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Research Article

Simple Modules for Modular Lie Superalgebras $W(0 \mid n)$, $S(0 \mid n)$, and $K(n)$

Zhu Wei,[1] Qingcheng Zhang,[1] Yongzheng Zhang,[1] and Chunyue Wang[2]

[1] *School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China*
[2] *School of Applied Sciences, Jilin Teachers Institute of Engineering and Technology, Changchun 130052, China*

Correspondence should be addressed to Qingcheng Zhang; zhangqc569@nenu.edu.cn

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This paper constructs a series of modules from modular Lie superalgebras $W(0 \mid n)$, $S(0 \mid n)$, and $K(n)$ over a field of prime characteristic $p \neq 2$. Cartan subalgebras, maximal vectors of these modular Lie superalgebras, can be solved. With certain properties of the positive root vectors, we obtain that the sufficient conditions of these modules are irreducible L-modules, where $L = W(0 \mid n)$, $S(0 \mid n)$, and $K(n)$.

1. Introduction

Giving a broad overview of the present situation, the representation theories of Lie algebras and Lie superalgebras over a field of characteristic 0 have been a remarkable evolution. Kac (see [1]) worked on classification of infinite dimensional simple linearly compact Lie superalgebras. Shchepochkina (see [2]) studied the five exceptional simple Lie superalgebras of vector fields and their fourteen regradings. More detailed description of one of the five simple exceptional Lie superalgebras of vector fields was given (see [3]). The complete proof of the recognition theorem for graded Lie algebras in prime characteristic was given (see [4]). Strade (see [5]) studied simple Lie algebras over fields of positive characteristic and obtained some important results. The classification of finite dimensional modular Lie superalgebras with indecomposable Cartan matrix was given, and the prolongations of the simple finite dimensional Lie algebras and Lie superalgebras with Cartan matrix are studied over algebraically closed fields of characteristic $p > 2$ (see [6, 7]).

Su (see [8–12]) got a new class of nongraded simple Lie algebras. It was proved that an irreducible quasifinite module was a module of the intermediate series. He also got some important results about the representation of classical Lie superalgebras. Zhang (see [13, 14]) worked on the graded module of W, S, H over fields of characteristic 0.

There are also a great deal of representative results of Lie algebras over fields of prime characteristic (see [15–18]). For a restricted Cartan-type Lie algebra, restricted simple modules have been determined in the sense that their isomorphism classes have been parametrized. And their dimensions have been computed. Shen (see [19–21]) constructed the graded modules for the Witt, special, and Hamiltonian Lie algebras. Shen determined those simple modules with fundamental dominant weights, except the contact algebra. Holmes (see [15]) solved the remaining problem about the contact algebra. He showed that the simple restricted modules were induced from the restricted universal enveloping algebra for the homogeneous component of degree zero extended trivially to positive components. Hu (see [22]) investigated the graded modules for the graded contact Cartan algebras $K(m, n)$ and $K(n)$. Shu (see [23]) worked on the generalized restricted representations of graded Lie algebras of Cartan type.

However, there are few results about the representations of Lie superalgebras over a field of prime characteristic $p \neq 2$, called modular Lie superalgebras. Liu (see [24]) established the dimension formula of induced modules and obtained some properties of induced modules. In this paper, we construct modules from Lie superalgebras $W(0 \mid n)$, $S(0 \mid n)$, and $K(n)$, induced from homogeneous components of their restricted universal enveloping superalgebras. We intend to show that the generator $1 \otimes m$ of these constructed modules belongs to their nonzero submodules.

2. Preliminaries

In this paper, $\mathbb{F}$ always denotes an algebraically closed field of prime characteristic $p \neq 2$.

Recall the definition of the restricted Lie superalgebras (see [6]) and the restricted universal enveloping superalgebra (see [25]).

Let L be a Lie algebra of characteristic $p \neq 2$. Then, for every $x \in L$, the operator ad_x^p is a derivation of L. If it is an inner derivation for every $x \in L$, that is, if $ad_x^p = ad_{x^{[p]}}$ for some element denoted $x^{[p]}$, then the corresponding map $[p] : x \rightarrow x^{[p]}$ is called a p-structure on L, and the Lie algebra L endowed with a p-structure is called a restricted Lie algebra. If L has no center, then L can have not more than one p-structure. The Lie algebra $\mathrm{gl}(n)$ possesses a p-structure, unique up to the contribution of the center; this p-structure is used in the next definition. The notion of a p-representation is naturally defined as a linear map $\rho : L \rightarrow \mathrm{gl}(V)$ such that $\rho(x^{[p]}) = \rho(x)^{[p]}$; in this case V is said to be a p-module. Passing to superalgebras, we see that, for any odd $D \in \mathrm{Der}(A)$, we have

$$D^{2n}([a,b]) = \sum \binom{n}{l} \left[D^{2l}(a), D^{2n-2l}(b)\right] \tag{1}$$

for any $a, b \in A$. So if char $p \neq 2$, then D^{2p} is always an even derivation for any odd $D \in \mathrm{Der}(A)$. Now, let L be a Lie superalgebra of characteristic $p \neq 2$. Then, for every $x \in L_{\bar{0}}$, the operator ad_x^p is a derivation of L; that is, $L_{\bar{0}}$-action on $L_{\bar{1}}$ is a p-representation. For every $x \in L_{\bar{1}}$, the operator $ad_x^{2p} = ad_{x^2}^p$ is a derivation of L. So if, for every $x \in L_{\bar{0}}$, there is $x^{[p]} \in L_{\bar{0}}$ such that $ad_x^p = ad_{x^{[p]}}$ for any $x \in L_{\bar{0}}$, then we can define $x^{[2p]} := (x^2)^{[p]}$ for any $x \in L_{\bar{1}}$. We demand that, for any $x \in L_{\bar{0}}$, we have $ad_x^p = ad_{x^{[p]}}$ as operators on the whole L; that is, $L_{\bar{1}}$ is a restricted $L_{\bar{0}}$-module. Then, the pair of maps

$$\begin{aligned} [p] &: L_{\bar{0}} \longrightarrow L_{\bar{0}} \left(x \longmapsto x^{[p]}\right), \\ [2p] &: L_{\bar{1}} \longrightarrow L_{\bar{0}} \left(x \longmapsto x^{[2p]}\right) \end{aligned} \tag{2}$$

is called a $p \mid 2p$-structure or just p-structure on L, and the Lie superalgebra L endowed with a p-structure is called a restricted Lie superalgebra.

A pair $(u(L), i)$ consisting of an associative $\mathbb{F}$-superalgebra with unity and a restricted homomorphism $i : L \rightarrow u(L)^-$ is called a restricted universal enveloping superalgebra if given any associative $\mathbb{F}$-superalgebra A with unity and any restricted homomorphism $f : L \rightarrow A^-$; there exists a unique homomorphism $\overline{f} : u(L) \rightarrow A$ of associative $\mathbb{F}$-superalgebra such that $\overline{f} \circ i = f$. The category of $u(L)$-modules and that of restricted L-modules are equivalent. According to the PBW theorem, then the following statement holds: let $(L, [p])$ be a restricted Lie superalgebra. If $(u(L), i)$ is a restricted universal enveloping superalgebra and $(l_j)_{j\in J_0} \cup (f_j)_{j \in J_1}$ is an ordered basis of L over $\mathbb{F}$, where $l_j \in L_{\bar{0}}$, $f_j \in L_{\bar{1}}$, then the elements $i(l_{j_1})^{s_1} i(l_{j_2})^{s_2} \cdots i(l_{j_n})^{s_n} i(f_{i_1}) i(f_{i_2}) \cdots i(f_{i_m})$, $j_1 < \cdots < j_n$, $0 \leq s_k \leq p-1$, $1 \leq k \leq n$, and $i_1 < \cdots < i_m$, consist of a basis of $u(L)$ over $\mathbb{F}$. Sometimes, with no confusion, we will identify L with its image $i(L)$ in $u(L)$.

Definition 1. Let L be a $\mathbb{Z}$-graded Lie superalgebra over a field of characteristic p. Suppose that H is a Cartan subalgebra of L_0, where L_0 is the set of the 0th homogenous elements of $\mathbb{Z}$-graded Lie superalgebra L. For $\lambda \in H^*$ and a $u(L_0)$-module V, we set $V_\lambda := \{v \in V \mid h \cdot v = \lambda(h)v, \forall h \in H\}$. If $V_\lambda \neq 0$, then λ is called a weight and a nonzero vector v in V_λ is called a weight vector (of weight λ). A nonzero $v \in V_\lambda$ is called a maximal vector (of weight λ), provided $x \cdot v = 0$, where x is any positive root vector of L_0.

Let $L = \Sigma_{i \in \mathbb{Z}} L_i$ be a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$. Set $N^+ := \sum_{i>0} L_i$, where L_i denotes the homogeneous component of degree i in the $\mathbb{Z}$-graded Lie superalgebra L. Then, $N^+ \triangleleft N^+ + L_0 := L^+$ and $L^+/N^+ \cong L_0$. In particular, any L_0-module becomes a L^+-module by letting N^+ act trivially. Define $M_L(S) := u(L) \otimes_{u(L^+)} S$, where $u(L)$ and $u(L^+)$ denote the restricted universal enveloping superalgebras of L and L^+, respectively, and S is a simple $u(L_0)$-module. According to the classical theory, for each weight λ, there exists a simple $u(L_0)$-module $S(\lambda)$ which is generated by a maximal vector of weight λ.

In the following, L denotes one of three classes of Cartan-type Lie superalgebras $W(0 \mid n)$, $S(0 \mid n)$, or $K(n)$. Each of these classes will be described in detail in the following paper.

Remark 2. In this paper, if A is a subset of some linear space, then $\langle A \rangle$ denotes the subspace spanned by the set A over $\mathbb{F}$.

3. Simple Modules of the Lie Superalgebra $W(0 \mid n)$

We begin by describing the Lie superalgebra $W(0 \mid n)$, drawing most of notations and standard results from [26].

Let $\Lambda(n)$ be an exterior algebra over $\mathbb{F}$ in n variables $x_1, \ldots, x_n$. Fix $n \in \mathbb{N}$, and then $\Lambda(n)$ becomes $\mathbb{Z}_2$-graded if we set $\deg x_i = \bar{1}$, $i = 1, \ldots, n$. Then, $\Lambda(n)$ is an associative superalgebra. The multiplication satisfies the rule $x_i x_j = -x_j x_i$, in particular, $x_i x_i = 0$. For $k = 1, \ldots, n$, put $B_k := \{(i_1, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. Let $B(n) = \bigcup_{i=0}^n B_k$, where $B_0 = \emptyset$. If $u = (i_1, i_2, \ldots, i_r) \in B_r$, where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, then we set $x^u = x_{i_1} x_{i_2} \cdots x_{i_r}$. Put $x^\emptyset = 1$. Note that $\Lambda(n)$ is $\mathbb{Z}$-graded by $\Lambda(n)_r := \langle \{x^u \mid u \in B_r\} \rangle$, where $r = 0, 1, \ldots, n$.

For each $1 \leq i \leq n$, let D_i denote the derivation of $\Lambda(n)$ uniquely determined by the property $D_i(x_j) = (\partial/\partial x_i)(x_j) = \delta_{ij}$ (=Kronecker delta). Let ε_k, $1 \leq k \leq n$, be the n tuple with jth component δ_{jk} (=Kronecker delta), and then $W = W(0 \mid n) := \{\sum_{i=1}^n a_i D_i \mid a_i \in \Lambda(n)\}$ is a Lie superalgebra, which has a $\mathbb{F}$-basis $\{x^u D_i \mid u \in B(n)\}$. The bracket product in W satisfies

$$\begin{aligned} \left[x^u D_i, x^v D_j\right] &= x^u D_i\left(x^v\right) D_j \\ &\quad - (-1)^{\deg(x^u D_i)\deg(x^v D_j)} x^v D_j\left(x^u\right) D_i, \end{aligned} \tag{3}$$

where $x^u D_i, x^v D_j$ are $\mathbb{Z}_2$-homogeneous elements of the Lie superalgebra W, and $\deg(x^u D_i) = \deg(x^u) + 1$, $\deg(x^v D_j) = \deg(x^v) + 1$. Note that $\deg(D_i) = \overline{1} \in \mathbb{Z}_2$. W inherits a $\mathbb{Z}$-gradation from $\Lambda(n)$ by means of $W_k = \Sigma_j \Lambda(n)_{k+1} D_j$. Consequently, $W = \sum_{i=-1}^{n-1} W_i$.

Lemma 3. *Let $H = \sum_{i=1}^n \mathbb{F} x_i D_i$ be a Cartan subalgebra of W_0. Then, positive root vectors of W_0 are $\{x_i D_j \mid 1 \le i < j \le n\}$.*

Proof. Introduce a homomorphism $\varphi : W_0 \to \mathrm{gl}_n(\mathbb{F})$, where $\mathrm{gl}_n(\mathbb{F})$ is the general linear Lie algebra that sends $x_i D_j$ to E_{ij} ($=n \times n$-matrix with 1 in the (i, j)-position and zeros elsewhere). Obviously, φ is an isomorphism of Lie algebras. We know that the Cartan subalgebra H_1 of $\mathrm{gl}_n(\mathbb{F})$ is $\langle\{E_{ii} \mid i = 1, \ldots, n\}\rangle$. Define linear function Λ_j on the vector space $\langle\{E_{11}, \ldots, E_{nn}\}\rangle$, such that $\Lambda_j(E_{ii}) = \delta_{ji}$. Then, the positive roots of $\mathrm{gl}_n(\mathbb{F})$ are $\{\Lambda_i - \Lambda_j\}_{i<j}$. Correspondingly, the positive root vectors are E_{ij}, $1 \le i < j \le n$. By the isomorphism $W_0 \cong \mathrm{gl}_n(\mathbb{F})$ via $x_i D_j \mapsto E_{ij}$, we may therefore obtain that the Cartan subalgebra H of W_0 is $\langle\{x_i D_i \mid i = 1, \ldots, n\}\rangle$. And the positive root vectors are $\{x_i D_j \mid 1 \le i < j \le n\}$. □

According to Definition 1 and Lemma 3, the following statement holds: if H is a Cartan subalgebra, V is a $u(W_0)$-module, and $\lambda \in H^*$, then $V_\lambda = \{v \in V \mid x_i D_i \cdot v = \lambda(x_i D_i) v, 1 \le i \le n\}$. Write $\lambda_i := \lambda(x_i D_i)$. A nonzero $v \in V_\lambda$ is a maximal vector (of weight λ), provided $x_i D_j \cdot v = 0$, $\forall 1 \le i < j \le n$.

Theorem 4. *If there exist $i, j \in \{1, \ldots, n\}$ and $j < i$ such that $\lambda_i \ne 1$ and $\lambda_j \ne 0$, then $M_W(S(\lambda))$ is simple.*

Proof. Let M' be a nonzero submodule of $M_W(S(\lambda)) = u(W) \otimes_{u(W^+)} S(\lambda)$. Now choose $0 \ne v \in M'$. By virtue of $u(W^+) \cdot S(\lambda) = u(W_0) \cdot S(\lambda)$, we obtain $u(W) \cdot S(\lambda) = (u(W_0) + u(W_{-1})) \cdot S(\lambda) \subseteq u(W_{-1}) \cdot S(\lambda)$. Hence, $M_W(S(\lambda)) = u(W) \otimes_{u(W^+)} S(\lambda) = (u(W_{-1}) + u(W_0)) \otimes_{u(W^+)} S(\lambda) \subseteq u(W_{-1}) \otimes_{u(W^+)} S(\lambda)$; namely, $M_W(S(\lambda)) = u(W_{-1}) \otimes_{u(W^+)} S(\lambda)$. Therefore, we can describe the element form of v; that is,

$$v = \Sigma_{\beta \in A} c(\beta)\, i(D_1)^{\beta_1} \cdots i(D_n)^{\beta_n} \otimes s_\beta, \tag{4}$$

where $A := \{\beta = (\beta_1, \ldots, \beta_n) \mid \beta_i = 0 \text{ or } 1, i = 1, \ldots, n\} \subset Z^n$, $c(\beta) \in \mathbb{F}$. Formula (3) shows that $[D_k, D_j] = 0$. Then, we obtain the equality in $u(W)$:

$$i(D_k)\, i(D_j) = -i(D_j)\, i(D_k); \tag{5}$$

in particular,

$$i(D_j)^2 = 0. \tag{6}$$

In the following, we simply write $i(D_j) = D_j$. Define an order of A satisfying that $\beta = (\beta_1, \ldots, \beta_n) < \beta' = (\beta'_1, \ldots, \beta'_n)$ if and only if there exists $k \in \{1, 2, \ldots, n\}$ such that $\beta_i = \beta'_i \forall i > k$ and $\beta_k < \beta'_k$. We set $B := \{\beta \in A \mid c(\beta) \ne 0\}$, where $c(\beta)$ comes from the right side of equality (4). According to the order A, we choose the least element $\eta = (\eta_1, \ldots, \eta_n) \in B$. Obviously, $c(\eta) \ne 0$. Put $y := \prod_{j=1}^n D_j^{1-\eta_j}$. By (5) and (6), we obtain

$$\begin{aligned} yv &= \prod_{j=1}^n D_j^{1-\eta_j} \left[\Sigma_{\beta \in A} c(\beta)\, D_1^{\beta_1} \cdots D_n^{\beta_n} \otimes s_\beta\right] \\ &= \alpha c(\eta) \prod_{j=1}^n D_j \otimes s_\eta \in M', \end{aligned} \tag{7}$$

where $\alpha = 1$ or -1. Consequently, $\prod_{j=1}^n D_j \otimes s_\eta \in M'$.

We will show that $\prod_{j=1}^n D_j \otimes S(\lambda)$ is a $u(W_0)$-module. By virtue of (3), we obtain $[x_k D_l, D_i] = -\delta_{ik} D_l$. Thus, we have

$$(x_k D_l)\, D_i = D_i\, (x_k D_l) - \delta_{ik} D_l, \tag{8}$$

in $u(W)$. We note that the elements $x_k D_l$, $\forall 1 \le k, l \le n$, are $\mathbb{F}$-basis of W_0. Using (8), a straightforward computation shows that

$$\begin{aligned} & x_k D_l \cdot \prod_{j=1}^n D_j \otimes S(\lambda) \\ &= \prod_{j=1}^{k-1} D_j \left(-D_l + D_k\, (x_k D_l)\right) D_{k+1} \cdots D_n \otimes S(\lambda) \\ &= \left[-\prod_{j=1}^{k-1} D_j D_l \prod_{j=k+1}^{n} D_j + \prod_{j=1}^{n} D_j\, (x_k D_l) \right] \otimes S(\lambda) \\ &\subseteq \prod_{j=1}^n D_j \otimes S(\lambda). \end{aligned} \tag{9}$$

Since $S(\lambda)$ is a simple $u(W_0)$-module, we can get that $\prod_{j=1}^n D_j \otimes S(\lambda)$ is a simple $u(W_0)$-module. Since $\prod_{j=1}^n D_j \otimes s_\eta \in M' \cap \prod_{j=1}^n D_j \otimes S(\lambda)$, we can conclude that $M' \cap \prod_{j=1}^n D_j \otimes S(\lambda)$ is nontrivial. $\prod_{j=1}^n D_j \otimes S(\lambda)$ must be contained in M'. Thereby, there exists a maximal vector m of weight λ such that $\prod_{j=1}^n D_j \otimes m \in M'$.

Noting that $i \ne 1$, this yields $(x_1 x_i D_i) \prod_{j=1}^n D_j \otimes m \in M'$. According to (3), we have $[x_1 x_i D_i, D_j] = \delta_{1j} x_i D_i - \delta_{ij} x_1 D_i$. Then,

$$(x_1 x_i D_i)\, D_j = -D_j\, (x_1 x_i D_i) + \delta_{1j} x_i D_i - \delta_{ij} x_1 D_i \tag{10}$$

holds in $u(W)$. By (10), we have

$$\begin{aligned} & (x_1 x_i D_i) \prod_{j=1}^n D_j \otimes m \\ &= \left[-D_1\, (x_1 x_i D_i) + x_i D_i\right] D_2 \cdots D_n \otimes m \\ &= (-1)^{i-1} D_1 \cdots D_{i-1}\, (x_1 x_i D_i)\, D_i \cdots D_n \otimes m \\ &\quad + D_2 \cdots D_{i-1}\, (x_i D_i)\, D_i \cdots D_n \otimes m \\ &= (-1)^n D_1 \cdots D_n\, (x_1 x_i D_i) \otimes m \\ &\quad + (-1)^i D_1 \cdots \widehat{D_i} \cdots D_n\, (x_1 D_i) \otimes m \\ &\quad + D_2 \cdots D_n\, (x_i D_i) \otimes m - D_2 \cdots D_n \otimes m, \end{aligned} \tag{11}$$

where $\widehat{D_i}$ means that D_i is deleted. Since $S(\lambda)$ is a $u(W_0)$-module and m is a maximal vector of weight λ, the first two terms vanish. Formula (11) implies that $(x_1x_iD_i)\prod_{j=1}^{n}D_j\otimes m = (\lambda_i - 1)D_2\cdots D_n\otimes m$. Since $\lambda_i \neq 1$, we have $D_2\cdots D_n\otimes m \in M'$.

If $D_2\cdots D_n\otimes m$ multiplied on the left by the elements $x_2x_iD_i, x_3x_iD_i, \ldots, x_{i-1}x_iD_i$, successively, then it yields $D_i\cdots D_n\otimes m \in M'$. By the hypothesis of the theorem, there exists $1 \leq j < i \leq n$ such that $\lambda_j \neq 0$, and we thus have

$$\begin{aligned}&\left(x_ix_jD_j\right)D_i\cdots D_n\otimes m\\&\quad=\left[-D_i\left(x_ix_jD_j\right)+x_jD_j\right]D_{i+1}\cdots D_n\otimes m.\end{aligned}\tag{12}$$

Observe that the first term vanishes by the $\mathbb{Z}$-graded degree of $x_ix_jD_j$ which is 1, furthermore $x_ix_jD_j\cdot m = 0$, and the second term is equal to $\lambda_jD_{i+1}\cdots D_n\otimes m$. Since $\lambda_j \neq 0$, we have $D_{i+1}\cdots D_n\otimes m \in M'$. If we multiply $D_{i+1}\cdots D_n\otimes m$ by the elements $x_{i+1}x_jD_j, \ldots, x_nx_jD_j$, consecutively, then we see that $1\otimes m \in M'$. Since $u(W_0)$-module $M_W(S(\lambda))$ is generated by $1\otimes m$, $1\otimes m \in M'$ indicates that $M_W(S(\lambda)) = M'$. Hence, $M_W(S(\lambda))$ is simple. □

4. Simple Modules of the Lie Superalgebra $S(0 \mid n)$

In the above section, we give the definition of the exterior algebra $\Lambda(n)$. We begin by describing $S = S(n) := \langle\{D_{ij}(a) \mid a \in \Lambda(n), i, j = 1, \ldots, n\}\rangle$, where

$$D_{ij}(a) = D_i(a)D_j + D_j(a)D_i.\tag{13}$$

Putting $S_i = W_i \cap S$, we have $W_{-1} = S_{-1}$.

Lemma 5. *Let $H = \sum_{i=1}^{n-1}\mathbb{F}D_{i+1,i}(x_{i+1}x_i)$ be a Cartan subalgebra of S_0. Then positive root vectors of S_0 are $\{x_iD_j \mid 1 \leq i < j \leq n\}$.*

Proof. Recall the isomorphism $W_0 \cong \mathrm{gl}_n(\mathbb{F})$ via $x_iD_j \mapsto E_{ij}$, described above. It induces the isomorphism $\psi : S_0 \cong \mathrm{sl}(n,\mathbb{F})$, $D_{ij}(x_kx_l) \mapsto \delta_{ik}E_{lj} - \delta_{il}E_{kj} + \delta_{jk}E_{li} - \delta_{jl}E_{ki}$, where $\mathrm{sl}(n,\mathbb{F})$ is the special linear Lie algebra.

By calculation, we obtain that the Cartan subalgebra H_1 of $\mathrm{sl}(n,\mathbb{F})$ is equal to $\langle\{E_{ii} - E_{i+1,i+1} \mid i = 1, \ldots, n-1\}\rangle$. We define linear function Λ_j on the vector space $\langle\{E_{11}, \ldots, E_{nn}\}\rangle$ as before in Lemma 3. Then, the positive roots of $\mathrm{sl}(n,\mathbb{F})$ are $\{\Lambda_i - \Lambda_j\}_{i<j}$. Correspondingly, the positive root vectors of $\mathrm{sl}(n,\mathbb{F})$ are E_{ij}, $1 \leq i < j \leq n$. The isomorphism ψ sends $-(1/2)D_{jj}(x_ix_j) = x_iD_j$ to E_{ij} and $D_{i+1,i}(x_{i+1}x_i) = x_iD_i - x_{i+1}D_{i+1}$ to $E_{i,i} - E_{i+1,i+1}$, respectively. Therefore, the Cartan subalgebra of S_0 is $\langle\{x_iD_i - x_{i+1}D_{i+1} \mid i = 1, \ldots, n-1\}\rangle$. The positive root vectors are $\{x_iD_j \mid 1 \leq i < j \leq n\}$. □

In terms of Definition 1 and Lemma 5, the following facts hold: if H is a Cartan subalgebra of S_0, V is a $u(S_0)$-module and $\lambda \in H^*$. Then, $V_\lambda = \{v \in V \mid D_{i+1,i}(x_{i+1}x_i)\cdot v = \lambda_iv, 1 \leq i \leq n-1\}$. A nonzero $v \in V_\lambda$ is a maximal vector (of weight λ), provided $x_iD_j\cdot v = 0$, whenever $1 \leq i < j \leq n$.

Theorem 6. *If there exist i, j, such that $|j - i| > 1$ and $\lambda_i \neq 0, \lambda_j \neq 0$, then $M_S(S(\lambda))$ is simple.*

Proof. Let M' be a nonzero submodule of $M_S(S(\lambda))$. Choose $0 \neq v$ contained in M'. It owns the same form as in W; that is, $v = \sum_{\beta\in A}c(\beta)i(D_1)^{\beta_1}\cdots i(D_n)^{\beta_n}\otimes s_\beta$, where $\beta = (\beta_1, \ldots, \beta_n)$ and $c(\beta) \in \mathbb{F}$. The same discussion described above about the Lie superalgebra W applies to the Lie superalgebra S. We can get a maximal vector m such that $\prod_{j=1}^{n}D_j\otimes m \in M'$.

Using (3) and (13), we see that

$$\left[D_k, D_{ij}(a)\right] = -D_{ij}\left(D_k(a)\right).\tag{14}$$

Thus, $D_{ij}(a)D_k = (-1)^{d(a)}[D_kD_{ij}(a) + D_{ij}(D_k(a))]$ in $u(S)$. If $i \neq 1$, we have

$$\begin{aligned}&D_{i,i+1}\left(x_1x_ix_{i+1}\right)\cdot\prod_{j=1}^{n}D_j\otimes m\\
&=-D_1\cdot D_{i,i+1}\left(x_1x_ix_{i+1}\right)\cdot D_2\cdots D_n\otimes m\\
&\quad-D_{i,i+1}\left(x_ix_{i+1}\right)\cdot D_2\cdots D_n\otimes m\\
&=(-1)^{i-1}D_1\cdots D_{i-1}D_{i,i+1}\left(x_1x_ix_{i+1}\right)D_i\cdots D_n\otimes m\\
&\quad-D_2\cdots D_{i-1}D_{i,i+1}\left(x_ix_{i+1}\right)D_i\cdots D_n\otimes m\\
&=(-1)^{i}D_1\cdots D_iD_{i,i+1}\left(x_1x_ix_{i+1}\right)D_{i+1}\cdots D_n\otimes m\\
&\quad+(-1)^{i+1}D_1\cdots D_{i-1}D_{i,i+1}\left(x_1x_{i+1}\right)D_{i+1}\cdots D_n\\
&\quad\otimes m-D_2\cdots D_iD_{i,i+1}\left(x_ix_{i+1}\right)D_{i+1}\cdots D_n\otimes m\\
&\quad-D_2\cdots D_{i-1}D_{i,i+1}\left(x_{i+1}\right)D_{i+1}\cdots D_n\otimes m\\
&=(-1)^{i+1}D_1\cdots D_{i+1}D_{i,i+1}\left(x_1x_ix_{i+1}\right)D_{i+2}\cdots D_n\\
&\quad\otimes m+(-1)^{i+1}D_1\cdots D_iD_{i,i+1}\left(x_1x_i\right)D_{i+2}\cdots D_n\\
&\quad\otimes m+(-1)^{i+2}D_1\cdots D_{i-1}D_{i,i+1}\left(x_1\right)D_{i+2}\cdots D_n\\
&\quad\otimes m-D_2\cdots D_{i+1}D_{i,i+1}\left(x_ix_{i+1}\right)D_{i+2}\cdots D_n\otimes m\\
&\quad+(-1)^{i+1}D_1\cdots\widehat{D_i}D_{i+1}D_{i,i+1}\left(x_1x_{i+1}\right)D_{i+2}\cdots D_n\\
&\quad\otimes m+D_2\cdots D_iD_{i,i+1}\left(x_i\right)D_{i+2}\cdots D_n\otimes m\\
&\quad-D_2\cdots D_n\otimes m.\end{aligned}\tag{15}$$

Since the $\mathbb{Z}$-graded degree of $x_1x_ix_{i+1}$is 1, it implies that the first term vanishes. By the definition of a maximal vector m, it implies that the second and the forth terms vanish. It yields

$$\begin{aligned}&D_{i,i+1}\left(x_1x_ix_{i+1}\right)\cdot\prod_{i=1}^{n}D_i\otimes m\\
&\quad=-D_2\cdots D_nD_{i,i+1}\left(x_ix_{i+1}\right)\otimes m\\
&\quad=\lambda_iD_2\cdots D_n\otimes m.\end{aligned}\tag{16}$$

Since $\lambda_i \neq 0$, we have $D_2\cdots D_n\otimes m \in M'$.

Multiply $D_2\cdots D_n\otimes m$ by $D_{i,i+1}(x_2x_ix_{i+1}), \ldots, D_{i,i+1}(x_{i-1}x_ix_{i+1})$, in turn. By the same calculation as above, it yields $D_i\cdots D_n\otimes m \in M'$, where $1 < i < n$.

If $j < i-1$, then we have

$$\begin{aligned} & D_{j,j+1}\left(x_i x_j x_{j+1}\right) D_i \cdots D_n \otimes m \\ & \quad = (-1)^{n-i+1} D_i D_{i+1} \cdots D_n D_{j,j+1}\left(x_i x_j x_{j+1}\right) \otimes m \\ & \qquad - D_{i+1} \cdots D_n D_{j,j+1}\left(x_j x_{j+1}\right) \otimes m. \end{aligned} \tag{17}$$

It also can be found that the first term vanishes and the second term can conclude $D_{i+1} \cdots D_n \otimes m \in M'$ by the hypothesis of the Theorem $\lambda_j \neq 0$. Multiplying $D_{i+1} \cdots D_n \otimes m$ on the left by $D_{j,j+1}(x_{i+1}x_j x_{j+1}), \ldots, D_{j,j+1}(x_n x_j x_{j+1})$, then $1 \otimes m \in M'$ holds. If $j > i+1$, then we have

$$\begin{aligned} & D_{j,j+1}\left(x_i x_j x_{j+1}\right) D_i \cdots D_n \otimes m = (-1)^{j-i} \\ & \quad \cdot D_i \cdots D_{j-1} D_{j,j+1}\left(x_i x_j x_{j+1}\right) D_j \cdots D_n \otimes m \\ & \quad - D_{j,j+1}\left(x_j x_{j+1}\right) D_{i+1} \cdots D_n \otimes m = (-1)^{j-i+1} \\ & \quad \cdot D_i \cdots D_j D_{j,j+1}\left(x_i x_j x_{j+1}\right) D_{j+1} \cdots D_n \otimes m + (-1)^{j-i} \\ & \quad \cdot D_i \cdots D_{j-1} D_{j,j+1}\left(x_i x_{j+1}\right) D_{j+1} \cdots D_n \otimes m \\ & \quad - D_{i+1} \cdots D_{j-1} D_j D_{j,j+1}\left(x_j x_{j+1}\right) D_{j+1} \cdots D_n \otimes m \\ & \quad - D_{i+1} \cdots D_{j-1} D_{j,j+1}\left(x_{j+1}\right) D_{j+1} \cdots D_n \otimes m \\ & \quad = (-1)^{n-i+1} D_i \cdots D_n D_{j,j+1}\left(x_i x_j x_{j+1}\right) \\ & \quad \otimes m (-1)^{j-i+2} D_i \cdots \widehat{D_{j+1}} \cdots D_n D_{j,j+1}\left(x_i x_j\right) \\ & \quad \cdot (-1)^{j-i+1} D_i \cdots \widehat{D_j} \cdots D_n \left(x_i D_j\right) \otimes m \\ & \quad - D_{i+1} \cdots D_n D_{j,j+1}\left(x_j x_{j+1}\right) \otimes m = \lambda_j D_{i+1} \cdots D_n \\ & \quad \otimes m. \end{aligned} \tag{18}$$

Obviously, we can obtain $D_{i+1} \cdots D_n \otimes m \in M'$.

Moreover, $D_{j,j+1}(x_{i+1}x_j x_{j+1}) D_{i+1} \cdots D_n \otimes m$, for $j \neq i+1$, implies that $D_{i+2} \cdots D_n \otimes m \in M'$. And so on, multiply $D_{i+2} \cdots D_n \otimes m \in M'$ on the left by $D_{i,i+1}(x_{i+2}x_i x_{i+1}), \ldots, D_{i,i+1}(x_n x_i x_{i+1})$, consecutively. Finally, it yields $1 \otimes m \in M'$.

If $i = 1$, we have $D_{j,j+1}(x_1 x_j x_{j+1}) \cdot \prod_{i=1}^{n} D_i \otimes m = \lambda_j D_2 \cdots D_n \otimes m \in M'$; furthermore, $D_2 \cdots D_n \otimes m \in M'$. Imitating the process of calculation, we have $1 \otimes m \in M'$. We get the conclusion. □

5. Simple Modules of the Lie Superalgebra $K(n)$

Given a linear mapping $\widetilde{D_k} : \Lambda(n) \to W(n)$ satisfies

$$\widetilde{D_k}(a) = \sum_{i=1}^{n-1} a_i D_i + a_n x_n D_n, \tag{19}$$

where $a \in \Lambda(n)_\alpha$, $a_i = (-1)^\alpha (x_i x_n D_n(a) + D_i(a))$, and $a_n = 2a - \sum_{i=1}^{n-1} x_i D_i(a)$. We can obtain that $\widetilde{D_k} : \Lambda(n) \to \widetilde{D_k}(\Lambda(n))$ is an isomorphism of linear spaces. By computation, we know that $[\widetilde{D_k}(a), \widetilde{D_k}(b)] = \widetilde{D_k}(\langle a, b\rangle)$, where $\langle a, b\rangle = \widetilde{D_k}(a)(b) - (-1)^{\alpha\beta} 2b x_n D_n(a)$, $a \in \Lambda(n)_\alpha$, $b \in \Lambda(n)_\beta$, α, and $\beta \in \mathbb{Z}_2$. We define a bracket product in $\Lambda(n)$ by means of

$$\begin{aligned} [a, b] = & \left(2a - \sum_{i=1}^{n-1} x_i D_i(a)\right) x_n D_n(b) \\ & - (-1)^{\alpha\beta}\left(2b - \sum_{i=1}^{n-1} x_i D_i(b)\right) x_n D_n(a) \\ & + \sum_{i=1}^{n-1} (-1)^\alpha D_i(a) D_i(b), \end{aligned} \tag{20}$$

where $a \in \Lambda(n)_\alpha$, $b \in \Lambda(n)_\beta$, α, and $\beta \in \mathbb{Z}_2$. Pertaining to this bracket product, $\Lambda(n)$ becomes a Lie superalgebra which is denoted by $K(n)$ (see [27]).

Then, $K(n) = \sum_{j \geq -2} K_j$ is a $\mathbb{Z}$-graded Lie superalgebra, where $K_j := \langle \{x^u \mid j = |u| + \delta(u, n) - 2\}\rangle$, and

$$\delta(u, n) := \begin{cases} 0, & n \notin u, \\ 1, & n \in u. \end{cases} \tag{21}$$

Put

$$j^\nu = \begin{cases} j + q, & 1 \leq j \leq q, \\ j - q, & q + 1 \leq j \leq 2q. \end{cases} \tag{22}$$

Write $K := K(n)$.

Lemma 7. *$K(n)$ is a restricted Lie superalgebra.*

Proof. Since $\widetilde{D_k}$ is an isomorphism of $\Lambda(n) \to \widetilde{D_k}(\Lambda(n))$, we can regard $K(n) = \{\widetilde{D_k}(a) \mid a \in \Lambda(n)\}$. Obviously, $K(n)$ is a subalgebra of $W(0 \mid n)$. For any $a \in \Lambda(n)_{\bar{0}}$, then $\widetilde{D_k}(a) \in K_{\bar{0}} \subseteq W_{\bar{0}}$. Then, the $\mathbb{Z}$-graded degree of a is an even number. If $|x^u| = 2$, namely, $x^u = x_k x_l$, $k \neq l \neq n$, or $x^u = x_k x_n$, $k \neq n$. By direct calculation, $(\widetilde{D_k}(x_k x_l))^p = c\widetilde{D_k}(x_k x_l)$, where $c = 1$ or -1. $(\widetilde{D_k}(x_k x_n))^p = 0$. If $|x^u| = 2t$, $t \in \{2, 3, \ldots\}$, we have $(\widetilde{D_k}(x^u))^p = 0$. In a word, $(\widetilde{D_k}(a))^p \in K_{\bar{0}}$, where $a \in \Lambda(n)_{\bar{0}}$. Since $K(n)$ is a subalgebra of $W(n)$ and $W(n)$ is a restricted Lie superalgebra, we can obtain that $K(n)$ is a superalgebra. □

First we consider the case where $n = 2q + 1$ is an odd number.

Lemma 8. *Let $H = \langle \{\mu x_j x_{j^\nu}, x_n \mid \mu \in \mathbb{F}, j = 1, \ldots, q\}\rangle$ be a Cartan subalgebra of K_0. Then, positive root vectors of K_0 are $\{(1/2)x_j x_i + (\mu/2)x_{j^\nu} x_i + (\mu/2)x_{i^\nu} x_j + (1/2)x_{j^\nu} x_{i^\nu}, x_j x_i + \mu x_{i^\nu} x_j + x_i x_{j^\nu} + x_{i^\nu} x_{j^\nu} \mid 1 \leq i < j \leq q\}$, where $\nu\mu^2 = -1$.*

Proof. Suppose that $\varphi : K_0 \to \mathbf{L} = \{A \in M_{n-1}(\mathbb{F}) \mid A^t + A = 0\} \oplus \mathbb{F}$, given by

$$\begin{aligned} & x_i x_j \longmapsto E_{ji} - E_{ij}, \quad (i \neq j, i, j \neq n), \\ & x_n \longmapsto 1 \in \mathbb{F}, \end{aligned} \tag{23}$$

is a mapping of vector spaces. It can be verified that φ is an isomorphism. Let $P = \begin{bmatrix} I_q & (1/2)I_q \\ -\mu I_q & (\mu/2)I_q \end{bmatrix}$, where $\mu^2 = -1$. Let $M := P^t P = \begin{bmatrix} 0 & I_q \\ I_q & 0 \end{bmatrix}$. Set $\mathbf{L}(P) = \{P^{-1}AP \mid A \in \mathbf{L}\}$. Then, $A \in \mathbf{L}(P) \Leftrightarrow MA + A^t M = 0$; namely, $A \in \mathbf{o}(q)$, the orthogonal algebra. We can conclude $K_0 \cong \mathbf{L} \oplus \mathbb{F} \cong \mathbf{o}(q) \oplus \mathbb{F}$.

By calculation, we obtain that the Cartan subalgebra H_1 of $\mathbf{o}(q)$ is $H_1 = \langle \{E_{jj} - E_{j^\nu j^\nu} \mid j = 1, \ldots, q\} \rangle$.

We will define linear function Λ_j the same as before. Then, the positive roots of $\mathbf{o}(q)$ are $\{\Lambda_i - \Lambda_j\}_{1 \le i < j \le q}$ and $\{\Lambda_i + \Lambda_j\}_{1 \le i < j \le q}$. Correspondingly, the positive root vectors are $E_{ij} - E_{j^\nu i^\nu}, E_{ij^\nu} - E_{ji^\nu}, 1 \le i < j \le q$, respectively. By the isomorphism, we get the positive root vectors of $\mathbf{L}$:

$$\begin{aligned} P\left(E_{ij} - E_{j^\nu i^\nu}\right)P^{-1} &= \frac{1}{2}\left(E_{ij} - E_{ji}\right) + \frac{\mu}{2}\left(E_{ij^\nu} - E_{j^\nu i}\right) \\ &\quad + \frac{\mu}{2}\left(E_{ji^\nu} - E_{i^\nu j}\right) \\ &\quad + \frac{1}{2}\left(E_{i^\nu j^\nu} - E_{j^\nu i^\nu}\right), \\ P\left(E_{ij^\nu} - E_{ji^\nu}\right)P^{-1} &= \left(E_{ij} - E_{ji}\right) + \mu\left(E_{ji^\nu} - E_{i^\nu j}\right) \\ &\quad + \mu\left(E_{j^\nu i} - E_{ij^\nu}\right) \\ &\quad + \left(E_{j^\nu i^\nu} - E_{i^\nu j^\nu}\right), \end{aligned} \tag{24}$$

where $1 \le i < j \le n$. So we can obtain that positive root vectors of K_0 are

$$\begin{gathered} \frac{1}{2}x_j x_i + \frac{\mu}{2}x_{j^\nu} x_i + \frac{\mu}{2}x_{i^\nu} x_j + \frac{1}{2}x_{j^\nu} x_{i^\nu}, \\ x_j x_i + \mu x_{i^\nu} x_j + x_i x_{j^\nu} + x_{i^\nu} x_{j^\nu}, \end{gathered} \tag{25}$$

for $1 \le i < j \le q$. The Cartan subalgebra of K_0 is $\langle \{\mu x_{i^\nu} x_i, x_n \mid 1 \le i \le q\} \rangle$. □

In view of Definition 1 and Lemma 8, the following fact holds: suppose H is a Cartan subalgebra of K_0, V is a $u(K_0)$-module, and $\lambda \in H^*$. Then, $V_\lambda = \{w \in V \mid (x_j x_{j^\nu}) \cdot w = \lambda_j w, x_n \cdot w = \lambda_n w, j = 1, \ldots, q\}$. A nonzero $w \in V_\lambda$ is a maximal vector (of weight λ), provided $(x_i - \mu x_{i^\nu}) x_j \cdot w = 0$, whenever $1 \le i < j \le q$.

Theorem 9. *If $\lambda_n \ne 0$, then $M_K(S(\lambda))$ is simple.*

Proof. Let M' be a nonzero submodule of $M_K(S(\lambda))$. Take $w \in M'$ and $w \ne 0$. We note

$$w = \Sigma_{\beta \in A} c(\beta)\, i(x_1)^{\beta_1} \cdots i(x_{n-1})^{\beta_{n-1}}\, i(1)^{\beta_0} \otimes s_\beta, \tag{26}$$

where $\beta = (\beta_1, \ldots, \beta_{n-1}, \beta_0)$, $c(\beta) \in \mathbb{F}$, and $A := \{a = \Sigma_k a_k \varepsilon_k \mid a_k = 0, \text{ or } 1 \text{ for } 0 \le k \le n-1, 0 \le a_n \le p-1\} \subset \mathbb{Z}^n$. Write $i(x_j) = x_j$, $i(1) = x_0$ in $u(K)$.

By formula (20), we have $[x_i, x_j] = 0$, for $i \ne j$ and $i, j \ne n$. Obviously, $[x_i, x_i] = -1$, for $i \ne n$. And $[1, x_i] = 0$, for $i \ne n$. Thus,

$$x_i x_j = -x_j x_i, \tag{27}$$

in $u(K)$, where $i \ne j$ and $i, j \ne n$. In particular,

$$2x_i^2 = -x_0, \tag{28}$$

where $i \ne n$. And

$$x_0 x_i = x_i x_0, \tag{29}$$

where $i \ne n$.

Put $\alpha_0 = \min\{t_0 \mid w = \Sigma_{\beta \in A} c(\beta) x_1^{t_1} \cdots x_{n-1}^{t_{n-1}} x_0^{t_0} \otimes s_\beta, c(\beta) \ne 0\}$. By (28), we can get $x_0^{p-1-\alpha_0} \cdot w = \Sigma_{\beta' \in A} c(\beta') x_1^{t_1} \cdots x_{n-1}^{t_{n-1}} x_0^{p-1} \otimes s_{\beta'} \in M'$, where $\beta' = (t_1, \ldots, t_{n-1}, \alpha_0)$. Put $\alpha_1 = \min\{t_1 \mid \Sigma_{\beta' \in A} c(\beta') x_1^{t_1} \cdots x_{n-1}^{t_{n-1}} x_0^{p-1} \otimes s_{\beta'}, c(\beta') \ne 0\}$.

Multiplying $\Sigma_{\beta' \in A} c(\beta') x_1^{t_1} \cdots x_{n-1}^{t_{n-1}} x_0^{p-1} \otimes s_{\beta'}$ by $x_1^{1-\alpha_1}$ and then by (27) and (29), we can obtain $\Sigma_{\beta'' \in A} c(\beta'') x_1 \cdots x_{n-1}^{t_{n-1}} x_0^{p-1} \otimes s_{\beta''} \in M'$, where $\beta'' = (\alpha_1, t_2, \ldots, t_{n-1}, \alpha_0)$. And so on, we can conclude that there exists η such that $c(\eta) x_1 x_2 \cdots x_{n-1} x_0^{p-1} \otimes s_\eta \in M'$, where $\eta = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_0)$, $c(\eta) \ne 0$. Imitating the discussion as W, there exists a maximal vector m such that $x_1 x_2 \cdots x_{n-1} x_0^{p-1} \otimes m \in M'$. By (20), we get

$$[x_i, x^u x_n] = x_i x^u x_n, \tag{30}$$

where x_i does not appear in x^u and $x_i \ne x_n$. Also we have

$$[x_i, x^u x_n] = x^{u - \varepsilon_i} x_n, \tag{31}$$

where x_i occurs in x^u. We will prove $x_2 x_3 \cdots x_{n-1} x_0^{p-1} \otimes m \in M'$ in two steps.

First, by (30) and (31), we have

$$\begin{aligned} &(x_1 x_n)\, x_1 x_2 \cdots x_{n-1} x_0^{p-1} \otimes m \\ &\quad = [x_1 (x_1 x_n) - x_n]\, x_2 \cdots x_{n-1} x_0^{p-1} \otimes m \\ &\quad = x_1 x_2 (x_1 x_n)\, x_3 \cdots x_{n-1} x_0^{p-1} \otimes m \\ &\qquad + x_1 (x_1 x_2 x_n)\, x_3 \cdots x_{n-1} x_0^{p-1} \otimes m \\ &\qquad + x_2 x_n x_3 \cdots x_{n-1} x_0^{p-1} \otimes m \\ &\qquad - (x_2 x_n)\, x_3 \cdots x_{n-1} x_0^{p-1} \otimes m. \end{aligned} \tag{32}$$

Observing these terms, it remains the third nonzero term. With $[1, x^u x_n] = 2x^u x_n$, where x_n does not appear in x^u, then

$$x_0 (x^u x_n) - (x^u x_n)\, x_0 = 2x^u x_n, \tag{33}$$

in $u(K)$. Hence, by (30), (31), and (33), then (32) can be adjusted to

$$x_2 x_3 \cdots x_{n-1} x_n x_0^{p-1} \otimes m \in M'. \tag{34}$$

Secondly, for $[1, x_n] = 2x_n$, then $x_n x_0 = (x_0 - 2) x_n$ in $u(K)$. Using the induction hypothesis, we get

$$x_n x_0^{p-1} = (x_0 - 2)^{p-1} x_n. \tag{35}$$

Hence, we get

$$\begin{aligned} & x_2x_3\cdots x_{n-1}x_nx_0^{p-1}\otimes m \\ &\quad = x_2x_3\cdots x_{n-1}\left(x_0-2\right)^{p-1}x_n\otimes m \\ &\quad = x_2x_3\cdots x_{n-1}\left(x_0-2\right)^{p-1}\otimes\lambda_nm\in M'. \end{aligned} \tag{36}$$

Since $\lambda_n\neq 0$, we have $x_2x_3\cdots x_{n-1}(x_0-2)^{p-1}\otimes m\in M'$. Multiplying $x_2x_3\cdots x_{n-1}(x_0-2)^{p-1}\otimes m$ on the left by x_0^{p-1}, thus we find that $x_2\cdots x_{n-1}x_0^{p-1}\otimes m\in M'$.

Imitate the way above on $x_2\cdots x_{n-1}x_0^{p-1}\otimes m$. First, multiply $x_2\cdots x_{n-1}x_0^{p-1}\otimes m$ on the left with x_2x_n. By computation, we can obtain that $x_3\cdots x_nx_0^{p-1}\otimes m\in M'$. Repeating the second step, we can get $x_3\cdots x_{n-1}x_0^{p-1}\otimes m\in M'$.

Repeating the process above, we can get $x_{n-1}x_0^{p-1}\otimes m\in M'$. Multiplying $x_{n-1}x_0^{p-1}\otimes m$ on the left by $x_{n-1}x_n$, we then have

$$\begin{aligned} & \left(x_{n-1}x_n\right)x_{n-1}x_0^{p-1}\otimes m \\ &\quad = \left[x_{n-1}\left(x_{n-1}x_n\right)x_0^{p-1}+x_nx_0^{p-1}\right]\otimes m \\ &\quad = x_nx_0^{p-1}\otimes m = \left(x_0-2\right)^{p-1}x_n\otimes m\in M'. \end{aligned} \tag{37}$$

For $\lambda_n\neq 0$, we know that $(x_0-2)^{p-1}\otimes m\in M'$. Following $x_0^{p-1}(x_0-2)^{p-1}\otimes m\in M'$, it implies that $x_0^{p-1}\otimes m\in M'$. Thus,

$$x_nx_0^{p-1}\otimes m = \left(x_0-2\right)^{p-1}x_n\otimes m\in M'. \tag{38}$$

Moving and expanding these terms of (38), then we have the following identity:

$$\begin{aligned} (-2)^{p-1}\lambda_n\otimes m &= x_nx_0^{p-1}\otimes m \\ &\quad -\left(x_0^{p-1}+\cdots+(-p)\,x_0\,(-2)^{p-2}\right) \\ &\quad \otimes m. \end{aligned} \tag{39}$$

Multiplying

$$\begin{aligned} (-2)^{p-1}\lambda_n\otimes m &= x_nx_0^{p-1}\otimes m \\ &\quad -\left(x_0^{p-1}+\cdots+(-p)\,x_0\,(-2)^{p-2}\right) \\ &\quad \otimes m \end{aligned} \tag{40}$$

on the left by x_0^{p-2}, we obtain that $x_0^{p-2}\otimes m\in M'$. Using the induction hypothesis, finally, we can get the desired formula; namely, $1\otimes m\in M'$. Therefore, $M_K(S(\lambda))$ is simple. □

If $n = 2q+2$ is an even number, then we let $P_1 = [{}^{P}{}_{1}]$. Imitating the proof of Lemma 7 and Theorem 9, we can obtain the following theorem.

Theorem 10. *If $\lambda_n\neq 0$, then $M_K(S(\lambda))$ is simple.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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