Factorizations of Möbius gyrogroups†

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Abstract

In this paper we consider a Möbius gyrogroup on a real Hilbert space (of finite or infinite dimension) and we obtain its factorization by gyro-subgroups and subgroups. It is shown that there is a duality relation between the quotient spaces and the orbits obtained. As an example we will present the factorization of the Möbius gyrogroup of the unit ball in $\mathbb{R}^n$ linked to the proper Lorentz group $\mathrm{Spin}^+(1,n)$.

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1 Introduction

Gyrogroups are group-like structures that first arose in the study of Einstein's velocity addition in the special theory of relativity [21, 22]. They have been studied intensively by A. Ungar since 1998. The first known gyrogroup is the relativistic gyrogroup $(B^3, \oplus_E)$ [21], consisting of the unit ball B^3 of Euclidean 3-space, endowed with Einstein's velocity addition $\oplus_E$. Einstein's addition of relativistically admissible velocities is a binary operation in B^3, where the vacuum speed of light is normalized to $c = 1$. Counter-intuitively, the Einstein velocity addition is neither commutative nor associative. The group structure that has been lost in the transition from the group $(\mathbb{R}^3, +)$ to the groupoid $(B^3, \oplus_E)$ is replaced by a loop structure using a peculiar rotation called Thomas precession. The gyrogroup notion follows by abstraction in which the abstract Thomas precession, called Thomas gyration, plays a central role. Gyrogroups are classified in gyrocommutative and non-gyrocommutative gyrogroups. Gyrogroups, both gyrocommutative and non-gyrocommutative, finite or infinite, abound in the theory of groups, loops, quasigroups, and Lie groups. Gyrocommutative gyrogroups have strong relations with the theory of homogeneous spaces [15, 22]. The relationship between gyrogroups and other structures

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in geometry and in non-associative algebras like K-loops, Bruck loops and Bol loops has been pointed out by many authors [15, 19, 22].

Every gyrogroup is a twisted subgroup of some specified group and some twisted subgroups are gyrogroups [6]. The gyrosemidirect product of a gyrogroup and any one of its gyroautomorphisms groups gives a group [23]. Moreover, the existence of a gyrocommutative gyrogroup in a group is linked to the existence of an involutory automorphism of the group that decomposes it [6, 7].

The factorization of gyrogroups is not a trivial question due to the presence of gyrations. This makes difficult to obtain, in general, a partition of the gyrogroup. In [6] it was proved that if $(G, \oplus)$ is a gyrogroup then it has a normal subgroup N such that G/N is a gyrocommutative gyrogroup (Theorem 4.11).

The best way to introduce the gyrogroup notion is via the proper Lorentz group in $\mathbb{R}^n$. It is a group of pairs parameterized by a velocity in the ball $B_c = \{x \in \mathbb{R}^n : ||x|| \leq c\}$ and an orientation parameter, i.e. $L = \{(v, A) : v \in B_c\,, A \in SO(n)\}$, with group operation given by the gyrosemidirect product

$$(u, A) \times_{\mathrm{gyr}} (v, D) = (u \oplus_E Av, \mathrm{gyr}[u, Av]AD),$$

where $\oplus_E$ is the Einstein velocity addition [22] and $SO(n)$ is the special orthogonal group in $\mathbb{R}^n$. Both Einstein's addition $\oplus_E$ of relativistically admissible velocities and Möbius addition $\oplus_M$ on the ball are important in modern physics and they are gyro-isomorphic to each other [23].

In this paper we are mainly interested in Möbius gyrogroups on a real Hilbert space. Möbius addition in the ball of a Hilbert space H was introduced by Ungar [23] as a generalization of the Möbius addition on the unit disc of the complex plane. We will introduce a Clifford algebra structure on the Hilbert space H and we will take advantage of the new algebraic structure to obtain factorizations of the gyrogroup of the unit ball of H. For some references about Möbius transformations see e.g. [1, 2, 3].

The paper is organized as follows. In Section 2 we will introduce the structure of gyrogroups and some important results concerning the factorization of gyrogroups in general. In Section 3 we define the Clifford algebra structure over an arbitrary finite or infinite dimensional quadratic space. We discuss the Spin groups in finite and infinite dimensional cases. In Section 4 we define the Möbius addition on the s-ball of a Hilbert space H. Using the Clifford algebra structure we will describe gyrations in a geometrical way. Considering a Lorentzian form on the Hilbert space $H \oplus \mathbb{R}$ we will establish a one to one correspondence between the proper Lorentz group in Minkowski space and the set of Möbius transformations acting on the unit ball of H. Section 5 contains the main results of this paper, concerning the factorization of the gyrogroup of the unit ball of H by the gyro-subgroups $(D_\omega, \oplus)$ and the subgroups $(L_\omega, \oplus)$. In Section 6 we establish a duality relation between the orbits of the quotient spaces obtained and in Section 7 we construct the orbits for the case of the Lorentz group $\mathrm{Spin}^+(1, n)$. In [5], one of these homogeneous spaces was used in order to establish a theory of wavelets for the unit sphere in $\mathbb{R}^n$, via the general construction method of coherent states through square-integrable group representations over a homogeneous space of the group.

2 Gyrogroups

Definition 2.1 *([23]) A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms:*

(G1) *There is at least one element 0 satisfying $0 \oplus a = a$, for all $a \in G$ (left identity);*

(G2) *For each $a \in G$ there is an element $\ominus a \in G$ such that $\ominus a \oplus a = 0$ (left inverse of a);*

(G3) *For any $a, b, c \in G$ there exists a unique element $gyr[a, b]c \in G$ such that the binary operation satisfies the left gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c; \tag{2.1}$$

(G4) *The map $gyr[a, b] : G \to G$ given by $c \mapsto gyr[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is $gyr[a, b] \in Aut(G, \oplus)$;*

(G5) *The gyroautomorphism $gyr[a, b]$ possesses the left loop property*

$$gyr[a, b] = gyr[a \oplus b, b]. \tag{2.2}$$

Definition 2.2 *A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation satisfies $a \oplus b = gyr[a, b](b \oplus a)$, for all $a, b \in G$.*

Left and right gyrotranslations are defined by $\tau_a^l(b) = a \oplus b$ and $\tau_a^r(b) = b \oplus a$, respectively. The solution of the equations $\tau_a^l(x) = b$ and $\tau_a^r(x) = b$ in a gyrogroup are given uniquely by $x = \ominus a \oplus b$ and $x = b \ominus gyr[b, a]a$, respectively, [23].

One of the most important results of this theory is that the gyrosemidirect product of a gyrogroup $(G, \oplus)$ with a gyroautomorphism group $H \subset Aut(G, \oplus)$ is a group.

Theorem 2.3 *([23]) Let $(G, \oplus)$ be a gyrogroup, and let $Aut_0(G, \oplus)$ be a gyroautomorphism group of G. Then the gyrosemidirect product $G \times Aut_0(G)$ is a group, with group operation given by the gyrosemidirect product*

$$(x, X)(y, Y) = (x \oplus Xy, gyr[x, Xy]XY). \tag{2.3}$$

This is a generalization of the external semidirect product of groups (c.f. [12]).

Let G be a group possessing the unique decomposition $G = BK$ in the sense that every element can be written uniquely as $g = bk$, where $b \in B$ and $k \in K$. This decomposition is said to be a gyrodecomposition of G if

(i) K is a subgroup of G;

(ii) B is a subset of G such that $kBk^{-1} = B$, for all $k \in K$;

(iii) $1 \in B$ and $B = B^{-1}$.

Let G be a group with a gyrodecomposition $G = BK$. The gyrodecomposition of $g_1 g_2 \in G$ gives the unique decomposition

$$g_1 g_2 = (g_1 \oplus g_2) k(g_1, g_2)$$

where $g_1 \oplus g_2 \in B$ and $k(g_1, g_2) \in K$. Foguel and Ungar [6] proved that $(B, \oplus)$ is a left gyrogroup with gyrations given by

$$\mathrm{gyr}[g_1, g_2]g = k(g_1, g_2)g(k(g_1, g_2))^{-1}.$$

Definition 2.4 *A nonempty subset X of a gyrogroup $(G, \oplus)$ is a subgroup (of a gyrogroup) if it is a group under the restriction of $\oplus$ to X.*

Definition 2.5 *A subgroup X of a gyrogroup G is normal in G if*

(i) *$gyr[a, x] = 1$ for all $x \in X$ and $a \in G$;*

(ii) *$gyr[a, b](X) \subset X$ for all $a, b \in G$;*

(iii) *$a \oplus X = X \oplus a$ for all $a \in G$.*

The factorization of a gyrogroup by a normal subgroup is well understood in the following propositions.

Proposition 2.6 *([6]) If X is a normal subgroup of a gyrogroup G, then G/X is a factor gyrogroup.*

Proposition 2.7 *([6]) If $(G, \oplus)$ is a gyrogroup, then G has a normal subgroup K such that G/K is a gyrocommutative gyrogroup.*

3 Clifford algebras

The traditional definition of Clifford algebra (or geometric algebra) is carried out in the context of vector spaces with an inner product, or more generally a quadratic form. Clifford algebras can be constructed over finite or infinite dimensional quadratic spaces (see e.g. [11, 10, 4, 17]). We assume that (V, Q) is a quadratic space (of finite or infinite dimension) over the field of real or complex numbers, with Q a nondegenerate quadratic form and $B_Q(v, w) = \frac{1}{2}(Q(v+w) - Q(v) - Q(w))$ the bilinear form associated to Q. The Clifford (or geometric) algebra $\mathrm{Cl}(V, Q)$ over the vector space V with quadratic form Q is an algebra with unit e_0 defined as the quotient of the tensor algebra $T(V) = \bigoplus_{k=0}^{\infty} \bigotimes^k V$ by the ideal $I_Q(V) = \{A \otimes (v \otimes v + Q(v)) \otimes B : v \in V, A, B \in T(V)\}$, i.e. $\mathrm{Cl}(V, Q) := T(V)/I_Q(V)$.

There is a canonical injection $i_Q : V \to \mathrm{Cl}(V, Q)$ by which V can be identified to a subspace of $\mathrm{Cl}(V, Q)$. The Z-gradation on the tensor algebra of V induces a Z_2-grading on $\mathrm{Cl}(V, Q)$, and the corresponding decomposition will be denoted by $\mathrm{Cl}(V, Q) = \mathrm{Cl}^+(V, Q) \oplus \mathrm{Cl}^-(V, Q)$. On $\mathrm{Cl}(V, Q)$ there is defined the *principal automorphism* α and the *principal antiautomorphism* β. Both α and β are involutions on $\mathrm{Cl}(V, Q)$. The restriction of α to $\mathrm{Cl}^+(V, Q)$ is the identity, the restriction of α to $\mathrm{Cl}^-(V, Q)$ is minus the identity and the restriction of β to V is the identity. We will denote the bar-map $v \mapsto \overline{v} := \beta(\alpha(v)) = \alpha(\beta(v))$, $v \in \mathrm{Cl}(V, Q)$. Note that the involutions α and β commute.

Let E be a subspace of V and let Q_E be the restriction of Q to E. The injection $E \to V$ induces an injection $\mathrm{Cl}(E, Q_E) \to \mathrm{Cl}(V, Q)$ by which $\mathrm{Cl}(E, Q_E)$ can be identified to a subalgebra of $\mathrm{Cl}(V, Q)$. Let $(e_i)_{i \in I}$ be a basis of V, and suppose I is a totally ordered set. Let $S(I)$ be the set of strictly increasing finite sequences of elements of I. For $s = (i_1, \ldots, i_k) \in S(I)$, let e_s be the product $e_{i_1} \cdots e_{i_k}$ in $\mathrm{Cl}(V, Q)$. Then $(e_s)_{s \in S(I)}$ is a basis for the vector space $\mathrm{Cl}(V, Q)$. The identity e_0 of $\mathrm{Cl}(V, Q)$ corresponds to the empty sequence.

The product in V, called the geometric product, is inherited from the tensor product in $T(V)$ and is defined by $uv = [u \otimes v]$, for all $u, v \in V$. This product is bilinear and associative and it satisfies the identity $uv + vu = -2B_Q(u, v)$. A non-zero vector such that $Q(v) \neq 0$ is invertible in $\mathrm{Cl}(V, Q)$ and the inverse is given by $v^{-1} := \frac{\overline{v}}{Q(v)^2}$. As it was shown by de la Harpe [11] there exists a unique linear form ζ on $\mathrm{Cl}(V, Q)$ such that

(i) $\zeta(e_0) = 1$;

(ii) $\zeta(uv) = \zeta(vu)$, for all $u, v \in \mathrm{Cl}(V, Q)$;

(iii) $\zeta(\alpha(u)) = \zeta(u)$, for all $u \in \mathrm{Cl}(V, Q)$.

From these properties and the definition of the Clifford algebra $\mathrm{Cl}(V, Q)$ it is easy to conclude that $\zeta(uv) = \zeta(u)\zeta(v)$, for any $u, v \in V$. We define a scalar product on $\mathrm{Cl}(V, Q)$ by $\langle u, v \rangle := \zeta(u\overline{v})$ and a respective norm $|u|^2 := \zeta(u\overline{u})$, for $u, v \in \mathrm{Cl}(V, Q)$. At this moment we have not made any assumptions on the dimension of V.

Now we want to describe the orthogonal group of the space V, its subgroups, and the respective double covering groups constructed using the Clifford algebra structure. In what follows we will omit the quadratic form Q and we will write only $\mathrm{Cl}(V)$ instead of $\mathrm{Cl}(V, Q)$.

Let dim $V = n < \infty$ and $e_1, \ldots, e_n$ an orthonormal basis of signature (p,q) with $p + q = n$, that is, $B(e_i, e_i) = 1$, $i = 1, \ldots p$ and $B(e_i, e_i) = -1, i = p+1, \ldots p+q$. A basis for $\mathrm{Cl}(V)$ is given by $\{1, e_{i_1} e_{i_2} \cdots e_{i_k}\}_{1 \leq i_1 < i_2 < i_k \leq n}$, with 1 the identity element in $\mathrm{Cl}(V)$. The decomposition of $\mathrm{Cl}(V)$ by α originates the even subalgebra $\mathrm{Cl}^+(V) = \{x \in \mathrm{Cl}(V) : \alpha(a) = a\}$. Let $\Gamma(p,q)$ denote the Clifford group, that is, the group of all finite products of invertible elements: $\Gamma(p,q) = \left\{\prod_{i=1}^k s_i, s_i \in V, s_i^2 \neq 0, i = 1, \ldots, k, \, k \in \mathbb{N}\right\}$. For $s \in \Gamma(p,q)$ and $x \in V$, the transformation $\chi(s)x := sx\alpha(s)^{-1}$ is an orthogonal transformation of V and is a group homomorphism. This automorphism on V extends to an automorphism on $\mathrm{Cl}(V)$ known as Bogoliubov automorphism [17]. Introducing the spinorial norm $||x|| := x\overline{x}, x \in \mathrm{Cl}(V)$ we arrive at the definition of the Pin and Spin groups given by $\mathrm{Pin}(p,q) = \{s \in \Gamma(p,q) : ||s||^2 = 1\}$ and $\mathrm{Spin}(p,q) = \{s \in \Gamma(p,q) \cap \mathrm{Cl}^+(V) : ||s||^2 = 1\}$, respectively. These are double covering groups of the orthogonal group $O(V)$, and of the special orthogonal group $SO(V)$. The connected component of $\mathrm{Spin}(p,q)$ denoted by $\mathrm{Spin}^+(p,q) = \{s \in \mathrm{Spin}(p,q) : ||s|| = 1\}$ corresponds to the proper Lorentz group and it is a double covering group of $SO_0(V)$.

In the infinite dimensional case there are some difficulties arising in the definition of the Pin and Spin groups since we need some extra topological structure of V (see [11]). In the case of finite dimension the set of all orthogonal matrices breaks into two closed manifolds (those of determinant $+1$ and -1, respectively), which have no point in common. But, in the infinite dimensional case the sets of rotations and reflections of V are not open nor closed in the set of the orthogonal matrices of V with the infinite norm [16]. Therefore, the construction of double covering groups in infinite dimension is a non trivial problem. A successful approach was obtained by de la Harpe in [11], through the general theory of Banach-Lie groups.

Let H be an infinite dimensional real Hilbert space (we can start by considering an infinite dimensional real pre-Hilbert space and then its completion relatively to its norm [11]). Let $\mathrm{Cl}(H)$ denote the Clifford algebra over H and $\mathrm{Cl}_2(H)$ the algebra of bounded elements in $\mathrm{Cl}(H)$. For any $A \in \mathrm{Cl}_2(H)$, let L_A denote the bounded operator on $\mathrm{Cl}(H)$ which acts as left multiplication. Similarly for the right multiplication R_A.

Let $O(H)$ be the group of all orthogonal operators on H. If $U \in O(H)$, then by the universal property of Clifford algebras, U extends to an automorphism $\widetilde{U}$ of $\mathrm{Cl}(H)$, known as Bogoliubov automorphism. The following lemma and corollary can be found in [11].

Lemma 3.1 *Let $U \in O(H)$. Then the following are equivalent*

(*i*) *There exists an invertible $u \in Cl_2(H)$ such that $\widetilde{U} = L_u R_{u^{-1}}$;*

(*ii*) *There exists a unique (up to multiplication by a scalar) invertible $u \in Cl_2(H)$ such that $\widetilde{U} = L_u R_{u^{-1}}$, and one has either $u \in Cl_2^+(H)$ or $u \in Cl_2^-(H)$.*

Corollary 3.2 *Let Spin(H) be the set of all unitary even elements $u \in Cl_2^+(H)$ such that $uHu^{-1} = H$. Then Spin(H) is a subgroup in the (abstract) group of all invertible elements of $Cl_2(H)$.*

For $u \in \mathrm{Spin}(H)$, let $\rho_u : H \to H$ be the orthogonal projector defined by $\rho_u(x) = uxu^{-1}$. Then the image of ρ_u is a normal subgroup of $O(H)$, and its kernel, consisting of e_0 and $-e_0$, is isomorphic to $\mathbb{Z}_2$.

The above corollary indicates that $\mathrm{Spin}(H)$ must support the covering group of some topological subgroup of $O(H)$. The construction of this double covering group is realized through the general theory of Banach-Lie groups via an extension of the Clifford algebra $\mathrm{Cl}_2(H)$ to a C^*-algebra [11, 18].

4 Möbius addition in the ball

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ and corresponding norm $||x|| = \langle x, x \rangle^{1/2}$, for $x \in H$. For any subset $W \subseteq H$ we will denote by $W^\perp$ the orthogonal complement of W in H. Let v_0 be a distinguished unit vector in H and let $W = v_0^\perp$ be the orthogonal complement in H of v_0. Thus, elements of the orthogonal direct sum $H = \mathbb{R}v_0 \oplus W$ will be denoted by $v = x_0 v_0 + u$, where $x_0 = \langle v, v_0 \rangle$ and $u = v - x_0 v_0$. For any closed subspace $E \subseteq H$, let $\mathcal{B}(E)$ denote the set of all bounded linear operators on E and $O(E) = \{A \in \mathcal{B}(E) : A^T A = I\}$ denote the orthogonal group of E, where A^T denotes the dual operator of A defined by $\langle A^T x, y \rangle = \langle x, Ay \rangle$, for any $x, y \in E$. We will identify $O(v_0^\perp)$ with a particular subgroup of $O(H)$, namely $O(v_0^\perp) = \{A \in O(H) : Av_0 = v_0\}$.

Definition 4.1 *[23](**Möbius addition in the ball**)*
Let $(H, +, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with a binary operation $+$ and a positive definite inner product $\langle \cdot, \cdot \rangle$. Let $\mathbb{B}_s = \{a \in H : ||a|| < s\}$ be the s-ball of H, for some $s \geq 0$. Möbius addition $\oplus_s$ in $\mathbb{B}_s$ is a binary operation defined by

$$a \oplus_s b = \frac{(1 + \frac{2}{s^2}\langle a, b \rangle + \frac{1}{s^2}||b||^2)a + (1 - \frac{1}{s^2}||a||^2)b}{1 + \frac{2}{s^2}\langle a, b \rangle + \frac{1}{s^4}||a||^2||b||^2}. \tag{4.1}$$

In the limit of large s ($s \to +\infty$), the ball $\mathbb{B}_s$ expands to the whole of its space H, and Möbius addition reduces to vector addition in H.

Möbius gyrations $\mathrm{gyr}[a, b] : \mathbb{B}_s \to \mathbb{B}_s$ are automorphisms of the Möbius gyrogroup $(B_s, \oplus_s)$ since $\mathrm{gyr}[a, b](c \oplus d) = \mathrm{gyr}[a, b]c \oplus \mathrm{gyr}[a, b]d$. Solving the left gyroassociative law (2.1) we obtain

$$\mathrm{gyr}[a, b]c = \ominus_s(a \oplus_s b) \oplus_s (a \oplus_s (b \oplus_s c)). \tag{4.2}$$

Gyrations preserve the inner product in the Hilbert space H, i.e.

$$\langle \mathrm{gyr}[a, b]c, \mathrm{gyr}[a, b]d \rangle = \langle c, d \rangle ,$$

for all $a, b, c, d \in \mathbb{B}_s$ (see [23]).

Next we introduce a Clifford algebra $\mathrm{Cl}(H)$ over H. The Möbius addition (4.1) on the unit ball $\mathbb{B}_1$ can be written as a Möbius transformation of type

$$a \oplus_1 b = (a + b)(1 - ab)^{-1} := \varphi_a(b), \quad a, b \in \mathbb{B}_1. \tag{4.3}$$

Indeed, as $aba = (\frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba))a = (-\langle a, b \rangle + \frac{1}{2}(ab - ba))a = -\langle a, b \rangle a + \frac{1}{2}aba + \frac{1}{2}||a||^2 b$ we have that $aba = -2\langle a, b \rangle a + ||a||^2 b$ and thus it follows

$$\varphi_a(b) = \frac{(a + b)(1 - ba)}{|1 - ab|^2} = \frac{(1 + ||b||^2 + 2\langle a, b \rangle)a + (1 - ||a||^2)b}{1 + 2\langle a, b \rangle + ||a||^2||b||^2}. \tag{4.4}$$

As pointed out by Friedman [8], introducing the mappings $\Phi : \mathbb{B}_1 \to \mathbb{B}_s$, $\Phi(a) = \frac{2sa}{1+||a||^2}$ and $\Phi^{-1} : \mathbb{B}_s \to \mathbb{B}_1$, $\Phi^{-1}(b) = \frac{sb}{1+\sqrt{1-\frac{||b||^2}{s^2}}}$ we can describe the Möbius addition (4.1) on $\mathbb{B}_s$ by

$$a \oplus_s b = \Phi(\Phi^{-1}(b) \oplus_1 \Phi^{-1}(a)), \quad a, b \in \mathbb{B}_s. \tag{4.5}$$

Therefore, from now on we restrict ourselves to the unit ball $\mathbb{B}_1$ and we denote $\mathbb{S} = \{a \in H : ||a|| = 1\}$ the unit sphere in H. The left gyroassociative law in $\mathbb{B}_1$ is given by

$$a \oplus_1 (b \oplus_1 c) = (a \oplus_1 b) \oplus_1 (qc\overline{q}), \quad \text{with} \quad q = \frac{1-ab}{|1-ab|}, \text{ for all } a,b,c \in \mathbb{B}_1. \tag{4.6}$$

In this compact form gyrations (4.2) are described by

$$\mathrm{gyr}[a,b]c = \frac{1-ab}{|1-ab|}c\frac{\overline{1-ab}}{|1-ab|}, \tag{4.7}$$

for all $a,b,c \in \mathbb{B}_1$. It is easy to see that gyr $\in \mathrm{Aut}(\mathbb{B}_1)$ since $\frac{1-ab}{|1-ab|} = \frac{a}{|a|}\frac{a^{-1}-b}{|a^{-1}-b|}$ is an element of the group $\mathrm{Spin}(H)$.

In the case of Möbius gyrogroups associativity happens in some special situations.

Lemma 4.2 *If $a,b,c \in \mathbb{B}_1$ such that $a//b$ or ($a \perp c$ and $b \perp c$) then the operation $\oplus_1$ is associative, i.e.*

$$a \oplus_1 (b \oplus_1 c) = (a \oplus_1 b) \oplus_1 c.$$

Proof: We have to solve the equation $qc\overline{q} = c$. Computing the left hand side we have:

$$qc\overline{q} = \frac{1-ab}{|1-ab|}c\frac{\overline{1-ab}}{|1-ba|} = \frac{c - abc - cba + abcba}{1 + 2\langle a,b\rangle + ||a||^2||b||^2}. \tag{4.8}$$

As

$$\begin{aligned} abc &= -2\langle a,b\rangle c - bac \\ &= -2\langle a,b\rangle c - (-2\langle a,c\rangle b - bca) \\ &= -2\langle a,b\rangle c - (-2\langle a,c\rangle b - (-2\langle b,c\rangle a - cba)) \\ &= -2\langle a,b\rangle c + 2\langle a,c\rangle b - 2\langle b,c\rangle a - cba \end{aligned}$$

and

$$\begin{aligned} abcba &= a(-2\langle b,c\rangle b + ||b||^2 c)a \\ &= -2\langle b,c\rangle(-2\langle a,b\rangle a + ||a||^2 b) + ||b||^2(-2\langle a,c\rangle a + ||a||^2 c) \\ &= 4\langle a,b\rangle\langle b,c\rangle a - 2\langle b,c\rangle ||a||^2 b - 2\langle a,c\rangle ||b||^2 a + ||a||^2||b||^2 c \end{aligned}$$

we find that

$$\begin{aligned} qc\overline{q} &= \frac{(1 + 2\langle a,b\rangle + ||a||^2||b||^2)c - 2(\langle a,c\rangle + \langle b,c\rangle ||a||^2)b}{1 + 2\langle a,b\rangle + ||a||^2||b||^2} + \\ &\quad \frac{2(\langle b,c\rangle(1 + 2\langle a,b\rangle) - \langle a,c\rangle ||b||^2)a}{1 + 2\langle a,b\rangle + ||a||^2||b||^2}. \end{aligned} \tag{4.9}$$

Thus, $qc\overline{q} = c$ if and only if

$$(\langle a,c\rangle + \langle b,c\rangle ||a||^2)b = (\langle b,c\rangle(1 + 2\langle a,b\rangle) - \langle a,c\rangle ||b||^2)a\,.$$

The last equality can only be satisfied when $a//b$, i.e. $a = t_1\omega$ and $b = t_2\omega$ for some $-1 < t_1, t_2 < 1$ and $\omega \in \mathbb{S}$ or when $c \perp a$ and $c \perp b$.

The gyrogroup $(\mathbb{B}_1, \oplus_1)$ is gyrocommutative since it satisfies the relation

$$b \oplus_1 a = q(a \oplus_1 b)\overline{q}, \;\; \text{with} \;\; q = \frac{1 - ab}{|1 - ab|}. \tag{4.10}$$

It is easy to see that $b \oplus_1 a = a \oplus_1 b$ if and only if $ab = ba$, i.e. if the vectors a and b are colinear.

By definition $\mathrm{Spin}(H)$ is an automorphism group that contains all the gyrations (4.7). From Theorem 2.3, we obtain that $\mathrm{Spin}(H) \times \mathbb{B}_1$ is a group for the gyrosemidirect product given by

$$(s_1, a) \times (s_2, b) = (s_1 s_2 q, b \oplus_1 (\overline{s_2} a s_2)), \;\; \text{with} \;\; q = \frac{1 - \overline{s_2} a s_2 b}{|1 - \overline{s_2} a s_2 b|}. \tag{4.11}$$

This is a generalization of the external semidirect product of groups (see [12]). We need to establish some properties between Möbius transformations and the group $\mathrm{Spin}(H)$.

Lemma 4.3 *For $s \in \mathrm{Spin}(H)$ and $a, b \in \mathbb{B}_1$ we have*

$$(i) \quad \varphi_a(sb\overline{s}) = s\varphi_{\overline{s}as}(b)\overline{s}; \tag{4.12}$$

$$(ii) \quad s\varphi_a(b)\overline{s} = \varphi_{sa\overline{s}}(sb\overline{s}). \tag{4.13}$$

Corollary 4.4 *For $s \in \mathrm{Spin}(H)$ and $a, b \in \mathbb{B}_1$ we have*

$$(i) \quad (sa\overline{s}) \oplus_1 b = s(a \oplus_1 (\overline{s}bs))\overline{s}; \tag{4.14}$$

$$(ii) \quad s(a \oplus_1 b)\overline{s} = (sa\overline{s}) \oplus_1 (sb\overline{s}). \tag{4.15}$$

The relation $s(a \oplus_1 b)\overline{s} = (sa\overline{s}) \oplus_1 (sb\overline{s})$ defines a homomorphism of $\mathrm{Spin}(H)$ onto the gyrogroup $(\mathbb{B}_1, \oplus_1)$. The left and right cancelation laws are given by

$$(-b) \oplus_1 (b \oplus_1 a) = a; \tag{4.16}$$

$$(a \oplus_1 b) \oplus_1 (q(-b)\overline{q}) = a, \tag{4.17}$$

for all $a, b \in \mathbb{B}_1$, and $q = \frac{1-ab}{|1-ab|}$.

Let $v_1, v_2 \in H$ be two unitary and orthogonal vectors. A rotation over an angle θ in the plane defined by the vectors v_1 and v_2 can be defined using the rotor given by $s = e^{\frac{\theta}{2} v_1 v_2} := \cos\left(\frac{\theta}{2}\right) + v_1 v_2 \sin\left(\frac{\theta}{2}\right)$. It is easy to see that for $s \in \mathrm{Spin}(H)$ and for $v = x_1 v_1 + x_2 v_2$ in the v_1v_2-plane we have

$$sv\overline{s} = (x_1 \cos\theta - x_2 \sin\theta)v_1 + (x_1 \sin\theta + x_2 \cos\theta)v_2.$$

Now let $(e_i)_{i \in \mathcal{I}}$ be an orthonormal basis of H. Thus, every element $v \in H$ can be written as $v = \sum_{i \in \mathcal{I}} x_i e_i$ with $x_i = \langle v, e_i \rangle$, $i \in \mathcal{I}$. By the orthogonal projection of a vector onto a subspace we have the following lemma.

Lemma 4.5 *The Hilbert space H admits the decomposition $H = span\{e_1, e_2\} \oplus^{\perp} V$, with $V = (span\{e_1, e_2\})^{\perp}$, that is, for each $0 \neq v \in H$, there exist $r > 0$, $\theta_1 \in [0, 2\pi[$, $\theta_2 \in [0, \pi[$, and $v_3 \in V$ such that $v = r\cos\theta_1 e_1 + r\sin\theta_1 \cos\theta_2 e_2 + r\sin\theta_1 \sin\theta_2 v_3$.*

Corollary 4.6 *Each $0 \neq v \in H$ can be written as $s_* s_1 r e_1 \overline{s_1}\, \overline{s_*}$, with $r = ||v||$, $s_1 = e^{\frac{\theta_1}{2} e_1 e_2}$ and $s_* = e^{\frac{\theta_2}{2} e_2 v_3}$, for some $\theta_1 \in [0, 2\pi[$, and $\theta_2 \in [0, \pi[$.*

Proof: Applying first the rotor s_1 to the vector re_1 we obtain:

$$s_1 r e_1 \overline{s_1} = r \cos\theta_1 e_1 + r \sin\theta_1 e_2.$$

Finally, applying the rotor s_* to the vector $s_1 r e_1 \overline{s_1}$ we obtain:

$$s_* s_1 r e_1 \overline{s_1} s_* = r \cos\theta_1 e_1 + r \sin\theta_1 \cos\theta_2 e_2 + r \sin\theta_1 \sin\theta_2 v_3. \tag{4.18}$$

It is easy to see that the rotor s_* leaves invariant the vector e_1 and therefore $s_* \in \mathrm{Spin}(e_1^\perp)$. Thus, the decomposition (4.18) is a polar decomposition of the group $\mathrm{Spin}(H)$, i.e. $\mathrm{Spin}(H) = \mathrm{Spin}(2)\,\mathrm{Spin}(e_1^\perp)$. We remark that this decomposition can be obtained for an arbitrary direction. For the sake of simplicity we choose the direction e_1. This result together with (4.12) allow us to obtain a polar decomposition of the Möbius transformation $\varphi_a(b)$.

Lemma 4.7 *For $a = s_* s_1 r e_n \overline{s_1}\, \overline{s_*} \in \mathbb{B}_1$ it follows*

$$\varphi_a(b) = s_* s_1 \varphi_{re_1}(\overline{s_*}\, \overline{s_1} b s_1 \overline{s_*}) \overline{s_1}\, \overline{s_*}. \tag{4.19}$$

These Möbius transformations are associated to Lorentz boosts on the Minkowski space constructed from the Hilbert space H. We will make such correspondence in order to identify φ_{re_1} with Lorentz boosts in the e_1-direction. For more details see for example [14]. We consider the vector space direct sum $H \oplus \mathbb{R}$ and we define a Lorentzian form $\langle \cdot, \cdot \rangle_L$ on $H \oplus \mathbb{R}$ by

$$\langle u_1, u_2 \rangle_L := \langle x, y \rangle_H - ts, \qquad \text{where } u_1 = (x, t) \text{ and } u_2 = (y, s).$$

The second component is usually called the time component. The time-reversal operator J is given by

$$J : H \oplus \mathbb{R} \to H \oplus \mathbb{R}, \quad J(x, t) = (x, -t).$$

A bounded linear mapping $A : H \oplus \mathbb{R} \to H \oplus \mathbb{R}$ is said to be a pseudo-orthogonal transformation if it is bijective and $\langle Au_1, Au_2 \rangle_L = \langle u_1, u_2 \rangle_L$, for all $u_1, u_2 \in H \oplus \mathbb{R}$. We note that A^{-1} preserves the Lorentzian form and by the Banach Open Mapping Theorem A^{-1} is also continuous.

The set of all pseudo-orthogonal transformations forms a group with respect to composition, denoted by $O(H, 1)$, and it is sometimes called the general Lorentz group. The topology in $O(H, 1)$ (and its subgroups) is the relative topology arising from the usual operator norm computed from the Hilbert space $H \oplus \mathbb{R}$.

The set $K_0 := \{u \in H \oplus \mathbb{R} : \langle u, u \rangle_L = 0\}$ is a (non-convex) cone in $H \oplus \mathbb{R}$ with vertex at $(0, 0)$ and is called the light cone. It divides $H \oplus \mathbb{R}$ into three open connected regions, the external region where $\langle u, u \rangle_L > 0$, and two internal regions where $\langle (x, t), (x, t) \rangle_L < 0$, with $t > 0$ and where $\langle (x, t), (x, t) \rangle_L < 0$, with $t < 0$. Any pseudo-orthogonal transformation transforms the external region, the light-cone, and the internal region (where $\langle u, u \rangle_L < 0$) into themselves. If each of the two open connected internal regions are carried into themselves, then the pseudo-orthogonal transformation is called a *Lorentz transformation*. We call the subgroup of Lorentz transformations the (homogeneous) Lorentz group, and we will denote it by $O^+(H, 1)$. We have $O(H, 1) = O^+(H, 1) \cup JO^+(H, 1)$ and hence the Lorentz group is normal of index 2 in $O(H,1)$. Let $O_0^+(H,1)$ denote the identity component of $O^+(H,1)$ called the *proper Lorentz group*. The action of the proper Lorentz group decomposes $H \oplus \mathbb{R}$ into disjoint orbits (see [9, 14]).

If $A \in O^+(H,1)$ fixes the vector $(0,1)$ then it must leave invariant its orthogonal complement with respect to $\langle \cdot, \cdot \rangle_L$, the Hilbert space H. Since the Lorentzian form and the inner product agree on H, A must be an orthogonal transformation on H. Conversely, any orthogonal transformation on H extends uniquely to a Lorentz transformation that fixes $(0,1)$. Thus, we identify the orthogonal group $O(H)$ of H with the isotropy subgroup of $O^+(H,1)$ for $(0,1)$. In the finite-dimensional case, it is a standard result that $O(H)$ is a maximal compact subgroup of $O^+(H,1)$.

Proposition 4.8 *[14] For H finite dimensional, the Lorentz group consists of two connected components, the proper Lorentz subgroup $O_0^+(H,1) = SO^+(H,1)$ consisting of Lorentz transformations of determinant 1 and its coset of Lorentz transformation of determinant -1. For H infinite-dimensional, the Lorentz group is connected (and hence) equal to the proper Lorentz group.*

Next we will introduce a Clifford algebra structure on the Minkowski space $(H \oplus \mathbb{R}, \langle \cdot, \cdot \rangle_L)$. Let ϵ be a unit that spans the time axis such that $\epsilon^2 = +1$ and it anticommutes with all the elements of the basis of the Hilbert space, regarded as elements of $\mathrm{Cl}(H)$. A pure boost (or hyperbolic rotation) in the direction $\omega \in \mathbb{S}$ is defined by

$$s_\omega = \cosh\left(\frac{\alpha}{2}\right) + \epsilon\omega \sinh\left(\frac{\alpha}{2}\right)$$

and it acts on space-time vectors $X = x + t\epsilon$, with $x \in H$, $t \in \mathbb{R}$ via the transformation $X \mapsto Y = s_\omega X \overline{s_\omega}$.

Proposition 4.9 *The action of the boost s_ω on $x \in \mathbb{S}$ yields the point*

$$\xi = \frac{x + ((\cosh\alpha - 1)\langle \omega, x\rangle - \sinh\alpha)\omega}{\cosh\alpha - \sinh\alpha \langle \omega, x\rangle}. \tag{4.20}$$

Proof: We extend the point $x \in H$ to the Minkowski space $H \oplus \mathbb{R}$ by considering the point $X = x + \epsilon$ in the intersection of the Null Cone with the hyperplane $T = 1$. Since $\overline{\epsilon\omega} = -\epsilon\omega$, $\epsilon x = -x\epsilon$ and $\epsilon^2 = +1$, we obtain

$$\begin{aligned} s_\omega X \overline{s_\omega} &= \left(\cosh\frac{\alpha}{2} + \epsilon\omega\sinh\frac{\alpha}{2}\right)(x+\epsilon)\left(\cosh\frac{\alpha}{2} - \epsilon\omega\sinh\frac{\alpha}{2}\right) \\ &= \cosh^2\left(\frac{\alpha}{2}\right)x + \cosh\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\alpha}{2}\right)(x\omega + \omega x)\epsilon + \left(\cosh^2\frac{\alpha}{2} + \sinh^2\frac{\alpha}{2}\right)\epsilon \\ &\quad -2\sinh\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\alpha}{2}\right)\omega - \sinh^2\left(\frac{\alpha}{2}\right)\omega x\omega. \end{aligned}$$

As $\omega x \omega = (-\langle \omega, x\rangle + \frac{1}{2}(\omega x - x\omega))\omega = -\langle \omega, x\rangle\omega + \frac{1}{2}\omega x\omega + \frac{1}{2}|\omega|^2 x$ we obtain that $\omega x\omega = -2\langle \omega, x\rangle\omega + x$. Moreover, $x\omega + \omega x = -2\langle \omega, x\rangle$. Therefore,

$$\begin{aligned} Y = s_\omega X \overline{s_\omega} &= x + \left(2\sinh^2\left(\frac{\alpha}{2}\right)\langle \omega, x\rangle - \sinh\alpha\right)\omega + (\cosh\alpha - \sinh\alpha\langle \omega, x\rangle)\epsilon \\ &= x + ((\cosh\alpha - 1)\langle \omega, x\rangle - \sinh\alpha)\omega + (\cosh\alpha - \sinh\alpha\langle \omega, x\rangle)\epsilon. \end{aligned}$$

By homogeneity, i.e. by restricting this point to the hyperplane $T = 1$ we obtain the desired result:

$$\xi = \frac{x + ((\cosh\alpha - 1)\langle \omega, x\rangle - \sinh\alpha)\omega}{\cosh\alpha - \sinh\alpha\langle \omega, x\rangle}.$$

There is an isomorphism between the subgroup of Lorentz boosts on a fixed direction $\omega \in \mathbb{S}$ and the subgroup of Möbius transformations $\varphi_{t\omega}$.

Proposition 4.10 *[3] Let $\omega \in \mathbb{S}$ and $a = t\omega$, with $-1 < t < 1$. Then transformations (4.4) and (4.20) are related by*

$$\cosh\alpha = \frac{1+t^2}{1-t^2} \quad \textit{and} \quad \sinh\alpha = \frac{2t}{1-t^2} \tag{4.21}$$

$$\alpha = \ln\left(\frac{1+t}{1-t}\right) \quad \textit{and} \quad t = \frac{e^\alpha - 1}{e^\alpha + 1} = \tanh\left(\frac{\alpha}{2}\right). \tag{4.22}$$

We will consider the subgroup Spin$(1,1)$ as the subgroup of Lorentz boosts on the e_1-direction.

The elements of Spin(H) that fix e_1 are the elements of Spin$(e_1^\perp)$. Thus, the centralizer C of $A = \text{Spin}(1,1)$, in Spin(H), i.e., $C = \{s \in \text{Spin}(H) : \bar{s}\varphi_{te_1}(x)s = \varphi_{te_1}(\bar{s}xs)\}$ corresponds to the subgroup Spin$(e_1^\perp)$. Therefore, the decomposition (4.19) is not unique. The polar decomposition of the Möbius transformation φ_a induces the Cartan decomposition of the group $(\text{Spin}(n) \times B_1, \times)$.

A gyrogroup contains in general different types of substructures like subgroups or gyro-subgroups. In the case of a Möbius gyrogroup, the substructures of gyro-subgroups and subgroups are of foremost importance.

Definition 4.11 *Let $(G, \oplus)$ be a gyrogroup and K a non-empty subset of G. K is a gyro-subgroup of $(G, \oplus)$ if it is a gyrogroup for the operation induced from G and $gyr[a,b] \in Aut(K)$ for all $a, b \in K$.*

For a fixed $\omega \in \mathbb{S}$, we consider the subsets $L_\omega = \{x \in \mathbb{B}_1 : x = t\omega,\ -1 < t < 1\}$ and $D_\omega = \{x \in \mathbb{B}_1 : \langle x, \omega \rangle = 0\}$. Clearly, $D_\omega = (L_\omega)^\perp$ and $\mathbb{B}_1$ is the direct sum of L_ω and D_ω, i.e. $\mathbb{B}_1 = L_\omega \oplus D_\omega$.

Proposition 4.12 *The sets D_ω and L_ω endowed with the operation $\oplus_1$ are gyro-subgroups of $(\mathbb{B}_1, \oplus_1)$. Moreover, $(L_\omega, \oplus_1)$ is a subgroup.*

Proof: Let a and b be two arbitrary points of D_ω. Then $\langle a, \omega \rangle = 0$ and $\langle b, \omega \rangle = 0$. By (4.4) we have that $\langle a \oplus_1 b, \omega \rangle = 0$. In an analogous way, it is easy to conclude that $\langle b \oplus_1 a, \omega \rangle = 0$. Thus, $a \oplus_1 b \in D_\omega$ and $b \oplus_1 a \in D_\omega$. The identity element 0 belongs to D_ω and for each $a \in D_\omega$ the inverse element $-a$ belongs to D_ω. Finally, by (4.9) it follows that $\text{gyr}[a,b]c = qc\bar{q} \in D_\omega$, for all $a, b, c \in D_\omega$. Thus, $(D_\omega, \oplus_1)$ is a gyro-subgroup of $(\mathbb{B}_1, \oplus_1)$.

For the case of L_ω, let c and d be two arbitrary points of L_ω. Then $c = t_1\omega$ and $d = t_2\omega$, for some $t_1, t_2 \in]-1, 1[$. As

$$c \oplus_1 d = (c+d)(1-dc)^{-1} = \frac{t_1 + t_2}{1 + t_1 t_2}\omega = d \oplus_1 c$$

and $-1 < \frac{t_1+t_2}{1+t_1t_2} < 1$, we obtain that $c \oplus_1 d \in L_\omega$. Moreover, $-c \in L_\omega$ for each $c \in L_\omega$ and $\text{gyr}[a,b]c = qc\bar{q} = c$, for all $a, b, c \in L_\omega$. Thus, $(L_\omega, \oplus_1)$ is a subgroup of $(\mathbb{B}_1, \oplus_1)$.

We remark that the subgroups $(L_\omega, \oplus_1)$ are not normal in $\mathbb{B}_1$, according to the Definition 2.5. Thus, the factorizations presented below do not follow the general theory.

5 Factorizations of the gyrogoup of the unit ball

At a first look it is readily seen that the equivalence relation used in the factorization of a group by a subgroup cannot be applied for the factorization of a gyrogroup by a gyro-subgroup since the operation $\oplus_1$ is neither commutative nor associative. Thus, one possible approach would be to construct first a convenient partition of $\mathbb{B}_1$. The following theorem is the basis of our construction. It gives us a unique decomposition for each point $c \in \mathbb{B}_1$ with respect to the operation $\oplus_1$.

Theorem 5.1 *For each $c \in \mathbb{B}_1$ there exist unique $b, u \in L_\omega$ and $a, v \in D_\omega$ such that $c = a \oplus_1 b$ and $c = u \oplus_1 v$.*

Proof: First we prove the existence of the decomposition $c = a \oplus_1 b$. Let $c \in \mathbb{B}_1$ be arbitrary. Since $\mathbb{B}_1 = L_\omega \oplus D_\omega$ there exist unique $c_1 \in L_\omega$ and $c_2 \in D_\omega$ such that $c = c_1 + c_2$. Let $c_1 = t_1\omega$ and $c_2 = \lambda_1\omega^*$ with $\omega^* \in \mathbb{S}$ such that $\langle \omega^*, \omega \rangle = 0$ and $t_1, \lambda_1 \in]-1, 1[$. If $c_1 = 0$ then it suffices to consider $b = 0$ and $a = c_2$; otherwise, we consider $b = t_2\omega \in L_\omega$ and $a = \lambda_2\omega^* \in D_\omega$ such that

$$c = a \oplus_1 b = \frac{(1 + ||b||^2)a + (1 - ||a||^2)b}{1 + ||a||^2||b||^2}$$

that is,

$$c_1 + c_2 = t_1\omega + \lambda_1\omega^* = \frac{(1 - \lambda_2^2)t_2}{1 + (t_2\lambda_2)^2}\omega + \frac{(1 + t_2^2)\lambda_2}{1 + (t_2\lambda_2)^2}\omega^*. \tag{5.1}$$

We have to find t_2 and λ_2 satisfying (5.1). The system of equations (5.1) has a unique solution given by

$$t_2 = \frac{t_1^2 + \lambda_1^2 - 1 + \sqrt{((1 + \lambda_1)^2 + t_1^2)((1 - \lambda_1)^2 + t_1^2)}}{2t_1} \quad \text{and} \quad \lambda_2 = \frac{\lambda_1}{1 + t_1 t_2}. \tag{5.2}$$

As $||c||^2 = c_1^2 + c_2^2 = \lambda_1^2 + t_1^2 < 1$ we have that $\lambda_1 = r\sin\theta$ and $t_1 = r\cos\theta$, for some $r \in [0, 1[$ and $\theta \in [0, 2\pi[$. From this it is easy to see that $\lambda_2, t_2 \in]-1, 1[$ and thus the existence is proved.

To prove the uniqueness of the decomposition we suppose that there exist $a, d \in D_\omega$ and $b, f \in L_\omega$ such that $c = a \oplus_1 b = d \oplus_1 f$. Then $b = (-a) \oplus_1 (d \oplus_1 f)$, by (4.16). As $a \perp f$ and $d \perp f$ we have $b = ((-a) \oplus_1 d) \oplus_1 f$, by Lemma 4.2. Since by hypothesis $b, f \in L_\omega$ then $(-a) \oplus_1 d$ must be an element of L_ω. This is true if and only if $(-a) \oplus_1 d = 0$. This implies $a = d$ and consequently $b = 0 \oplus_1 f = f$, as we wish to prove.

To prove the decomposition $c = u \oplus_1 v$ we have again two cases: if $c_2 = 0$ then we consider $v = 0$ and $u = c_1$, otherwise we consider $u = t_3\omega \in L_\omega$ and $v = \lambda_3\omega^* \in D_\omega$ such that

$$c = u \oplus_1 v = \frac{(1 + ||v||^2)u + (1 - ||u||^2)v}{1 + ||u||^2||v||^2}$$

that is,

$$c_1 + c_2 = t_1\omega + \lambda_1\omega^* = \frac{(1 + \lambda_3^2)t_3}{1 + (t_3\lambda_3)^2}\omega + \frac{(1 - t_3^2)\lambda_3}{1 + (t_3\lambda_3)^2}\omega^*. \tag{5.3}$$

In this case we have to find t_3 and λ_3 satisfying (5.3). The system of equations (5.3) has an unique solution given by

$$t_3 = \frac{t_1^2 + \lambda_1^2 - 1 + \sqrt{((1 + t_1)^2 + \lambda_1^2)((1 - t_1)^2 + \lambda_1^2)}}{2\lambda_1} \quad \text{and} \quad \lambda_3 = \frac{t_1}{1 + \lambda_1 t_3}. \tag{5.4}$$

The proof of the uniqueness of this decomposition is analogous to the previous one.

5.1 Factorizations of type I

The factorization of the gyrogroup $(\mathbb{B}_1, \oplus_1)$ by a given gyro-subgroup $(D_\omega, \oplus_1)$ will be called a factorization of type I. We will construct left and right cosets arising from convenient partitions of $\mathbb{B}_1$. For each $b \in L_\omega$ left and right equivalence classes will be denoted by S_b^l and S_b^r, respectively.

Proposition 5.2 *The family* $\{S_b^l : b \in L_\omega\}$*, where* $S_b^l = \{b \oplus_1 a : a \in D_\omega\}$*, is a disjoint partition of* $\mathbb{B}_1$.

Proof: We first prove that this family is indeed disjoint. Let $b = t_1\omega \in L_\omega$ and $c = t_2\omega \in L_\omega$ with $t_1 \neq t_2$ and assume that $S_b^l \cap S_c^l \neq \emptyset$. Then there exists $f \in \mathbb{B}_1$ such that $f = b \oplus_1 a$ and $f = c \oplus_1 d$ for some $a, d \in D_\omega$. By (4.16) and (4.6) we have

$$a = (-b) \oplus_1 (c \oplus d) = ((-b) \oplus_1 c) \oplus_1 (qd\overline{q}), \quad \text{with} \quad q = \frac{1 + bc}{|1 + bc|}.$$

As $q = \frac{1+bc}{|1+bc|} = \frac{1-t_1t_2}{|1-t_1t_2|} = 1$, then $a = ((-b) \oplus_1 c) \oplus_1 d$. Since $a, d \in D_\omega$ then $(-b) \oplus_1 c \in D_\omega$. Therefore, $(-b) \oplus_1 c = 0$, i.e. $b = c$. But this contradicts our assumption. Thus, $S_b^l \cap S_c^l = \emptyset$, for $b \neq c$. Finally, by Theorem 5.1 we have that $\cup_{b \in L_\omega} S_b^l = \mathbb{B}_1$.

This partition induces a left equivalence relation $\sim_l$ on $\mathbb{B}_1$:

$$\forall\, c, d \in \mathbb{B}_1, \quad c \sim_l d \;\Leftrightarrow\; \exists\, b \in L_\omega,\ \exists\, a, f \in D_\omega :\ c = b \oplus_1 a \ \text{ and } \ d = b \oplus_1 f, \tag{5.5}$$

which is equivalent to

$$\forall\, c, d \in \mathbb{B}_1, \quad c \sim_l d \;\Leftrightarrow\; \exists\, b \in L_\omega,\ \exists\, a, f \in D_\omega :\ c \oplus_1 (q_1(-a)\overline{q_1}) = d \oplus_1 (q_2(-f)\overline{q_2}), \tag{5.6}$$

with $q_1 = \frac{1-ab}{|1-ab|}$ and $q_2 = \frac{1-fb}{|1-fb|}$.

By Proposition 5.2 we obtain the following isomorphism:

$$\mathbb{B}_1/(D_\omega, \sim_l) \cong L_\omega.$$

We wish to give a characterization of the surfaces S_b^l, with $b \in L_\omega$.

Proposition 5.3 *For each* $b = t\omega \in L_\omega$*,* S_b^l *is the intersection of* $\mathbb{B}_1$ *with the sphere orthogonal to* $\mathbb{S}$*, with center in the point* $C = \frac{1+t^2}{2t}\omega$ *and radius* $\tau = \frac{1-t^2}{2|t|}$.

Proof: Let $b = t\omega \in L_\omega$, $c = \lambda\omega^* \in D_\omega$ and

$$P_\lambda := b \oplus_1 c = \frac{t(1+\lambda^2)}{1+\lambda^2t^2}\omega + \frac{\lambda(1-t^2)}{1+\lambda^2t^2}\omega^*.$$

Let $C_b = \{b \oplus_1 c : -1 < \lambda < 1\}$ be an arc inside $\mathbb{B}_1$ in the $\omega\omega^*$-plane. As each $a \in D_\omega$ can be described as $a = s_* c\overline{s_*}$ with $s_* \in \text{Spin}(\omega^\perp)$ we have by (4.14) that

$$b \oplus_1 (s_* c\overline{s_*}) = s_*((\overline{s_*}bs_*) \oplus_1 c)\overline{s_*} = s_*(b \oplus_1 c)\overline{s_*}.$$

Thus, $S_b^l = \{b \oplus_1 a : a \in D_\omega\}$ is obtained by the action of the group $\text{Spin}(\omega^\perp)$ on the arc C_b. For all $\lambda \in]-1, 1[$, we have that $||P_\lambda - C||^2 = \tau^2$, with $C = \frac{1+t^2}{2t}\omega$ and $\tau = \frac{1-t^2}{2|t|}$. Thus, S_b^l belongs to a sphere centered at C and radius τ. Moreover, as t tends to zero the radius of this sphere tends to infinity thus proving that S_0^l coincides with D_ω.

Each S_b^l is orthogonal to $\mathbb{S}$ because $||C||^2 = 1 + \tau^2$. We recall that two spheres, S_1 and S_2, with centers m_1 and m_2 and radii τ_1 and τ_2, respectively, intersect orthogonally if and only if $\langle m_1 - y, m_2 - y\rangle = 0$, for all $y \in S_1 \cap S_2$, or equivalently, if $||m_1 - m_2||^2 = \tau_1^2 + \tau_2^2$.

Since $(\mathbb{B}_1, \oplus_1)$ is a gyrocommutative gyrogroup we can consider right coset spaces arising from the decomposition of $\mathbb{B}_1$ by the gyro-subgroups D_ω. Analogously as for the left action we obtain the following results for the right action.

Proposition 5.4 *The family* $\{S_b^r : b \in L_\omega\}$, *where* $S_b^r = \{a \oplus_1 b : a \in D_\omega\}$, *is a disjoint partition of* $\mathbb{B}_1$.

From Proposition 5.4 we obtain the isomorphism $\mathbb{B}_1/(D_\omega, \sim_r) \cong L_\omega$. In the next proposition we characterize the equivalence classes S_b^r, with $b \in L_\omega$.

Proposition 5.5 *For each* $b = t\omega \in L_\omega$, S_b^r *is the intersection of* $\mathbb{B}_1$ *with the sphere centered in* $C^r = \frac{t^2-1}{2t}\omega$ *and radius* $\tau = \frac{1+t^2}{2|t|}$.

Let us remark that in Proposition 5.5 the spheres S_b^r are not orthogonal to $\mathbb{S}$ because they do not satisfy the relation $||C^r||^2 = 1 + \tau^2$.

5.2 Factorizations of type II

The factorization of the gyrogroup $(\mathbb{B}_1, \oplus_1)$ by a given gyro-subgroup $(L_\omega, \oplus_1)$ will be called a factorization of type II. We will construct left and right cosets arising from convenient partitions of $\mathbb{B}_1$. Left and right equivalence classes will be denoted by T_b^l and T_b^r, respectively, for each $a \in D_\omega$. Some proofs will be omitted.

Proposition 5.6 *The family* $T^l = \{T_a^l : a \in D_\omega\}$, *with* $T_a^l = \{a \oplus_1 b : b \in L_\omega\}$ *is a disjoint partition of* $\mathbb{B}_1$.

This partition induces the following equivalence relation on $\mathbb{B}_1$:

$$\forall\, c, d \in \mathbb{B}_1, \quad c \sim_l d \;\Leftrightarrow\; \exists\, a \in D_\omega,\ \exists\, b, f \in L_\omega :\ c = a \oplus_1 b \ \text{ and }\ d = a \oplus_1 f. \tag{5.7}$$

Corollary 5.7 *The isomorphism* $\mathbb{B}_1/(L_\omega, \sim_l) \;\cong\; D_\omega$ *holds.*

Proposition 5.8 *For an arbitrary* $a = \lambda\omega^* \in D_\omega$, *the curve* T_a^l *is obtained from the intersection between* $\mathbb{B}_1$ *and the circumference of radius* $\tau = \frac{1-\lambda^2}{2|\lambda|}$ *and center in the point* $C_0^l = \frac{1+\lambda^2}{2\lambda}\omega^*$.

Proof: Let $a = \lambda\omega^* \in D_\omega$, $b = t\omega \in L_\omega$, with $-1 < \lambda, t < 1$, and

$$P_t := a \oplus_1 b = \frac{\lambda(1+t^2)}{1+\lambda^2 t^2}\omega^* + \frac{t(1-\lambda^2)}{1+\lambda^2 t^2}\omega.$$

Let $T_a^l = \{a \oplus_1 b : b \in L_\omega\}$ be a curve inside the unit ball in the $\omega\omega^*$-plane. For all $t \in]-1, 1[$, we have $||P_t - C_0^l||^2 = \tau^2$. Thus, the curve T_a^l lies on the circumference with center in C_0^l and radius τ, in the $\omega\omega^*$-plane. When λ tends to zero, the radius of this circumference tends to infinity, thus proving that the curve T_0^l coincides with L_ω.

Now we will consider the right coset space $\mathbb{B}_1/(L_\omega, \sim_r)$.

The family $T^r = \{T_a^r : a \in D_\omega\}$, where $T_a^r = \{b \oplus a : b \in L_\omega\}$, is again a partition of $\mathbb{B}_1$ and it induces the following equivalence relation on $\mathbb{B}_1$

$$\forall c, d \in \mathbb{B}_1, \quad c \sim_r d \;\Leftrightarrow\; \exists\, a \in D_\omega,\ \exists\, b, f \in L_\omega :\ c = b \oplus_1 a \ \text{ and } \ d = f \oplus_1 a. \tag{5.8}$$

Proposition 5.9 *For an arbitrary $a = \lambda\omega^* \in D_\omega$, T_a^r is the intersection of $\mathbb{B}_1$ with the circumference of center in the point $C_0^r = \frac{\lambda^2-1}{2\lambda}\omega^*$, with radius $\tau = \frac{1+\lambda^2}{2|\lambda|}$, in the $\omega\omega^*$-plane. Moreover, we have that each T_a^r is orthogonal to $\mathbb{S}$.*

The proof is analogous to the proof of Proposition 5.8. To see that each curve T_a^r is orthogonal to $\mathbb{S}$ it suffices to verify that the relation $||C_0^r||^2 = 1 + \tau^2$ holds. We will summarize in Table 1 the left and right orbits constructed by factorizations of types I and II.

Quotient spaces	Orbits $v = \lambda_3\omega^*,\ u = t_3\omega,\ a = \lambda_2\omega^*,\ b = t_2\omega,\ \langle\omega, \omega^*\rangle = 0\,.$
$\mathbb{B}_1/(D_\omega, \sim_l)$	$S_b^l = \{s_*(u \oplus_1 v)\overline{s_*} : v \in D_\omega, s_* \in \text{Spin}(\omega^\perp)\}$ $= \left\{\frac{(1-t_3^2)\lambda_3}{1+(t_3\lambda_3)^2} s_*\omega^*\overline{s_*} + \frac{(1+\lambda_3^2)t_3}{1+(t_3\lambda_3)^2}\omega : -1 < \lambda_3 < 1, s_* \in \text{Spin}(\omega^\perp)\right\}$
$\mathbb{B}_1/(D_\omega, \sim_r)$	$S_b^r = \{s_*(a \oplus_1 b)\overline{s_*} : a \in D_\omega, s_* \in \text{Spin}(\omega^\perp)\}$ $= \left\{\frac{(1+t_2^2)\lambda_2}{1+(t_2\lambda_2)^2} s_*\omega^*\overline{s_*} + \frac{(1-\lambda_2^2)t_2}{1+(t_2\lambda_2)^2}\omega : -1 < \lambda_2 < 1, s_* \in \text{Spin}(\omega^\perp)\right\}$
$\mathbb{B}_1/(L_\omega, \sim_l)$	$T_a^l = \{a \oplus_1 b : b \in L_\omega\} = \left\{\frac{(1+t_2^2)\lambda_2}{1+(t_2\lambda_2)^2}\omega^* + \frac{(1-\lambda_2^2)t_2}{1+(t_2\lambda_2)^2}\omega : -1 < t_2 < 1\right\}$
$\mathbb{B}_1/(L_\omega, \sim_r)$	$T_v^r = \{u \oplus_1 v : u \in L_\omega\} = \left\{\frac{(1-t_3^2)\lambda_3}{1+(t_3\lambda_3)^2}\omega^* + \frac{(1+\lambda_3^2)t_3}{1+(t_3\lambda_3)^2}\omega : -1 < t_3 < 1\right\}$

Table 1: Orbits of factorizations of types I and II

6 Duality relations

For each factorization obtained previously we can define a fiber bundle structure, global and local sections. For instance, for the quotient space $X_1 = \mathbb{B}_1/(D_\omega, \sim_l)$ we can define the projection mapping $\pi : \mathbb{B}_1 \to X_1$, $\pi(a) = [a]$, where $[a]$ is the equivalence class of $a \in \mathbb{B}_1$ on X_1, which coincides with S_b^l, for some $b \in L_\omega$. The 4$-$tuple $(\mathbb{B}_1, X_1, \pi, S_b^l)$ is a fiber bundle. By the bijection $\mathbb{B}_1/(D_\omega, \sim_l) \cong L_\omega$ we can define a second projection $\widetilde{\pi} : \mathbb{B}_1 \to L_\omega$, $\widetilde{\pi}(a) = b$, with $[a] = S_b^l$, for some $b \in L_\omega$. Thus, the fibers generated by π and $\widetilde{\pi}$ coincide. A (global) *section* on X_1 is a mapping $\sigma : X_1 \to \mathbb{B}_1$ such that $\pi(\sigma(x)) = x$, for all $x \in X_1$. In general, bundles may not have globally defined sections and, therefore, we may only define local sections. In our case, we can define both type of sections. For the quotient space $\mathbb{B}_1/(D_\omega, \sim_l)$ we consider L_ω as the fundamental section σ_0. From Proposition 5.2 an entire class of sections $\sigma : \mathbb{B}_1/(D_\omega, \sim_l) \to \mathbb{B}_1$ can be obtained from L_ω by considering

$$\sigma(t\omega) = t\omega \oplus f(t)\omega^* = \frac{t(1+f(t)^2)}{1+(tf(t))^2}\omega + \frac{f(t)(1-t^2)}{1+(tf(t))^2}\omega^* \tag{6.1}$$

where $f :]-1,1[\to]-1,1[$ is the generating function of the section. Depending on the properties of f we can obtain sections that are Borel maps and also smooth sections. If $f \in C^k(]-1,1[), k \in \mathbb{N}$, then the section generates a C^k-curve inside the unit ball. For instance, for $f(t) = \lambda$, for some $\lambda \in]-1,1[$ we obtain the section $\sigma_\lambda(t\omega) = t\omega \oplus f(t)\omega^* = \frac{t(1+\lambda^2)}{1+(t\lambda)^2}\omega + \frac{\lambda(1-t^2)}{1+(t\lambda)^2}\omega^*$ which belongs to the set of orbits of $\mathbb{B}_1/(L_\omega, \sim_r)$.

There is an interesting duality relation between the orbits of the quotient spaces of Table 1.

Theorem 6.1 *The following duality relations hold:*

1. *The orbits of $\mathbb{B}_1/(L_\omega, \sim_r)$ are global sections for the quotient spaces $\mathbb{B}_1/(D_\omega, \sim_r)$ and $\mathbb{B}_1/(D_\omega, \sim_l)$, and vice versa.*
2. *The orbits of $\mathbb{B}_1/(L_\omega, \sim_l)$ are global sections for the quotient space $\mathbb{B}_1/(D_\omega, \sim_r)$ and vice versa.*
3. *The orbits of $\mathbb{B}_1/(L_\omega, \sim_l)$ are local sections for the quotient space $\mathbb{B}_1/(D_\omega, \sim_l)$ and vice versa.*

Proof: To prove the theorem we have to find the intersection points between orbits of different quotient spaces. First we will prove the statement 1. The intersection point of two arbitrary orbits T_u^r and S_b^r is obtained for

$$\lambda_2 = \frac{(1+t_2^2)(1-\lambda_3^2) - \sqrt{((1+\lambda_3^2)(1+t_2^2)+4t_2\lambda_3)((1+t_2^2)(1+\lambda_3^2)-4t_2\lambda_3)}}{2\lambda_3(t_2^2-1)}$$

and

$$t_3 = \frac{\lambda_2 t_2(1-\lambda_3^2)}{\lambda_2(\lambda_3^2 - t_2^2) + \lambda_3(1-t_2^2)}.$$

This can be interpreted in the following way: for each t_2 fixed and $\lambda_3 \in]-1,1[$ we obtain that λ_2 lies on the interval $]-1,1[$, which means that the orbit S_b^r intersects the set of orbits $A = \{T_u^r : u = \lambda_3\omega^*, -1 < \lambda_3 < 1\}$. Since the orbits of $\mathbb{B}_1/(D_\omega, \sim_r)$ are invariant under the action of $\mathrm{Spin}(\omega^\perp)$ and the orbits $(\mathbb{B}_1/(L_\omega, \sim_r))\backslash A$ are obtained from the action of $\mathrm{Spin}(\omega^\perp)$ on the orbits of the set A, we conclude that the orbits of $\mathbb{B}_1/(D_\omega, \sim_r)$ are global sections for the quotient space $\mathbb{B}_1/(L_\omega, \sim_r)$. Analogously, for each λ_3 fixed t_3 runs the interval $]-1,1[$ which allows us to conclude the converse, i.e. the orbits of $\mathbb{B}_1/(L_\omega, \sim_r)$ are global sections for the quotient space $\mathbb{B}_1/(D_\omega, \sim_r)$.

The second part of statement 1 and statement 2 are proved in the same way. In both cases the intersection point of two arbitrary orbits is easily obtained from Table 1.

Finally we prove the statement 3. The intersection of two arbitrary orbits T_a^l and S_b^l is given by

$$t_2 = \frac{(1+t_3^2)(1-\lambda_2^2) - \sqrt{(\lambda_2^2(t_3-1)^2 - (1+t_3)^2)(\lambda_2^2(1+t_3)^2 - (1-t_3)^2)}}{2t_3(1+\lambda_2^2)}$$

and

$$\lambda_3 = \frac{t_2\lambda_2(1+t_3^2)}{t_2(\lambda_2^2 - t_3^2) + t_3(1+\lambda_2^2)}.$$

As for each λ_2 fixed and $t_3 \in]-1,1[$, the parameter t_2 only runs a proper subset of the interval $]-1,1[$ we conclude that the orbits of $\mathbb{B}_1/(L_\omega, \sim_l)$ are local sections for the quotient space $\mathbb{B}_1/(D_\omega, \sim_l)$. The converse statement is also true.

7 The proper Lorentz group $\mathrm{Spin}^+(1,n)$

The full Lorentz group $G = SO(1,n)$ consists of linear homogeneous transformations of the $(n+1)-$dimensional space under which the quadratic form $|x|^2 = |\underline{x}|^2 - x_0^2,\ x = (x_0, \underline{x}),\ \underline{x} = (x_1, \ldots, x_n)$ is invariant. Here we identify x_0 as the time component and the $\underline{x} = (x_1, \ldots, x_n)$ as the spatial component. The group of all Lorentz transformations preserving both orientation and the direction of time is called the proper *orthochronous* Lorentz group and it is denoted by $SO_0(1,n)$. It is generated by spatial rotations of the maximal compact subgroup $K = SO(n)$ and hyperbolic rotations of the subgroup $A = SO(1,1)$, according to the Cartan decomposition KAK of $SO_0(1,n)$ (see [20, 24]).

The group $SO_0(1,n)$ is connected and locally compact. The coset space $X = SO_0(1,n)/K$ is the Lobachevsky space of n dimensions. It can be realized in various manners, e.g. by the upper sheet of the hyperboloid $H^+ = \{x = (x_0, \underline{x}) : x_0^2 - |\underline{x}|^2 = 1,\ x_0 > 0\}$ or by the unit ball $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$.

The double covering group of $SO_0(1,n)$ is the group $\mathrm{Spin}^+(1,n)$. In Clifford Analysis it can be described by Vahlen matrices (see [3, 13]). These matrices can be decomposed into the maximal compact subgroup $\mathrm{Spin}(n)$ and the set of Möbius transformations of the form $\varphi_a(x) = (x-a)(1+ax)^{-1},\ a \in B^n$, which map the closed unit ball $\overline{B^n}$ onto itself.

The unit ball in $\mathbb{R}^n$ endowed with the operation $\oplus_1$ becomes a gyrogroup. The gyrosemidirect product between $(B^n, \oplus_1)$ and $\mathrm{Spin}(n)$ is the group of pairs (s,a) where $a \in B^n$ and $s \in \mathrm{Spin}(n)$, with operation $\times$ given by the gyrosemidirect product

$$(s_1, a) \times (s_2, b) = (s_1 s_2 q, b \oplus_1 (\overline{s_2} a s_2)), \quad \text{with } q = \frac{1 - \overline{s_2} a s_2 b}{|1 - \overline{s_2} a s_2 b|}. \tag{7.1}$$

The group $\mathrm{Spin}^+(1,n)$ admits a Cartan or KAK decomposition, where $K = \mathrm{Spin}(n)$ and $A = \mathrm{Spin}(1,1)$ is the subgroup of Lorentz boosts in a fixed direction. We choose the direction $e_n = (0, \ldots, 0, 1)$.

Lemma 7.1 *Each $a \in \mathbb{B}_1$ can be written as $a = s r e_n \overline{s}$, where $r = |a| \in [0,1[$ and $s = s_1 \cdots s_{n-1} \in \mathrm{Spin}(n-1)$ with*

$$s_i = \cos\frac{\theta_i}{2} + e_{i+1} e_i \sin\frac{\theta_i}{2}, \quad i = 1, \ldots, n-1, \tag{7.2}$$

where $0 \leq \theta_1 < 2\pi,\ 0 \leq \theta_i < \pi,\ i = 2, \ldots, n-1$.

This follows from the description of $a \in \mathbb{B}_1$ in spherical coordinates using the rotors (7.2). These rotors describe rotations in coordinate planes. For $s = \cos\left(\frac{\theta}{2}\right) + \mathrm{e}_i \mathrm{e}_j \sin\left(\frac{\theta}{2}\right),\ i \neq j$ we have

$$s x \overline{s} = (\cos\theta\, x_i - \sin\theta\, x_j)\mathrm{e}_i + (\cos\theta\, x_j + \sin\theta\, x_i)\mathrm{e}_j + \sum_{k \neq i,j} x_k e_k\,,$$

which represents a rotation of angle θ in the $\mathrm{e}_{i+1}\mathrm{e}_i-$plane. In general, we have $s_i s_j \neq s_j s_i,\ i \neq j$. It is easy to see that $s_* = s_1 \ldots s_{n-2} \in \mathrm{Spin}(n-1)$ leaves the x_n-axis invariant.

For each $\omega \in S^{n-1}$ we consider the hyperplane $H_\omega = \{x \in \mathbb{R}^n : \langle \omega, x \rangle = 0\}$, the hyperdisc $D_\omega = H_\omega \cap B^n$ and the segment $L_\omega = \{x \in B^n : x = t\omega,\ -1 < t < 1\}$. The particular group $(L_{e_n}, \oplus_1)$ is isomorphic to the $\mathrm{Spin}(1,1)$ group.

We end this paper showing the orbits of the quotient spaces obtained from the factorizations of the Möbius gyrogroup of the unit ball B^n, considering the direction e_n. We will present a projection of the orbits on the $e_{n-1}e_n$-plane in Table 2.

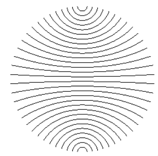
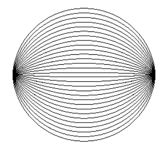
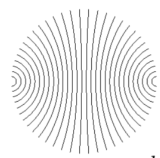
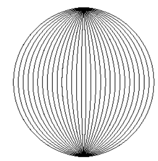
Factorizations	Orbits	
Type I	 Surfaces S_b^l	 Surfaces S_b^r
Type II	 Curves T_a^l	 Curves T_a^r

Table 2: Orbits of factorizations of types I and II for B^n.

The quotient space $B^n/(D_{e_n}, \sim_l)$ was used in [5] for the construction of continuous wavelet transforms on the unit sphere S^{n-1}. It encodes some important information regarding the action of Möbius transformations on the unit sphere.

In our opinion the quotient spaces constructed in this paper play an important role in the comprehension of the action of Möbius transformations.

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