given by an object $C \in \mathbb{C}$ equipped with two equivalence relations $(\pi_1, \pi_2) : C \rightrightarrows R$ and $(p_1, p_2) : C \rightrightarrows S$ such that the following diagram

$$\begin{array}{ccc} C & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & R \\ {\scriptstyle p_1}\downarrow\downarrow{\scriptstyle p_2} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} \\ S & \overset{s_1}{\underset{s_2}{\rightrightarrows}} & X \end{array}$$

commutes (in the "obvious" way).

A double equivalence relation C on R and S is called a **centralizing relation** [9] when the square

$$\begin{array}{ccc} C & \xrightarrow{\pi_1} & R \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1} \\ S & \xrightarrow[s_1]{} & X \end{array}$$

is a pullback. Under this assumption it follows that any of the commutative squares in the definition of a centralizing relation is a pullback.

The following lemma gives the relationship between connectors and centralizing relations.

Lemma 3.6 [4]**.** *If* $\mathbb{C}$ *is a category with finite limits and* R *and* S *are two equivalence relations on the same object* X*, then the following conditions are equivalent:*

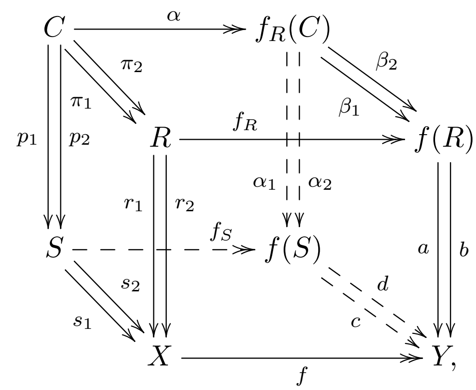
(i) *there exists a connector between* R *and* S*;*
(ii) *there exists a centralizing relation on* R *and* S*.*

When $\mathbb{C}$ is a Mal'tsev category, R and S are equivalence relations on an object X with a connector and $i : I \rightarrowtail X$ is a monomorphism, then the inverse images $i^{-1}(R)$ and $i^{-1}(S)$ also have a connector [4]. We establish a similar property for Goursat categories, with respect to regular epimorphisms:

Proposition 3.7. *Let* $\mathbb{C}$ *be a Goursat category,* R *and* S *two equivalence relations on an object* X*, and let* $f : X \twoheadrightarrow Y$ *be a regular epimorphism. If there exists a connector between* R *and* S*, then there exists a connector between the regular images* $f(R)$ *and* $f(S)$*.*

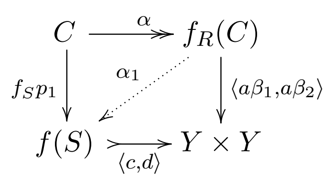
Proof. Suppose that there exists a connector between R and S. This implies that there exists a centralizing relation $(C, (\pi_1, \pi_2), (p_1, p_2))$ on R and S. Consider the regular image

$(f(R), a, b)$ and $(f(S), c, d)$ of R and S along f. We obtain the following diagram

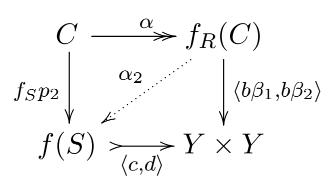


(3.1)

where $(f_R(C), \beta_1, \beta_2)$ is the regular image of the equivalence relation (C, π_1, π_2) along the regular epimorphism f_R. The fact that the square



commutes, α is a strong epimorphism and $\langle c, d\rangle$ is a monomorphism, implies the existence of an arrow $\alpha_1 : f_R(C) \longrightarrow f(S)$ making the above diagram commute. Similarly, from the commutativity of the third diagram



we obtain an arrow $\alpha_2 : f_R(C) \longrightarrow f(S)$.

The relations $(f_R(C), \beta_1, \beta_2)$, $(f(R), a, b)$ and $(f(S), c, d)$ are all equivalence relations by Theorem 1.4. It is then easy to check that the relation $(f_R(C), \alpha_1, \alpha_2)$ is an equivalence relation on $f(S)$. In fact, the morphism $\langle \alpha_1, \alpha_2\rangle \colon f_R(C) \to f(S) \times f(S)$ is a monomorphism since $\langle c \times c, d \times d\rangle \circ \langle \alpha_1, \alpha_2\rangle = \langle a, b\rangle \times \langle a, b\rangle \circ \langle \beta_1, \beta_2\rangle$. So, $\langle \alpha_1, \alpha_2\rangle$ is the regular image of $\langle p_1, p_2\rangle$ along f_S, thus it is an equivalence relation on $f(S)$ by Theorem 1.4.

By assumption all the left squares of (3.1) are pullbacks, so it follows that all the right squares of (3.1) are pullbacks as well by Theorem 1.6 (ii). Then $(f_R(C), (\alpha_1, \alpha_2), (\beta_1, \beta_2))$ is a centralizing relation on $f(R)$ and $f(S)$. By Lemma 3.6 there is a connector between $f(R)$ and $f(S)$. □

We are now going to show that the category whose objects are pairs of equivalence relations equipped with a connector is a Goursat category whenever the base category is a Goursat category. For this, let us first fix some notation: if $\mathbb{C}$ is a finitely complete category, we write 2-Eq($\mathbb{C}$) for the category whose objects (R, S, X) are pairs of equivalence relations

R and S on the same object X

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \underset{s_2}{\overset{s_1}{\leftleftarrows}} S$$

and arrows are triples (f_R, f_S, f) making the following diagram commute:

$$\begin{array}{ccccc} R & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & X & \underset{s_2}{\overset{s_1}{\leftleftarrows}} & S \\ {\scriptstyle f_R}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_S} \\ \bar{R} & \underset{\bar{r_2}}{\overset{\bar{r_1}}{\rightrightarrows}} & \bar{X} & \underset{\bar{s_2}}{\overset{\bar{s_1}}{\leftleftarrows}} & \bar{S}. \end{array} \tag{3.2}$$

We write Conn($\mathbb{C}$) for the category whose objects (R, S, X, p) are pairs of equivalence relations R and S on an object X with a given connector $p : R \times_X S \to X$; arrows in Conn($\mathbb{C}$) are arrows in 2-Eq($\mathbb{C}$) respecting the connectors. This means that, given a diagram (3.2) where both (R, S, X) and $(\bar{R}, \bar{S}, \bar{X})$ are in Conn($\mathbb{C}$), with $p : R \times_X S \to X$ and $\bar{p} : \bar{R} \times_Y \bar{S} \to Y$ the corresponding connectors, then the diagram

$$\begin{array}{ccc} R \times_X S & \xrightarrow{\bar{f}} & \bar{R} \times_{\bar{X}} \bar{S} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ X & \xrightarrow[f]{} & \bar{X} \end{array}$$

commutes, where $\bar{f}$ is the natural map induced by the universal property of the pullback $\bar{R} \times_{\bar{X}} \bar{S}$.

We say that a subcategory $\mathbb{P}$ is *closed under (regular) quotients* in a category $\mathbb{Q}$ if, for any regular epimorphism $f : A \twoheadrightarrow B$ in $\mathbb{Q}$ such that $A \in \mathbb{P}$, then $B \in \mathbb{P}$.

Proposition 3.8. *If* $\mathbb{C}$ *is a Goursat category, then* Conn($\mathbb{C}$) *is a full subcategory of 2-*Eq($\mathbb{C}$)*, that is closed in 2-*Eq($\mathbb{C}$) *under quotients.*

Proof. The fullness of the forgetful functor Conn($\mathbb{C}$) $\to$ 2-Eq($\mathbb{C}$) follows from Corollary 5.2 in [6], by taking into account the fact that any Goursat category satisfies the Shifting Property.

Let us then consider a regular epimorphism in 2-Eq($\mathbb{C}$)

$$\begin{array}{ccccc} R & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & X & \underset{s_2}{\overset{s_1}{\leftleftarrows}} & S \\ {\scriptstyle f_R}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_S} \\ \bar{R} & \underset{\bar{r_2}}{\overset{\bar{r_1}}{\rightrightarrows}} & \bar{X} & \underset{\bar{s_2}}{\overset{\bar{s_1}}{\leftleftarrows}} & \bar{S} \end{array}$$

(this means that f, f_R and f_S are regular epimorphisms in $\mathbb{C}$) such that its domain (R, S, X) belongs to Conn($\mathbb{C}$). The equalities $f(R) = \bar{R}$ and $f(S) = \bar{S}$, together with Proposition 3.7, imply that there exists a connector between $\bar{R}$ and $\bar{S}$. □

Lemma 3.9. *Let* $\mathbb{D}$ *be a finitely complete category, and* $\mathbb{C}$ *a full subcategory of* $\mathbb{D}$ *closed in* $\mathbb{D}$ *under finite limits and quotients. Then:*

(1) $\mathbb{C}$ *is regular whenever* $\mathbb{D}$ *is regular.*
(2) $\mathbb{D}$ *is a Goursat category whenever* $\mathbb{C}$ *is a Goursat category.*

Proof. The (regular epimorphism, monomorphism) factorisation in $\mathbb{D}$ of an arrow in $\mathbb{C}$ is also its factorisation in $\mathbb{C}$, since $\mathbb{C}$ is closed in $\mathbb{D}$ under quotients. Since finite limits in $\mathbb{C}$ are calculated as in $\mathbb{D}$, it follows that regular epimorphisms are stable under pullbacks. Now the second statement easily follows from the fact that the composition of relations is computed in the same way in $\mathbb{C}$ and in $\mathbb{D}$. □

Theorem 3.10. *If $\mathbb{C}$ is a Goursat category then* $\mathrm{Conn}(\mathbb{C})$ *is a Goursat category.*

Proof. Using similar arguments as those given in the proof of Proposition 2.2 with respect to $\mathrm{Equiv}(\mathbb{C})$, one may deduce that $2\text{-}\mathrm{Eq}(\mathbb{C})$ is a Goursat category. The result then follows from Proposition 3.8 and Lemma 3.9. □

We finally prove that internal categories and groupoids can be used to characterise Goursat categories. Recall that an **internal category** in a category $\mathbb{C}$ with pullbacks is a reflexive graph with a multiplication $m\colon X_1 \times_{X_0} X_1 \to X_1$

$$X_1 \times_{X_0} X_1 \underset{p_2}{\overset{p_1}{\underset{m}{\rightrightarrows}}} X_1 \underset{c}{\overset{d}{\underset{\leftarrow e -}{\rightrightarrows}}} X_0,$$

(where $X_1 \times_{X_0} X_1$ is the pullback of d and c) such that:

- $d \circ m = d \circ p_2$, $c \circ m = c \circ p_1$, $m \circ \langle e \circ d, 1_{X_1} \rangle = 1_{X_1} = m \circ \langle 1_{X_1}, e \circ c \rangle$;
- $m \circ (1 \times m) = m \circ (m \times 1)$.

An internal category

$$X_1 \times_{X_0} X_1 \underset{p_2}{\overset{p_1}{\underset{m}{\rightrightarrows}}} X_1 \underset{c}{\overset{d}{\underset{\leftarrow e -}{\rightrightarrows}}} X_0,$$

is a **groupoid** when there is an additional morphism $i\colon X_1 \longrightarrow X_1$ satisfying the axioms:

- $d \circ i = c$, $c \circ i = d$;
- $m \circ \langle i, 1_{X_1} \rangle = e \circ c$ and $m \circ \langle 1_{X_1}, i \rangle = e \circ d$.

We write $\mathrm{Cat}(\mathbb{C})$ for the category of internal categories in $\mathbb{C}$ (and internal functors as morphisms), $\mathrm{Grpd}(\mathbb{C})$ for the category of groupoids in $\mathbb{C}$, and $\mathrm{RG}(\mathbb{C})$ for the category of reflexive graphs in $\mathbb{C}$ (with obvious morphisms).

An equivalence relation is a special kind of groupoid, where its domain and codomain morphisms are jointly monomorphic; also any reflexive and transitive relation is in particular an internal category. If $\mathbb{C}$ is a Goursat category, then any reflexive and transitive relation is an equivalence relation or, equivalently, any internal category is a groupoid (Theorem 1 in [22]). Then Theorem 1.4, which could equivalently be stated through the property that $\mathrm{Equiv}(\mathbb{C})$ (or the category of reflexive and transitive relations in $\mathbb{C}$) is closed in the category of reflexive relations in $\mathbb{C}$ under quotients, has an *extended* counterpart given below. This characterisation leads to the observation that the structural aspects of Goursat categories mainly concern groupoids (rather than equivalence relations).

Theorem 3.11. *Let $\mathbb{C}$ be a regular category. Then the following conditions are equivalent:*

(i) $\mathbb{C}$ *is a Goursat category;*
(ii) $\mathrm{Grpd}(\mathbb{C})$ *is closed in* $\mathrm{RG}(\mathbb{C})$ *under quotients;*
(iii) $\mathrm{Cat}(\mathbb{C})$ *is closed in* $\mathrm{RG}(\mathbb{C})$ *under quotients.*

Proof.

(i) $\Rightarrow$ (ii) Let

$$\begin{array}{ccc} X_1 & \overset{g}{\twoheadrightarrow} & X_1' \\ {\scriptstyle d}\downarrow\uparrow{\scriptstyle e}\downarrow{\scriptstyle c} & & {\scriptstyle d'}\downarrow\uparrow{\scriptstyle e'}\downarrow{\scriptstyle c'} \\ X_0 & \underset{f}{\twoheadrightarrow} & X_0' \end{array}$$

be a regular epimorphism (f, g) in $\mathrm{RG}(\mathbb{C})$ (which means that f and g are regular epimorphisms in $\mathbb{C}$), with

$$X_1 \underset{c}{\overset{d}{\rightrightarrows}} X_0 \quad (e : X_0 \to X_1)$$

a groupoid in $\mathbb{C}$. By Proposition 3.4, there exists a connector between $\mathrm{Eq}(d)$ and $\mathrm{Eq}(c)$. Let $\mathrm{Eq}(d)$, $\mathrm{Eq}(c)$, $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$ be the kernel pairs of the arrows d, c, d' and c', respectively. Let $\lambda : \mathrm{Eq}(d) \to \mathrm{Eq}(d')$ and $\beta : \mathrm{Eq}(c) \to \mathrm{Eq}(c')$ be the arrows induced by the universal property of kernel pairs $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$, respectively. By Theorem 1.5, λ and β are regular epimorphisms, so that $g(\mathrm{Eq}(d)) = \mathrm{Eq}(d')$ and $g(\mathrm{Eq}(c)) = \mathrm{Eq}(c')$. By Proposition 3.7 there is then a connector between $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$, thus

$$X_1' \underset{c'}{\overset{d'}{\rightrightarrows}} X_0' \quad (e' : X_0' \to X_1')$$

is a groupoid (Proposition 3.4).

(ii) $\Rightarrow$ (i) This implication follows from Theorem 1.4 and the fact that equivalence relations are in particular groupoids (whose domain and codomain morphisms are jointly monomorphic).

(i) $\Rightarrow$ (iii) This implication follows from (i) $\Rightarrow$ (ii) and the fact that $\mathrm{Grpd}(\mathbb{C}) \cong \mathrm{Cat}(\mathbb{C})$ in a Goursat context, as recalled above.

(iii) $\Rightarrow$ (i) Let (R, r_1, r_2) be an equivalence relation on X, $f : X \twoheadrightarrow Y$ a regular epimorphism and $(f(R), t_1, t_2)$ the regular image of R along f

$$\begin{array}{ccc} R & \overset{g}{\twoheadrightarrow} & f(R) \\ {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle t_1}\downarrow\downarrow{\scriptstyle t_2} \\ X & \underset{f}{\twoheadrightarrow} & Y. \end{array}$$

$(f(R), t_1, t_2)$ is reflexive and symmetric being the image of the equivalence relation R along a regular epimorphism f. By assumption, $(f(R), t_1, t_2)$ is an internal category, thus it is an equivalence relation. It follows that $\mathbb{C}$ is a Goursat category (by Theorem 1.4). $\square$

Remark 3.12. Observe that Theorem 3.11 also implies that $\mathrm{Grpd}(\mathbb{C})$ and $\mathrm{Cat}(\mathbb{C})$ are Goursat categories whenever $\mathbb{C}$ is, again thanks to Lemma 3.9, the category $\mathrm{RG}(\mathbb{C})$ obviously being a Goursat category. This simplifies and slightly extends Proposition 4.3 in [14], where the existence of coequalizers in $\mathbb{C}$ was assumed.

Remark 3.13. A result analogous to Theorem 3.11 holds in the Mal'tsev context: a category $\mathbb{C}$ is a Mal'tsev category if and only if $\mathrm{Grpd}(\mathbb{C})$ (or, equivalently, $\mathrm{Cat}(\mathbb{C})$) is closed in $\mathrm{RG}(\mathbb{C})$ under subobjects [3]. Together with the comments made before Proposition 3.7 we observe the existence of a sort of duality between Mal'tsev categories and Goursat categories.

Similar results hold for Mal'tsev categories with respect to monomorphisms and for Goursat categories with respect to regular epimorphisms.

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